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Quantitative profile decomposition and stability for a nonlocal Sobolev inequality

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Abstract

In this paper, we focus on studying the quantitative stability of the nonlocal Sobolev inequality given by

$$S_{HL}\left(\int\limits_{\mathbb{R}^N}\left(|x|^{-\mu}*|u|^{2^*_\mu}\right)|u|^{2^*_\mu}dx\right)^{\frac{1}{2^*_\mu}}\leq\int\limits_{\mathbb{R}^N}|\nabla u|^2dx,\quad\forall\, u\in\mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $*$ denotes the convolution of functions, $2^*_\mu:=\frac{2N-\mu}{N-2}$ and S_{HL} are positive constants that depends solely on N and μ. For $N\geq 3$ and $0<\mu<N$, it is well-known that, up to translation and scaling, the nonlocal Sobolev inequality possesses a unique extremal function $W[\xi,\lambda]$ that is positive and radially symmetric.

Our research consists of three main parts. Firstly, we prove a result that provides quantitative stability of the nonlocal Sobolev inequality with the level of gradients. Secondly, we establish the stability of profile decomposition in relation to the Euler-Lagrange equation of the aforementioned inequality for nonnegative

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functions. Lastly, we investigate the quantitative stability of the nonlocal Sobolev inequality in the following form:

$$\Big\|\nabla u-\sum_{i=1}^{\kappa}\nabla W[\xi_i,\lambda_i]\Big\|_{L^2}\leq C\Big\|\Delta u+\Big(\frac{1}{|x|^{\mu}}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}},$$

where the parameter region satisfies $\kappa\geq 2$, $3\leq N<6-\mu$, $\mu\in(0,N)$ with $0<\mu\leq 4$, or in the case of dimension $N\geq 3$ and $\kappa=1$, $\mu\in(0,N)$ with $0<\mu\leq 4$.

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1. Introduction and main results

The goal of the present paper is to study quantitative profile decomposition and stability related to the nonlocal Soblev inequality, i.e., an improved functional inequality involving the classical Sobolev and Hardy-Littlewood-Sobolev inequalities.

The classical Hardy-Littlewood-Sobolev (HLS) inequality was initially introduced by Hardy and Littlewood [18] in the context of $\mathbb{R}$, and subsequently extended to $\mathbb{R}^N$ by Sobolev [24]. It can be formulated as follows:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)|x-y|^{-\mu}g(y)dxdy\leq C(N,r,t,\mu)\|f\|_{L^r}\|g\|_{L^t} \tag{1.1}$$

with $\mu\in(0,N)$, $1<r,t<\infty$ and $\frac{1}{r}+\frac{1}{t}+\frac{\mu}{N}=2$. Through the process of rearrangement and symmetrization, the problem of optimality is reduced to a class of simple functions, such as radial functions. This strategy was effectively employed by Lieb in [21] to establish the existence of the extremal function for the HLS inequality with its sharp constant. More precisely, in the general diagonal case $t=r=\frac{2N}{2N-\mu}$, the best constant has the following form

$$C(N,r,t,\mu)=C(N,\mu)=\pi^{\mu/2}\frac{\Gamma((N-\mu)/2)}{\Gamma(N-\mu/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{1-\frac{\mu}{N}}, \tag{1.2}$$

where classified the extremal function of HLS inequality is given by

$$f(x)=cg(x)=a\Big(\frac{1}{1+\lambda^2|x-x_0|^2}\Big)^{\frac{2N-\mu}{2}}$$

for some $a\in\mathbb{C}$, $\lambda\in\mathbb{R}\backslash\{0\}$ and $x_0\in\mathbb{R}^N$.

Remark 1.1. Inequality (1.1) is equivalent to the following Riesz potential estimate for all $f\in L^r(\mathbb{R}^N)$,

$$\left\| f * |x|^{-\mu} \right\|_{L^t} \leq C(N, r, t, \mu) \| f \|_{L^r}, \quad \text{with} \quad \frac{1}{r} + \frac{\mu}{N} = 1 + \frac{1}{t}. \tag{1.3}$$

Consequently, a natural question arises as to whether a remainder term can be incorporated into the left-hand side of inequalities (1.1). This question was answered in the case $t = r = \frac{2N}{2N-\mu}$ by Carlen in [6]: there exists some constant $K_{HL,\mu}$ dependent only on N and μ such that for any $u \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$, there holds

$$C(N, \mu) \| f \|^2_{L^{\frac{2N}{2N-\mu}}} - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x) f(y)}{|x-y|^\mu} dy dx \geq K_{HL,\mu} \inf_{g \in \mathcal{M}_{HL,\mu}} \| f - g \|^2_{L^{\frac{2N}{2N-\mu}}},$$

where $\mathcal{M}_{HL,\mu}$ denotes the manifold of all optimal functions,

$$\mathcal{M}_{HL,\mu} := \left\{ v(x) = c \left(\frac{1}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{2N-\mu}{2}}, \ \lambda > 0, \ x_0 \in \mathbb{R}^N, \ c \in \mathbb{R} \right\}.$$

We remark that $K_{HL,\mu}$ of Carlen's result or even its lower bound is still unknown. Recently, Chen et al. in [7] made significant progress on the quantitative stability of Hardy-Littlewood-Sobolev inequality by giving an explicit lower bound of $K_{HL,\mu}$, which are important progress in the quantitative stability of geometric inequality.

Using differentiation under integral sign, for any N-dimensional unit vector ω, one has

$$f(x) = -\int_0^\infty \frac{\partial}{\partial r} f(x + \omega r) dr, \quad \text{for any} \ f \in \mathcal{C}^1_c(\mathbb{R}^N).$$

Integrating on the unit sphere $\mathbb{S}^{N-1}$ yields

$$f(x) = \frac{1}{\omega_{N-1}} \int_{\mathbb{R}^N} \frac{(x-y) \cdot \nabla f(y)}{|x-y|^N} dy,$$

and so

$$|f(x)| = \frac{1}{\omega_{N-1}} \int_{\mathbb{R}^N} \frac{|\nabla f(y)|}{|x-y|^{N-1}} dy, \quad i.e., \quad |f(x)| \leq C(N) \left| |x|^{1-N} * |\nabla f| \right|.$$

As a result, the HLS inequality (1.3) tells us that the classical Sobolev inequality holds

$$\| f \|_{L^{\frac{Np}{N-p}}} \leq \mathcal{S}(p, N) \| \nabla f \|_{L^p} \quad \text{for all} \ 1 < p < N, \quad \text{and} \quad p^* = Np/(N-p) \tag{1.4}$$

for every function f from the homogeneous Sobolev space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ of functions $f \in L^{p^*}(\mathbb{R}^N)$ such that $\nabla f \in L^p(\mathbb{R}^N)$. The classical Sobolev inequality with exponent 2 states that, for any $N \geq 3$, there exists a dimensional constant $S = S(N) > 0$ such that

$$\| \nabla u \|^2_{L^2} \geq S \| u \|^2_{L^{2^*}}, \quad \forall \, u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{1.5}$$

It is well known that the Euler-Lagrange equation associated to (1.5) is given by

$$\Delta u + |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

The best constant S in the Sobolev inequality is achieved by the Aubin-Talenti bubbles [26] $u = U[\xi, \lambda](x)$ defined by

$$U[\xi,\lambda](x) = [N(N-2)]^{\frac{N-2}{4}}\left(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\right)^{\frac{N-2}{2}}, \quad \lambda \in \mathbb{R}^+, \quad \xi \in \mathbb{R}^N.$$

In fact, Caffarelli et al. [5] and Gidas et al. [17] proved that all the positive solutions to equation (1.6) are the Aubin-Talenti bubbles. In other words, the smooth manifold of extremal function in (1.5)

$$\mathcal{M} := \left\{cU[\xi,\lambda] : \ c \in \mathbb{R}, \ \xi \in \mathbb{R}^N, \ \lambda > 0\right\}$$

consists of all nonnegative solutions to equation (1.6). Later on, in [4], Brezis and Lieb raised the question of stability for Sobolev inequality, whether a remainder term proportional to the quadratic distance between a function u and the manifold $\mathcal{M}$ can be added to the right hand side of (1.5). This question was first answered by Bianchi and Egnell [3] (see also [1]), that is,

$$\inf_{c\in\mathbb{R},\ \xi\in\mathbb{R}^N,\ \lambda\in\mathbb{R}^+}\left\|\nabla\big(u - cU[\xi,\lambda]\big)\right\|_{L^2}^2 \le C(N)\Big(\|\nabla u\|_{L^2}^2 - S^2\|u\|_{L^{2^*}}^2\Big).$$

A natural and more challenging problem is to consider the stability question for Euler-Lagrange equation (1.6). Roughly speaking, this amounts to determining whether a function u that almost solves (1.6) must be quantitatively close to Aubin-Talenti bubbles. In fact, a seminal work of Struwe [25] proved the well-known stability of profile decompositions of (1.6), that is u sufficiently close to a sum of weakly-interacting bubbles even if we restrict to nonnegative functions. Starting with Struwe [25], an alternative approach has been developed to study the stability of critical functional inequalities in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Several notable contributions have been made in this area, including works by Aryan [2], Ciraolo et al. [9], Figalli and Glaudo [14], Wei and Wu [27], and Deng, Sun and Wei [10]. It is important to highlight the pioneering work of Ciraolo et al. [9], who obtained the first quantitative stability estimate for equation (1.6) in the case of a single bubble and $N \ge 3$. Subsequently, Figalli and Glaudo [14] established the stability of the Sobolev inequality in a critical point setting for any finite number of bubbles. However, they also constructed counterexamples that demonstrate that when $N \ge 6$ and there are $\kappa \ge 2$ bubbles, the $\mathcal{D}^{1,2}(\mathbb{R}^N)$-distance between u and the manifold composed of sums of Aubin-Talenti bubbles can be significantly larger than the right side of equation (1.6), that is,

$$\inf_{\substack{(z_1,\cdots,z_\kappa)\in\mathbb{R}^N,\\ \lambda_1,\cdots,\lambda_\kappa>0}}\left\|\nabla\Big(u - \sum_{i=1}^{\kappa} W[\xi_i,\lambda_i]\Big)\right\|_{L^2}^2 \gg \left\|\Delta u + |u|^{2^*-2}u\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}.$$

Furthermore, in their work, they put forth conjectures regarding the case of higher dimensions ($N \ge 6$). Notably, Deng et al. provided a significant advancement in this direction by proving sharp quantitative estimates of Struwe's decomposition for the Sobolev inequality (1.5) in their recent publication [10]. Their approach utilizes the finite-dimensional reduction method, leading to a comprehensive solution for the remaining dimension $N \geq 6$.

On the other hand, the classical Sobolev inequality and HLS inequality are dual inequalities. There exists $\mathcal{S}(N,s) > 0$ such that the fractional Sobolev inequality can be stated as:

$$\left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^2}^2 \geq \mathcal{S}(N,s)\|f\|^2_{L^{2N/(N-2s)}} \quad \text{for all} \quad f \in \dot{H}^s(\mathbb{R}^N). \tag{1.7}$$

Inequality (1.7) holds for all function f in the *homogeneous Sobolev space* $\dot{H}^s(\mathbb{R}^N)$ consisting of tempered distributions whose Fourier transform $\hat{f}$ satisfies:

$$\hat{f} \in L^1_{loc}(\mathbb{R}^N) \quad \text{and} \quad \left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^2}^2 = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{f}|^2 < \infty.$$

In particular, by duality it is straightforward to check that inequality (1.1) corresponding to Sobolev inequality (1.5) is given by, for $\mu = N-2$ and $r = t$,

$$\int_{\mathbb{R}^N} f(-\Delta)^{-1} f \leq \frac{1}{\pi N(N-2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{\frac{2}{N}} \|f\|^2_{L^{2N/(N+2)}}. \tag{1.8}$$

It is interesting to study the question of stability for the Sobolev inequality (1.7), or (1.8). Encouragingly, Chen, Frank, and Weth addressed this query affirmatively in their work [8], specifically regarding the stability of inequality (1.7). They established the following stability result:

$$\|(-\Delta)^{\frac{s}{2}} f\|^2_{L^2} - \mathcal{S}(N,s)\|f\|^2_{L^t} \geq K_{N,S} \inf_{g \in \mathcal{M}_{N,S}} \|(-\Delta)^{\frac{s}{2}}(f-g)\|^2_{L^2} \quad \text{for all } s \in (0, N/2), \tag{1.9}$$

where $\mathcal{M}_{N,S}$ represents the manifold of all optimal functions. This manifold is generated from $v(x) = (1+x^2)^{(2s-N)/2}$ through operations such as multiplication by a constant, translations, and scalings. Furthermore, explicit stability results for the HLS inequality are obtained as direct consequences of the duality approach. These results are presented in [12,19] and are based on flow methods.

Let $N \geq 3$ and $\mu \in (0, N)$. We define the Coulomb space $\mathcal{Q}^{\mu,q}(\mathbb{R}^N)$ as the vector space of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_{\mathcal{Q}^{\mu,q}} = \left(\int_{\mathbb{R}^N} \left(|x|^{-\mu} * u^q\right) u^q \, dx\right)^{\frac{1}{2q}} < \infty.$$

It is easy to see that for every measurable function $u \in \mathcal{Q}^{\mu,q}(\mathbb{R}^N)$ if and only if $|u|^q \in \mathcal{Q}^{\mu,1}(\mathbb{R}^N)$. By the HLS inequality we find

$$\int_{\mathbb{R}^N} \left(|x|^{-\mu} * u^q\right) u^q \leq C\left(\int_{\mathbb{R}^N} |u|^{\frac{2Nq}{2N-\mu}}\right)^{\frac{2N-\mu}{N}}$$

and thus $L^{\frac{2Nq}{2N-\mu}}(\mathbb{R}^N) \subset Q^{\mu,q}(\mathbb{R}^N)$. It is not difficult to see that the Coulomb space $Q^{\mu,q}(\mathbb{R}^N)$ is a Banach space under the norm $\|\cdot\|_{Q^{\mu,q}}$. See [22]. The HLS inequality tells us that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)|^q |x|^{-\mu} |v(y)|^q dy dx$$

is well-defined for $u \in L^{qt}(\mathbb{R}^N)$ satisfying $\frac{2}{t} + \frac{\mu}{N} = 2$. Hence, for $u \in H^1(\mathbb{R}^N)$, the continuous Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ for every $\gamma \in \left[2, \frac{2N}{N-2}\right]$ implies that $(2N-\mu)/N \leq q \leq (2N-\mu)/(N-2)$. Hence we may call $\frac{2N-\mu}{N}$ the low critical exponent and $\frac{2N-\mu}{N-2} =: 2_\mu^*$ the upper critical exponent due to the HLS inequality. Consider the upper critical case, we are concerned with the following nonlocal Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq S_{HL} \left(\int_{\mathbb{R}^N} (|x|^{-\mu} * |u|^{2_\mu^*}) |u|^{2_\mu^*} dx \right)^{\frac{1}{2_\mu^*}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.10}$$

for some positive constant S_{HL} depending only on N and μ, where $N \geq 3$, $0 < \mu < N$. It is well-know that the optimal constant in (1.10) is given by

$$S_{HL} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} (|x|^{-\mu} * |u|^{2_\mu^*}) |u|^{2_\mu^*} dx\right)^{\frac{1}{2_\mu^*}}}.$$

What's more, the equality is achieved in (1.10) if and only if by

$$\begin{aligned} W[\xi,\lambda](x) = {} & S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} [C(N,\mu)]^{\frac{2-N}{2(N-\mu+2)}} [N(N-2)]^{\frac{N-2}{4}} \\ & \times \left(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\right)^{\frac{N-2}{2}}, \ \lambda \in \mathbb{R}^+, \ \xi \in \mathbb{R}^N, \end{aligned} \tag{1.11}$$

which satisfies the Euler-Lagrange equation of (1.10)

$$\Delta u + (|x|^{-\mu} * |u|^{2_\mu^*}) |u|^{2_\mu^*-2} u = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.12}$$

See [13,16] for details.

To the best of our knowledge, there are limited findings regarding the stability of the nonlocal Sobolev inequality, making it a compelling subject for investigation. Recently, Deng et al. [11] made an important contribution by presenting the first result on the gradient-type remainder term of inequality (1.10), which can be expressed as follows:

$$B_2 \mathrm{dist}(u, \widetilde{\mathcal{M}})^2 \geq \int_{\mathbb{R}^N} |\nabla u|^2 dx - S_{HL} \left(\int_{\mathbb{R}^N} \left(|x|^{-\mu} * |u|^{2_\mu^*}\right) |u|^{2_\mu^*} dx \right)^{\frac{1}{2_\mu^*}} \geq B_1 \mathrm{dist}(u, \widetilde{\mathcal{M}})^2,$$

where

$$\widetilde{\mathcal{M}}=\left\{cW[\xi,\lambda]:\ c\in\mathbb{R},\ \xi\in\mathbb{R}^N,\ \lambda>0\right\}$$

is an $(N+2)$-dimensional manifold, and $\operatorname{dist}(u,\widetilde{\mathcal{M}}):=\inf_{c\in\mathbb{R},\lambda>0,z\in\mathbb{R}^N}\|u-cW[\xi,\lambda]\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$.

We first establish the following quantitative estimate for (1.10) at the level of gradients.

Theorem 1.2. *Fix $N\geq 3$ and $\mu\in(0,N)$. There exist constants $K,L>0$, depending only on N and μ, such that for all $u\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ and for any $v\in\widetilde{\mathcal{M}}$ with $\|u\|_{Q^{\mu,2^*_\mu}}=\|v\|_{Q^{\mu,2^*_\mu}}$,*

$$\|\nabla u-\nabla v\|_{L^2}^2\leq K(N,\mu)\widetilde{\delta}(u)+L(N,\mu)\,\|u\|_{Q^{\mu,2^*_\mu}}\,\|u-v\|_{Q^{\mu,2^*_\mu}}\,.$$

Here $\widetilde{\delta}(u):=\int_{\mathbb{R}^N}|\nabla u|^2dx-S_{HL}\left(\int_{\mathbb{R}^N}(|x|^{-\mu}*|u|^{2^*_\mu})|u|^{2^*_\mu}dx\right)^{\frac{1}{2^*_\mu}}$ and

$$S_{HL}=S\left[\frac{\Gamma((N-\mu)/2)\pi^{\mu/2}}{\Gamma(N-\mu/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{1-\frac{\mu}{N}}\right]^{(2-N)/(2N-\mu)}.$$

Inspired by the spirit of Struwe in [25], a nonlocal version of the stability of profile decompositions to (1.12) for nonnegative functions can be formulated as follows:

Theorem 1.3. *Let $N\geq 3$ and $\kappa\geq 1$ be positive integers. Let $(u_m)_{m\in\mathbb{N}}\subseteq\mathcal{D}^{1,2}(\mathbb{R}^N)$ be a sequence of nonnegative functions such that*

$$(\kappa-\frac{1}{2})S_{HL}^{\frac{2N-\mu}{N+2-\mu}}\leq\left\|u_m\right\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\leq(\kappa+\frac{1}{2})S_{HL}^{\frac{2N-\mu}{N+2-\mu}}$$

with $S_{HL}=S_{HL}(N,\mu)$ as in (1.10), and assume that

$$\left\|\Delta u_m+\left(\frac{1}{|x|^\mu}*|u_m|^{2^*_\mu}\right)|u_m|^{2^*_\mu-2}u_m\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}\to 0\quad as\quad m\to\infty.$$

Then, there exist κ-tuples of points $(\xi_1^{(m)},\cdots,\xi_\kappa^{(m)})_{m\in\mathbb{N}}$ in $\mathbb{R}^N$ and κ-tuples of positive real numbers $(\lambda_1^{(m)},\cdots,\lambda_\kappa^{(m)})_{m\in\mathbb{N}}$ such that

$$\left\|\nabla\Big(u_m-\sum_{i=1}^{\kappa}W[\xi_i^{(m)},\lambda_i^{(m)}]\Big)\right\|_{L^2}\to 0\quad as\quad m\to\infty.$$

Remark 1.4. Note that,

$$\left\|\Delta u_m+\left(\frac{1}{|x|^\mu}*|u_m|^{2^*_\mu}\right)|u_m|^{2^*_\mu-2}u_m\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}\to 0\quad\text{as}\quad m\to\infty,$$

is equivalent to saying that u_m is Palais-Smale sequence for the function E_0 corresponding to (1.12)

$$E_0(u;\mathbb{R}^N) := \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2\cdot 2_\mu^*}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{u^{2_\mu^*}(x)u^{2_\mu^*}(y)}{|x-y|^\mu}, \qquad u \in H_0^1(\mathbb{R}^N).$$

Studying the stability of the nonlocal Sobolev inequality through the Euler-Lagrange equation (1.12) poses a significantly greater challenge. In this paper, our primary objective is to investigate the stability of the nonlocal Sobolev equality and make an initial attempt to estimate the stability of the nonlocal Sobolev inequality using the Euler-Lagrange equation (1.12) in the case $3 \le N < 6-\mu$.

To facilitate our discussion and understanding of the results, it is essential to introduce the concept of the interaction of Talenti bubbles.

Definition 1.5 *([10,14]).* Let $W[\xi_i,\lambda_i]$ and $W[\xi_j,\lambda_j]$ be two bubbles. Define the interaction of them by

$$Q(\xi_i,\xi_j,\lambda_i,\lambda_j) = \min\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|\xi_i-\xi_j|^2\right)^{-\frac{N-2}{2}}.$$

Let $(W[\xi_i,\lambda_i])_{1\le i\le\kappa}$ be a family of Talenti bubbles. We say that the family is δ-interacting if

$$\max\left\{Q(\xi_i,\xi_j,\lambda_i,\lambda_j):\ i,j=1,\cdots,\kappa\right\} < \delta. \tag{1.13}$$

Our main result is in the following:

Theorem 1.6. *For any dimension $3 \le N < 6-\mu$ and $\kappa \ge 2$, $\mu \in (0,N)$ satisfying $0<\mu\le 4$, there exist a small constant $\delta=\delta(N,\kappa)>0$ and a large constant $C=C(N,\kappa)>0$ such that the following statement holds. Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a function such that*

$$\left\|\nabla u - \sum_{i=1}^{\kappa}\nabla \widetilde{W}[\xi_i,\lambda_i]\right\|_{L^2} \le \delta,$$

where $\left(\widetilde{W}[\xi_i,\lambda_i]\right)_{1\le i\le\kappa}$ is a δ-interacting family of Talenti bubbles. Then there exist κ Talenti bubbles $(W[\xi_1,\lambda_1],\cdots,W[\xi_\kappa,\lambda_\kappa])$ such that

$$dist_{\mathcal{D}^{1,2}}\big(u,\widetilde{\mathcal{M}}_0^\kappa\big) \le C\left\|\Delta u + \left(\frac{1}{|x|^\mu} * |u|^{2_\mu^*}\right)|u|^{2_\mu^*-2}u\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}},$$

where

$$\widetilde{\mathcal{M}}_0^\kappa = \left\{\sum_{i=1}^{\kappa} W[\xi_i,\lambda_i]:\ \xi_i\in\mathbb{R}^N,\ \lambda_i>0\right\}.$$

As a direct consequence of Theorem 1.6, we derive the following two corollaries.

Corollary 1.7. *For dimensions $3 \le N < 6-\mu$, $\mu \in (0,N)$ satisfying $0<\mu\le 4$, and $\kappa \in \mathbb{N}$, there exists a positive constant $C=C(N,\kappa)$ such that the following statement holds. For any nonnegative function $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfying*

$$\big(\kappa - \frac{1}{2}\big)S_{HL}^{\frac{2N-\mu}{N+2-\mu}} \le \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \le \big(\kappa + \frac{1}{2}\big)S_{HL}^{\frac{2N-\mu}{N+2-\mu}},$$

there exist κ Talenti bubbles $(W[\xi_1,\lambda_1],\cdots,W[\xi_\kappa,\lambda_\kappa])$ such that

$$dist_{\mathcal{D}^{1,2}}\big(u,\widetilde{\mathcal{M}}_0^\kappa\big) \le C\left\| \Delta u + \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|u|^{2^*_\mu-2}u \right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}.$$

Moreover, for any $i \neq j$, the interaction between the bubbles can be estimated as

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * W^{2^*_\mu}[\xi_i,\lambda_i]\Big)W^{2^*_\mu-1}[\xi_i,\lambda_i]W[\xi_j,\lambda_j]$$
$$\le C\left\| \Delta u + \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|u|^{2^*_\mu-2}u \right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}.$$

Corollary 1.8. *For dimensions $N \ge 3$ and $\kappa = 1$, $\mu \in (0,N)$ satisfying $0 < \mu \le 4$, there exists a large constant $C = C(N,\kappa)$ such that the following statement holds. Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a nonnegative function satisfying*

$$\frac{1}{2}S_{HL}^{\frac{2N-\mu}{N+2-\mu}} \le \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \le \frac{3}{2}S_{HL}^{\frac{2N-\mu}{N+2-\mu}}. \tag{1.14}$$

Then there exists a Talenti bubble $W[\xi,\lambda]$ such that

$$dist_{\mathcal{D}^{1,2}}\big(u,\widetilde{\mathcal{M}}_0^1\big) \le C\left\| \Delta u + \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|u|^{2^*_\mu-2}u \right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}, \tag{1.15}$$

where $\widetilde{\mathcal{M}}_0^1 = \big\{W[\xi,\lambda] : \xi \in \mathbb{R}^N, \lambda > 0\big\}$.

Here we note that in the case that $N < 6-\mu$, the method in [10] is ineffective in proving Theorem 1.6 due to the appearance of nonlocal interaction parts. Later, we will analyze the stability of the solution using a finite dimensional reduction method in [10] for every dimension $N \ge 6-\mu$. To prove the main results in Theorem 1.6 and Corollaries 1.7–1.8, the arguments rely heavily on the nondegeneracy property of positive solutions of equation (1.12), which can be summarized as follows:

Proposition 1.9. *Assume $N \ge 3$, $0 < \mu < N$ with $0 < \mu \le 4$. Let $W[\xi,\lambda]$ be as in (1.11). Then the linearized operator of equation (1.12) at $W[\xi,\lambda]$ defined by*

$$L[\phi] := -\Delta\phi - 2^*_\mu\left(|x|^{-\mu} * (W^{2^*_\mu-1}[\xi,\lambda]\phi)\right)W^{2^*_\mu-1}[\xi,\lambda] - (2^*_\mu - 1)$$
$$\times\left(|x|^{-\mu} * W^{2^*_\mu}[\xi,\lambda]\right)W^{2^*_\mu-2}[\xi,\lambda]\phi$$

only admits solutions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ of the form

$$\phi = aD_\lambda W[\xi,\lambda] + \mathbf{b}\cdot\nabla W[\xi,\lambda],$$

where $a\in\mathbb{R}$, $\mathbf{b}\in\mathbb{R}^N$.

The topic of non-degeneracy of the ground state solution to the nonlinear Hartree equation has generated considerable interest in recent years. It is a key ingredient in the stability analysis of functional inequalities and the Lyapunov-Schmidt reduction method for constructing blow-up solutions of the equation. Du and Yang [13] established the nondegeneracy of $W[\xi,\lambda]$ defined in (1.11) for μ close to N with $N=3$ or 4. They proved that solutions of the linearized equation of (1.12) for $x\in\mathbb{R}^N$ and $\phi\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ (linear combinations of functions $\frac{N-2}{2}W[\xi,\lambda]+x\cdot\nabla W[\xi,\lambda]$ and $\partial_{x_j}W[\xi,\lambda]$, $j=1,\ldots,N$) satisfy the nondegeneracy condition. However, investigating the expansion of the nonlocal term by spherical harmonics becomes challenging when the working space is $\mathcal{D}^{1,2}(\mathbb{R}^N)$ for the critical case. Therefore, alternative approaches are required to analyze the nondegeneracy property for the critical problem. In this context, Gao et al. [15] demonstrated the nondegeneracy result at $W[\xi,\lambda]$ for (1.12) when $N=6$ and $\mu=4$, but the remaining range of N and μ is still an open problem. Recently, Li et al. [20] provided a comprehensive description of the nondegeneracy property by using the spherical harmonic decomposition and the Funk-Hecke formula of the spherical harmonic functions.

The paper is organized as follows. In Section 2, we prove Theorem 1.2. We start by establishing the stability of profile decompositions and then complete the proof of Theorem 1.3 in Section 4. In Section 4, we provide a detailed proof of Theorem 1.6 and Corollaries 1.7-1.8. The proof of Theorem 1.6 relies on several crucial propositions, which we present and prove in Section 5.

Notations. In this paper, we use the symbols C and $\widetilde{C}$ to represent different positive constants. The specific values of these constants may vary from one occurrence to another, but they are always chosen to be positive. We use the notation $a\lesssim b$ to indicate that a is less than or equal to Cb, where C is one of these positive constants. Similarly, $a\approx b$ means that $a\lesssim b$ and $a\gtrsim b$, implying that a is bounded both above and below by a multiple of b.

2. Quantitative estimate for nonlocal Sobolev inequality

In this section, we prove Theorem 1.2 using a version of Clarkson's inequality for vector-valued functions [23], which we recall here:

Proposition 2.1. *Let* $X,Y:\mathbb{R}^N\to\mathbb{R}^N$ *with* $|X|,|Y|\in L^2(\mathbb{R}^N)$. *Then*

$$\left\|\frac{X+Y}{2}\right\|_{L^2}^2+\left\|\frac{X-Y}{2}\right\|_{L^2}^2\le\frac{1}{2}\|X\|_{L^2}^2+\frac{1}{2}\|Y\|_{L^2}^2.$$

The proof of Theorem 1.2. Applying the Clarkson's inequality in Proposition with $X=\nabla u$ and $Y=\nabla v$, we have

$$\left\|\frac{\nabla u-\nabla v}{2}\right\|_{L^2}^2\le\frac{1}{2}\|\nabla u\|_{L^2}^2+\frac{1}{2}\|\nabla v\|_{L^2}^2-\left\|\frac{\nabla u+\nabla v}{2}\right\|_{L^2}^2.$$

Combining u, v satisfy the nonlocal Sobolev inequality (1.10), we find

$$\|\nabla v\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2$$

and

$$\left\|\nabla u + \nabla v\right\|_{L^2}^2 \ge S_{HL}\left(\int_{\mathbb{R}^N} \left(|x|^{-\mu} * |u+v|^{2_\mu^*}\right)|u+v|^{2_\mu^*}\right)^{\frac{1}{2_\mu^*}}.$$

Hence we obtain

$$\left\|\frac{\nabla u - \nabla v}{2}\right\|_{L^2}^2 \le \left\|\nabla u\right\|_{L^2}^2 - S_{HL}\left(\int_{\mathbb{R}^N} \left(|x|^{-\mu} * \left|\frac{u+v}{2}\right|^{2_\mu^*}\right)\left|\frac{u+v}{2}\right|^{2_\mu^*}\right)^{\frac{1}{2_\mu^*}}.$$

By the semigroup property of the Riesz potential, we omit the coefficients of for convenience and we find that

$$\int_{\mathbb{R}^N} \left(|x|^{-\mu} * \left|\frac{u+v}{2}\right|^{2_\mu^*}\right)\left|\frac{u+v}{2}\right|^{2_\mu^*} = \int_{\mathbb{R}^N} \left(|x|^{-\frac{N+\mu}{2}} * \left|\frac{u+v}{2}\right|^{2_\mu^*}\right)^2$$

and

$$\int_{\mathbb{R}^N} \left(|x|^{-\mu} * \left|\frac{u-v}{2}\right|^{2_\mu^*}\right)\left|\frac{u-v}{2}\right|^{2_\mu^*} = \int_{\mathbb{R}^N} \left(|x|^{-\frac{N+\mu}{2}} * \left|\frac{u-v}{2}\right|^{2_\mu^*}\right)^2.$$

Combining Minkowski's inequality, then we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}\right)^2 &= \left(\int_{\mathbb{R}^N} \left|\frac{\frac{u+v}{2}}{|x-y|^{\frac{N+\mu}{2}\cdot 2_\mu^*}} + \frac{\frac{u-v}{2}}{|x-y|^{\frac{N+\mu}{2}\cdot 2_\mu^*}}\right|^{2_\mu^*}\right)^{\frac{1}{2_\mu^*}\cdot 2\cdot 2_\mu^*} \\ &\le \left(\left\|\frac{\frac{u+v}{2}}{|x-y|^{\frac{N+\mu}{2}\cdot 2_\mu^*}}\right\|_{L^{2_\mu^*}} + \left\|\frac{\frac{u-v}{2}}{|x-y|^{\frac{N+\mu}{2}\cdot 2_\mu^*}}\right\|_{L^{2_\mu^*}}\right)^{2\cdot 2_\mu^*}.\end{aligned}$$

By Minkowski's inequality again, we find that

$$\begin{aligned}\left(\int_{\mathbb{R}^N} \left(|x-y|^{-\frac{N+\mu}{2}} * |u(y)|^{2_\mu^*}\right)^2\right)^{\frac{1}{2\cdot 2_\mu^*}} &\le \left\||x-y|^{-\frac{N+\mu}{2}} * \left|\frac{u+v}{2}\right|^{2_\mu^*}\right\|_{L^2}^{\frac{1}{2_\mu^*}} \\ &+ \left\||x-y|^{-\frac{N+\mu}{2}} * \left|\frac{u-v}{2}\right|^{2_\mu^*}\right\|_{L^2}^{\frac{1}{2_\mu^*}}.\end{aligned}$$

Therefore, we get

$$\left(\int_{\mathbb{R}^N}\left(|x|^{-\mu}*\left|\frac{u+v}{2}\right|^{2_\mu^*}\right)\left|\frac{u+v}{2}\right|^{2_\mu^*}\right)^{\frac{1}{2_\mu^*}} \geq \left(\|u\|_{Q^{\mu,2_\mu^*}}-\left\|\frac{u-v}{2}\right\|_{Q^{\mu,2_\mu^*}}\right)^2.$$

Furthermore, by the convexity of the function $f(x)=|x|^2$, $f(x+y)\geq f(x)+f'(x)y$, and so

$$\left(\|u\|_{Q^{\mu,2_\mu^*}}-\left\|\frac{u-v}{2}\right\|_{Q^{\mu,2_\mu^*}}\right)^2 \geq \|u\|^2_{Q^{\mu,2_\mu^*}}-2\|u\|_{Q^{\mu,2_\mu^*}}\left\|\frac{u-v}{2}\right\|_{Q^{\mu,2_\mu^*}}.$$

These two inequalities imply:

$$\left\|\frac{\nabla u-\nabla v}{2}\right\|^2_{L^2} \leq \|\nabla u\|^2_{L^2}-S_{HL}\|u\|^2_{Q^{\mu,2_\mu^*}}+2S_{HL}\|u\|_{Q^{\mu,2_\mu^*}}\left\|\frac{u-v}{2}\right\|_{Q^{\mu,2_\mu^*}},$$

and the conclusion follows. □

3. Spectrum of the linear operator

For the simplicity of notations, we write W instead of $W[\xi,\lambda]$ defined in (1.11). Firstly, we study the following eigenvalue problem

$$\begin{aligned}-\Delta\omega+\left(|x|^{-\mu}*W^{2_\mu^*}\right)W^{2_\mu^*-2}\omega=\bar{\nu}\Big[&\Big(|x|^{-\mu}*(W^{2_\mu^*-1}\omega)\Big)W^{2_\mu^*-1}\\ &+\Big(|x|^{-\mu}*W^{2_\mu^*}\Big)W^{2_\mu^*-2}\omega\Big],\end{aligned} \tag{3.1}$$

$\omega\in\mathcal{D}^{1,2}(\mathbb{R}^N)$. By a straightforward computation, for all $\omega\in\mathcal{D}^{1,2}(\mathbb{R}^N)$, we get that

$$\int_{\mathbb{R}^N}\Big(|x|^{-\mu}*(W^{2_\mu^*-1}\omega)\Big)W^{2_\mu^*-1}\omega \leq C(N,\mu)\|W\|^{2(2_\mu^*-1)}_{L^{2^*}}\|\omega\|^2_{L^{2^*}} \leq \|\omega\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}. \tag{3.2}$$

Similarly, we have that

$$\int_{\mathbb{R}^N}\Big(|x|^{-\mu}*W^{2_\mu^*}\Big)W^{2_\mu^*-2}\omega^2 \leq \|\omega\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}. \tag{3.3}$$

Therefore, one could consider the eigenvalues of problem (3.1) defined as follows:

Definition 3.1. The Rayleigh quotient characterization of first eigenvalue implies

$$\bar{\nu}_1:=\inf_{v\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla\omega|^2+\int_{\mathbb{R}^N}(|x|^{-\mu}*W^{2_\mu^*})\omega^2}{\int_{\mathbb{R}^N}(|x|^{-\mu}*(W^{2_\mu^*-1}\omega))W^{2_\mu^*-1}\omega+\int_{\mathbb{R}^N}(|x|^{-\mu}*W^{2_\mu^*})\omega^2}. \tag{3.4}$$

In addition, for any $l\in\mathbb{N}$ the eigenvalues can be characterized as follows:

$$\bar{\nu}_{l+1}:=\inf_{v\in\mathbb{W}_{l+1}\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla\omega|^2+\int_{\mathbb{R}^N}(|x|^{-\mu}*W^{2_\mu^*})\omega^2}{\int_{\mathbb{R}^N}(|x|^{-\mu}*(W^{2_\mu^*-1}\omega))W^{2_\mu^*-1}\omega+\int_{\mathbb{R}^N}(|x|^{-\mu}*W^{2_\mu^*-1})\omega^2}, \tag{3.5}$$

where

$$\mathbb{W}_{l+1} := \left\{ \omega \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \nabla\omega \cdot \nabla v_j = 0, \quad \text{for all} \quad j = 1, \ldots, l. \right\}, \tag{3.6}$$

and v_j is the corresponding eigenfunction to $\bar{\nu}_j$.

We have the following discrete spectral information of operator $\mathcal{L}[u]$ (see (3.7)).

Proposition 3.2. *Let $\bar{\nu}_j$, $j = 1, 2, \ldots$, denote the eigenvalues of (3.1) in increasing order as in Definition 3.1. Then operator $\mathcal{L}[u]$ from (3.7) has a discrete spectrum $\{\bar{\nu}_j\}_1^\infty$, with $0 < \bar{\nu}_j < \bar{\nu}_{j+1}$ for all j, and*

$$\bar{\nu}_1 = 1, \qquad X_1 = span\{W\},$$
$$\bar{\nu}_2 = 2^*_\mu, \qquad X_2 = span\left\{\partial x_1 W, \ldots, \partial x_N W, \quad x \cdot \nabla W + \frac{N-2}{2} W\right\},$$

where X_j denotes the eigenfunction space corresponding to $\bar{\nu}_j$.

Proof. The result follows from the non-degeneracy of Talenti bubbles for linearized equation of (1.12) at $W[\xi, \lambda]$. Here we omit the details. □

Let $\rho = u - \sigma$ where $\sigma = \sum_{i=1}^{\kappa} \alpha_i W[\xi_i, \lambda_i]$ be the linear combination of Talenti bubbles that is closet to u in the $\mathcal{D}^{1,2}$-norm, such that,

$$\left\| \nabla u - \nabla\sigma \right\|_{L^2} = \min_{\substack{\bar{\alpha}_1, \ldots, \bar{\alpha}_\kappa \in \mathbb{R}, \\ \bar{\xi}_1, \ldots, \bar{\xi}_\kappa \in \mathbb{R}^N, \\ \bar{\lambda}_1, \ldots, \bar{\lambda}_\kappa \in \mathbb{R}}} \left\| \nabla u - \nabla\left(\sum_{i=1}^{\kappa} \bar{\alpha}_i W[\bar{\xi}_i, \bar{\lambda}_i] \right) \right\|_{L^2}.$$

Here we note that the family together with the coefficients $\alpha_1, \cdots, \alpha_\kappa \in \mathbb{R}$ is δ-interacting if (1.13) holds, and we have that

$$\max_{1 \le i \le \kappa} |\alpha_i - 1| \le \delta.$$

Moreover, for any $1 \le i \le \kappa$, ρ also satisfies the following orthogonality conditions:

$$\int_{\mathbb{R}^N} \nabla\rho \nabla W[\xi_i, \lambda_i] = 0, \qquad \int_{\mathbb{R}^N} \nabla\rho \nabla \frac{\partial W[\xi_i, \lambda_i]}{\partial\lambda} = 0,$$
$$\int_{\mathbb{R}^N} \nabla\rho \nabla \frac{\partial W[\xi_i, \lambda_i]}{\partial\xi_i} = 0 \quad \text{for any} \quad 1 \le i \le N.$$

In the sequel, we write W_i instead of $W[\xi_i,\lambda_i]$ for simplicity. In view of the eigenvalue problem, we know that the functions W_i, $\frac{\partial W_i}{\partial\lambda}$ and $\frac{\partial W_i}{\partial\xi_i}$ are eigenfunctions for the eigenvalue problem

$$\begin{cases}\mathcal{L}[u]=\bar{\nu}\mathcal{R}[u], \quad \text{in} \quad \mathbb{R}^N,\\ u\in\mathcal{D}^{1,2}(\mathbb{R}^N),\end{cases} \tag{3.7}$$

where we denote

$$\mathcal{L}[u]:=-\Delta u+\left(|x|^{-\mu}*W_i^{2_\mu^*}\right)W_i^{2_\mu^*-2}u$$

and

$$\mathcal{R}[u]:=\left(|x|^{-\mu}*\left(W_i^{2_\mu^*-1}u\right)\right)W_i^{2_\mu^*-1}+\left(|x|^{-\mu}*W_i^{2_\mu^*}\right)W_i^{2_\mu^*-2}u.$$

Then the orthogonal conditions are equivalent to the following

$$\int\limits_{\mathbb{R}^N}\left(|x|^{-\mu}*\left(W_i^{2_\mu^*-1}\rho\right)\right)W_i^{2_\mu^*}=0, \tag{3.8}$$

$$2_\mu^*\int\limits_{\mathbb{R}^N}\left(|x|^{-\mu}*\left(W_i^{2_\mu^*-1}\rho\right)\right)W_i^{2_\mu^*-1}\frac{\partial W_i}{\partial\lambda}+(2_\mu^*-1)\int\limits_{\mathbb{R}^N}\left(|x|^{-\mu}*W_i^{2_\mu^*}\right)W_i^{2_\mu^*-2}\frac{\partial W_i}{\partial\lambda}\rho=0 \tag{3.9}$$

and

$$2_\mu^*\int\limits_{\mathbb{R}^N}\left(|x|^{-\mu}*\left(W_i^{2_\mu^*-1}\rho\right)\right)W_i^{2_\mu^*-1}\frac{\partial W_i}{\partial\xi_i}+(2_\mu^*-1)\int\limits_{\mathbb{R}^N}\left(|x|^{-\mu}*W_i^{2_\mu^*}\right)W_i^{2_\mu^*-2}\frac{\partial W_i}{\partial\xi_i}\rho=0, \tag{3.10}$$

for any $1\le i\le N$.

The following inequality is an immediate consequence of Proposition 3.2, which will be used in the proof of Proposition 4.2.

Proposition 3.3. *For every $\varphi\in(\widetilde{\mathcal{Z}})^\perp$ is almost orthogonal to the functions W_i, $\frac{\partial W_i}{\partial\lambda}$ and $\frac{\partial W_i}{\partial\xi_i}$, we have that*

$$\int\limits_{\mathbb{R}^N}\left[\left(|x|^{-\mu}*\left(W_i^{2_\mu^*-1}\varphi\right)\right)W_i^{2_\mu^*-1}\varphi+\left(|x|^{-\mu}*W_i^{2_\mu^*}\right)W_i^{2_\mu^*-2}\varphi^2\right]$$
$$\le\frac{1}{\hbar}\Big[\int\limits_{\mathbb{R}^N}|\nabla\varphi|^2+\int\limits_{\mathbb{R}^N}\left(|x|^{-\mu}*W_i^{2_\mu^*}\right)W_i^{2_\mu^*-2}\varphi^2\Big],$$

where $\hbar>2_\mu^*$ is the least eigenvalue of $\frac{\mathcal{L}[u]}{\mathcal{R}[u]}$ and the set $\widetilde{\mathcal{Z}}$ is the eigenfunction space associated to eigenvalues of $\frac{\mathcal{L}[u]}{\mathcal{R}[u]}$.

4. Stability through Euler-Lagrange equation

In this section we prove Theorems 1.3 and 1.6.

4.1. A stability of profile decompositions

Building upon the renowned profile decomposition results of Struwe in [25] for the equation (1.6), we draw inspiration and leverage these findings to establish a profile decomposition for nonnegative solutions of the nonlocal Hartree-type equation (1.12).

Proof the Theorem 1.3. We define the concentration function by

$$Q_m(r) := \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |w_m(x)|^2.$$

Since

$$(\kappa - \frac{1}{2}) S_{HL}^{\frac{2N-\mu}{N+2-\mu}} \leq \|w_m\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \leq (\kappa + \frac{1}{2}) S_{HL}^{\frac{2N-\mu}{N+2-\mu}}$$

for some $\kappa \in \mathbb{N}$, we choose $r_m > 0$, and $y_m \in \mathbb{R}^N$ such that

$$Q_m(r) = \int_{B_{r_m}(y_m)} |w_m|^2 = \frac{1}{2L_{R_\epsilon}} S_{HL}^{\frac{2N-\mu}{N+2-\mu}},$$

where L_{R_ϵ} is the number such that the ball $B_{2R_\epsilon}(0)$ is covered by L_{R_ϵ} balls of radius R_ϵ. Let

$$w_m \mapsto \tilde{w}_m(x) = \left(\frac{r_m}{R_\epsilon}\right)^{(N-2)/2} w_m\left(\frac{r_m}{R_\epsilon} x\right)$$

such that

$$\widetilde{Q}_m(R_\epsilon) := \sup_{y\in\mathbb{R}^N} \int_{B_{R_\epsilon}(y)} |\nabla \tilde{w}_m|^2 = \int_{B_{R_\epsilon}(r_m^{-1} R_\epsilon y_m)} |\nabla \tilde{w}_m|^2 = \frac{1}{2L_{R_\epsilon}} S_{HL}^{\frac{2N-\mu}{N+2-\mu}}. \tag{4.1}$$

By invariance of the $\mathcal{D}^{1,2}(\mathbb{R}^N)$ norms under translation and dilation, we get that

$$\|\tilde{w}_m\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \|w_m\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{w}_m(x)|^{2^*_\mu} |\tilde{w}_m(y)|^{2^*_\mu}}{|x-y|^\mu} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_m(x)|^{2^*_\mu} |w_m(y)|^{2^*_\mu}}{|x-y|^\mu}.$$

Hence we may assume that there exists $w^0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$\tilde{w}_m \rightharpoonup w^0, \quad \text{weakly in } \in \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{as } m \to \infty.$$

For any $y \in \mathbb{R}^N$, let ψ_m be given by

$$\psi_m = (\tilde{w}_m - w^0)\tilde{\xi},$$

where $\tilde{\xi} \in C_c^\infty$ with $\tilde{\xi} \equiv 1$ for $x \in B_{R_\epsilon}(y)$ and $\tilde{\xi} \equiv 0$ in $B^c_{2R_\epsilon}(y)$. Then we have

$$\|\psi_m\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \le C\|\tilde{w}_m - w^0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + C \int\limits_{B_{2R_\epsilon}(y)\setminus B_{R_\epsilon}(y)} |\tilde{w}_m - w^0|^2 \le C$$

and

$$\int\limits_{B_{2R_\epsilon}(y)\setminus B_{R_\epsilon}(y)} |\tilde{w}_m - w^0|^2 \to 0 \quad \text{as } m \to \infty.$$

Combining $DE_0(w_m; \mathbb{R}^N) \to 0$ in $(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}$ as $m \to \infty$, $DE_0(w^0; \mathbb{R}^N) = 0$, the Brezis-Lieb lemma and HLS inequality, we deduce that

$$\begin{aligned}
o_m(1) &= \Big\langle \psi_m\xi, DE_0(\tilde{w}_m; \mathbb{R}^N) - DE_0(w^0; \mathbb{R}^N)\Big\rangle \\
&= \int\limits_{\mathbb{R}^N} \nabla(\tilde{w}_m - w^0)\nabla(\psi_m\tilde{\xi}) \\
&\quad - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\frac{|\tilde{w}_m(y)|^{2^*_\mu}|\tilde{w}_m(x)|^{2^*_\mu-1}\psi_m\tilde{\xi}}{|x-y|^\mu} - \frac{|w^0(y)|^{2^*_\mu}|w^0(x)|^{2^*_\mu-1}\psi_m\tilde{\xi}}{|x-y|^\mu}\Big] \\
&= \int\limits_{\mathbb{R}^N} |\nabla\psi_m|^2 - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\tilde{w}_m - w^0)(y)|^{2^*_\mu}|(\tilde{w}_m - w^0)(x)|^{2^*_\mu-2}|\psi_m|^2}{|x-y|^\mu} + o_m(1) \qquad (4.2) \\
&= \|\psi_m\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \int\limits_{B_{R_\epsilon}}\int\limits_{B_{R_\epsilon}} \frac{|(\tilde{w}_m - w^0)(y)|^{2^*_\mu}|(\tilde{w}_m - w^0)(x)|^{2^*_\mu-2}|\psi_m|^2}{|x-y|^\mu} + o_m(1) \\
&\ge \|\psi_m\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|\psi_m(y)|^{2^*_\mu}|\psi_m(x)|^{2^*_\mu}}{|x-y|^\mu} + o_m(1)
\end{aligned}$$

as $m \to \infty$. Moreover, HLS and Sobolev inequalities give that

$$\|\psi_m\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|\psi_m(y)|^{2^*_\mu}|\psi_m(x)|^{2^*_\mu}}{|x-y|^\mu} \ge \|\psi_m\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\Big(1 - S_{HL}^{-\frac{2N-\mu}{N-2}}\|\psi_m\|^{\frac{2(N-\mu+2)}{N-2}}_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\Big). \qquad (4.3)$$

On the other hand, we note that

$$\int_{\mathbb{R}^N} |\nabla\psi_m|^2 = \int_{B_{R_\epsilon}} |\nabla(\tilde{w}_m - w^0)|^2 + o_m(1) \le \int_{B_{2R_\epsilon}} |\nabla\tilde{w}_m|^2.$$

It follows from (4.1), (4.2) and (4.3) that $\psi_m \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, that is, $\tilde{w}_m \to w^0$ in $\mathcal{D}^{1,2}(B_{R_0}(y))$ for any $y \in \mathbb{R}^N$. Hence by standard covering argument, we obtain $\tilde{w}_m \to w^0$ in $\mathcal{D}^{1,2}_{loc}(\mathbb{R}^N)$ as $m \to \infty$. Clearly, we have that

$$\int_{B_{R_\epsilon}(x)} |w^0|^2 = \frac{1}{2L_{R_\epsilon}} S_{HL}^{\frac{2N-\mu}{N+2-\mu}} > 0,$$

hence $w^0 \not\equiv 0$. Since $w^0 \ge 0$, we have $w^0(x) = W[\xi,\lambda](x)$, which means that

$$w_m \rightharpoonup \Big(\frac{r_m}{R_\epsilon}\Big)^{\frac{N-2}{2}} W[\xi,\lambda]\Big(\frac{r_m}{R_\epsilon}x\Big)$$

in $\in \mathcal{D}^{1,2}(\mathbb{R}^N)$ as $m \to \infty$. Hence for some $r_{m,1} >$ and $\epsilon_1 > 0$, we have that

$$w_m \rightharpoonup \Big(\frac{r_m}{R_\epsilon}\Big)^{\frac{N-2}{2}} W[\xi,\lambda]\Big(\frac{r_m}{R_\epsilon}x\Big) + \Big(\frac{r_m r_{m,1}}{R_\epsilon R_{\epsilon_1}}\Big)^{\frac{N-2}{2}} W[\xi,\lambda]\Big(\frac{r_m r_{m,1}}{R_{\epsilon_1} R_\epsilon}x\Big),$$

weakly in $\in \mathcal{D}^{1,2}(\mathbb{R}^N)$ as $m \to \infty$. In view of W is the unique nonnegative solution of (1.12) and iterating the above process κ times, then the result easily follows. □

4.2. A quantitative version of the stability result

The proof of Theorem 1.6 is based on several crucial propositions. However, for clarity and coherence, the proofs of Propositions 4.2 and 4.3 will be deferred and presented in Section 5.

Proposition 4.1. *Let $N \ge 3$, $W[\xi_i,\lambda_i]$ and $W[\xi_j,\lambda_j]$ be two bubbles. Then, for any fixed $\varepsilon > 0$ and any nonnegative exponents such that $p+q = 2^*$, it holds that*

$$\int_{\mathbb{R}^N} W[\xi_i,\lambda_i]^p W[\xi_j,\lambda_j]^q \approx \begin{cases} Q^{\min(p,q)}, & \text{if} \quad |p-q| \ge \varepsilon, \\ Q^{\frac{N}{N-2}} \log(\frac{1}{Q}), & \text{if} \quad p = q, \end{cases}$$

where the quantity

$$Q := Q(\xi_i,\xi_j,\lambda_i,\lambda_j) = \min\Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|\xi_i - \xi_j|^2\Big)^{-\frac{N-2}{2}}.$$

Proof. It is similar to that of Proposition B.2 in [14], so is omitted. □

Proposition 4.2. *Let $N \ge 3$ and $\kappa \in \mathbb{N}$. There exists a positive constant $\delta = \delta(N,\kappa) > 0$ such that if $\sigma = \sum_{i=1}^{\kappa} \alpha_i W_i$ is a linear combination of δ-interacting Talenti bubbles and $\rho \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfies (3.8), (3.9) and (3.10). Then we have that*

$$(2_\mu^* - 1)\sum_{i=1}^{\kappa}\left|\alpha_i\right|^{2_\mu^*}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * W_i^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho^2 + 2_\mu^*\sum_{i=1}^{\kappa}\left|\alpha_i\right|^{2_\mu^*-1}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \big(\sigma^{2_\mu^*-1}\rho\big)\Big)W_i^{2_\mu^*-1}\rho \le \nu\int_{\mathbb{R}^N}|\nabla\rho|^2,$$

where ν is a constant strictly less than 1 which depends only on N and κ.

Proposition 4.3. *Let $N \ge 3$ and $\kappa \in \mathbb{N}$. For any $\epsilon > 0$ there exists $\delta = \delta(N,\kappa,\epsilon) > 0$ such that the following statement holds. Let $u = \sum_{i=1}^{\kappa}\alpha_i W_i + \rho$, where the family $(\alpha_i, W_i)_{1\le i\le\kappa}$ is δ-interacting, and ρ satisfies both the orthogonality conditions satisfies (3.8), (3.9), (3.10) and the bound $\|\nabla\rho\|_{L^2} \le 1$. Then, for any $1 \le i \le \kappa$, it holds that*

$$|\alpha_i - 1| \lesssim \epsilon\left\|\nabla\rho\right\|_{L^2} + \left\|\Delta u + \Big(\frac{1}{|x|^\mu} * |u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}} + \left\|\nabla\rho\right\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})}, \tag{4.4}$$

and for any pair of indices $i \neq j$ it holds

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * W_i^{2_\mu^*}\Big)W_i^{2_\mu^*-1}W_j \lesssim \epsilon\left\|\nabla\rho\right\|_{L^2} + \left\|\Delta u + \Big(\frac{1}{|x|^\mu} * |u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}} + \left\|\nabla\rho\right\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})}.$$

Now, we are ready to prove the main results.

4.3. *Proof of Theorem 1.6*

First, we assume Propositions 4.2 and 4.3 and proceed to prove Theorem 1.6. Proposition 4.3 provides important insights indicating that the optimal coefficients in the linear combination of Talenti bubbles approach 1. Let us define

$$\widetilde{\mathcal{M}}_\kappa = \left\{\sum_{i=1}^{\kappa}\widetilde{W}[\xi_i,\lambda_i] : \xi_i \in \mathbb{R}^N, \lambda_i > 0\right\}.$$

It follows from $dist_{\mathcal{D}^{1,2}}\big(u, \widetilde{\mathcal{M}}\kappa\big) \le \delta$ that $\|\nabla\rho\|_{L^2} \le \delta$.

Moreover, as $(\widetilde{W}[\xi_i,\lambda_i])_{1\le i\le\kappa}$ constitutes a δ-interacting family of Talenti bubbles, we can choose $\tilde{\delta} \le \delta$ such that $(W[\xi_i,\lambda_i])_{1\le i\le\kappa}$ forms a $\tilde{\delta}$-interacting family of Talenti bubbles.

Now, our objective is to estimate the left-hand side of (1.12) in the $(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}$-norm, which should provide control over the $\mathcal{D}^{1,2}(\mathbb{R}^N)$-distance between u and the manifold composed of sums of Talenti bubbles. Taking into account the orthogonality condition, we obtain:

$$\int_{\mathbb{R}^N}|\nabla\rho|^2 = \int_{\mathbb{R}^N}\nabla u\nabla\rho = \int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * |u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\rho - \int_{\mathbb{R}^N}\Big[\Delta u + \Big(\frac{1}{|x|^\mu} * |u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big]\rho$$

$$\leq \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|u|^{2^*_\mu-2}u\rho + \left\|\nabla\rho\right\|_{L^2}\Big\|\Delta u + \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|u|^{2^*_\mu-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}. \tag{4.5}$$

Calculating the first term on the right-hand side, we know

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|u|^{2^*_\mu-2}u\rho &\leq \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big||u|^{2^*_\mu} - |\sigma|^{2^*_\mu}\Big|\right)\Big||u|^{2^*_\mu-2}u - |\sigma|^{2^*_\mu-2}\sigma\Big|\rho \\
&+ \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big||u|^{2^*_\mu} - |\sigma|^{2^*_\mu}\Big|\right)\Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2^*_\mu-2}\sum_{i=1}^{\kappa}\alpha_i W_i - \sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2^*_\mu-2}\alpha_i\big|W_i\big|^{2^*_\mu-2}W_i\Big|\rho \\
&+ \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2^*_\mu} - \sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2^*_\mu}\big|W_i\big|^{2^*_\mu}\Big|\right)\Big||u|^{2^*_\mu-2}u - |\sigma|^{2^*_\mu-2}\sigma\Big|\rho \\
&+ \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2^*_\mu} - \sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2^*_\mu}\big|W_i\big|^{2^*_\mu}\Big|\right)\Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2^*_\mu-2}\sum_{i=1}^{\kappa}\alpha_i W_i \\
&- \sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2^*_\mu-2}\alpha_i\big|W_i\big|^{2^*_\mu-2}W_i\Big|\rho \\
&+ \sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big||u|^{2^*_\mu} - |\sigma|^{2^*_\mu}\Big|\right)\big|\alpha_i\big|^{2^*_\mu-2}\alpha_i\big|W_i\big|^{2^*_\mu-2}W_i\rho \\
&+ \sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2^*_\mu} - \sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2^*_\mu}\big|W_i\big|^{2^*_\mu}\Big|\right)\big|\alpha_i\big|^{2^*_\mu-2}\alpha_i\big|W_i\big|^{2^*_\mu-2}W_i\rho \\
&+ \sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big|\alpha_i W_i\Big|^{2^*_\mu}\right)\Big||u|^{2^*_\mu-2}u - |\sigma|^{2^*_\mu-2}\sigma\Big|\rho \\
&+ \sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big|\alpha_i W_i\Big|^{2^*_\mu}\right)\Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2^*_\mu-2}\sum_{i=1}^{\kappa}\alpha_i W_i - \sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2^*_\mu-2}\alpha_i\big|W_i\big|^{2^*_\mu-2}W_i\Big|\rho \\
&+ \sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2^*_\mu}\right)\big|\alpha_i\big|^{2^*_\mu-2}\alpha_i\big|W_i\big|^{2^*_\mu-2}W_i\rho.
\end{aligned}$$

We divide our argument into several cases.

Case 1. The exponents μ and N satisfy

$$\begin{cases} N=3, & \mu\in(0,N),\\ N=4, & \mu\in(0,2],\\ N\geq 5, & \mu\in(0,N) \text{ and } \mu\in(0,1]. \end{cases}$$

Thus we have that

$$\Big||u|^{2_\mu^*}-|\sigma|^{2_\mu^*}\Big|\leq 2_\mu^*|\sigma|^{2_\mu^*-1}|\rho|+\widetilde{C}_1\big(|\sigma|^{2_\mu^*-2}|\rho|^2+|\rho|^{2_\mu^*}\big),\quad \text{for } 2_\mu^*\geq 2, \tag{4.6}$$

and

$$\Big||u|^{2_\mu^*-2}u-|\sigma|^{2_\mu^*-2}\sigma\Big|\leq (2_\mu^*-1)|\sigma|^{2_\mu^*-2}|\rho|+\widetilde{C}_2\big(|\sigma|^{2_\mu^*-3}|\rho|^2+|\rho|^{2_\mu^*-1}\big),\quad \text{for } 2_\mu^*-1\geq 2. \tag{4.7}$$

Combining the elementary inequalities

$$\Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2_\mu^*}-\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_\mu^*}\big|W_i\big|^{2_\mu^*}\Big|\lesssim \sum_{1\leq i\neq j\leq\kappa}\big|\alpha_i\big|^{2_\mu^*-1}\alpha_j\big|W_i\big|^{2_\mu^*-1}W_j, \tag{4.8}$$

and

$$\Big|\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2_\mu^*-2}\sum_{i=1}^{\kappa}\alpha_i W_i-\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_\mu^*-2}\alpha_i\big|W_i\big|^{2_\mu^*-2}W_i\Big|\lesssim \sum_{1\leq i\neq j\leq\kappa}\big|\alpha_i\big|^{2_\mu^*-2}\alpha_j\big|W_i\big|^{2_\mu^*-2}W_j, \tag{4.9}$$

then we can evaluate separately the following various terms by using the basic inequalities. We first have that

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(|\sigma|^{2_\mu^*-1}|\rho|\big)\Big)|\sigma|^{2_\mu^*-2}|\rho|^2 &\leq \big\||\sigma|^{2_\mu^*-1}|\rho|\big\|_{L^{\frac{2N}{2N-\mu}}}\big\||\sigma|^{2_\mu^*-2}|\rho|^2\big\|_{L^{\frac{2N}{2N-\mu}}}\\ &\leq \|\sigma\|_{L^{2^*}}^{\frac{N-2\mu+6}{N-2}}\|\rho\|_{L^{2^*}}^3\lesssim\|\nabla\rho\|_{L^2}^3.\end{aligned}$$

We similarly compute and get

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(|\sigma|^{2_\mu^*-2}|\rho|^2+|\rho|^{2_\mu^*}\big)\Big)|\sigma|^{2_\mu^*-2}|\rho|^2\lesssim\|\nabla\rho\|_{L^2}^4+\|\nabla\rho\|_{L^2}^{\frac{4N-\mu-4}{N-2}}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^\mu}*\Big(|\sigma|^{2_\mu^*-1}|\rho|+|\sigma|^{2_\mu^*-2}|\rho|^2+|\rho|^{2_\mu^*}\Big)\Big]\big(|\sigma|^{2_\mu^*-3}|\rho|^3+|\rho|^{2_\mu^*}\big)\\ &\quad\lesssim\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}^4+\|\nabla\rho\|_{L^2}^5+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{4N-\mu-4}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{4N-2\mu}{N-2}}.\end{aligned}$$

By a direct computation, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\mu}}*\left||u|^{2_\mu^*}-|\sigma|^{2_\mu^*}\right|\right)\left|\left|\sum_{i=1}^{\kappa}\alpha_i W_i\right|^{2_\mu^*-2}\sum_{i=1}^{\kappa}\alpha_i W_i-\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*-2}\alpha_i|W_i|^{2_\mu^*-2}W_i\right|\rho\\
&\quad\lesssim\sum_{1\le i\neq j\le\kappa}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{\mu}}*\left(|\sigma|^{2_\mu^*-1}|\rho|+|\sigma|^{2_\mu^*-2}|\rho|^2+|\rho|^{2_\mu^*}\right)\right]W_i^{2_\mu^*-2}W_j|\rho| \qquad (4.10)\\
&\quad\lesssim\sum_{1\le i\neq j\le\kappa}\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}\right)\|W_i^{2_\mu^*-2}W_j\|_{L^{\frac{2N}{N-\mu+2}}}=:(I_1).
\end{aligned}$$

Similar to the calculation of (4.10), we also get that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\mu}}*\left|\left|\sum_{i=1}^{\kappa}\alpha_i W_i\right|^{2_\mu^*}-\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*}|W_i|^{2_\mu^*}\right|\right)\left||u|^{2_\mu^*-2}u-|\sigma|^{2_\mu^*-2}\sigma\right|\rho\\
&\quad\lesssim\sum_{1\le i\neq j\le\kappa}\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}\right)\|W_i^{2_\mu^*-1}W_j\|_{L^{\frac{2N}{2N-\mu}}}=:(I_2),
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\mu}}*\left|\left|\sum_{i=1}^{\kappa}\alpha_i W_i\right|^{2_\mu^*}-\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*}|W_i|^{2_\mu^*}\right|\right)|\alpha_i|^{2_\mu^*-2}\alpha_i|W_i|^{2_\mu^*-2}W_i\rho\\
&\le\widetilde{C}_3\sum_{1\le i\neq j\le\kappa}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{\mu}}*\left(|W_i|^{2_\mu^*-1}W_j\right)\right]\sum_{i=1}^{\kappa}W_i^{2_\mu^*-1}|\rho|\lesssim\sum_{1\le i\neq j\le\kappa}\|\nabla\rho\|_{L^2}\|W_i^{2_\mu^*-1}W_j\|_{L^{\frac{2N}{2N-\mu}}}\\
&=:(I_3),
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\mu}}*\left|\alpha_i W_i\right|^{2_\mu^*}\right)\left|\left|\sum_{i=1}^{\kappa}\alpha_i W_i\right|^{2_\mu^*-2}\sum_{i=1}^{\kappa}\alpha_i W_i-\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*-2}\alpha_i|W_i|^{2_\mu^*-2}W_i\right|\rho\\
&\quad\le\widetilde{C}_4\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\mu}}*|W_i|^{2_\mu^*}\right)\sum_{1\le i\neq j\le\kappa}W_i^{2_\mu^*-2}W_j|\rho|\lesssim\sum_{1\le i\neq j\le\kappa}\|\nabla\rho\|_{L^2}\|W_i^{2_\mu^*-2}W_j\|_{L^{\frac{2N}{N-\mu+2}}}\\
&\quad=:(I_4),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\frac{1}{|x|^{\mu}}*\left|\left|\sum_{i=1}^{\kappa}\alpha_i W_i\right|^{2_\mu^*}-\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*}|W_i|^{2_\mu^*}\right|\right)\\
&\quad\times\left|\left|\sum_{i=1}^{\kappa}\alpha_i W_i\right|^{2_\mu^*-2}\sum_{i=1}^{\kappa}\alpha_i W_i-\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*-2}\alpha_i|W_i|^{2_\mu^*-2}W_i\right|\rho\\
&\qquad\lesssim\sum_{1\le i\neq j\le\kappa}\|\nabla\rho\|_{L^2}\|W_i^{2_\mu^*-2}W_j\|_{L^{\frac{2N}{N-\mu+2}}}\|W_i^{2_\mu^*-1}W_j\|_{L^{\frac{2N}{2N-\mu}}}=:(I_5).
\end{aligned}$$

By straight computations show that

$$\begin{aligned}
&\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\Big|\alpha_i W_i\Big|^{2_{\mu}^{*}}\Big)\Big||u|^{2_{\mu}^{*}-2}u-|\sigma|^{2_{\mu}^{*}-2}\sigma\Big|\rho\\
&\quad\le\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_{\mu}^{*}}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\big|W_i\big|^{2_{\mu}^{*}}\Big)\Big((2_{\mu}^{*}-1)|\sigma|^{2_{\mu}^{*}-2}|\rho|^2+\widetilde{C}_2\big(|\sigma|^{2_{\mu}^{*}-3}|\rho|^3+|\rho|^{2_{\mu}^{*}}\big)\Big)\\
&\quad\le\widetilde{C}_5\Big(\|\nabla\rho\|_{L^2}^{3}+\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}\Big)+(2_{\mu}^{*}-1)\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_{\mu}^{*}}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\big|W_i\big|^{2_{\mu}^{*}}\Big)|\sigma|^{2_{\mu}^{*}-2}|\rho|^2,
\end{aligned}$$

and analogously

$$\begin{aligned}
&\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\Big||u|^{2_{\mu}^{*}}-|\sigma|^{2_{\mu}^{*}}\Big|\Big)\big|\alpha_i\big|^{2_{\mu}^{*}-2}\alpha_i\big|W_i\big|^{2_{\mu}^{*}-2}W_i\rho\\
&\quad\le\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_{\mu}^{*}-1}\int_{\mathbb{R}^N}\Big[\frac{1}{|x|^{\mu}}*\Big(2_{\mu}^{*}|\sigma|^{2_{\mu}^{*}-1}|\rho|+\widetilde{C}_1\big(|\sigma|^{2_{\mu}^{*}-2}|\rho|^2+|\rho|^{2_{\mu}^{*}}\big)\Big)\Big]W_i^{2_{\mu}^{*}-1}|\rho|\\
&\quad\le\widetilde{C}_6\Big(\|\nabla\rho\|_{L^2}^{3}+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}\Big)+2_{\mu}^{*}\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_{\mu}^{*}-1}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\big(|\sigma|^{2_{\mu}^{*}-1}|\rho|\big)\Big)W_i^{2_{\mu}^{*}-1}|\rho|.
\end{aligned}$$

Using (3.8) we have

$$\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\Big|\sum_{i=1}^{\kappa}\alpha_i W_i\Big|^{2_{\mu}^{*}}\Big)|\alpha_i|^{2_{\mu}^{*}-2}\alpha_i|W_i|^{2_{\mu}^{*}-2}W_i\rho=0.$$

Substituting these estimates into the first term on the right-hand side of (4.5), we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*|u|^{2_{\mu}^{*}}\Big)|u|^{2_{\mu}^{*}-2}u\rho\le(2_{\mu}^{*}-1)\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_{\mu}^{*}}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\big|W_i\big|^{2_{\mu}^{*}}\Big)|\sigma|^{2_{\mu}^{*}-2}|\rho|^2\\
&+2_{\mu}^{*}\sum_{i=1}^{\kappa}\big|\alpha_i\big|^{2_{\mu}^{*}-1}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^{\mu}}*\big(|\sigma|^{2_{\mu}^{*}-1}|\rho|\big)\Big)W_i^{2_{\mu}^{*}-1}|\rho|+\widetilde{C}_{N,1}\Big(\|\nabla\rho\|_{L^2}^{3}+\|\nabla\rho\|_{L^2}^{4}+\|\nabla\rho\|_{L^2}^{5}\\
&+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{4N-\mu-4}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{4N-2\mu}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}\Big)\\
&+\widetilde{C}_{N,2}\Big((I_1)+(I_2)+(I_3)+(I_4)+(I_5)\Big).
\end{aligned}\tag{4.11}$$

We consider (I_1). For any $i \neq j$, $3 \leq N < 6-\mu$, by Proposition 4.1, it is clear that

$$\begin{aligned}\|W_i^{2_\mu^*-2}W_j\|_{L^{\frac{2N}{N-\mu+2}}} &= \Big(\int_{\mathbb{R}^N} W_i^{\frac{2N(2_\mu^*-2)}{N-\mu+2}} W_j^{\frac{2N}{N-\mu+2}}\Big)^{\frac{N-\mu+2}{2N}} \\ &\approx \min\Big(\frac{\lambda_i}{\lambda_j}, \frac{\lambda_j}{\lambda_i}, \frac{1}{\lambda_i\lambda_j|\xi_i-\xi_j|^2}\Big)^{\frac{N-2}{2}} = \int_{\mathbb{R}^N} W_i^{2^*-1}W_j.\end{aligned} \tag{4.12}$$

Combining this estimate we obtain

$$(I_1) \lesssim \sum_{1\leq i\neq j\leq\kappa} \Big(\|\nabla\rho\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^3 + \|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}\Big) \int_{\mathbb{R}^N} W_i^{2^*-1}W_j \tag{4.13}$$

and

$$(I_4) \lesssim \sum_{1\leq i\neq j\leq\kappa} \|\nabla\rho\|_{L^2} \int_{\mathbb{R}^N} W_i^{2^*-1}W_j.$$

Similar to the above argument, for any $i \neq j$, it holds that

$$\|W_i^{2_\mu^*-1}W_j\|_{L^{\frac{2N}{2N-\mu}}} \approx \int_{\mathbb{R}^N} W_i^{2^*-1}W_j. \tag{4.14}$$

So

$$(I_2) \lesssim \sum_{1\leq i\neq j\leq\kappa} \Big(\|\nabla\rho\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^3 + \|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}\Big) \int_{\mathbb{R}^N} W_i^{2^*-1}W_j$$

and

$$(I_3) \lesssim \sum_{1\leq i\neq j\leq\kappa} \|\nabla\rho\|_{L^2} \int_{\mathbb{R}^N} W_i^{2^*-1}W_j.$$

Then, we find from (4.12) and (4.14) that

$$(I_5) \lesssim \sum_{1\leq i\neq j\leq\kappa} \|\nabla\rho\|_{L^2}\Big(\int_{\mathbb{R}^N} W_i^{2^*-1}W_j\Big)^2.$$

Now note that the identity

$$\int_{\mathbb{R}^N} \frac{1}{|x-y|^{2\gamma}}\Big(\frac{1}{1+|y|^2}\Big)^{N-\gamma} dy = I(\gamma)\Big(\frac{1}{1+|x|^2}\Big)^{\gamma}, \ \ 0<\gamma<\frac{N}{2}, \tag{4.15}$$

where

$$I(\gamma)=\frac{\pi^{\frac{N}{2}}\Gamma(\frac{N-2\gamma}{2})}{\Gamma(N-\gamma)} \text{ with } \Gamma(\gamma)=\int_0^{+\infty} x^{\gamma-1}e^{-x}\,dx, \quad \gamma>0,$$

then we get

$$|x|^{-\mu}*|W_i|^{2^*_\mu}=\int_{\mathbb{R}^N}\frac{W_i^{2^*_\mu}(y)}{|x-y|^{\mu}}dy=\widetilde{\mathcal{Q}}W_i^{2^*-2^*_\mu}(x),$$

where

$$\widetilde{\mathcal{Q}}=I(\gamma)S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}}[C(N,\mu)]^{\frac{2-N}{2(N-\mu+2)}}[N(N-2)]^{\frac{N-2}{4}}.$$

Therefore, by applying the Proposition 4.3 implies that

$$\begin{aligned}
&\sum_{1\le i\ne j\le\kappa}\Big(\|\nabla\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}\Big)\int_{\mathbb{R}^N}W_i^{2^*-1}W_j\\
&\lesssim\epsilon\Big(\|\nabla\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}\Big)+\Big(\|\nabla\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}\\
&+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}\Big)\Big\|\Delta u+\Big(\frac{1}{|x|^{\mu}}*|u|^{2^*_\mu}\Big)|u|^{2^*_\mu-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}+\Big\|\nabla\rho\Big\|_{L^2}^{\min(4,\frac{3N-\mu+2}{N-2})}\\
&+\Big\|\nabla\rho\Big\|_{L^2}^{\min(5,\frac{4N-\mu-4}{N-2})}+\Big\|\nabla\rho\Big\|_{L^2}^{\min(\frac{4N-\mu-4}{N-2},\frac{3N-2\mu+2}{N-2})}+\Big\|\nabla\rho\Big\|_{L^2}^{\min(\frac{4}{5N-\mu-2},\frac{4N-2\mu+4}{N-2})},
\end{aligned}$$

using also that

$$\begin{aligned}
&\sum_{1\le i\ne j\le\kappa}\|\nabla\rho\|_{L^2}\Big(\int_{\mathbb{R}^N}W_i^{2^*-1}W_j\Big)^2\\
&\qquad\lesssim\epsilon^2\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}\Big\|\Delta u+\Big(\frac{1}{|x|^{\mu}}*|u|^{2^*_\mu}\Big)|u|^{2^*_\mu-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}^2\\
&\qquad+\Big\|\nabla\rho\Big\|_{L^2}^{\min(5,\frac{3N-2\mu+2}{N-2})},
\end{aligned}$$

and

$$\sum_{1\le i\neq j\le\kappa}\|\nabla\rho\|_{L^2}\int_{\mathbb{R}^N}W_i^{2^*-1}W_j$$
$$\lesssim\epsilon\|\nabla\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}$$
$$+\Big\|\nabla\rho\Big\|_{L^2}^{\min(3,\frac{2N-\mu}{N-2})}.$$

Now we are in a position to prove Theorem 1.6. To this end, choosing $\epsilon>0$ such that

$$\nu(N,\kappa)+\epsilon\widetilde{C}_{N,3}\widetilde{C}_{N,5}+\epsilon\widetilde{C}_{N,4}\widetilde{C}_{N,5}<1,$$

we are able to conclude that

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\right)|u|^{2_\mu^*-2}u\rho\le\Big[\nu(N,\kappa)+\epsilon\widetilde{C}_{N,3}\widetilde{C}_{N,5}+\epsilon\widetilde{C}_{N,4}\widetilde{C}_{N,5}\Big]\|\nabla\rho\|_{L^2}^2$$
$$+(J_1)+(J_2)+(J_3)+(J_4)+(J_5)+(J_6)+(J_7),$$

where

$$(J_1):=\widetilde{C}_{N,6}\|\nabla\rho\|_{L^2}\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}},$$
$$(J_2):=\widetilde{C}_{N,7}\|\nabla\rho\|_{L^2}\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}^2,$$
$$(J_3):=\widetilde{C}_{N,8}\Big(\|\nabla\rho\|_{L^2}^2+\|\nabla\rho\|_{L^2}^3\Big)\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}},$$
$$(J_4):=\widetilde{C}_{N,9}\Big(\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}\Big)\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}},$$
$$(J_5):=\widetilde{C}_{N,10}\Big(\|\nabla\rho\|_{L^2}^3+\|\nabla\rho\|_{L^2}^4+\|\nabla\rho\|_{L^2}^5+\|\nabla\rho\|_{L^2}^{\frac{3N-\mu-2}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{4N-\mu-4}{N-2}}+\|\nabla\rho\|_{L^2}^{\frac{4N-2\mu}{N-2}}$$
$$+\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}\Big),$$
$$(J_6):=\widetilde{C}_{N,11}\Big(\Big\|\nabla\rho\Big\|_{L^2}^{\min(5,\frac{3N-2\mu+2}{N-2})}+\Big\|\nabla\rho\Big\|_{L^2}^{\min(4,\frac{3N-\mu+2}{N-2})}+\Big\|\nabla\rho\Big\|_{L^2}^{\min(5,\frac{4N-\mu-4}{N-2})}$$
$$+\Big\|\nabla\rho\Big\|_{L^2}^{\min(5,\frac{3N-2\mu+2}{N-2})}\Big),$$
$$(J_7):=\widetilde{C}_{N,12}\Big(\Big\|\nabla\rho\Big\|_{L^2}^{\min(\frac{4N-\mu-4}{N-2},\frac{3N-2\mu+2}{N-2})}+\Big\|\nabla\rho\Big\|_{L^2}^{\min(\frac{4}{5N-\mu-2},\frac{4N-2\mu+4}{N-2})}\Big).$$

As a consequence, together with (4.5) and $\|\nabla\rho\|_{L^2}\ll 1$,

$$\Big[1-\nu(N,\kappa)-\epsilon\widetilde{C}_{N,3}\widetilde{C}_{N,5}-\epsilon\widetilde{C}_{N,4}\widetilde{C}_{N,5}\Big]\big\|\nabla\rho\big\|_{L^2}^2$$
$$\lesssim\Big\|\nabla\rho\Big\|_{L^2}\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}$$
$$+(J_4)+(J_5)+(J_6)+(J_7).$$

Combining this inequality with $\|\nabla\rho\|_{L^2}\ll 1$ yields the conclusion

$$\big\|\nabla\rho\big\|_{L^2}\lesssim\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}.$$

On the other hand, by (4.4) of the Proposition 4.3, the conclusion follows.

Case 2. The exponents μ and N satisfy

$$\begin{cases} N=4, & \mu\in(2,N),\\ N\geq 5, & \mu\in(0,N) \text{ and } \mu\in(1,4]. \end{cases}$$

We find the elementary inequality

$$\Big|u|u|^{2_\mu^*-2}-\sigma|\sigma|^{2_\mu^*-2}\Big|\leq(2_\mu^*-1)|\sigma|^{2_\mu^*-2}|\rho|+\widetilde{C}_3|\rho|^{2_\mu^*-1},\quad \text{for } 1<2_\mu^*-1\leq 2.$$

The argument of the statement is identical to the proof of case 1. Combining this equality with (4.6) yields the conclusion. □

Remark 4.4. In the proof of Theorem 1.6, it is pointed out that the evaluations when $N\geq 6-\mu$ is not sufficient to obtain the desired conclusion. In fact, for $N>6-\mu$, we have that

$$\|W_i^{2_\mu^*-2}W_j\|_{L^{\frac{2N}{N-\mu+2}}}=\Big(\int_{\mathbb{R}^N}W_i^{\frac{2N(2_\mu^*-2)}{N-\mu+2}}W_j^{\frac{2N}{N-\mu+2}}\Big)^{\frac{N-\mu+2}{2N}}$$
$$\approx\Big(\int_{\mathbb{R}^N}W_i^{2^*-1}W_j\Big)^{2_\mu^*-2}\gg\int_{\mathbb{R}^N}W_i^{2^*-1}W_j.$$

For $N=6-\mu$, we get that

$$\|W_i^{2_\mu^*-2}W_j\|_{L^{\frac{2N}{N-\mu+2}}}\approx\Big(\int_{\mathbb{R}^N}W_i^{2^*-1}W_j\Big)^{\frac{3(8-\mu)}{24}}\Big|\log\Big(\int_{\mathbb{R}^N}W_i^{2^*-1}W_j\Big)\Big|^{\frac{8-\mu}{12}}\gg\int_{\mathbb{R}^N}W_i^{2^*-1}W_j.$$

Proof of Corollary 1.7. We can now prove Corollary 1.7 by using Theorems 1.6 and 1.3. In view of Theorem 1.3 there exists $\epsilon>0$ such that

$$\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}\leq\epsilon.$$

Then, for any $\delta>0$, we have that

$$\Big\|\nabla u - \sum_{i=1}^{\kappa} \nabla W[\xi_i, \lambda_i]\Big\|_{L^2} \leq \delta,$$

where $(W[\xi_i, \lambda_i])_{1\leq i\leq\kappa}$ is a δ-interacting family of Talenti bubbles. Combining Theorem 1.6 and the conclusion follows. □

Proof of Corollary 1.8. For dimension $N \geq 3$ and a single bubble case, combining the arguments of Theorem 1.6 and Corollary 1.7, the result easily follows. □

5. Proofs of Propositions 4.2 and 4.3

We conclude the proof of Theorem 1.6 by establishing Propositions 4.2 and 4.3. The subsequent propositions serve as analogs to Lemmas 3.8 and 3.9 of [14], playing a crucial role in our subsequent analysis.

Proposition 5.1. *Let $N \geq 1$. Given a point $\tilde{x} \in \mathbb{R}^N$ and two radii $0 < r < R$, there exists a Lipschitz bump function $\psi = \psi_{\tilde{x},r,R} : \mathbb{R}^N \to [0, 1]$ such that $\psi \equiv 1$ in $B_r(\tilde{x})$, $\psi \equiv$ in $B_R(\tilde{x})$, and*

$$\int_{\mathbb{R}^N} |\nabla\psi|^N \lesssim \log\Big(\frac{R}{r}\Big)^{1-N}.$$

Proposition 5.2. *For any $N \geq 3$, $\kappa \in \mathbb{N}$, and $\epsilon > 0$, there exists $\delta = \delta(N, \kappa, \epsilon) > 0$ such that if $(W[\xi_i, \lambda_i])_{1\leq i\leq\kappa}$ is a δ-interacting family of Talenti bubbles. Then for any $1 \leq i \leq \kappa$ there exists a Lipschitz bump function $\Psi_i : \mathbb{R}^N \to [0, 1]$ such that the following hold:*

(1) *Almost all mass of $\big(|x|^{-\mu} * W_i^{2^*_\mu}\big)W_i^{2^*_\mu}$ is in the region $\{\Psi_i = 1\}$, that is*

$$\int_{\{\Psi_i=1\}} \Big(|x|^{-\mu} * W_i^{2^*_\mu}\Big)W_i^{2^*_\mu} = \widetilde{\mathcal{Q}}_0 S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} \int_{\{\Psi_i=1\}} W_i^{2^*} \geq \widetilde{\mathcal{Q}}_0(1-\epsilon) S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}+N},$$

where

$$\widetilde{\mathcal{Q}}_0 = I(\gamma)[C(N,\mu)]^{\frac{2-N}{2(N-\mu+2)}}[N(N-2)]^{\frac{N-2}{4}},$$

$$I(\gamma) = \frac{\pi^{\frac{N}{2}}\Gamma(\frac{N-2\gamma}{2})}{\Gamma(N-\gamma)},\ \Gamma(\gamma) = \int_0^{+\infty} x^{\gamma-1}e^{-x}\,dx,\quad \forall\gamma > 0,$$

$C(N,\mu)$ is the sharp constant of HLS inequality given as in (1.2).

(2) *In the region $\{\Psi_i > 0\}$ it holds $\epsilon W_i > W_j$ for any $j \neq i$.*

(3) *The L^N-norm of the gradient is small, that is*

$$\|\nabla\Psi_i\|_{L^N} \leq \epsilon.$$

(4) *For any $j \neq i$ such that $\lambda_j \leq \lambda_i$, it holds that*

$$\frac{\sup_{\{\Psi_i>0\}} W_j}{\inf_{\{\Psi_i>0\}} W_j} \leq 1+\epsilon.$$

Proof. The proof of this proposition closely resembles that of Lemma 3.9 in [14]. Thus, we will provide a brief sketch of the proof for part (1). Without loss of generality, we consider a single index, denoted by $W = W_\kappa$.

Suppose there exists $j \in 1, \cdots, \kappa-1$ such that $\lambda_j > 1$ and $|\xi_j| < 2r$. Let $r_j \in (0,\infty)$ be the unique positive real number satisfying:

$$\epsilon\Big(\frac{1}{1+r^2}\Big)\frac{N-2}{2} = \Big(\frac{\lambda_j}{1+\lambda_j^2|r_j|^2}\Big)^{\frac{N-2}{2}}.$$

Then δ is sufficiently small to tell us that $r_j \leq \epsilon^2$ for any $j \in \{1, \cdots, \kappa-1\}$. Notice that

$$|x|^{-\mu} * |W(x)|^{2_\mu^*} = \widetilde{\mathcal{Q}} S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} W^{2^*-2_\mu^*}(x),$$

where

$$\widetilde{\mathcal{Q}}_0 = I(\gamma)[C(N,\mu)]^{\frac{2-N}{2(N-\mu+2)}}[N(N-2)]^{\frac{N-2}{4}} \text{ with } I(\gamma) = \frac{\pi^{\frac{N}{2}}\Gamma(\frac{N-2\gamma}{2})}{\Gamma(N-\gamma)}.$$

Then we denote $\Psi = \Psi_\kappa$ and imply that

$$\begin{aligned}\int\limits_{\{\Psi<1\}} \Big(|x|^{-\mu} * W^{2_\mu^*}\Big)W^{2_\mu^*} &= \widetilde{\mathcal{Q}}_0 S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} \int\limits_{\{\Psi<1\}} W^{2^*}(x) \leq \widetilde{\mathcal{Q}}_0 S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} \int\limits_{B^c_{\varepsilon r}(0)} W^{2^*} \\ &\quad + \widetilde{\mathcal{Q}}_0 S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} \sum_j \int\limits_{B_{\epsilon^{-1}r_j}(\xi_j)} W^{2^*} \\ &\leq \widetilde{\mathcal{Q}}_0 \epsilon S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}+N},\end{aligned}$$

by ϵ is sufficient small and choosing $r = r(\epsilon)$ large enough. Thus we find that

$$\begin{aligned}\int\limits_{\{\Psi=1\}} \Big(|x|^{-\mu} * W^{2_\mu^*}\Big)W^{2_\mu^*} &= \widetilde{\mathcal{Q}}_0 S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} \int\limits_{\{\Psi=1\}} W^{2^*} \\ &\geq \widetilde{\mathcal{Q}}_0 S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} \int\limits_{\{\Psi=1\}} W^{2^*} + \widetilde{\mathcal{Q}}_0 S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} \Big(\int\limits_{\{\Psi<1\}} W^{2^*} - \varepsilon \mathcal{S}^N\Big).\end{aligned}$$

The conclusion follows. □

5.1. Proof of Proposition 4.2

Proposition 5.2-(2) can be used to deduce that for choosing $\epsilon = o(1)$,

$$\begin{aligned}(2_\mu^*-1)\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_j^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho^2 \\ \leq \big(1+o(1)\big)(2_\mu^*-1)\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(\sum_{j=1}^{\kappa}W_j^{2_\mu^*}\big)\Big)W_i^{2_\mu^*-2}(\Psi_i\rho)^2 \\ +(2_\mu^*-1)\sum_{j=1}^{\kappa}\int_{\{\sum\Psi_i<1\}}\Big(\frac{1}{|x|^\mu}*W_j^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho^2.\end{aligned} \tag{5.1}$$

In virtue of Proposition 5.2-(2), we get that

$$\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_1^{2_\mu^*}\Big)W_i^{2_\mu^*-2}(\Psi_i\rho)^2 \leq \big(1+o(1)\big)\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_1^{2_\mu^*}\Big)W_1^{2_\mu^*-2}(\Psi_1\rho)^2,$$

$$\cdots,$$

$$\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_\kappa^{2_\mu^*}\Big)W_i^{2_\mu^*-2}(\Psi_i\rho)^2 \leq \big(1+o(1)\big)\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_\kappa^{2_\mu^*}\Big)W_\kappa^{2_\mu^*-2}(\Psi_\kappa\rho)^2.$$

Hence we are able to conclude that

$$\begin{aligned}\big(1+o(1)\big)(2_\mu^*-1)\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(\sum_{j=1}^{\kappa}W_j^{2_\mu^*}\big)\Big)W_i^{2_\mu^*-2}(\Psi_i\rho)^2 \\ \leq \big(1+o(1)\big)(2_\mu^*-1)\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_i^{2_\mu^*}\Big)W_i^{2_\mu^*-2}(\Psi_i\rho)^2.\end{aligned} \tag{5.2}$$

Then, we similarly compute and get

$$\begin{aligned}\big(1+o(1)\big)2_\mu^*\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2_\mu^*-1}\Psi_i\rho\big)\Big)\big(\sum_{j=1}^{\kappa}W_j^{2_\mu^*-1}\big)\Psi_i\rho \\ \leq \big(1+o(1)\big)2_\mu^*\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2_\mu^*-1}\Psi_i\rho\big)\Big)W_i^{2_\mu^*-1}\Psi_i\rho.\end{aligned}$$

Thus application of Proposition 5.2-(2) ensures that

$$\begin{aligned}
&\sum_{j=1}^{\kappa}\left|\alpha_i\right|^{2_\mu^*-1}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}\rho\big)\Big)W_j^{2_\mu^*-1}\rho\\
&\le\big(1+o(1)\big)\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2_\mu^*-1}\Psi_i\rho\big)\Big)W_i^{2_\mu^*-1}\Psi_i\rho\\
&\quad+\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\int_{\{\sum\Psi_i<1\}}\frac{\sigma^{2_\mu^*-1}\rho(y)W_i^{2_\mu^*-1}(x)\Psi_i(x)\rho(x)}{|x-y|^\mu}\\
&\quad+\sum_{i=1}^{\kappa}\int_{\{\sum\Psi_i<1\}}\int_{\mathbb{R}^N}\frac{W_i^{2_\mu^*-1}(x)\Psi_i(x)\rho(x)W_i^{2_\mu^*-1}(y)\rho(y)}{|x-y|^\mu}\\
&\quad+\sum_{i=1}^{\kappa}\int_{\{\sum\Psi_i<1\}}\int_{\{\sum\Psi_i<1\}}\frac{\sigma^{2_\mu^*-1}\rho(y)W_i^{2_\mu^*-1}(x)\rho(x)}{|x-y|^\mu}.
\end{aligned}\tag{5.3}$$

Combined with Proposition 5.2-(1), we compute and get

$$\begin{aligned}
\sum_{j=1}^{\kappa}\int_{\{\sum\Psi_i<1\}}\Big(\frac{1}{|x|^\mu}*W_j^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho^2&\le\sum_{j=1}^{\kappa}\Big(\int_{\mathbb{R}^N}W_j^{2^*}\Big)^{\frac{2N-\mu}{2N}}\Big(\int_{\{\sum\Psi_i<1\}}\sigma^{2^*}\Big)^{\frac{4-\mu}{2N}}\left\|\nabla\rho\right\|_{L^2}^2\\
&\le o(1)\left\|\nabla\rho\right\|_{L^2}^2,
\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}
&\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N}\int_{\{\sum\Psi_i<1\}}\frac{\sigma^{2_\mu^*-1}\rho(y)W_i^{2_\mu^*-1}(x)\Psi_i(x)\rho(x)}{|x-y|^\mu}\\
&\qquad\le\sum_{i=1}^{\kappa}\Big(\int_{\mathbb{R}^N}W_i^{2^*}\Big)^{\frac{N-\mu+2}{2N}}\Big(\int_{\{\sum\Psi_i<1\}}\sigma^{2^*}\Big)^{\frac{N-\mu+2}{2N}}\left\|\nabla\rho\right\|_{L^2}^2\\
&\qquad\le o(1)\left\|\nabla\rho\right\|_{L^2}^2.
\end{aligned}\tag{5.5}$$

Similar arguments give that

$$\sum_{i=1}^{\kappa}\int_{\{\sum\Psi_i<1\}}\int_{\mathbb{R}^N}\frac{W_i^{2_\mu^*-1}(x)\Psi_i(x)\rho(x)W_i^{2_\mu^*-1}(y)\rho(y)}{|x-y|^\mu}\le o(1)\left\|\nabla\rho\right\|_{L^2}^2,\tag{5.6}$$

and

$$\sum_{i=1}^{\kappa}\int_{\{\sum\Psi_i<1\}}\int_{\{\sum\Psi_i<1\}}\frac{\sigma^{2_\mu^*-1}\rho(y)W_i^{2_\mu^*-1}(x)\rho(x)}{|x-y|^\mu}\le o(1)\left\|\nabla\rho\right\|_{L^2}^2.\tag{5.7}$$

We claim that there exists a constant $\nu(N,\kappa)$ less than 1 such that

$$\begin{aligned}(2^*_\mu-1)\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2^*_\mu}\big)\Big)W_i^{2^*_\mu-2}(\Psi_i\rho)^2+2^*_\mu\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2^*_\mu-1}\Psi_i\rho\big)\Big)W_i^{2^*_\mu-1}\Psi_i\rho\\ \le\nu(N,\kappa)\int\limits_{\mathbb{R}^N}|\nabla(\Psi_i\rho)|^2+o(1)\|\nabla\rho\|^2_{L^2}.\end{aligned}\tag{5.8}$$

To verify (5.8), our goal is to show that $\Psi_i\rho$ almost satisfies the orthogonality conditions. Let us first define $\widehat{f}:\mathbb{R}^N\to\mathbb{R}$ be, up to scaling, the functions W_i. Then

$$\begin{aligned}\Big|\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_i^{2^*_\mu}\Big)W_i^{2^*_\mu-2}(\Psi_i\rho)\widehat{f}\Big|&=\Big|\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_i^{2^*_\mu}\Big)W_i^{2^*_\mu-2}\rho\widehat{f}(1-\Psi_i)\Big|\\&\le\Big|\int\limits_{\{\Psi_i<1\}}\Big(\frac{1}{|x|^\mu}*W_i^{2^*_\mu}\Big)W_i^{2^*_\mu-2}\rho\widehat{f}\Big|\\&\le o(1)\|\nabla\rho\|_{L^2},\end{aligned}\tag{5.9}$$

by the orthogonality conditions (3.8). Now, we define $\widehat{g}:\mathbb{R}^N\to\mathbb{R}$ be, up to scaling, the functions $\frac{\partial W_i}{\partial\lambda}$. We have that

$$\begin{aligned}&\Big|2^*_\mu\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2^*_\mu-1}\Psi_i\rho\big)\Big)W_i^{2^*_\mu-1}\widehat{g}+(2^*_\mu-1)\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_i^{2^*_\mu}\Big)W_i^{2^*_\mu-2}\widehat{g}\Psi_i\rho\Big|\\=&\Big|2^*_\mu\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2^*_\mu-1}\rho(1-\Psi_i)\big)\Big)W_i^{2^*_\mu-1}\widehat{g}+(2^*_\mu-1)\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_i^{2^*_\mu}\Big)W_i^{2^*_\mu-2}\widehat{g}\rho(1-\Psi_i)\Big|\\\le&o(1)\|\nabla\rho\|_{L^2},\end{aligned}\tag{5.10}$$

by the orthogonality conditions (3.9). Analogously, let $\widehat{h}:\mathbb{R}^N\to\mathbb{R}$ be, up to scaling, the functions $\frac{\partial W_i}{\partial\xi_i}$. Then we clearly have that

$$\Big|2^*_\mu\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(W_i^{2^*_\mu-1}\Psi_i\rho\big)\Big)W_i^{2^*_\mu-1}\widehat{h}+(2^*_\mu-1)\int\limits_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_i^{2^*_\mu}\Big)W_i^{2^*_\mu-2}\widehat{h}\Psi_i\rho\Big|\le o(1)\|\nabla\rho\|_{L^2},\tag{5.11}$$

by the orthogonality conditions (3.10). As a consequence, (5.9), (5.10) and (5.11) tell us that $\Psi_i\rho$ almost orthogonal to $\widehat{f}$, $\widehat{g}$ and $\widehat{h}$, as desired.

Combining Proposition 3.3 and the fact that an orthogonal basis of the eigenfunctions space of $\frac{\mathcal{L}[u]}{\mathcal{R}[u]}$ composed of the functions W_i, $\frac{\partial W_i}{\partial\lambda}$ and $\frac{\partial W_i}{\partial\xi_i}$, so that we get that

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * W_i^{2_\mu^*}\Big) W_i^{2_\mu^*-2}(\Psi_i\rho)^2 + \int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * \big(W_i^{2_\mu^*-1}\Psi_i\rho\big)\Big) W_i^{2_\mu^*-1}\Psi_i\rho$$
$$\le \frac{1}{\hbar}\Big[\int_{\mathbb{R}^N} |\nabla(\Psi_i\rho)|^2 + \int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * W_i^{2_\mu^*}\Big) W_i^{2_\mu^*-2}(\Psi_i\rho)^2\Big] + o(1)\|\nabla\rho\|_{L^2}^2.$$

This yields

$$\int_{\mathbb{R}^N} \Big[(2_\mu^*-1)\Big(\frac{1}{|x|^\mu} * W_i^{2_\mu^*}\Big) W_i^{2_\mu^*-2}(\Psi_i\rho)^2 + 2_\mu^*\Big(\frac{1}{|x|^\mu} * \big(W_i^{2_\mu^*-1}\Psi_i\rho\big)\Big) W_i^{2_\mu^*-1}\Psi_i\rho\Big]$$
$$\le \nu \int_{\mathbb{R}^N} |\nabla(\Psi_i\rho)|^2$$
$$+ o(1)\|\nabla\rho\|_{L^2}^2,$$

yielding the result.

Now, combining Hölder and Sobolev inequalities, one can infer from Proposition 5.2 that

$$\begin{aligned}\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} |\nabla(\Psi_i\rho)|^2 \le& \sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} |\nabla\rho|^2\Psi_i^2 + \sum_{i=1}^{\kappa} \|\nabla\Psi_i\|_{L^N}^2\|\rho\|_{L^{2^*}}^2 \\ &+ \sum_{i=1}^{\nu} \|\nabla\Psi_i\|_{L^N}\|\Psi_i\|_{L^\infty}\|\rho\|_{L^{2^*}}\|\nabla\rho\|_{L^2} \\ \le& \int_{\mathbb{R}^N} |\nabla\rho|^2 + o(1)\|\nabla\rho\|_{L^2}^2.\end{aligned} \tag{5.12}$$

Therefore, putting (5.1)-(5.12) together it follows that

$$(2_\mu^*-1)\sum_{i=1}^{\kappa}|\alpha_i|^{2_\mu^*}\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * W_j^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho^2 + 2_\mu^*\sum_{j=1}^{\kappa}|\alpha_i|^{2_\mu^*-1}\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * \big(\sigma^{2_\mu^*-1}\rho\big)\Big) W_j^{2_\mu^*-1}\rho$$
$$\begin{aligned}\le&\big(1+o(1)\big)\Big[(2_\mu^*-1)\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * W_i^{2_\mu^*}\Big) W_i^{2_\mu^*-2}(\Psi_i\rho)^2 \\ &+ 2_\mu^*\sum_{i=1}^{\kappa}\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * \big(W_i^{2_\mu^*-1}\Psi_i\rho\big)\Big) W_i^{2_\mu^*-1}\Psi_i\rho\Big] + o(1)\|\nabla\rho\|_{L^2}^2 \\ \le&\nu(N,\kappa)\int_{\mathbb{R}^N} |\nabla\rho|^2.\end{aligned}$$

Hence the result easily follows. □

5.2. *Proof of Proposition 4.3*

To simplify notation, we denote W_i as $\mathcal{W}$, α_i as α, and $\sum_{j\neq i}\alpha_j W_j$ as $\mathcal{Z}$. We also introduce $\Psi=\Psi_i$ as the bump function from Proposition 5.2, assuming there is no ambiguity.

We start by establishing the following identity:

$$
\begin{aligned}
&\big(\alpha-\alpha^{2\cdot 2_\mu^*-1}\big)\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}-2_\mu^*\alpha^{2(2_{\mu}*-1)}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*-1}\mathcal{Z}\Big)\mathcal{W}^{2_\mu^*-1}\\
&\quad-\big(2_\mu^*-1\big)\alpha^{2(2_{\mu}*-1)}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\mathcal{Z}\\
=&\Delta\rho+\Big[-\Delta u-\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big]-\sum\alpha_i\Big(\frac{1}{|x|^\mu}*W_i^{2_\mu^*}\Big)W_i^{2_\mu^*-1}\\
&+2_\mu^*\Big(\frac{1}{|x|^\mu}*\big((\alpha\mathcal{W})^{2_\mu^*-1}\rho\big)\Big)(\alpha\mathcal{W})^{2_\mu^*-1}+\big(2_\mu^*-1\big)\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W})^{2_\mu^*}\Big)(\alpha\mathcal{W})^{2_\mu^*-2}\rho\\
&+\Big[\Big(\frac{1}{|x|^\mu}*|\sigma+\rho|^{2_\mu^*}\Big)|\sigma+\rho|^{2_\mu^*-2}(\sigma+\rho)-\Big(\frac{1}{|x|^\mu}*\sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-1}\Big]\\
&+\Big[-2_\mu^*\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-1}-\big(2_\mu^*-1\big)\Big(\frac{1}{|x|^\mu}*\sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho\Big]\\
&+\Big[2_\mu^*\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-1}+\big(2_\mu^*-1\big)\Big(\frac{1}{|x|^\mu}*\sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho\Big]\\
&+\Big[-2_\mu^*\Big(\frac{1}{|x|^\mu}*\big((\alpha\mathcal{W})^{2_\mu^*-1}\rho\big)\Big)(\alpha\mathcal{W})^{2_\mu^*-1}-\big(2_\mu^*-1\big)\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W})^{2_\mu^*}\Big)(\alpha\mathcal{W})^{2_\mu^*-2}\rho\Big]\\
&+\Big[\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W}+\mathcal{Z})^{2_\mu^*}\Big)(\alpha\mathcal{W}+\mathcal{Z})^{2_\mu^*-1}-\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W})^{2_\mu^*}\Big)(\alpha\mathcal{W})^{2_\mu^*-1}\Big]\\
&+\Big[-2_\mu^*\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W})^{2_\mu^*-1}\mathcal{Z}\Big)(\alpha\mathcal{W})^{2_\mu^*-1}-\big(2_\mu^*-1\big)\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W})^{2_\mu^*}\Big)(\alpha\mathcal{W})^{2_\mu^*-2}\mathcal{Z}\Big].
\end{aligned}
\tag{5.13}
$$

Now let's evaluate some terms on the right-hand side of equation (5.13). Utilizing Proposition 5.2-(2), we can obtain the following expression in the region $\Psi>0$:

$$
\sum_{j\neq i}\alpha_j\Big(\frac{1}{|x|^\mu}*W_j^{2_\mu^*}\Big)W_j^{2_\mu^*-1}=o\Big[\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\mathcal{Z}\Big].
$$

By a straightforward computation

$$
\begin{aligned}
&2_\mu^*\Big[\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-1}-\Big(\frac{1}{|x|^\mu}*\big((\alpha\mathcal{W})^{2_\mu^*-1}\rho\big)\Big)(\alpha\mathcal{W})^{2_\mu^*-1}\Big]\\
&\quad+\big(2_\mu^*-1\big)\Big[\Big(\frac{1}{|x|^\mu}*\sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho-\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W})^{2_\mu^*}\Big)(\alpha\mathcal{W})^{2_\mu^*-2}\rho\Big]
\end{aligned}
$$

$$= o\Big[\Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}\Big] + o\Big[\Big(\frac{1}{|x|^{\mu}} * \big(\mathcal{W}^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}\Big]$$
$$+ o\Big[\Big(\frac{1}{|x|^{\mu}} * \sigma^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\rho\Big] + o\Big[\Big(\frac{1}{|x|^{\mu}} * \mathcal{W}^{2_\mu^*}\Big)W^{2_\mu^*-2}|\rho|\Big],$$

and

$$\Big(\frac{1}{|x|^{\mu}} * \big(\alpha\mathcal{W} + \mathcal{Z}\big)^{2_\mu^*}\Big)\big(\alpha\mathcal{W} + \mathcal{Z}\big)^{2_\mu^*-1} - \Big(\frac{1}{|x|^{\mu}} * \big(\alpha\mathcal{W}\big)^{2_\mu^*}\Big)\big(\alpha\mathcal{W}\big)^{2_\mu^*-1}$$
$$- 2_\mu^*\Big(\frac{1}{|x|^{\mu}} * \big((\alpha\mathcal{W})^{2_\mu^*-1}\mathcal{Z}\big)\Big)\big(\alpha\mathcal{W}\big)^{2_\mu^*-1} - (2_\mu^* - 1)\Big(\frac{1}{|x|^{\mu}} * \big(\alpha\mathcal{W}\big)^{2_\mu^*}\Big)\big(\alpha\mathcal{W}\big)^{2_\mu^*-2}\mathcal{Z}$$
$$= \Big[\frac{1}{|x|^{\mu}} * \Big((\alpha\mathcal{W} + \mathcal{Z})^{2_\mu^*} - (\alpha\mathcal{W})^{2_\mu^*} - 2_\mu^*(\alpha\mathcal{W})^{2_\mu^*-1}\mathcal{Z}\Big)\Big]\big(\alpha\mathcal{W} + \mathcal{Z}\big)^{2_\mu^*-1}$$
$$+ \Big(\frac{1}{|x|^{\mu}} * \big(\alpha\mathcal{W}\big)^{2_\mu^*}\Big)\Big[(\alpha\mathcal{W} + \mathcal{Z})^{2_\mu^*-1} - (\alpha\mathcal{W})^{2_\mu^*-1} - (2_\mu^* - 1)(\alpha\mathcal{W})^{2_\mu^*-2}\mathcal{Z}\Big]$$
$$+ 2_\mu^*\Big(\frac{1}{|x|^{\mu}} * \big((\alpha\mathcal{W})^{2_\mu^*-1}\mathcal{Z}\big)\Big)\Big[(\alpha\mathcal{W} + \mathcal{Z})^{2_\mu^*-1} - (\alpha\mathcal{W})^{2_\mu^*-1} - (2_\mu^* - 1)(\alpha\mathcal{W})^{2_\mu^*-2}\mathcal{Z}\Big]$$
$$+ 2_\mu^*(2_\mu^* - 1)\Big(\frac{1}{|x|^{\mu}} * \big((\alpha\mathcal{W})^{2_\mu^*-1}\mathcal{Z}\big)\Big)\big(\alpha\mathcal{W}\big)^{2_\mu^*-2}\mathcal{Z}$$
$$= o\Big[\Big(\frac{1}{|x|^{\mu}} * \big(\mathcal{W}^{2_\mu^*-1}\mathcal{Z}\big)\Big)\mathcal{W}^{2_\mu^*-1}\Big] + o\Big[\Big(\frac{1}{|x|^{\mu}} * \mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\mathcal{Z}\Big].$$

Furthermore, by applying an elementary inequality, we have that

$$\Big(\frac{1}{|x|^{\mu}} * |\sigma + \rho|^{2_\mu^*}\Big)|\sigma + \rho|^{2_\mu^*-2}(\sigma + \rho) - \Big(\frac{1}{|x|^{\mu}} * \sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-1} - 2_\mu^*\Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-1}$$
$$- \big(2_\mu^* - 1\big)\Big(\frac{1}{|x|^{\mu}} * \sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho$$
$$\lesssim \Big(\frac{1}{|x|^{\mu}} * \rho^{2_\mu^*}\Big)\rho^{2_\mu^*-1} + \Big(\frac{1}{|x|^{\mu}} * \sigma^{2_\mu^*}\Big)\rho^{2_\mu^*-1} + \Big(\frac{1}{|x|^{\mu}} * \sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-3}\rho^2$$
$$+ \Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-2}\rho + \Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-3}\rho^2 + \Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-1}\rho\big)\Big)\rho^{2_\mu^*-1}$$
$$+ \Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-2}\rho^2\big)\Big)\sigma^{2_\mu^*-2}\rho + \Big(\frac{1}{|x|^{\mu}} * \rho^{2_\mu^*}\Big)\sigma^{2_\mu^*-3}\rho^2 + \Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-2}\rho^2\big)\Big)\sigma^{2_\mu^*-3}\rho^2$$
$$+ \Big(\frac{1}{|x|^{\mu}} * \big(\sigma^{2_\mu^*-2}\rho^2\big)\Big)\rho^{2_\mu^*-1} + \Big(\frac{1}{|x|^{\mu}} * \rho^{2_\mu^*}\Big)\sigma^{2_\mu^*-1} + \Big(\frac{1}{|x|^{\mu}} * \rho^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho$$
$$=: \Gamma(\rho).$$

Therefore, combining (5.13) and the preceding estimates, we obtain the estimate

$$
\begin{aligned}
&\big(\alpha-\alpha^{2\cdot 2_\mu^*-1}\big)\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}-\big(2_\mu^*\alpha^{2(2_{\mu^*}-1)}+o(1)\big)\Big(\frac{1}{|x|^\mu}*\big(\mathcal{W}^{2_\mu^*-1}\mathcal{Z}\big)\Big)\mathcal{W}^{2_\mu^*-1}\\
&\quad-\big((2_\mu^*-1)\alpha^{2(2_{\mu^*}-1)}+o(1)\big)\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\mathcal{Z}-\Delta\rho\\
&\quad-\Big[-\Delta u-\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big]\\
&\quad-2_\mu^*\Big(\frac{1}{|x|^\mu}*\big((\alpha\mathcal{W})^{2_\mu^*-1}\rho\big)\Big)(\alpha\mathcal{W})^{2_\mu^*-1}-(2_\mu^*-1)\Big(\frac{1}{|x|^\mu}*(\alpha\mathcal{W})^{2_\mu^*}\Big)(\alpha\mathcal{W})^{2_\mu^*-2}\rho\\
&\lesssim\Gamma(\rho)+o\Big[\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}\Big]+o\Big[\Big(\frac{1}{|x|^\mu}*\big(\mathcal{W}^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}\Big]\\
&\quad+o\Big[\Big(\frac{1}{|x|^\mu}*\sigma^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}|\rho|\Big]+o\Big[\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}|\rho|\Big].
\end{aligned}
\tag{5.14}
$$

Now by letting η be either $\mathcal{W}$ or $\frac{\partial\mathcal{W}}{\partial\lambda}$, we see that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\nabla\rho\nabla\eta=0,\\
&2_\mu^*\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(\mathcal{W}^{2_\mu^*-1}\rho\big)\Big)\mathcal{W}^{2_\mu^*-1}\eta+(2_\mu^*-1)\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\eta\rho=0,
\end{aligned}
\tag{5.15}
$$

by orthogonality conditions. By testing (5.14) with $\eta\Psi$, we obtain

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\Big[\big(\alpha-\alpha^{2\cdot 2_\mu^*-1}\big)\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}-\big(2_\mu^*\alpha^{2(2_\mu^*-1)}+o(1)\big)\Big(\frac{1}{|x|^\mu}*\big(\mathcal{W}^{2_\mu^*-1}\mathcal{Z}\big)\Big)\mathcal{W}^{2_\mu^*-1}\\
&\quad-\big((2_\mu^*-1)\alpha^{2(2_\mu^*-1)}+o(1)\big)\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}V\Big]\eta\Psi\Big|\\
&\lesssim\Big|\int_{\mathbb{R}^N}\Gamma(\rho)\eta\Psi\Big|+\Big|\int_{\mathbb{R}^N}\nabla\rho\nabla(\eta\Psi)\Big|+\Big|\int_{\mathbb{R}^N}\Big[-\Delta u-\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big]\eta\Psi\Big|\\
&\quad+\Big|2_\mu^*\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(\mathcal{W}^{2_\mu^*-1}\rho\big)\Big)\mathcal{W}^{2_\mu^*-1}\eta\Psi+(2_\mu^*-1)\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\rho\eta\Psi\Big|\\
&\quad+o\Big[\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}|\eta|\Psi\Big]+o\Big[\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\big(\mathcal{W}^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}|\eta|\Psi\Big]\\
&\quad+o\Big[\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\sigma^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}|\rho||\eta|\Psi\Big]+o\Big[\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}|\rho||\eta|\Psi\Big].
\end{aligned}
\tag{5.16}
$$

Let us estimate each term of the right hand side of (5.16).

$$\Big|\int_{\mathbb{R}^N} \nabla\rho\nabla(\eta\Psi)\Big| = \Big|\int_{\mathbb{R}^N} \nabla\rho\nabla\big(\eta(\Psi-1)\big)\Big| \le \big\|\nabla\rho\big\|_{L^2}\big\|\nabla\big(\eta(\Psi-1)\big\|_{L^2},$$

$$\Big|\int_{\mathbb{R}^N} \Big[-\Delta u - \Big(\frac{1}{|x|^\mu} * |u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big]\eta\Psi\Big|$$
$$\le \Big\|\Delta u + \Big(\frac{1}{|x|^\mu} * |u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}\big\|\nabla(\eta\Psi)\big\|_{L^2}.$$

Combining (5.15) with HLS and Sobolev inequalities, we get that

$$\Big|2_\mu^*\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \big(\mathcal{W}^{2_\mu^*-1}\rho\big)\Big)\mathcal{W}^{2_\mu^*-1}\eta\Psi + (2_\mu^*-1)\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\rho\eta\Psi\Big|$$
$$\lesssim \|\mathcal{W}\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\|\nabla\rho\|_{L^2}\Big(\int_{\{\Psi<1\}}\big(\mathcal{W}^{2_\mu^*-1}|\eta|\big)^{\frac{2N}{2N-\mu}}\Big)^{\frac{2N-\mu}{2N}}$$
$$+\|\mathcal{W}\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\|\nabla\rho\|_{L^2}\Big(\int_{\{\Psi<1\}}\big(\mathcal{W}^{2_\mu^*-2}|\eta|\big)^{\frac{2N}{N-\mu+2}}\Big)^{\frac{N-\mu+2}{2N}},$$

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \big(\sigma^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}|\eta|\Psi \lesssim \|\sigma\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\big\|\mathcal{W}^{2_\mu^*-1}\eta\big\|_{L^{\frac{2N}{2N-\mu}}}\|\nabla\rho\|_{L^2},$$

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \big(\mathcal{W}^{2_\mu^*-1}|\rho|\big)\Big)\mathcal{W}^{2_\mu^*-1}|\eta|\Psi \lesssim \|\mathcal{W}\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\big\|\mathcal{W}^{2_\mu^*-1}\eta\big\|_{L^{\frac{2N}{2N-\mu}}}\|\nabla\rho\|_{L^2},$$

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \sigma^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}|\rho||\eta|\Psi \lesssim \|\sigma\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\big\|\mathcal{W}^{2_\mu^*-2}\eta\big\|_{L^{\frac{2N}{N-\mu+2}}}\|\nabla\rho\|_{L^2},$$

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}|\rho||\eta|\Psi \lesssim \|\mathcal{W}\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\big\|\mathcal{W}^{2_\mu^*-2}\eta\big\|_{L^{\frac{2N}{N-\mu+2}}}\|\nabla\rho\|_{L^2}.$$

It remains to estimate $\Big|\int_{\mathbb{R}^N}\Gamma(\rho)\eta\Psi\Big|$. Then by applying HLS inequality and Sobolev inequality to all terms, we have that

$$\Big|\Big(\frac{1}{|x|^\mu} * \rho^{2_\mu^*}\Big)\rho^{2_\mu^*-1}\eta\Psi\Big| \lesssim \|\nabla\rho\|_{L^2}^{\frac{3N-2\mu+2}{N-2}}\|\eta\|_{L^{2^*}},$$

$$\Big|\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \sigma^{2_\mu^*}\Big)\rho^{2_\mu^*-1}\eta\Psi + \int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * \sigma^{2_\mu^*}\Big)\sigma^{2_\mu^*-3}\rho^2\eta\Psi\Big|$$
$$\lesssim \|\sigma\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\|\nabla\rho\|_{L^2}^{\frac{N-\mu+2}{N-2}}\|\eta\|_{L^{2^*}} + \|\sigma\|_{L^{2^*}}^{\frac{N-2\mu+6}{N-2}}\|\nabla\rho\|_{L^2}^2\|\eta\|_{L^{2^*}},$$

$$\Big|\int_{\mathbb{R}^N}\Big[\Big(\frac{1}{|x|^\mu}*\rho^{2_\mu^*}\Big)\sigma^{2_\mu^*-3}\rho^2+\Big(\frac{1}{|x|^\mu}*\rho^{2_\mu^*}\Big)\sigma^{2_\mu^*-1}+\Big(\frac{1}{|x|^\mu}*\rho^{2_\mu^*}\Big)\sigma^{2_\mu^*-2}\rho\Big]\eta\Psi\Big|$$

$$\lesssim\|\sigma\|_{L^{2^*}}^{2_\mu^*-3}\|\nabla\rho\|_{L^2}^{2_\mu^*+2}\|\eta\|_{L^{2^*}}+\|\sigma\|_{L^{2^*}}^{2_\mu^*-1}\|\nabla\rho\|_{L^2}^{2_\mu^*}\|\eta\|_{L^{2^*}}+\|\sigma\|_{L^{2^*}}^{2_\mu^*-2}\|\nabla\rho\|_{L^2}^{2_\mu^*+1}\|\eta\|_{L^{2^*}},$$

$$\Big|\int_{\mathbb{R}^N}\Big[\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-2}\rho+\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}\rho\big)\Big)\sigma^{2_\mu^*-3}\rho^2+\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-1}\rho\big)\Big)\rho^{2_\mu^*-1}\Big]\eta\Psi\Big|$$

$$\lesssim\|\sigma\|_{L^{2^*}}^{\frac{N-2\mu+6}{N-2}}\|\nabla\rho\|_{L^2}^2\|\eta\|_{L^{2^*}}+\|\sigma\|_{L^{2^*}}^{\frac{8-2\mu}{N-2}}\|\nabla\rho\|_{L^2}^3\|\eta\|_{L^{2^*}}+\|\sigma\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\|\nabla\rho\|_{L^2}^{\frac{2N-\mu}{N-2}}\|\eta\|_{L^{2^*}},$$

and

$$\Big|\int_{\mathbb{R}^N}\Big[\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-2}\rho^2\big)\Big)\sigma^{2_\mu^*-2}\rho+\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-2}\rho^2\big)\Big)\sigma^{2_\mu^*-3}\rho^2$$

$$+\Big(\frac{1}{|x|^\mu}*\big(\sigma^{2_\mu^*-2}\rho^2\big)\Big)\rho^{2_\mu^*-1}\Big]\eta\Psi\Big|$$

$$\lesssim\|\sigma\|_{L^{2^*}}^{2(2_\mu^*-2)}\|\nabla\rho\|_{L^2}^3\|\eta\|_{L^{2^*}}+\|\sigma\|_{L^{2^*}}^{2\cdot2_\mu^*-5}\|\nabla\rho\|_{L^2}^4\|\eta\|_{L^{2^*}}+\|\sigma\|_{L^{2^*}}^{2_\mu^*-2}\|\nabla\rho\|_{L^2}^{2_\mu^*+1}\|\eta\|_{L^{2^*}}.$$

Moreover, by Proposition 5.2 and $|\eta|\lesssim\mathcal{W}$, we get that

$$\|\nabla\big(\eta(\Psi-1)\big)\|_{L^2}=o(1),\quad \|\nabla(\eta\Psi)\|_{L^2}\lesssim1,$$

$$\|\mathcal{W}\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\Big(\int_{\{\Psi<1\}}\big(\mathcal{W}^{2_\mu^*-1}|\eta|\big)^{\frac{2N}{2N-\mu}}\Big)^{\frac{2N-\mu}{2N}}=o(1),$$

$$\|\mathcal{W}\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\Big(\int_{\{\Psi<1\}}\big(\mathcal{W}^{2_\mu^*-2}|\eta|\big)^{\frac{2N}{N-\mu+2}}\Big)^{\frac{N-\mu+2}{2N}}=o(1),$$

$$\|\sigma\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\|\mathcal{W}^{2_\mu^*-1}\eta\|_{L^{\frac{2N}{2N-\mu}}}\lesssim1,\quad \|\mathcal{W}\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\|\mathcal{W}^{2_\mu^*-1}\eta\|_{L^{\frac{2N}{2N-\mu}}}\lesssim1,$$

$$\|\sigma\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\|\mathcal{W}^{2_\mu^*-2}\eta\|_{L^{\frac{2N}{N-\mu+2}}}\lesssim1,\quad \|\mathcal{W}\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\|\mathcal{W}^{2_\mu^*-2}\eta\|_{L^{\frac{2N}{N-\mu+2}}}\lesssim1,$$

$$\|\eta\|_{L^{2^*}}\lesssim1,\quad \|\sigma\|_{L^{2^*}}^{\frac{2N-\mu}{N-2}}\|\eta\|_{L^{2^*}}\lesssim1,\quad \|\sigma\|_{L^{2^*}}^{\frac{N-2\mu+6}{N-2}}\|\eta\|_{L^{2^*}}\lesssim1,$$

$$\|\sigma\|_{L^{2^*}}^{\frac{6-N-\mu}{N-2}}\|\eta\|_{L^{2^*}}\lesssim1,\quad \|\sigma\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\|\eta\|_{L^{2^*}}\lesssim1,\quad \|\sigma\|_{L^{2^*}}^{\frac{4-\mu}{N-2}}\|\eta\|_{L^{2^*}}\lesssim1,$$

$$\|\sigma\|_{L^{2^*}}^{\frac{N-2\mu+6}{N-2}}\|\eta\|_{L^{2^*}}\lesssim1,\quad \|\sigma\|_{L^{2^*}}^{\frac{8-2\mu}{N-2}},\|\eta\|_{L^{2^*}}\lesssim1,\quad \|\sigma\|_{L^{2^*}}^{\frac{N-\mu+2}{N-2}}\|\eta\|_{L^{2^*}}\lesssim1,$$

and

$$\|\sigma\|_{L^{2^*}}^{2(2_\mu^*-2)}\|\eta\|_{L^{2^*}}\lesssim1,\quad \|\sigma\|_{L^{2^*}}^{2\cdot2_\mu^*-5}\|\eta\|_{L^{2^*}}\lesssim1\quad ,\|\sigma\|_{L^{2^*}}^{2_\mu^*-2}\|\eta\|_{L^{2^*}}\lesssim1.$$

Therefore, combining (5.16) and the above all estimates, we eventually have

$$\begin{aligned}&\Bigg|\int_{\mathbb{R}^N}\Big[\big(\alpha-\alpha^{2\cdot 2_\mu^*-1}\big)\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}-\Big(2_\mu^*\alpha^{2(2_\mu^*-1)}+o(1)\Big)\Big(\frac{1}{|x|^\mu}*\big(\mathcal{W}^{2_\mu^*-1}\mathcal{Z}\big)\Big)\mathcal{W}^{2_\mu^*-1}\\&\quad-\Big(\big(2_\mu^*-1\big)\alpha^{2(2_\mu^*-1)}+o(1)\Big)\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\mathcal{Z}\Big]\eta\Psi\Bigg|\\&\lesssim o(1)\|\nabla\rho\|_{L^2}+\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}+\|\nabla\rho\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})}.\end{aligned}\tag{5.17}$$

Let us denote $\mathcal{Z}=\mathcal{Z}_1+\mathcal{Z}_2$, where $\mathcal{Z}_1:=\sum j<i\alpha_jW_j$ and $\mathcal{Z}_2:=\sum j>i\alpha_jW_j$.

By applying induction, we can assume that the statement of the proposition holds for all $j<i$. Now, we aim to show that the proposition also holds for all $j>i$.

By noting that

$$\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_i^{2_\mu^*}\Big)W_i^{2_\mu^*-1}W_j=\int_{\mathbb{R}^N}\nabla W_i\nabla W_j=\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W_j^{2_\mu^*}\Big)W_j^{2_\mu^*-1}W_i,$$

then we get that

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W^{2_\mu^*}\Big)W^{2_\mu^*-2}\mathcal{Z}_1|\eta|\Psi&\lesssim\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*W^{2_\mu^*}\Big)W^{2_\mu^*-1}\mathcal{Z}_1\Psi\\&\lesssim\epsilon\|\nabla\rho\|_{L^2}+\Big\|\Delta u+\Big(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}+\|\nabla\rho\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})},\end{aligned}\tag{5.18}$$

by $|\eta|\le\mathcal{W}$. Now we can proceed to prove (4.4) of Proposition 4.3. If $\alpha=1$, the proof is complete. Therefore, we will consider the case where $\alpha\neq 1$.

Let us begin by examining the left-hand side of inequality (5.17). Notice that, according to Proposition 5.2-(4), we have the following result:

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}-\big(1+o(1)\big)\hat{\xi}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\Big]\eta\Psi\\&\quad=\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}\eta-\hat{\xi}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\eta+o(1),\end{aligned}\tag{5.19}$$

where

$$\hat{\xi}:=\frac{(2_\mu^*-1)\alpha^{2(2_\mu^*-1)}\mathcal{Z}_2(0)}{\alpha-\alpha^{2\cdot 2_\mu^*-1}}.$$

Furthermore, choosing $\eta=W$ and $\eta=\partial_\lambda\mathcal{W}$, respectively, we have that

$$\frac{\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*}}{\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}}\neq\frac{\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-1}\partial_\lambda\mathcal{W}}{\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\Big)\mathcal{W}^{2_\mu^*-2}\partial_\lambda\mathcal{W}},$$

where we have used the fact that

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2^*_\mu}\Big)\mathcal{W}^{2^*_\mu-1}\partial_\lambda \mathcal{W} = \int_{\mathbb{R}^N} \nabla\mathcal{W}\nabla\partial_\lambda\mathcal{W} = 0.$$

By a straight computation,

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2^*_\mu}\Big)\mathcal{W}^{2^*_\mu-2}\partial_\lambda \mathcal{W} = \frac{\widetilde{\mathcal{Q}}}{2^*-1}\frac{d}{d\lambda}\Big|_{\lambda=1}\Big(\frac{1}{\lambda^{N-2/2}}\int_{\mathbb{R}^N} \mathcal{W}^{2^*-1}[0,1]\Big) \neq 0.$$

The preceding arguments indicate that the equality (5.19) cannot be arbitrarily small and can be maximized by choosing an appropriate value for η. Consequently, taking into account the fact that $\mathcal{Z}_2(x)\Psi(x) = (1+o(1))\mathcal{Z}_2(0)\Psi(x)$, along with Proposition 5.2-(2) and the inequalities (5.17)-(5.18), we obtain the following implication:

$$\begin{aligned}
&\Big|\alpha - \alpha^{2\cdot 2^*_\mu-1}\Big|\Big|\int_{\mathbb{R}^N}\Big[\Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2^*_\mu}\Big)\mathcal{W}^{2^*_\mu-1} - \big(1+o(1)\big)\hat{\xi}\Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2^*_\mu}\Big)\mathcal{W}^{2^*_\mu-2}\Big]\eta\Psi\Big| \\
&\quad\lesssim \Big|\int_{\mathbb{R}^N}\Big[(\alpha - \alpha^{2\cdot 2^*_\mu-1})\Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2^*_\mu}\Big)\mathcal{W}^{2^*_\mu-1} \\
&\qquad - \Big(2^*_\mu\alpha^{2(2^*_\mu-1)} + o(1)\Big)\Big(\frac{1}{|x|^\mu} * \big(\mathcal{W}^{2^*_\mu-1}\sum_{j\neq i}\alpha_j W_j\big)\Big)\mathcal{W}^{2^*_\mu-1} \\
&\qquad - \Big((2^*_\mu-1)\alpha^{2(2^*_\mu-1)} + o(1)\Big)\Big(\frac{1}{|x|^\mu} * \mathcal{W}^{2^*_\mu}\Big)\mathcal{W}^{2^*_\mu-2}\mathcal{Z}\Big]\eta\Psi\Big| \\
&\qquad + \big(2^*_\mu-1\big)\alpha^{2(2^*_\mu-1)}\int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\mu} * W^{2^*_\mu}\Big)W^{2^*_\mu-2}\mathcal{Z}_1|\eta|\Psi \\
&\quad\lesssim o(1)\big\|\nabla\rho\big\|_{L^2} + \Big\|\Delta u + \Big(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\Big)|u|^{2^*_\mu-2}u\Big\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}} \\
&\qquad + \big\|\nabla\rho\big\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})}.
\end{aligned} \tag{5.20}$$

Hence, the first conclusion of Proposition 4.3 follows directly from the inequalities (5.19) and (5.20).

Now, we are ready to finalize the proof of the second conclusion of Proposition 4.3. By applying Lemma 5.2-(2) and setting $\epsilon = o(1)$, we can deduce from (5.17), with $\eta = \mathcal{W}$, the following inequality for $j \neq i$:

$$
\begin{aligned}
&\Bigg|\left(\alpha-\alpha^{2\cdot 2_\mu^*-1}\right)\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\right)\mathcal{W}^{2_\mu^*}\Psi-o(1)\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\right)\mathcal{W}^{2_\mu^*}\Psi\\
&\quad-\left((2_\mu^*-1)\alpha^{2(2_\mu^*-1)}+o(1)\right)\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\right)\mathcal{W}^{2_\mu^*-1}\mathcal{Z}\Psi\Bigg| \qquad (5.21)\\
&\lesssim o(1)\|\nabla\rho\|_{L^2}+\left\|\Delta u+\left(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\right)|u|^{2_\mu^*-2}u\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}+\|\nabla\rho\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})}.
\end{aligned}
$$

Then from the first conclusion of Proposition 4.3 and (5.21) imply that, for $j\neq i$,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*\mathcal{W}^{2_\mu^*}\right)\mathcal{W}^{2_\mu^*-1}\mathcal{Z}\Psi\right| \qquad (5.22)\\
&\lesssim o(1)\|\nabla\rho\|_{L^2}+\left\|\Delta u+\left(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\right)|u|^{2_\mu^*-2}u\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}+\|\nabla\rho\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})}.
\end{aligned}
$$

Combining the estimate from [14],

$$
\int_{\mathbb{R}^N}\mathcal{W}_i^{2^*-1}\mathcal{W}_j=\int_{\mathbb{R}^N}\mathcal{W}_j^{2^*-1}\mathcal{W}_i\approx\int_{B(0,1)}\mathcal{W}_i^{2^*-1}\mathcal{W}_j \quad \text{for any} \quad j\neq i,
$$

then we get that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*\mathcal{W}_i^{2_\mu^*}\right)\mathcal{W}_i^{2_\mu^*-1}\mathcal{W}_j=\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*\mathcal{W}_j^{2_\mu^*}\right)\mathcal{W}_j^{2_\mu^*-1}\mathcal{W}_i\\
&\lesssim o(1)\|\nabla\rho\|_{L^2}+\left\|\Delta u+\left(\frac{1}{|x|^\mu}*|u|^{2_\mu^*}\right)|u|^{2_\mu^*-2}u\right\|_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^{-1}}+\|\nabla\rho\|_{L^2}^{\min(2,\frac{N-\mu+2}{N-2})},
\end{aligned}
$$

for any $j\neq i$ and imply the second conclusion of Proposition 4.3 holds for all $j>i$. Hence the conclusion can be deduced by the induction. □

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No data was used for the research described in the article.

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