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Additive maps preserving products equal to fixed elements on Cayley-Dickson algebras

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ABSTRACT

In this paper, we study additive maps $f : \mathcal{A} \to \mathcal{A}$ on an alternative Cayley-Dickson algebra $\mathcal{A}$ satisfying the product preserving property: $f(x)f(y) = m$ whenever $xy = k$, where k and m are constant elements of $\mathcal{A}$. We prove that f may be written as a multiple of a Jordan homomorphism. Unlike in the associative case, Jordan homomorphisms of strictly alternative division rings need not be homomorphisms or anti-homomorphisms, but we provide necessary and sufficient conditions for this to occur. We then extend these ideas to split alternative Cayley-Dickson algebras of characteristic not 2 and present some open questions when k or m is noninvertible. The final section handles the problem $k = m = 0$ and generalizes to biadditive maps that preserve the zero product.

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1. Introduction

This paper is primarily concerned with the characterization of additive maps on certain (not necessarily associative) algebras that preserve products equal to fixed elements. In other words, if $\mathcal{A}$ is an algebra with two fixed elements $k, m \in \mathcal{A}$, and $f : \mathcal{A} \to \mathcal{A}$ is an additive map such that $f(x)f(y) = m$, whenever $xy = k$, is it possible to canonically describe f, provided such a map exists? Clearly, if f is a (ring) homomorphism, then taking $m = f(k)$ yields a solution. Moreover, if $k = m = 1$, then (ring) homomorphisms and anti-homomorphisms are solutions. But must f be a homomorphism or anti-homomorphism?

This general problem originates in [5] in the context of associative division rings and has been investigated on other associative structures, such as those in [3, 4, 7, 12, 13, 17]. All cited results have drawn the same conclusion that, under mild conditions, $f(x) = \lambda\varphi(x)$ where $\varphi(x)$ is a homomorphism or anti-homomorphism and $\lambda = f(1)$ is some constant element. There is a vast literature on the so-called *zero product preserver problem*, concerning maps f satisfying $f(x)f(y) = 0$ whenever $xy = 0$ on some ring or algebra. Extensive references may be found in [1, 6, 12, 17]. Our aim is to extend these results for certain nonassociative structures; in particular, *alternative* rings and algebras.

Alternative division rings are, by the celebrated Bruck-Kleinfeld theorem [2], either associative division rings or octonion division algebras over their center. It is frequently the case that theorems and techniques for associative division rings carry over to the alternative setting; for example, the description of commuting maps and certain functional identities involving inverses—see [11]. But not always. In fact, we will exhibit examples of product preserving maps on octonion division algebras

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that are *not* homomorphisms or anti-homomorphisms (see Example 3.2) but are still so-called *Jordan homomorphisms*. This is a stark departure from the associative case, where the classical works of Herstein [15] and Smiley [20, 21] proving that every Jordan homomorphism of a simple, associative ring is a homomorphism or anti-homomorphism. Therefore these famous results do not hold for alternative rings and algebras.

But, keeping with the theme that alternativity is close to associativity, we prove in this paper that fixed product preserving maps on certain alternative algebras must still be, up to left multiplication by a fixed element, a *Jordan automorphism*[1] (which is the same conclusion as in the associative cases.

More precisely, Theorems 3.3 and 3.5 treat division structures. Let $\mathcal{A}$ be an alternative division ring. Theorem 3.3 shows that an additive and bijective map $f : \mathcal{A} \to \mathcal{A}$ satisfying $f(x)f(y) = m$ whenever $xy = k$, where k and m are fixed nonzero elements of $\mathcal{A}$, may be written in the form $f(x) = f(1)\varphi(x)$, where φ is a unital Jordan automorphism. The second result Theorem 3.5 gives necessary and sufficient conditions for φ to be an automorphism or anti-automorphism, and concludes that either $f(1)$ or $f(k)$ is a central element, respectively, fully generalizing a result of Catalano [3, Theorem 5].

Next, Theorem 4.3 extends these ideas to alternative Cayley-Dickson algebras of characteristic not 2 and gives the same conclusion as Theorem 3.3 whenever m is invertible. This holds even in the case when $\mathcal{A}$ is split (i.e., $\mathcal{A}$ contains zero divisors), and therefore our results hold for any alternative Cayley-Dickson algebra of characteristic not 2. The situation for a noninvertible fixed element m can be pathological and we leave some open questions.

Finally, in Section 5, we present Theorem 5.2 on the zero product preserver problem and its generalizations to biadditive maps with an arbitrary codomain. We show via Theorem 5.3 that the split octonions form a *zero product determined algebra*, as defined in [1].

A coherent study of product preserving maps has thus been carried out on associative division rings [3, 5], matrix algebras (including certain Banach algebras) [4, 7, 12, 13, 17], test cases on some alternative rings [10], and now alternative Cayley-Dickson algebras in this paper. One connecting idea is Hua's Identity, which luckily holds in any alternative ring with unity (Lemma 2.1). But characterizing product preserving maps on other, strictly non-alternative algebras will likely be a challenge since Hua's identity depends on certain associativity laws. Therefore, while we are optimistic that fixed product preserving maps have nice descriptions on other non-associative algebras, may be helpful to find new techniques going forward.

2. Preliminaries

Let $\mathcal{R}$ be a ring (not necessarily associative or commutative) and consider the following notation convention for its multiplication operation: that $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in \mathcal{R}$. We denote the *associator* of the elements $x, y, z \in \mathcal{R}$ by $(x, y, z) = xy \cdot z - x \cdot yz$ and the *commutator* of elements $x, y \in \mathcal{R}$ by $[x, y] = xy - yx$. The *center* of a ring $\mathcal{R}$ is defined by

$$\mathcal{Z}(\mathcal{R}) = \{r \in \mathcal{R} \mid [r, x] = 0 \text{ for all } x \in \mathcal{R}\}.$$

We write $\mathcal{Z}$ instead of $\mathcal{Z}(\mathcal{R})$ when the underlying ring is clear from context.

A ring $\mathcal{A}$ is said to be *alternative* if satisfies both *left alternative law*:

$$(x, x, y) = 0$$

and *right alternative law*:

$$(y, x, x) = 0$$

for all $x, y \in \mathcal{A}$. Clearly any associative ring is also alternative. It is well-known that every alternative ring $\mathcal{A}$ satisfies the *flexible identity*

$$(x, y, x) = 0, \tag{1}$$

[1]And simultaneously a so-called *semi-automorphism*. These kinds of maps coincide in characteristic not 2 and see Definition 2.3 for a precise formulation.

for all $x, y \in \mathcal{A}$, and also satisfies the *Moufang identities*:

(a) $(xax)y = x(a(xy))$ Moufang left identity,
(b) $y(xax) = ((yx)a)x$ Moufang right identity,
(c) $(xy)(ax) = x(ya)x$ Moufang middle identity,

for all $x, y, a \in \mathcal{A}$, where we use the flexible identity (1) to unambiguously write $xax = x \cdot ax = xa \cdot x$.

The structure of alternative rings is well-known; for example, Artin's Theorem [19, Theorem 3.1] states that every subring of an alternative ring that is generated by two elements is associative. We refer to this result as simply "Artin's Theorem" in what follows.

If a ring $\mathcal{R}$ has a multiplicative identity element 1, then $x \in \mathcal{R}$ is said to be *invertible* if there exists $y \in \mathcal{R}$ satisfying $xy = yx = 1$ and y is called an *inverse* of x. A ring $\mathcal{R}$ with 1 is called a *division ring* if every nonzero element in $\mathcal{R}$ has an inverse. Throughout this paper, $\mathcal{R}^\times$ denotes the set of invertible elements of $\mathcal{R}$. From the Moufang identities, every alternative ring $\mathcal{R}$ with 1 has uniqueness of multiplicative inverses and satisfies the inverse associator identity:

$$(x, x^{-1}, y) = 0 \tag{2}$$

for all $x \in \mathcal{R}^\times$ and $y \in \mathcal{R}$. It also can be shown that $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in \mathcal{R}^\times$, which we use without mention. Each of these basic properties of inverses may be found on [19, p. 38] for example. Putting them together, Hua's identity holds for alternative division rings.

Lemma 2.1 (Hua's identity). *If $\mathcal{A}$ is a unital, alternative ring and $a, b \in \mathcal{A}$, then*

$$a - \left(a^{-1} + (b^{-1} - a)^{-1}\right)^{-1} = aba, \tag{3}$$

whenever a, b, and $1 - ab$ are invertible.

The Moufang identities also yield another familiar property of inverses.

Lemma 2.2. *If $\mathcal{A}$ is a unital, alternative ring and $a, b \in \mathcal{A}$ are elements such that $ab = ba$ and ab is invertible, then both a and b are invertible.*

We recall some standard definitions for various kinds of additive maps.

Definition 2.3. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an additive map of alternative rings.

- If $\varphi(a^2) = \varphi(a)^2$ for all $a \in \mathcal{A}$, then we call φ a *Jordan homomorphism*. Equivalently, if the characteristic of $\mathcal{A}$ is not 2, $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$.
- If $\varphi(aba) = \varphi(a)\varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$, then we call φ a *semi-homomorphism.*
- As is customary, we use the word *automorphism* instead of homomorphism when $\mathcal{A} = \mathcal{B}$ and φ is bijective. We also call φ *unital* if $\varphi(1) = 1$. Homomorphisms and anti-homomorphisms are not assumed to be unital below.

Clearly, every homomorphism and anti-homomorphism is both a Jordan homomorphism and a semi-homomorphism. If $\mathcal{A}$ is an associative simple ring of characteristic not 2, every Jordan homomorphism is either a homomorphism or anti-homomorphism (see Theorem I in [15] and the Main Theorem in [21]). Example 3.2 shows that alternative division rings need not have this property.

While on the topic of semi-automorphisms, we mention the following known result and inspiration for this paper.

Proposition 2.4. [10, Proposition 2.1] *Let $\mathcal{A}$ be an alternative division ring. If $\varphi : \mathcal{A} \to \mathcal{A}$ is additive and satisfies $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in \mathcal{A}^\times$, then φ is a semi-automorphism.*

Indeed, the property $\varphi(a^{-1}) = \varphi(a)^{-1}$ can be equivalently written as $\varphi(x)\varphi(y) = 1$ whenever $xy = 1$. Consequently, additive maps preserving products equal to 1 are necessarily semi-automorphisms of $\mathcal{A}$. The proof of one of our main theorems below (Theorem 3.3) reduces the general product preserving problem back to maps preserving products equal to 1.

Lastly, while the notions of Jordan homomorphism and semi-homomorphism are kept separate in this paper, the distinction is only necessary for rings of characteristic 2. In characteristic not 2, the notions essentially coincide. Indeed, every Jordan homomorphism φ is a semi-automorphism. The reader may supply a proof by computing $\varphi((xy + yx)x + x(xy + yx)) = \varphi(xy + yx)\varphi(x) + \varphi(x)\varphi(xy + yx)$ and concluding that $2\varphi(xyx) = 2\varphi(x)\varphi(y)\varphi(x)$. Conversely, every semi-automorphism of a unital ring is essentially a Jordan homomorphism. This is easy enough to see. Just take $\varphi(a^2) = \varphi(a)\varphi(1)\varphi(a)$ and note that $\varphi(1)\varphi(x)$ is a Jordan homomorphism.

3. Product preserving maps on alternative division rings

The first product preserving result on alternative division rings is due to Barreiro, Ferreira, and de Araujo Smigly [10], who took k and m to be central elements. The problem for nonzero k, m on associative division rings is due to Catalano [3, Theorem 5], but with a characteristic not 2 assumption (that we ultimately remove). Both articles showed that the map must be a scalar multiple of a semi-automorphism, which then must be an automorphism or anti-automorphism.

For alternative division rings, there is a criterion for additive semi-automorphisms to be automorphisms or anti-automorphisms. In 2017, the first author Ferreira [8] characterized so-called *Jordan elementary maps* by introducing a multiplicative condition on elements of the form $ab \cdot c + c \cdot ba$, later used in [9] to prove:

Theorem 3.1. [9, Theorem 2.2] *Let $\mathcal{A}$ be an alternative division ring. Suppose $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ is a semi-automorphism satisfying*

$$\varphi(ab \cdot c + c \cdot ba) = \varphi(a)\varphi(b) \cdot \varphi(c) + \varphi(c) \cdot \varphi(b)\varphi(a), \qquad (*)$$

whenever $a, b, c \in \mathcal{A}$. Then φ is either an automorphism or an anti-automorphism.

For characteristic not 2, we can also drop the tacit assumption that φ is a semi-automorphism by taking $a = c$ in $(*)$.

But Condition $(*)$ is a necessary and sufficient requirement for a semi-automorphism to be an automorphism or anti-automorphism of a strictly alternative division ring. The sufficiency of $(*)$ is demonstrated through an example.

Example 3.2 (Modified from Example 4.1 in [10]). Consider the algebra of real octonions, $\mathbb{O}$, and fix elements $c = d = i_0$, the identity element. See Figure (1) in Section 4 for a complete multiplication table by taking $c_1 = c_2 = c_3 = -1$. Define the bijective $\mathbb{R}$-linear map $\varphi \colon \mathbb{O} \to \mathbb{O}$ via its basis as follows:

i_k	i_0	i_1	i_2	i_3	i_4	i_5	i_6	i_7
$\varphi(i_k)$	i_0	i_4	i_3	i_6	i_1	i_2	i_7	i_5

Straightforward calculations show that $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in \mathbb{O}^{\times}$; hence φ preserves products equal to i_0 (in fact, any permutation of basis elements fixing i_0 induces a linear map preserving products equal to the identity[2]). By [10, Theorem 4.1], $\varphi = \lambda\xi = \varphi(i_0)\xi = \xi$, where ξ is a semi-automorphism. But

$$\xi(i_1 i_2) = \xi(i_3) = i_6, \quad \xi(i_1) = i_4, \quad \text{and} \quad \xi(i_2) = i_3$$

[2]If $a = \sum_{k=1}^{7} \alpha_k i_k$ with each $\alpha_k \in \mathbb{R}$, then $\varphi(a) = \alpha_0 i_0 + \sum_{k=1}^{7} \alpha_k i_{\sigma(k)}$ where σ permutes the indices in accordance with φ. Also, $\bar{a} = \alpha_0 i_0 - \sum_{k=1}^{7} \alpha_k i_k$. Now, $a^{-1} = \bar{a}/\sum_{k=0}^{7} \alpha_k^2$ and the denominator is permutation invariant. Then φ permutes the coordinates of $\bar{a}$ analogously to a so that $\varphi(a)\varphi(a^{-1}) = i_0$.

while

$$i_6 = \xi(i_1 i_2) \neq \xi(i_1)\xi(i_2) = -i_7 \quad \text{and} \quad i_6 = \xi(i_1 i_2) \neq \xi(i_2)\xi(i_1) = i_7,$$

which means that ξ is neither an automorphism nor an anti-automorphism.

It is useful to see that Condition $(*)$ for alternative division rings is not satisfied. Indeed,

$$\varphi(i_3 i_4 \cdot i_5 + i_5 \cdot i_4 i_3) = \varphi(i_7 i_5 + i_5(-i_7)) = 2\varphi(i_2) = 2i_3$$

while

$$\varphi(i_3)\varphi(i_4) \cdot \varphi(i_5) + \varphi(i_5) \cdot \varphi(i_4)\varphi(i_3) = i_6 i_1 \cdot i_2 + i_2 \cdot i_1 i_6 = 2i_5.$$

Theorem 3.3. *Let $\mathcal{A}$ be an alternative division ring. Let $m, k \in \mathcal{A}^\times$ be fixed elements, and let $f : \mathcal{A} \to \mathcal{A}$ be a bijective additive map satisfying the identity*

$$f(x)f(y) = m$$

for every $x, y \in \mathcal{A}^\times$ such that $xy = k$. Then $f(x) = f(1)\varphi(x)$ for all $x \in \mathcal{A}$, where $\varphi : \mathcal{A} \to \mathcal{A}$ is a unital Jordan automorphism. Moreover, φ is also a semi-automorphism.

Proof. Let $a, b \in \mathcal{A}^\times$ with $ab \neq 1$ and consider $y = a - aba$. Using Lemma 2.1,

$$a - aba = (a^{-1} + (b^{-1} - a)^{-1})^{-1},$$

hence $y^{-1} = a^{-1} + (b^{-1} - a)^{-1}$. Using the inverse identity (2), the equation $k = (yy^{-1})k = y(y^{-1}k)$ implies that

$$\begin{aligned}
m &= f(y)f(y^{-1}k) \\
&= f(a - aba)f(a^{-1}k + (b^{-1} - a)^{-1}k) \\
&= \big(f(a) - f(aba)\big)\big(f(a^{-1}k) + f((b^{-1} - a)^{-1}k)\big) \\
&= f(a)f(a^{-1}k) + f(a)f((b^{-1} - a)^{-1}k) - f(aba)f(a^{-1}k) - f(aba)f((b^{-1} - a)^{-1}k).
\end{aligned}$$

Observe that $m = f(a)f(a^{-1}k)$, and therefore

$$0 = f(a)f((b^{-1} - a)^{-1}k) - f(aba)f(a^{-1}k) - f(aba)f((b^{-1} - a)^{-1}k).$$

But it is also true that $f(a^{-1}k) = f(a)^{-1}m$, hence we have the following:

$$0 = f(a) \cdot f(b^{-1} - a)^{-1}m - f(aba) \cdot f(a)^{-1}m - f(aba) \cdot f(b^{-1} - a)^{-1}m.$$

Multiply on the right side by $m^{-1}f(b^{-1} - a) = (f(b^{-1} - a)^{-1}m)^{-1}$ to obtain

$$\begin{aligned}
0 &= f(a) - \big(f(aba) \cdot f(a)^{-1}m\big)\big(m^{-1}f(b^{-1}) - m^{-1}f(a)\big) - f(aba) \\
&= f(a) - \big(f(aba) \cdot f(a)^{-1}m\big)\big(m^{-1}f(b^{-1})\big) + \big(f(aba) \cdot f(a)^{-1}m\big)\big(m^{-1}f(a)\big) - f(aba) \\
&= f(a) - \big(f(aba) \cdot f(a)^{-1}m\big)\big(m^{-1}f(b^{-1})\big) + f(aba) - f(aba) \\
&= f(a) - \big(f(aba) \cdot f(a)^{-1}m\big)\big(m^{-1}f(b^{-1})\big).
\end{aligned}$$

Thus,

$$f(a) = \big(f(aba) \cdot f(a)^{-1}m\big)\big(m^{-1}f(b^{-1})\big)$$

and as a consequence,

$$f(aba) = \big(f(a) \cdot f(b^{-1})^{-1}m\big)(m^{-1}f(a)).$$

Using the Moufang middle identity, $(xy)(ax) = x(ya)x$, and the inverse identity (2), we get

$$f(aba) = f(a)(f(b^{-1})^{-1}m \cdot m^{-1})f(a) = f(a)f(b^{-1})^{-1}f(a), \tag{4}$$

whenever $ab \neq 0$ and $ab \neq 1$. For any $x \neq 0$ and $x \neq 1$, let $a = x$, and $b = 1$ in (4). Hence

$$f(x^2) = f(x)f(1)^{-1}f(x). \tag{5}$$

It is easy to see that (5) holds for $x = 0$ and $x = 1$, so (5) is valid for all $x \in \mathcal{A}$. Define $\varphi(x) := f(1)^{-1}f(x)$. Multiplying equation (5) on the left by $f(1)^{-1}$ implies that $\varphi(x^2) = \varphi(x)^2$ for all $x \in \mathcal{A}$, and φ is clearly unital and additive. Thus φ is a Jordan homomorphism.

In addition, taking $a = 1$ in equation (4), $f(b) = f(1)f(b^{-1})^{-1}f(1)$ implies that

$$f(1)\varphi(b) = f(1)(\varphi(b^{-1})^{-1}f(1)^{-1})f(1).$$

Using the Moufang middle identity and canceling $f(1)$ from both sides, we conclude that

$$\varphi(b^{-1}) = \varphi(b)^{-1}.$$

Since b can be arbitrary, φ is also a semi-automorphism by Proposition 2.4. □

If one wishes to describe fixed product preserving maps in terms of automorphisms and anti-automorphisms, as done in [3], we must assume Condition (∗) holds for the additive map φ in question. We first present a simple lemma.

Lemma 3.4. *Let $\mathcal{O}$ denote an octonion division algebra over its center, $\mathcal{Z}$, of characteristic not 2. If $m \notin \mathcal{Z}$, then the centralizer $C_{\mathcal{O}}(m)$ is 2-dimensional and $C_{\mathcal{O}}(m) = \mathcal{Z} + \mathcal{Z}m$, the subalgebra generated by 1 and m. Moreover, if $\mathcal{A}$ is an alternative division ring of characteristic not 2 and $a, m \in \mathcal{A}^\times$ satisfy*

1. $[a, m] = 0$, and
2. $a^{-1}xa = m^{-1}xm$

for all $x \in \mathcal{A}$, then $a^{-1}m \in \mathcal{Z}$.

Proof. The fact that $\dim C_{\mathcal{O}}(m)$ is 2-dimensional follows at once from [Lemma 3.1(b), [18]]. For the next claim, suppose a and m satisfy (1) and (2). If $\mathcal{A}$ is associative, then $a^{-1}m \in \mathcal{Z}$ is immediate, so assume that $\mathcal{A}$ is strictly nonassociative; hence $\mathcal{A}$ is an octonion division algebra over its center $\mathcal{Z}$. The equation (2) rearranges to

$$x = a(m^{-1}xm)a^{-1} \tag{6}$$

for all $x \in \mathcal{A}$. By the first part of the lemma, since a^{-1} centralizes m by (1), then $a^{-1} \in \mathcal{Z} + \mathcal{Z}m$ whenever $m \notin \mathcal{Z}$. For any given $x \in \mathcal{A}$, the right-hand side of equation (6) lies in a subalgebra of $\mathcal{A}$ generated by m and x, which is associative by Artin's theorem. Hence the right-hand side of equation (6) may be associated to give $x = (a^{-1}m)^{-1}x(a^{-1}m)$, for all $x \in \mathcal{A}$, as desired. On the other hand, if $m \in \mathcal{Z}$, then equation (6) implies that $a \in \mathcal{Z}$ too, hence $a^{-1}m \in \mathcal{Z}$. □

We can now fully extend Catalano's result [3, Theorem 5] to alternative division rings of any characteristic.

Theorem 3.5. *Let $\mathcal{A}$ be an alternative division ring with center $\mathcal{Z}$. Let $m, k \in \mathcal{A}^\times$ be fixed elements, and let $f : \mathcal{A} \to \mathcal{A}$ be a bijective additive map such that*

(i) $f(x)f(y) = m$ whenever $xy = k$, and
(ii) $f(1)^{-1}f$ satisfies Condition (∗).

Then $f(x) = f(1)\varphi(x)$ for all $x \in \mathcal{A}$, where $\varphi : \mathcal{A} \to \mathcal{A}$ is either an automorphism or anti-automorphism. Moreover, we have that

(1) if φ is an automorphism, then $f(1) \in \mathcal{Z}$ and
(2) if φ is an anti-automorphism, then $f(k) \in \mathcal{Z}$.

Proof. The property (i) implies that $f(x) = f(1)\varphi(x)$ for some semi-automorphism φ by Theorem 3.3. Furthermore, if (ii) holds, then it is routine to check that φ is a semi-automorphism satisfying Theorem 3.1. Hence φ is an automorphism or anti-automorphism.

Let $\lambda = f(1)$. When φ is an automorphism and $xy = k$, then

$$\begin{aligned} m = f(x)f(y) &= \lambda\varphi(x) \cdot \lambda\varphi(y) \\ &= \lambda\varphi(x)(\lambda\varphi(y) - \varphi(y)\lambda + \varphi(y)\lambda) \\ &= \lambda\varphi(x)[\lambda, \varphi(y)] + \lambda\varphi(x) \cdot \varphi(y)\lambda \\ &= \lambda\varphi(x)[\lambda, \varphi(y)] + \lambda\varphi(x)\varphi(y)\lambda \\ &= \lambda\varphi(x)[\lambda, \varphi(y)] + \lambda\varphi(k)\lambda \\ &= \lambda\varphi(x)[\lambda, \varphi(y)] + m \end{aligned}$$

implies that $\lambda\varphi(x)[\lambda, \varphi(y)] = 0$, where the Moufang middle identity is used in the fourth line. But since $\lambda \neq 0$ and $\varphi(x) \neq 0$, it follows that the commutator is zero. Since $xy = k$ always has the solution $x = ky^{-1}$, it follows that λ commutes with the full range of φ; hence $\lambda \in \mathcal{Z}$.

Suppose φ is an anti-automorphism. Examining the rule $f(x^{-1}k) = f(x)^{-1}m$, along with the standard fact that $\varphi(x^{-1}) = \varphi(x)^{-1}$, implies that

$$\lambda \cdot \varphi(k)\varphi(x)^{-1} = \varphi(x)^{-1}\lambda^{-1} \cdot m.$$

Then

$$\lambda(\varphi(k)(\varphi(x)^{-1}\lambda^{-1}\lambda)) = \varphi(x)^{-1}\lambda^{-1} \cdot m.$$

Multiply on the right by λ and use the Moufang middle identity:

$$(\lambda\varphi(k))((\lambda\varphi(x))^{-1}\lambda^2) = (\varphi(x)^{-1}\lambda^{-1} \cdot m)\lambda$$

which becomes

$$f(k)(f(x)^{-1}\lambda^2) = (f(x)^{-1}m)\lambda$$

after substituting $f(x) = \lambda\varphi(x)$. Equivalently,

$$f(k)(f(x)^{-1}\lambda^2) = f(x^{-1}k)\lambda$$

From here, multiply on the left by λ:

$$\lambda(f(k)(f(x)^{-1}\lambda^2)) = \lambda f(x^{-1}k)\lambda$$

and, by the Moufang left identity and Artin's theorem,

$$\lambda(f(k)(\lambda\lambda^{-1}f(x)^{-1}\lambda^2)) = (\lambda f(k)\lambda)(\lambda^{-1}f(x)^{-1}\lambda^2) = \lambda f(x^{-1}k)\lambda.$$

But since λ commutes with $f(k)$,

$$(\lambda^2 f(k))(\lambda^{-1}f(x)^{-1}\lambda^2) = \lambda f(x^{-1}k)\lambda$$

and using the Moufang middle identity on the left-hand side,

$$\lambda^2(f(k) \cdot \lambda^{-1}f(x)^{-1})\lambda^2 = \lambda f(x^{-1}k)\lambda.$$

Cancel λ on the left and right from both sides, then use another middle identity to put it back together:

$$\lambda f(k) \cdot \lambda^{-1}f(x)^{-1}\lambda = f(x^{-1}k).$$

Notice that $\lambda f(k) = m$ and $f(x^{-1}k) = f(x)^{-1}m$, so the previous equation becomes

$$m \cdot \lambda^{-1}f(x)^{-1}\lambda = f(x)^{-1}m$$

and therefore $\lambda^{-1}f(x)^{-1}\lambda = m^{-1}f(x)^{-1}m$. Invert to get

$$\lambda^{-1}f(x)\lambda = m^{-1}f(x)m \tag{7}$$

for all $x \in \mathcal{A}$. By Lemma 3.4, it follows that $\lambda^{-1}m = \lambda^{-1}\lambda f(k) = f(k) \in \mathcal{Z}$. □

Note: Bijectivity can be weakened to injectivity in both of the previous theorems. Indeed, if $f(x)f(y) = m \neq 0$ whenever $xy = k$ in an alternative division ring, then $xx^{-1}k = k \implies f(x)f(x^{-1}k) = m$ for all $x \neq 0$. Therefore $f(x) \neq 0$ whenever $x \neq 0$, so f has a trivial kernel. We only use surjectivity in the previous proof to conclude that $f(1)$ or $f(k)$ is central. In general, $f(1)$ or $f(k)$ commutes with the range of f.

4. Product preserving maps on split Cayley-Dickson algebras

Here we state general properties of Cayley-Dickson algebras and defer to [19, Chapter III, Section 4] and [16, Section 7.6] for explicit details. Let $\mathbb{F}$ be a field. The Cayley-Dickson construction produces a sequence of algebras with underlying vector spaces $\mathbb{F}, \mathbb{F}^2, \mathbb{F}^4, \mathbb{F}^8 \ldots$, and so on, whose endowed multiplication makes $\mathbb{F}, \mathbb{F}^2$, and $\mathbb{F}^4$ associative and $\mathbb{F}^8$ strictly alternative (while $\mathbb{F}^{16}$ and higher fail to be alternative). Each algebra in the construction admits a nondegenerate quadratic form N that is either anisotropic ($N(x) \neq 0$ for all $x \neq 0$) or isotropic ($N(x) = 0$ for some $x \neq 0$), with the additional property that $N(xy) = N(x)N(y)$ for all elements x and y. If N is anisotropic, then the algebra is a division algebra over its center, and if N is isotropic, the algebra is said to be *split*. Any two split Cayley-Dickson algebras of the same dimension are isomorphic as algebras, provided that the quadratic forms N are equivalent [19, Theorem 3.23]. This allows us to provide a canonical description of product preserving maps on split Cayley-Dickson algebras (up to isomorphism) by selecting a basis.

For the rest of the section, assume the characteristic of $\mathbb{F}$ is not 2. Employing the usual Cayley-Dickson construction as outlined in [16, Section 7.6], we obtain the following multiplication table in Figure 1 for the basis $\{i_0, i_1, \ldots, i_7\}$ of any eight-dimensional Cayley-Dickson algebra $\mathcal{O}$ over $\mathbb{F}$. The basis has the nice property that each element squares to a scalar and all non-identity elements pairwise anti-commute; that is, $i_p i_q + i_q i_p = 0$ whenever $p \neq q$. The multiplication table is valid over any field of characteristic not 2 with c_1, c_2, and c_3 being constant *nonzero* elements of $\mathbb{F}$ selected at each stage of the construction.

Two lemmas are needed to prepare for Theorem 4.3 on product preserving maps.

Lemma 4.1. *Let $\mathcal{A}$ be an alternative Cayley-Dickson algebra over a field $\mathbb{F}$. If $ab \in \mathcal{A}^\times$, then $a \in \mathcal{A}^\times$ and $b \in \mathcal{A}^\times$.*

Proof. Follows from the fact that $a \in \mathcal{A}^\times$ if and only if $N(a) \neq 0$. □

$(\mathcal{O}, \times)$	i_0	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_0	i_0	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	i_1	$c_1 i_0$	i_3	$c_1 i_2$	i_5	$c_1 i_4$	$-i_7$	$-c_1 i_6$
i_2	i_2	$-i_3$	$c_2 i_0$	$-c_2 i_1$	i_6	i_7	$c_2 i_4$	$c_2 i_5$
i_3	i_3	$-c_1 i_2$	$c_2 i_1$	$-c_1 c_2 i_0$	i_7	$c_1 i_6$	$-c_2 i_5$	$-c_1 c_2 i_4$
i_4	i_4	$-i_5$	$-i_6$	$-i_7$	$c_3 i_0$	$-c_3 i_1$	$-c_3 i_2$	$-c_3 i_3$
i_5	i_5	$-c_1 i_4$	$-i_7$	$-c_1 i_6$	$c_3 i_1$	$-c_1 c_3 i_0$	$c_3 i_3$	$c_1 c_3 i_2$
i_6	i_6	i_7	$-c_2 i_4$	$c_2 i_5$	$c_3 i_2$	$-c_3 i_3$	$-c_2 c_3 i_0$	$-c_2 c_3 i_1$
i_7	i_7	$c_1 i_6$	$-c_2 i_5$	$c_1 c_2 i_4$	$c_3 i_3$	$-c_1 c_3 i_2$	$c_2 c_3 i_1$	$c_1 c_2 c_3 i_0$

Figure 1. A canonical octonion multiplication table, where $c_1, c_2, c_3 \in \mathbb{F}^\times$ and $\mathrm{char}(\mathbb{F}) \neq 2$. A table for a 2-dimensional or 4-dimensional Cayley-Dickson algebra is obtained by restricting to the first 2 or 4 rows and columns.

Lemma 4.2. *Let $\mathcal{A}$ denote a split, alternative Cayley-Dickson algebra over a field $\mathbb{F}$ of characteristic not 2. Let $\{i_k\}_{k=0}^{d-1}$ be a basis of $\mathcal{A}$ with a canonical multiplication table as in Figure (1), where $d = \dim \mathcal{A}$. For all nonzero scalars $\alpha, \beta \in \mathbb{F}^{\times}$,*

1. *if $\dim \mathcal{A} = 4$ or 8, then whenever $p \neq 0$ and $q \neq 0$, either $\alpha i_p + \beta i_q$ or $i_0 + \alpha i_p + \beta i_q$ is invertible,*
2. *if $\dim \mathcal{A} = 2, 4,$ or 8 and $\mathbb{F}$ has at least five elements, then whenever $q \neq 0$, either $\alpha i_0 + \beta i_q$ is invertible or there exists $\lambda \in \mathbb{F}^{\times}$ (depending on α) such that $(\lambda + \alpha) i_0 + \beta i_q$ and $\lambda i_0 + \beta i_q$ are both invertible.*
3. *if $\dim \mathcal{A} = 4$ or 8 and $\mathbb{F} = \mathbb{F}_3$, then either $\alpha i_0 + \beta i_q$ is invertible or every linear combination of the form $\alpha i_0 + \lambda i_p + \beta i_q$ is invertible, where $p \notin \{0, q\}$ and $\lambda \in \mathbb{F}^{\times}$.*

Proof. To prove part 1, suppose $\dim \mathcal{A} = 4$ or 8. Then for pairs of arbitrary scalars $\alpha, \beta \in \mathbb{F}$, $(\alpha i_p + \beta i_q)^2 = (\alpha^2 i_p^2 + \beta^2 i_q^2) i_0$ is a scalar. If $\alpha^2 i_p^2 + \beta^2 i_q^2 \neq 0$ we are done. If $\alpha^2 i_p^2 + \beta^2 i_q^2 = 0$, then $s = \alpha i_p + \beta i_q$ is a square-zero element. Since $(i_0 + s)(i_0 - s) = i_0$, it follows that $i_0 + s$ is invertible. Part 1 is proved.

Now suppose $\dim \mathcal{A} = 2, 4,$ or 8. Then we have $(\alpha i_0 + \beta i_q)(\alpha i_0 - \beta i_q) = (\alpha^2 - \beta^2 i_q^2) i_0$. Again, when the element $\alpha^2 - \beta^2 i_q^2$ is nonzero, then $\alpha i_0 + \beta i_q$ is invertible. Otherwise, replacing α with $\lambda + \alpha$ where λ is nonzero, invertibility now depends on the scalar $(\lambda + \alpha)^2 - \beta^2 i_q^2$. If zero, then $(\lambda + \alpha)^2 = \alpha^2$ implies that $\lambda + 2\alpha = 0$. So select $\lambda + 2\alpha \neq 0$ to guarantee that $\lambda i_0 + \alpha i_0 + \beta i_q$ is invertible. Clearly we may take $\lambda = -\alpha$, but we also want $\lambda i_0 + \beta i_q$ to be invertible. Therefore $\lambda = -\alpha$ does not do the job. Moreover, if $\mathbb{F} = \mathbb{F}_3$, then $\alpha + 2\alpha = 0$ shows that $\lambda = \alpha$ is the only choice. So we nontrivially use the assumption $|\mathbb{F}| \geq 5$ to select $\lambda \notin \{0, \alpha, -\alpha, -2\alpha\}$ so that $(\lambda i_0 + \beta i_q)(\lambda i_0 - \beta i_q) = (\lambda^2 - \beta^2 i_q^2) i_0$ is nonzero; hence $\lambda i_0 + \beta i_q$ is invertible too, proving part 2.

To finish, assume that $\mathbb{F} = \mathbb{F}_3$. Then $(\alpha i_0 + \beta i_q)(\alpha i_0 - \beta i_q) = (1 - i_q^2) i_0$ whenever $\alpha, \beta \in \mathbb{F}_3^{\times} = \{1, -1\}$. If $i_q^2 = -i_0$, then $\alpha i_0 + \beta i_q$ is clearly invertible, but if $i_q^2 = i_0$, then $\alpha i_0 + \beta i_q$ is not invertible. So when $\dim \mathcal{A} = 2$ over $\mathbb{F}_3$ and $\mathcal{A}$ is split, no element of the form $\alpha i_0 + \beta i_1$ with $\alpha \neq 0$ and $\beta \neq 0$ is invertible. But when $\dim \mathcal{A} = 4$ or 8 and $\mathcal{A}$ is split, whenever $i_q^2 = i_0$ simply select $p \neq q$ and $p > 1$ and observe that

$$(\alpha i_0 + i_p + \beta i_q)(\alpha i_0 - i_p - \beta i_q) = (1 - i_p^2 - i_q^2) i_0 = -i_p^2 i_0.$$

It is also easy to see that $(\alpha i_0 + i_p - \beta i_q)(\alpha i_0 - i_p + \beta i_q) = -i_p^2 i_0$. Therefore, any linear combination of the form $\alpha i_0 + \lambda i_p + \beta i_q$ is invertible for $\alpha, \beta, \lambda \in \mathbb{F}_3^{\times}$. This proves part 3. □

With the above in hand, the following theorem uses Theorem 3.3 in Section 3 to extend to split Cayley-Dickson algebras. See Example 4.4 for a pathological counterexample if m is not assumed invertible.

Theorem 4.3. *Let $\mathcal{A}$ be a Cayley-Dickson algebra over its center $\mathbb{F}$ of characteristic not 2. Let $m, k \in \mathcal{A}$ be fixed elements, where $m \in \mathcal{A}^{\times}$, and let $f : \mathcal{A} \to \mathcal{A}$ be an additive map satisfying the identity*

$$f(x)f(y) = m$$

for every $x, y \in \mathcal{A}$ such that $xy = k$. Then $f(x) = f(1)\varphi(x)$ where φ is a Jordan homomorphism of $\mathcal{A}$.

Proof. By Lemma 4.1, $m \in \mathcal{A}^{\times}$ implies that $f(x)$ is invertible whenever x is invertible since $f(x)f(x^{-1}k) = m$. Repeat the proof of Theorem 3.3 to conclude that $f(x^2) = f(x)f(1)^{-1}f(x)$, but only for all invertible $x \in \mathcal{A}$. Define an additive map φ via $f(x) = f(1)\varphi(x)$ to obtain $\varphi(x^2) = \varphi(x)^2$ for all invertible $x \in \mathcal{A}$. If $\mathcal{A}$ is a division algebra we are done. It remains to show that $\varphi(x)^2 = \varphi(x)^2$ holds for all elements in a split alternative Cayley-Dickson algebra, not just invertible elements. We take x to be a suitable linear combination of elements as provided by Lemma 4.2 in the equation $\varphi(x^2) = \varphi(x)^2$.

Assume that $\mathcal{A}$ is split of dimension $d = 4$ or $d = 8$ and consider the basis $\{i_k\}_{k=0}^{d-1}$ and its canonical multiplication table (as in Figure (1)) with $i_p^2 \in \mathbb{F} i_0$ for all $p = 0, 1, 2, \ldots, d-1$. By Lemma 4.2 part 1, either $\alpha i_p + \beta i_q$ or $i_0 + \alpha i_p + \beta i_q$ is invertible for any two scalars $\alpha, \beta \in \mathbb{F}^\times$. In the former case, say, we have

$$\begin{aligned}\varphi((\alpha^2 i_p^2 + \beta^2 i_q^2)i_0) &= \varphi((\alpha i_p + \beta i_q)^2)\\ &= \varphi(\alpha i_p + \beta i_q)^2\\ &= \varphi(\alpha i_p)^2 + \varphi(\alpha i_p) \circ \varphi(\beta i_q) + \varphi(\beta i_q)^2\end{aligned}$$

and since $\varphi(\alpha i_p)^2 = \varphi(\alpha^2 i_p^2)$ holds, this implies that

$$\varphi(\alpha i_p) \circ \varphi(\beta i_q) = 0 = \varphi(\alpha i_p \circ \beta i_q).$$

If $\alpha i_p + \beta i_q$ is not invertible, then analogously use that $i_0 + \alpha i_p + \beta i_q$ is invertible to obtain $\varphi(\alpha i_p) \circ \varphi(\beta i_q) = 0 = \varphi(\alpha i_p \circ \beta i_q)$ as well.

If $\alpha i_0 + \beta i_q$ is invertible, then $\varphi((\alpha i_0 + \beta i_q)^2) = \varphi(\alpha i_0 + \beta i_q)^2$ implies that $\varphi(\alpha i_0 \circ \beta i_q) = \varphi(\alpha i_0) \circ \varphi(\beta i_q)$. If $\alpha i_0 + \beta i_q$ is not invertible and $\mathbb{F}$ has at least five elements, choose $\lambda \in \mathbb{F}^\times$ according to Lemma 4.2 part 2; hence the previous sentence implies that

$$\varphi((\lambda + \alpha)i_0 \circ \beta i_q) = \varphi(\lambda i_0) \circ \varphi(\beta i_q) + \varphi(\alpha i_0) \circ \varphi(\beta i_q).$$

But since $\lambda i_0 + \beta i_q$ is invertible too, it follows that $\varphi(\lambda i_0 \circ \beta i_q) = \varphi(\lambda i_0) \circ \varphi(\beta i_q)$, hence

$$\varphi(\alpha i_0 \circ \beta i_q) = \varphi(\alpha i_0) \circ \varphi(\beta i_q) \tag{8}$$

as well. The same conclusion holds in the special case that $\mathbb{F} = \mathbb{F}_3$ by Lemma 4.2 part 3. Indeed, taking $x = \alpha i_0 + \lambda i_p + \beta i_q$ in $\varphi(x^2) = \varphi(x)^2$, we obtain

$$\varphi(\alpha i_0 \circ \lambda i_p) + \varphi(\alpha i_0 \circ \beta i_q) = \varphi(\alpha i_0) \circ \varphi(\lambda i_p) + \varphi(\alpha i_0) \circ \varphi(\beta i_q).$$

Another invertible linear combination is $x = \alpha i_0 + \lambda i_p - \beta i_q$, which analogously gives

$$\varphi(\alpha i_0 \circ \lambda i_p) - \varphi(\alpha i_0 \circ \beta i_q) = \varphi(\alpha i_0) \circ \varphi(\lambda i_p) - \varphi(\alpha i_0) \circ \varphi(\beta i_q).$$

Subtract these two equations and cancel a factor of 2 to obtain $\varphi(\alpha i_0 \circ \beta i_q) = \varphi(\alpha i_0) \circ \varphi(\beta i_q)$.

We have shown $\varphi(\alpha i_p) \circ \varphi(\beta i_q) = \varphi(\alpha i_p \circ \beta i_q)$ for all nonzero scalars α, β and pairs of basis elements i_p, i_q, including $p = 0$ or $q = 0$. Extending by additivity, then $\varphi(x) \circ \varphi(y) = \varphi(x \circ y)$ for all $x, y \in \mathcal{A}$. In particular, $\varphi(x^2) = \varphi(x)^2$ for all $x \in \mathcal{A}$. Thus φ is a Jordan homomorphism on all of $\mathcal{A}$ (not just $\mathcal{A}^\times$) for any 4- or 8-dimensional Cayley-Dickson algebra over any field of characteristic not 2. Also, when $\dim \mathcal{A} = 2$, then the previous computations for equation (8) apply whenever $\mathbb{F}$ has at least five elements by Lemma 4.2. Therefore φ is a Jordan homomorphism in this case too.

There is one more special case. When $\mathbb{F} = \mathbb{F}_3$ and $\dim \mathcal{A} = 2$, the split algebra under consideration is (up to isomorphism)

$$\mathcal{A} = \{\alpha + \beta i \mid \alpha, \beta \in \mathbb{F}_3, i^2 = 1\}, \tag{9}$$

a commutative and associative Cayley-Dickson algebra over $\mathbb{F}_3$. The basis of $\mathcal{A}$ is denoted $i_0 = 1$ and $i_1 = i$ for simplicity.

Observe that $(\alpha - \beta i)(\alpha + \beta i) = \alpha^2 - \beta^2 = 0$ whenever α and β are both nonzero. Among the nine elements of $\mathcal{A}$, it follows that only $1, i, -1$, and $-i$ are invertible. If $f : \mathcal{A} \to \mathcal{A}$ preserves products equal to $m \in \{1, i, -1, -i\}$, then $f(x) = f(1)\varphi(x)$ and φ has $\varphi(x^2) = \varphi(x)^2$ for each invertible element by the first part of the proof. Now, $f(1)f(k) = m$ implies that $f(1) = f(1)\varphi(1)$ is invertible; hence $\varphi(1) = 1$ and $\varphi(i)^2 = \varphi(i^2) = \varphi(1) = 1$. It is easy to check that *any* additive map $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi(1) = 1$ and $\varphi(i)^2 = 1$ is a homomorphism. The theorem follows. □

Example 4.4. Having $m \in \mathcal{A}^\times$ is essential for Theorem 4.3. For the two-dimensional split algebra $\mathcal{A}$ in (9), the only solutions to $xy = i$ are $x = 1$ and $y = i$, up to scalars. The map

$$f(\alpha + \beta i) = \alpha - \beta + \beta i \qquad (\alpha, \beta \in \mathbb{F}_3)$$

is an additive bijection such that $f(x)f(y) = -1 + i$ whenever $xy = i$, with $f(1) = 1$. Now, if φ is a Jordan homomorphism such that $f(x) = f(1)\varphi(x)$, then $f = \varphi$. But f is certainly not a Jordan homomorphism since $f(1 + i)^2 = i^2 = 1$ and $f((1 + i)^2) = f(-1 - i) = -i$.

Remark 4.5. If $\dim \mathcal{A} = 4$ and split, then $\mathcal{A}$ is isomorphic to a 2×2 matrix algebra. Having m non-invertible is not a problem here and all bijective fixed product preserving maps have standard descriptions; see [4].

There are two interesting follow-up questions.

Question 1. Theorem 3.3 holds for characteristic 2, and so does Theorem 4.3 up to the result $\varphi(x^2) = \varphi(x)^2$ for all invertible x. Can one still use linear combinations of invertible basis elements to extend φ to the algebra in characteristic 2?

Question 2. For a split octonion algebra $\mathcal{A}$ and noninvertible elements k, m, under what conditions can we guarantee that $f(x) = f(1)\varphi(x)$ for some Jordan homomorphism φ?

5. The zero product preserver problem on split Cayley-Dickson algebras

Solutions of the zero product preserver problem for 2- and 4-dimensional Cayley-Dickson algebras may be deduced via the theory of zero product determined algebras (see [1]) since they are associative and spanned by idempotents. So our focus narrows to the eight-dimensional case. Throughout the section, $\mathcal{O}$ always denotes a split octonion algebra over its center $\mathcal{Z}$ of any characteristic. It is well-known that $\mathcal{O}$ has nontrivial idempotent elements and is linearly spanned by them.

Let $f : \mathcal{O} \to \mathcal{R}$ be an additive map such that $f(x)f(y) = 0$ whenever $xy = 0$, where $\mathcal{R}$ is an arbitrary nonassociative ring. If $p \in \mathcal{O}$ is idempotent and $a, b \in \mathcal{O}$ are elements such that $(a, p, b) = 0$, then $ap \cdot (b - pb) = 0$ and $(a - ap) \cdot pb = 0$ imply that

$$f(ap)f(b) = f(ap)f(pb) = f(a)f(pb). \tag{10}$$

We call this the *idempotent balancing property*. On associative algebras generated by idempotents, (10) yields a quick approach to determine the form of zero product preserving maps. The property is used extensively in [1, Section 2.1], though the idea originates in [6]. It can be suitably adapted for a split octonion algebra as follows.

Lemma 5.1. *Suppose $f : \mathcal{O} \to \mathcal{R}$ is an additive, zero product preserving map. Then $f(\alpha)f(x) = f(1)f(\alpha x)$ for all $\alpha \in \mathcal{Z}$ and $x \in \mathcal{O}$. Furthermore, $f(1)f(x) = f(x)f(1)$ for all $x \in \mathcal{O}$.*

Proof. Fix $x \in \mathcal{O}$. By [14, Proposition 5.5], there is a split quaternion subalgebra of $\mathcal{O}$ containing x; denote it by $\mathcal{H}$. For every scalar element $\alpha 1 \in \mathcal{H}$, we claim that there are idempotents $e, p \in \mathcal{H}$ such that $\alpha 1 = epe + (1 - e)(1 - p)(1 - e)$. To see this, we are free to represent $\mathcal{H}$ by 2×2 matrices since all split quaternion algebras are isomorphic. Assume then without loss of generality that $\mathcal{H}$ is a matrix algebra with respect to a chosen basis over $\mathcal{Z}$. Any idempotent $p \in \mathcal{H}$ is of the form $p = \begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix}$ provided that $\alpha^2 + \beta\gamma = \alpha$. Since $\mathcal{Z}$ is a field, α can be arbitrary, and the claim follows by taking $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We therefore have $\alpha x = xepe + x(1 - e)(1 - p)(1 - e)$.

Then $f(\alpha x)f(y) = f(xepe)f(y) + f(x(1 - e)(1 - p)(1 - e))f(y)$ for all $y \in \mathcal{O}$. But if $y \in \mathcal{H}$, then y associates with sums and products of x, e, and p and $f(xepe)f(y) = f(xep)f(ey)$ by one application of the idempotent balancing property (10). A second and third application puts p, followed by e, into the right argument, giving $f(xepe)f(y) = f(x)f(epey)$. Identical reasoning holds for $f(x(1-e)(1-p)(1-e))f(y) = f(x)f((1 - e)(1 - p)(1 - e)y)$ when $y \in \mathcal{H}$ as well. The additivity of f yields

$$f(\alpha x)f(y) = f(x)f(\alpha y) \tag{11}$$

for all $\alpha \in \mathcal{Z}$ and $x, y \in \mathcal{H}$ [3]. In particular, (11) implies that

$$f(\alpha)f(x) = f(1)f(\alpha x) \qquad \text{and} \qquad f(x)f(\alpha) = f(\alpha x)f(1) \tag{12}$$

for all $x \in \mathcal{O}$ and $\alpha \in \mathcal{Z}$. The first part of the lemma is proved.

For every idempotent $p \in \mathcal{O}$, then $f(\alpha p)f(1) \overset{(10)}{=} f(p)f(\alpha) \overset{(12)}{=} f(1)f(\alpha p)$. Since $\mathcal{O}$ is linearly spanned by idempotents, a typical $x \in \mathcal{O}$ is a sum of eight scalar multiples of idempotents. Hence $f(1)f(x) = f(x)f(1)$ for all $x \in \mathcal{O}$. □

Theorem 5.2. *Let $\mathcal{O}$ denote a split octonion algebra over its center $\mathcal{Z}$ and $\mathcal{R}$ an arbitrary nonassociative ring. Let $f : \mathcal{O} \to \mathcal{R}$ be an additive map such that $f(x)f(y) = 0$ whenever $xy = 0$. Then $f(x)f(y) = f(1)f(xy)$ for all $x, y \in \mathcal{O}$ and $f(1)$ centralizes the range of f.*

If $\mathcal{R}$ is unital and alternative, and there is an invertible element in the range of f, then $f(x) = f(1)\varphi(x)$, where φ is a unital ring homomorphism.

Proof. Let $\{p_1, p_2, \ldots, p_8\}$ denote a basis of $\mathcal{O}$ such that each p_i is idempotent. Express $x = \sum_{i=1}^{8} \alpha_i p_i$, where $\alpha_i \in \mathcal{Z}$. Since $(\alpha_i, p_i, y) = 0$ for $i = 1, \ldots, 8$, then

$$\begin{aligned} f(x)f(y) &= \sum_{i=1}^{8} f(\alpha_i p_i)f(y) \\ &\overset{(10)}{=} \sum_{i=1}^{8} f(\alpha_i)f(p_i y) \\ &\overset{(12)}{=} \sum_{i=1}^{8} f(1)f(\alpha_i p_i y) \\ &= f(1)f(xy) \end{aligned}$$

for all $y \in \mathcal{O}$. We know $f(1)$ centralizes the range of f by Lemma 5.1.

Suppose $\mathcal{R}$ is unital and alternative. If $f(a)$ is invertible for some $a \in \mathcal{O}$, then so is $f(a)^2$. Hence $f(1)f(a^2) = f(a)^2 = f(a^2)f(1)$ gives that $f(1)$ is invertible in light of Lemma 2.2. Therefore, define $\varphi(x) = f(1)^{-1}f(x)$ for all $x \in \mathcal{O}$. Since $f(1)$ commutes with every range element,

$$\begin{aligned} \varphi(x)\varphi(y) &= f(1)^{-1}f(x) \cdot f(1)^{-1}f(y) \\ &= f(1)^{-1}f(x) \cdot f(y)f(1)^{-1} \\ &= f(1)^{-1}(f(x)f(y))f(1)^{-1} \\ &= f(1)^{-1}(f(1)f(xy))f(1)^{-1} \\ &= f(xy)f(1)^{-1} \\ &= f(1)^{-1}f(xy) \\ &= \varphi(xy) \end{aligned}$$

for all $x, y \in \mathcal{O}$, since the third equality holds by the Moufang middle identity. □

The codomain plays no significant role in Theorem 5.2 for the conclusion $f(x)f(y) = f(1)f(xy)$. The codomain must obviously be a ring in order for f to be additive and preserve the zero product. However, consider an arbitrary additive group, G, and let $\psi(\cdot,\cdot) : \mathcal{O} \times \mathcal{O} \to G$ be a biadditive map with the property that $\psi(x, y) = 0$ whenever $xy = 0$. Notice that the idempotent balancing property, $\psi(ap, b) = \psi(a, pb)$, is obtained identically as it was for the specific biadditive map $(x, y) \mapsto f(x)f(y)$. It

[3]While any two elements generate an associative subalgebra of dimension at most 4 over $\mathcal{Z}$ [Lemma 2.6, [18]], it is not generally true that the subalgebra is split quaternion, so we do require $x, y \in \mathcal{H}$. See [Remark 1.10, [18]].

is easy to see that Lemma 5.1 transfers directly to biadditive maps vanishing on zero products simply by changing notation. Hence repeating the proof of Theorem 5.2 with the new notation proves the result:

Theorem 5.3. *Let $\mathcal{O}$ be a split octonion algebra over its center $\mathcal{Z}$ and let G be an arbitrary additive group. If $\psi(\cdot,\cdot) : \mathcal{O} \times \mathcal{O} \to G$ is a biadditive map such that $\psi(x,y) = 0$ whenever $xy = 0$, then there exists an additive map $\tau : \mathcal{O} \to G$ such that $\psi(x,y) = \tau(xy)$ for all $x, y \in \mathcal{O}$. In other words, a split octonion algebra is zero product determined.*

Proof. Repeat the above with $\psi(x,y)$ in place of $f(x)f(y)$ and $\tau(x) := \psi(1,x)$. □

Since the theory of zero product determined algebras [1] can be fruitfully applied to associative Cayley-Dickson algebras (specifically, via [Theorem 2.3, [1]]), Theorem 5.3 essentially proves that all split alternative Cayley-Dickson algebras are zero product determined, too.

Finally, it is clear that Lemma 5.1 is immediate if the underlying mapping is linear over the base field (as opposed to just additive). In that case, there is no apparent need to use the structure of $\mathcal{O}$; just that it is linearly spanned by idempotents. Thus, the idempotent balancing property (10) is enough to adapt Theorem 5.2 to bilinear maps on *any* finite-dimensional alternative algebra.

Theorem 5.4. *Let $\mathcal{A}$ be a finite-dimensional alternative algebra over a field $\mathbb{F}$ that is linearly generated by nontrivial idempotents. If V is a vector space and $\psi : \mathcal{A} \times \mathcal{A} \to V$ is an $\mathbb{F}$-bilinear map such that $\psi(x,y) = 0$ whenever $xy = 0$, then $\psi(x,y) = \psi(1,xy)$ for all $x, y \in \mathcal{A}$.*

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