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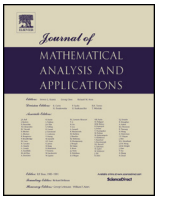

Regular Articles

Elliptic orthogonal polynomials and OPRL

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ABSTRACT

We explore a class of meromorphic functions on elliptic curves, termed *elliptic orthogonal a-polynomials* (a-EOPs), which extend the classical notion of orthogonal polynomials to compact Riemann surfaces of genus one. Building on Bertola's construction of orthogonal sections, we study these functions via non-Hermitian orthogonality on the torus, establish their recurrence properties, and derive an analogue of the Christoffel–Darboux formula. We demonstrate that, under real-valued orthogonality conditions, a-EOPs exhibit interlacing and simplicity of zeros similar to orthogonal polynomials on the real line (OPRL). Furthermore, we construct a general correspondence between families of OPRL and elliptic orthogonal functions, including a decomposition into multiple orthogonality relations, and identify new interlacing phenomena induced by rational deformations of the orthogonality weight.

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1. Introduction

Orthogonal polynomials on the real line (OPRL) play a central role in analysis, mathematical physics, and approximation theory. In recent years, there has been growing interest in generalizing this concept to higher-dimensional and more complex settings, particularly to functions defined on compact Riemann surfaces. Among these, *elliptic orthogonal functions*, defined on tori (genus one Riemann surfaces), provide a rich and tractable class of objects that bridge classical analysis with algebraic geometry.

A foundational approach to such generalizations was developed by Bertola [1], who introduced *orthogonal sections*: meromorphic functions on compact Riemann surfaces satisfying orthogonality relations defined via generalized Cauchy kernels and Weyl–Stieltjes transforms. These constructions, although technically involved, are intimately connected with Padé approximation and enable the formulation of Riemann–Hilbert problems suitable for asymptotic analysis [2].

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In this work, we focus on the simplest nontrivial case: the genus $g = 1$ setting of elliptic curves, or equivalently, tori. We define a natural analogue of orthogonal polynomials on these surfaces, which we call *elliptic orthogonal a-polynomials* (abbreviated as a-EOPs). These are meromorphic functions on the torus with prescribed poles at two points: the "point at infinity" and a fixed "anchor point" a (which is the parameter a in the acronym a-EOPs). They satisfy orthogonality conditions with respect to a complex weight on a closed contour Γ on the torus, and form a basis in appropriate function spaces.

In 1937, J. Dunham [11] studied orthogonal polynomials on a planar algebraic curve. Much more recent is the work of Spicer, Nijhoff, and Kamp [17], where they considered the orthogonalization of the sequence

$$1, x, y, x^2, xy, x^3, x^3y, x^4, x^4y, x^5, x^5y, \dots\dots$$

subject to the constraint that x and y are related by the elliptic equation

$$y^2 = 4x^3 - g_2x - g_3;$$

they called the corresponding orthogonal sequence the *two-variable elliptic orthogonal polynomials*.

In another direction, Fasondini, Olver and Xu also developed many bivariate orthogonal polynomials in planar algebraic curves of form $y^m = p(x)$ where $m = 1, 2$ and p is a polynomial, and quadratic surfaces of revolution [8,9,14–16].

The most recent and similar construction to ours appeared in Desiraju, Latimer and Roffelsen [4], where they consider elliptic orthogonal polynomials without the "anchor point" a (which forces them to skip the "polynomials" of degree 1). See also [3] for an investigation on their corresponding recurrence relations and Christoffel-Darboux formula.

Our main contributions are threefold:

- We provide a simplified construction of a-EOPs, establish their structural properties, and derive recurrence relations and a Christoffel–Darboux formula tailored to the elliptic setting.
- We investigate the case of real-valued orthogonality, proving that under natural symmetry and positivity assumptions, the zeros of a-EOPs are simple and interlace, thus mirroring classical results from the theory of OPRL.
- We develop a lifting mechanism from OPRL to the elliptic setting, and a decomposition framework that expresses a-EOPs in terms of multiple orthogonality conditions on the real line. This leads to novel interlacing results for rationally modified OPRL families, including examples involving Jacobi polynomials.

The paper is organized as follows. In Section 2, we recall the basic theory of elliptic functions and define the relevant function spaces. Section 3 introduces non-Hermitian orthogonality on the torus and derives foundational results. In Section 4, we analyze the real-valued setting, proving zero interlacing theorems. Section 5 is devoted to establishing connections between a-EOPs and classical OPRL, with explicit examples and a general decomposition theorem.

2. Polynomials on a torus

Elliptic functions are doubly-periodic meromorphic functions on $\mathbb{C}$. If we denote[1] their periods by $2\omega_1$ and $2\omega_3$, with the standard assumption that they are $\mathbb{R}$-linearly independent, then they can be regarded as meromorphic functions on a torus $\mathcal{T}$ defined as the quotient space of $\mathbb{C}$ by the integer lattice of periods:

[1] We are following the notation of [6].

$$\mathcal{T} := \mathbb{C}/\Lambda, \quad \Lambda := 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}. \tag{2.1}$$

It is known that $\mathcal{T}$ can be endowed with a complex structure, becoming a compact Riemann surface of genus 1. We can establish a one-to-one correspondence between $\mathcal{T}$ and the *fundamental parallelogram*

$$\Omega := 2\omega_1[0,1) + 2\omega_3[0,1) \tag{2.2}$$

by identifying the opposite edges. The set $\mathcal{L}(n\cdot 0)$ of elliptic functions holomorphic on $\Omega\setminus\{0\}$ with a pole at 0 of order $\le n \in \mathbb{Z}_{\ge 0} := \mathbb{N}\cup\{0\}$ is a vector space. By a well-known property of elliptic functions, the set $\mathcal{L}(1\cdot 0)$ contains only constant functions.

Since a polynomial of degree $\le n \in \mathbb{Z}_{\ge 0}$ is a meromorphic function on the Riemann sphere $\overline{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ with its only pole of degree at most n, at ∞, the set $\mathcal{L}(n\cdot 0)$ is a natural analogue on $\mathcal{T}$ of polynomials of degree $\le n$.

A well-know representative of elliptic functions is the *Weierstrass $\wp$ function*

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right),$$

which is even, and its only poles are double poles at Λ (so that $\wp\in\mathcal{L}(2\cdot 0)$), with

$$\wp(z) = \frac{1}{z^2} + \mathcal{O}(z^2), \quad z\to 0. \tag{2.3}$$

The function $\wp$ satisfies a first-order differential equation [6, §23.3]

$$(\wp'(z))^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3) = 4(\wp(z))^3 - g_2\,\wp(z) - g_3 \tag{2.4}$$

with constants

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_1+\omega_3), \quad e_3 = \wp(\omega_3), \tag{2.5}$$

and the *Weierstrass invariants*

$$g_2 = 60\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^4}, \qquad g_3 = 140\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^6}. \tag{2.6}$$

Function $\wp$, and hence, the lattice Λ, do not uniquely determine the half-periods ω_1, ω_2. However, by (2.4) and (2.6), for each $\wp$, the values e_1, e_2, e_3 and the Weierstrass invariants are unique, and we can write $\wp(\cdot;\Lambda)$ or $\wp(\cdot; g_2, g_3)$ when we need to specify the parameters.

Periods ω_1, ω_3 and constants e_j in (2.5) are related by

$$\begin{aligned} 2\omega_1 &= \int_{e_1}^{\infty} \frac{du}{\sqrt{(u-e_1)(u-e_2)(u-e_3)}} = \int_{e_3}^{e_2} \frac{du}{\sqrt{(e_1-u)(e_2-u)(u-e_3)}}, \\ 2\omega_3 &= i\int_{e_2}^{e_1} \frac{du}{\sqrt{(e_1-u)(u-e_2)(u-e_3)}} = i\int_{-\infty}^{e_3} \frac{du}{\sqrt{(e_1-u)(e_2-u)(e_3-u)}}, \end{aligned} \tag{2.7}$$

see [6].

Equation (2.4) allows the identification of $\mathcal{T}$ with an elliptic (cubic) plane curve $\mathcal{C}$, the compact Riemann surface of

$$y^2 = 4x^3 - g_2\, x - g_3.$$

Indeed,

$$z \quad \mapsto \quad (x, y) = \left(\wp(z), -\frac{\wp'(z)}{2}\right), \quad \wp(z) = \wp(z; g_2, g_3), \tag{2.8}$$

is a biholomorphic map (*Weierstrass' parametrization*) from $\mathcal{T}$ onto $\mathcal{C}$.

Addressing the lack of non-constant functions in $\mathcal{L}(1 \cdot 0)$, we could use a formal primitive of $\wp$. Recall that the *Weierstrass zeta function* is defined as

$$\zeta(z) := \frac{1}{z} - \int_0^z \left[\wp(u) - \frac{1}{u^2}\right] \mathrm{d}u = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left[\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right], \quad z \in \mathbb{C},$$

where the path of integration in $\mathbb{C}$ does not intersect $\Lambda \setminus \{0\}$. It is easy to verify that ζ is an odd function, with a simple pole at $z = 0$,

$$\zeta(z) = \frac{1}{z} + \mathcal{O}(z), \quad z \to 0,$$

that satisfies that $\zeta' = -\wp$. However, it is no longer doubly periodic (and, thus, not an elliptic function). The following identities are well-known (e.g., [6, §23.2]):

$$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i, \quad z \in \mathbb{C}, \quad \text{where} \quad \eta_i = \zeta(\omega_i), \quad i = 1, 3.$$

Since the additive increments do not depend on z, a difference of two zeta functions is elliptic, so that for any $a \in \Omega \setminus \{0\}$,

$$\zeta(z) - \zeta(z - a) + \text{const}$$

is an elliptic function with two simple poles in Ω: one at $z = 0$ and another one, at $z = a$.

The previous discussion motivates the following notion:

Definition 2.1. Let $a \in \Omega \setminus \{0\}$. For $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{L}(n \cdot 0 + a)$ denotes the vector space of elliptic functions with periods (2.1), holomorphic in $\Omega \setminus \{0, a\}$, with a possible pole at 0 of order $\leq n$, and at most a simple pole at a. Elements of $\mathcal{L}(n \cdot 0 + a)$ are *elliptic a-polynomials* on $\mathcal{T}$.

Moreover, if for $f \in \mathcal{L}(n \cdot 0 + a)$,

$$f(z) = \frac{c}{z^n} + \mathcal{O}\left(\frac{1}{z^{n-1}}\right), \quad z \to 0, \quad c \in \mathbb{C} \setminus \{0\},$$

we say that n is the *polynomial degree* of f, and c is its *leading coefficient*. When $c = 1$, f is said to be *monic*.

Remark 2.2. By Definition 2.1, the number of poles of f on $\mathcal{T}$, counted with multiplicity, can exceed at most in 1 its polynomial degree; these values actually match if and only if f is holomorphic at a.

Notice also that the limit case $a \to 0$ implies excluding the functions with polynomial degree 1 from our construction, building a theory that resembles that of the exceptional orthogonal polynomials on $\mathbb{C}$. This construction was considered in [3,4], where the authors study elliptic orthogonal functions with poles at 0 only.

Clearly,

$$b_0(z) := 1, \quad b_1(z;a) := \zeta(z) - \zeta(z-a) - \zeta(a) = -\frac{1}{2}\frac{\wp'(z)+\wp'(a)}{\wp(z)-\wp(a)}, \tag{2.9}$$

form a monic basis of $\mathcal{L}(0+a)$, see [1]; function b_1 is an analogue of the Cauchy kernel on $\mathcal{T}$, see [4], and the alternative expression for b_1 follows from the addition theorem for the ζ function, [6, formula (23.10.2)]. We can complement these functions with

$$b_{2k}(z) := \wp^k(z), \quad b_{2k+1}(z) := -\frac{1}{2}\wp'(z)\wp^{k-1}(z), \quad k \in \mathbb{N}, \tag{2.10}$$

so that $b_j \in \mathcal{L}(j \cdot 0)$, $j \geq 2$. Obviously, $\{b_j\}_{j\geq 0}$ forms a basis of $\mathcal{L}(\infty \cdot 0 + a) := \bigcup_{j\geq 0} \mathcal{L}(j \cdot 0 + a)$.

Remark 2.3. Another option is to use the derivatives of the $\wp$ function, and complete b_0 and b_1 in (2.9) with $c_k\wp^{(k-2)}(z)$, where c_k are some normalizing constants. Identities [6, §23.3]

$$\wp''(z) = 6\wp^2(z) - \frac{1}{2}g_2, \quad \wp'''(z) = 12\wp(z)\wp'(z)$$

show that these two choices are equivalent, and that we can restrict our attention to the set $\{b_j\}_{j\geq 0}$.

3. Non-Hermitian orthogonality on a torus

We want to address the notion of orthogonality on $\mathcal{T}$ with respect to a weight. The ingredients are a closed contour Γ on the torus $\mathcal{T}$ and a weight function W defined on Γ. The only assumptions on Γ and W so far are that all the integrals below are well-defined and finite; in what follows, we also impose that $a \in \Omega \setminus \{0\}$, $a \notin \Gamma$.

Given $n \in \mathbb{Z}_{\geq 0}$, an elliptic a-polynomial $F_n \in \mathcal{L}(n \cdot 0 + a)$ is *orthogonal* with respect to W (shortly, a-EOP) if

$$\int_\Gamma F_n(s)Q(s)W(s)\,\mathrm{d}s = 0 \tag{3.1}$$

for any $Q \in \mathcal{L}((n-1)\cdot 0 + a)$. Notice that this is a non-hermitian orthogonality, which implies that the existence of $F_n \not\equiv 0$ is not guaranteed a priori, and that Γ in the integral is freely deformable on $\Omega \setminus \{0, a\}$. From the general theory,

$$F_n(z) = \det \begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n-1} & \mu_{0,n} \\ \mu_{1,0} & \mu_{1,1} & \cdots & \mu_{1,n-1} & \mu_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n-1} & \mu_{n-1,n} \\ b_0(z) & b_1(z) & \cdots & b_{n-1}(z) & b_n(z) \end{pmatrix}, \tag{3.2}$$

where the bi-moments $\mu_{i,j}$ are given by integrals of products of elements of the basis (2.9)–(2.10),

$$\mu_{i,j} := \int_\Gamma b_i(s)b_j(s)W(s)\,\mathrm{d}s.$$

Moreover, the existence of F_n of the polynomial degree exactly n is guaranteed if all minors

$$D_k = D_k(a) := \det\left(\mu_{i,j}\right)_{i,j=0}^{k-1} \neq 0, \quad k \in \mathbb{N}.$$

An equivalent expression for D_k is Andréief's integration formula [18, Eq. (2.1.10)],

$$D_k = \frac{1}{k!}\int_{\Gamma^k}\cdots\int\left(\det\left[b_{j-1}(x_i)\right]_{i,j=1}^{k}\right)^2\prod_{i=1}^{k}W(x_i)\,dx_1\,\cdots\,dx_k. \tag{3.3}$$

In the case when all $D_k \neq 0$, we can introduce also the "orthonormal" a-polynomial

$$f_n(z) := \frac{1}{\sqrt{D_{n-1}D_n}}\,F_n(z) \in \mathcal{L}(n\cdot 0 + a), \tag{3.4}$$

where we choose any branch of the square root.

An analogue of the well-known three-term recurrence relation satisfied for non-Hermitian orthonormal polynomials on the complex plane is the following result:

Lemma 3.1. *Under the assumption that $D_n \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$, for each $k \geq 0$, there exist constants $A_k, B_k, C_k \in \mathbb{C}$ such that*

$$\wp(z)f_k(z) = A_k f_{k+2}(z) + B_k f_{k+1}(z) + C_k f_k(z) + B_{k-1}f_{k-1}(z) + A_{k-2}f_{k-2}(z), \tag{3.5}$$

where $f_{-1} \equiv f_{-2} \equiv 0$. Moreover,

$$\begin{aligned} A_k &= \int_\Gamma \wp(z)f_k(z)f_{k+2}(z)W(z)\,dz,\\ B_k &= \int_\Gamma \wp(z)f_k(z)f_{k+1}(z)W(z)\,dz,\\ C_k &= \int_\Gamma \wp(z)f_k(z)^2W(z)\,dz. \end{aligned} \tag{3.6}$$

Proof. Since $\wp f_k \in \mathcal{L}((k+2)\cdot 0 + a)$ and (2.9)–(2.10) is a basis of $\mathcal{L}(\infty\cdot 0 + a)$, we can write

$$\wp(z)f_k(z) = \sum_{j=0}^{k+2}\lambda_{j,k}f_j(z), \quad \lambda_{j,k} = \int_\Gamma \wp(z)f_k(z)f_j(z)W(z)\,dz, \quad j = 0,1,\ldots,k+2.$$

For $j < k-2$, $\wp f_j \in \mathcal{L}((k-1)\cdot 0 + a)$, and by orthogonality of f_j to $\mathcal{L}((k-1)\cdot 0 + a)$, $\lambda_{j,k} = 0$. Clearly, $\lambda_{j,k} = \lambda_{k,j}$ for $k-2 \leq j \leq k+2$ and (3.5) follows. □

The function

$$K_n(z,u) := \sum_{j=0}^{n-1} f_j(z)f_j(u)$$

is the reproducing kernel for $\mathcal{L}((n-1)\cdot 0 + a)$. Indeed, for $Q = \sum_{i=0}^{n-1}\lambda_i f_i \in \mathcal{L}((n-1)\cdot 0 + a)$,

$$\int_\Gamma Q(u)K_n(z,u)W(u)du = \sum_{i,j=0}^{n-1}\lambda_i f_j(z)\int_\Gamma f_i(u)f_j(u)W(u)du = \sum_{i=0}^{n-1}\lambda_i f_i(z) = Q(z).$$

As in the classical case, the existence of a finite recurrence relation implies the existence of a simplified expression for the kernel (the *Christoffel-Darboux formula*[2]):

Lemma 3.2. *If $z^2 \neq u^2$, the reproducing kernel satisfies the identity*

$$\begin{aligned} K_n(z,u) = A_{n-1}\frac{f_{n+1}(z)f_{n-1}(u) - f_{n+1}(u)f_{n-1}(z)}{\wp(z)-\wp(u)} \\ + A_{n-2}\frac{f_n(z)f_{n-2}(u) - f_n(u)f_{n-2}(z)}{\wp(z)-\wp(u)} + B_{n-1}\frac{f_n(z)f_{n-1}(u) - f_n(u)f_{n-1}(z)}{\wp(z)-\wp(u)}, \end{aligned} \tag{3.7}$$

with A_k's and B_k's given in (3.6).

If $z \in \Omega \setminus \{\omega_1, \omega_2, \omega_3\}$,

$$\begin{aligned} K_n(z,\pm z) = A_{n-1}\frac{f'_{n+1}(z)f_{n-1}(z) - f_{n+1}(z)f'_{n-1}(z)}{\wp'(z)} \\ + A_{n-2}\frac{f'_n(z)f_{n-2}(z) - f_n(z)f'_{n-2}(z)}{\wp'(z)} + B_{n-1}\frac{f'_n(z)f_{n-1}(z) - f_n(z)f'_{n-1}(z)}{\wp'(z)}. \end{aligned} \tag{3.8}$$

Finally, if $z \in \{\omega_1, \omega_2, \omega_3\}$,

$$\begin{aligned} K_n(z,\pm z) = A_{n-1}\frac{f''_{n+1}(z)f_{n-1}(z) - f_{n+1}(z)f''_{n-1}(z)}{\wp''(z)} \\ + A_{n-2}\frac{f''_n(z)f_{n-2}(z) - f_n(z)f''_{n-2}(z)}{\wp''(z)} + B_{n-1}\frac{f''_n(z)f_{n-1}(z) - f_n(z)f''_{n-1}(z)}{\wp''(z)}. \end{aligned} \tag{3.9}$$

Proof. Let $G_{j,k}(z,u) := f_{k+j}(z)f_k(u) - f_{k+j}(u)f_k(z)$. We multiply equation (3.5) by $f_k(u)$ and subtract the corresponding identity with the roles of z and u exchanged to obtain

$$[\wp(z)-\wp(u)]f_k(z)f_k(u) = A_kG_{2,k}(z,u) + B_kG_{1,k}(z,u) - B_{k-1}G_{1,k-1}(z,u) - A_{k-2}G_{2,k-2}(z,u).$$

A summation of the identities above from $k=0$ to $n-1$ gives us

$$K_n(z,u) = \frac{B_{n-1}G_{1,n-1}(z,u) + A_{n-1}G_{2,n-1}(z,u) + A_{n-2}F_{2,n-2}(z,u)}{\wp(z)-\wp(u)},$$

equivalent to (3.7). Equations (3.8) and (3.9) are obtained from (3.7) taking limits as $u \to \pm z$. □

4. Real-valued orthogonality on a torus

The non-Hermitian orthogonality on the complex plane reduces to the Hermitian one when the path of integration belongs to $\mathbb{R}$ and the weight function is real-valued and positive. In this case, the behavior of zeros of the orthogonal polynomials is well known.

In analogy with these considerations, we can assume that $\Gamma \subset \mathcal{T}$ is a contour on which both the elements of the basis $\{b_j\}$ and the weight W are real-valued, and $W > 0$. Andréief's integration formula (3.3) shows that $D_n > 0$ for all $n \in \mathbb{Z}_{\geq 0}$, which, as we have seen, implies the existence of the orthonormal basis $\{f_n\}$ (3.4), $f_n \in \mathcal{L}(n \cdot 0 + a)$.

By (2.9)–(2.10), for the real-valuedness of the basis $\{b_j\}$ it is sufficient to assume that $\wp(a) \in \mathbb{R}$ and that $\wp$ and $\wp'$ are real-valued on Γ. By (3.2), in this case f_j are real-valued on Γ.

[2] See the recent work [3] containing a similar result.

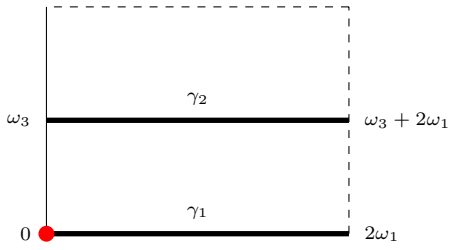


Fig. 1. Contours γ_1 and γ_2 on Ω.

A natural choice of Γ is the analogue of the real line on $\mathcal{T}$, namely, the interval $[0, 2\omega_1) \in \Omega$. Since the Weierstrass functions take real values on the real axis if and only if the invariants $g_2, g_3 \in \mathbb{R}$, it is natural to impose this condition. As a consequence of (2.5), $e_1 \in \mathbb{R}$. For simplicity, we consider the case when all three values e_i are real and distinct. This is the case of the discriminant

$$\Delta := g_2^3 - 27g_3^2 = 16\,(e_2 - e_3)^2\,(e_3 - e_1)^2\,(e_1 - e_2)^2 > 0,$$

and

$$e_3 < e_2 < e_1, \quad e_3 < 0 < e_1, \tag{4.1}$$

see [6, §23.9]. By (2.7), $\omega_1, \tau := \omega_3/i > 0$, and Ω is a rectangle. Properties of the Weierstrass functions show that both $\wp$ and $\wp'$ are real-valued on Γ when $\Gamma = \gamma_i$, $i = 1, 2$, with

$$\gamma_1 := [0, 2\omega_1), \quad \gamma_2 := \omega_3 + \gamma_1 = i\tau + \gamma_1 = [i\tau, i\tau + 2\omega_1)\,, \tag{4.2}$$

defined on Ω and lifted to $\mathcal{T}$. These will be the two contours of orthogonality (as well as equivalent contours freely deformable on $\Omega \setminus \{0, a\}$ into γ_1 or γ_2) that we consider henceforth, see Fig. 1. Additionally, we assume that $a \in (\gamma_1 \cup \gamma_2) \setminus \Gamma$, and that the leading coefficient of each f_n in (3.4) is strictly positive. Notice that with these settings, all b_j are real-valued on $\gamma_1 \cup \gamma_2$.

One of the main features of orthogonal polynomials on the real line (OPRL) is the property of their zeros: they are real, simple, and interlace with polynomials of the lower degree. In the rest of this section, we discuss the corresponding analogues. When discussing the interlacing of zeros of elliptic functions (i.e., meromorphic functions on $\mathcal{T}$), we restrict our attention to the zeros in the fundamental parallelogram Ω. This convention also applies to statements about the order of the zeros on a contour of $\mathcal{T}$.

Definition 4.1 *(Interlacing).* Let F, G be functions defined on the curve $\gamma \subset \mathbb{C}$, and let $z_1, \dots, z_n$ and $w_1, \dots, w_m$ be the zeros of F and G in γ, respectively, with account of their multiplicity. We say that G **interlaces** F **in** γ and denote it $F \preccurlyeq G$ in γ, if

$$m = n \quad \text{and} \quad \operatorname{Re} z_1 \le \operatorname{Re} w_1 \le \operatorname{Re} z_2 \le \operatorname{Re} w_2 \le \cdots \le \operatorname{Re} z_n \le \operatorname{Re} w_n, \tag{4.3}$$

or if

$$m = n - 1 \quad \text{and} \quad \operatorname{Re} z_1 \le \operatorname{Re} w_1 \le \operatorname{Re} z_2 \le \cdots \le \operatorname{Re} z_{n-1} \le \operatorname{Re} w_{n-1} \le \operatorname{Re} z_n. \tag{4.4}$$

Furthermore, we use the notation $F \prec$ in γ when all inequalities in (4.3) or (4.4) are strict.

The following theorem states that the zeros of the a-EOPs mirror the properties of the zeros of OPRL.

Theorem 4.2. *Let W be a weight function on $\Gamma \in \{\gamma_1, \gamma_2\}$, $a \in (\gamma_1 \cup \gamma_2) \setminus \Gamma$, and (f_n) be the sequence of elliptic orthogonal a-polynomials associated to W.*

(i) For $\Gamma = \gamma_1$, f_n has exactly n simple zeros on γ_1. Additionally, it has a simple zero at γ_2 if, and only if, it has a pole at a. Moreover, $f_{n+1} \prec f_n$ in γ_1.

(ii) For $\Gamma = \gamma_2$:

- *if n is even, f_n has exactly n simple zeros on γ_2. Additionally, it has a (simple) zero on γ_1 if, and only if, it has a pole at a;*
- *if n is odd, f_n has exactly $n+1$ simple zeros on γ_2, and $f_n \prec f_{n+1}$ in γ_2 or $f_{n+1} \prec f_n$ in γ_2.*

In particular, the zeros of f_n are analytic functions of the parameter a.

Remark 4.3. Theorem 4.2 is an extension of [2, Theorem 2.3], where only the case $\Gamma = \gamma_2$ was considered. However, the interlacing property in this case is new.

The rest of this section is devoted to the proof of Theorem 4.2. We start by studying the amount of zeros of f_n on the support of orthogonality.

The next lemma and proposition are a simple generalization of [2, Theorem 2.3]:

Lemma 4.4. *Let $\Phi \in \mathcal{L}(n \cdot 0 + a)$ for some $n \in \mathbb{N}$ be such that $\Phi(x) \in \mathbb{R}$ for $x \in \gamma_1 \setminus \{a\}$. Then, the amount of zeros of Φ on γ_2, with account of multiplicity, is*

(i) even, if $a \in \gamma_1$ or if $a \in \gamma_2$ but Φ is analytic at $z = a$;

(ii) odd, if $a \in \gamma_2$ and Φ has a pole at $z = a$.

Proof. Recall that for any elliptic function Φ with periods $2\omega_1 > 0$ and $2\omega_3 = 2i\tau$, $\tau > 0$, we have that

$$\sum_{\substack{\Phi(p)=\infty \\ p\in\Omega}} p = \sum_{\substack{\Phi(z)=0 \\ z\in\Omega}} z, \mod \Lambda \tag{4.5}$$

(meaning that the difference of the left and right-hand sides belongs to the lattice Λ), where all zeros and poles are enumerated with account of multiplicity. Taking the imaginary part, this identity is reduced to

$$\sum_{\substack{\Phi(p)=\infty \\ p\in\Omega}} \operatorname{Im}(p) = \sum_{\substack{\Phi(z)=0 \\ z\in\Omega}} \operatorname{Im}(z), \mod 2\tau. \tag{4.6}$$

If $\Phi \in \mathcal{L}(n \cdot 0 + a)$, then

$$\sum_{\substack{\Phi(p)=\infty \\ p\in\Omega}} \operatorname{Im}(p) = \begin{cases} 0 \mod 2\tau, & \text{if } a \in \gamma_1 \text{ or if } \Phi \text{ analytic at } a, \\ \tau \mod 2\tau, & \text{if } a \in \gamma_2 \text{ and } \Phi \text{ has a pole at } a. \end{cases}$$

Note that $z, w \in \Omega$ are such that $w = \overline{z} \mod \Lambda$, then $z \neq w \mod \Lambda$ if and only if $z, w \notin \gamma_1 \cup \gamma_2$. In addition, for such a pair z, w, by assumption on Φ, $\Phi(z) = 0$ if and only if $\Phi(w) = 0$. However, zeros of Φ on γ_1, or pairs of zeros (z, w) not on $\gamma_1 \cup \gamma_2$ do not contribute to the sum in

$$\sum_{\substack{\Phi(z)=0 \\ z\in\Omega}} \operatorname{Im}(z),$$

so that

$$\sum_{\substack{\Phi(z)=0\\ z\in\Omega}} \operatorname{Im}(z) = \sum_{\substack{\Phi(z)=0\\ z\in\gamma_2}} \operatorname{Im}(z).$$

This yields that

$$\sum_{\substack{\Phi(z)=0\\ z\in\gamma_2}} \operatorname{Im}(z) = \begin{cases} 0 \mod 2\tau, & \text{if } a\in\gamma_1 \text{ or } \Phi \text{ analytic at } a,\\ \tau \mod 2\tau, & \text{if } a\in\gamma_2 \text{ and } \Phi \text{ has a pole at } a, \end{cases}$$

which is equivalent to the statement of the lemma. □

We now investigate the location of the zeros. In the following statement, we denote by $[\cdot]$ the integer part of a real number.

Proposition 4.5. *Let f_n be the n-th orthogonal elliptic a-polynomial with respect to W, as in (3.4).*

(i) If $\Gamma=\gamma_2$ and $a\in\gamma_1$ then all zeros of f_n are simple, and $2[(n+1)/2]$ of them belong to γ_2. Moreover,
- *if n is odd, then f_n has no other zeros on Ω and has a pole at a.*
- *if n is even, f_n can have at most one additional zero, which is simple and on γ_1, if and only if it has a pole at a.*

(ii) If $\Gamma=\gamma_1$ and $a\in\gamma_2$ then all zeros of f_n are simple, and n of them belong to γ_1. In addition, f_n has a zero on γ_2 if and only if it has a pole at a.

Proof. The proof mimics the classical arguments for OPRL but needs an additional fact about the existence of a particular elliptic function. Let us prove *(i)*; the statement of *(ii)* is established in the same fashion.

Assume that $\Gamma=\gamma_2$, $a\in\gamma_1$, and let $z_1,\dots,z_m$ be the distinct zeros of f_n on γ_2 with odd multiplicities. Since $f_n\in\mathcal{L}(n\cdot 0+a)$, we have that $m\le n+1$. Inspired by (4.5), let $y\in\Omega$ be such that

$$y+\sum_{k=1}^{m} z_k = a = m\cdot 0 + a \mod \Lambda.$$

By Lemma 4.4, m is even, which shows that $y\in\gamma_1$. By Abel's Theorem [13, Chapter V, Theorem 2.8], there exists an elliptic function Φ with simple zeros at $z_1,\dots,z_m$ and at y, a simple pole at a, and a pole of multiplicity m at 0. Moreover, $\Phi(z)\in\mathbb{R}$ for $z\in\gamma_1\cup\gamma_2$. Indeed, since $\Phi\in\mathcal{L}(m\cdot 0+a)$, we can write $\Phi=\sum_{j=0}^{m}\lambda_j b_j$, and it is sufficient to show that all $\lambda_j\in\mathbb{R}$. The vector $\lambda^T=(\lambda_1,\dots,\lambda_m)$ is a non-trivial solution of the homogeneous system $A\lambda=0$, with $A=(b_j(z_i))_{i,j=0,1,\dots,m}\in\mathbb{R}^{(m+1)\times(m+1)}$, and thus, we can always choose $\lambda\in\mathbb{R}^m$.

Since the product Φf_n is real-valued and has constant sign on γ_2, the positivity of W implies that

$$\int_{\gamma_2}\Phi(z)f_n(z)W(z)\mathrm{d}z\neq 0.$$

This contradicts the orthogonality of f_n, unless $m\ge n$. Thus, $n\le m\le n+1$. Since m is even, we conclude that m is equal to the only even integer in the set $\{n,n+1\}$. In other words, if n is odd, then $m=n+1$; it means that all zeros $z_1,\dots,z_m$ are simple and that f_n must have a pole at $z=a$. On the other hand, if n is even, then $m=n$. Once again, all zeros $z_1,\dots,z_m$ must be simple, but f_n still can have another (simple) zero, this time on γ_1, which occurs if and only if f_n has a pole at $z=a$. □

Remark 4.6. On Section 5 we present choices of W and a for which the sequence f_{2n} never have a pole at a.

To finish the proof of Theorem 4.2, it only remains to verify the interlacing property. The lack of the three-term recurrence relation satisfied by $\{f_n\}$ impedes the application of the standard arguments based on the Christoffel-Darboux kernel (see, e.g., [10, Theorem 2.2.3]).

Proposition 4.7. *Let f_n, $n \geq 1$, be the n-th orthogonal elliptic a-polynomial with respect to W, as in (3.4).*

(i) If $\Gamma = \gamma_1$ and $a \in \gamma_2$, then $f_{n+1} \prec f_n$ in γ_1;
(ii) If $\Gamma = \gamma_2$, $a \in \gamma_1$ and n is odd, then $f_n \prec f_{n+1}$ in γ_2 or $f_{n+1} \prec f_n$ in γ_2.

Proof. Consider a linear combination $G = Af_n + Bf_{n+1}$, with $A, B \in \mathbb{R}$, $A^2 + B^2 \neq 0$. Since $G \in \mathcal{L}((n+1)\cdot 0 + a)$, the total number of its zeros in Ω, with the account of multiplicity, is $\leq n+2$. Moreover, by equation (4.5),

$$\sum_{\substack{G(z)=0 \\ z\in\Omega}} z = \varepsilon a, \quad \mod \Lambda, \tag{4.7}$$

where $\varepsilon = 1$ if G has a pole at a, and $\varepsilon = 0$ otherwise.

We address the case *(i)* first. By Lemma 4.4, *(ii)*, the number of zeros of G in Ω is exactly $n+2$ if and only if G has a pole at a, meaning that $\varepsilon = 1$ in (4.7). This shows that all zeros of G cannot be on γ_1, and thus, the number of zeros on Γ, with account of multiplicity, is $\leq n+1$.

Let $z_1, \ldots, z_m$ be the distinct zeros of odd order of $G = Af_n + Bf_{n+1}$ on Γ. Reasoning as in the proof of Proposition 4.5, taking $y \in \Omega$ such that

$$y + \sum_{k=1}^{m} z_k = a = m\cdot 0 + a \quad \mod \Lambda, \tag{4.8}$$

and invoking Abel's Theorem, we can claim the existence of an elliptic function $\Phi \in \mathcal{L}(m\cdot 0 + a)$ with simple zeros at $z_1, \ldots, z_m$ and at y, a simple pole at a, and a pole of multiplicity m at 0. Moreover, $y \in \gamma_2$, Φ is real-valued on Γ, and $z_1, \ldots, z_m$ are the only (simple) zeros of Φ on Γ. Thus,

$$\int_{\Gamma} \Phi(z) G(z) W(z)\, \mathrm{d}z \neq 0.$$

Since G is orthogonal to $\mathcal{L}((n-1)\cdot 0 + a)$, we conclude that $m \geq n$. This implies that all zeros of G on Γ are simple (otherwise, the total number of zeros would be at least $n+2$).

We aim at a similar conclusion in the case *(ii)*. Now Lemma 4.4, *(i)*, yields that G must have an even number of zeros of G on γ_2, and hence, there are at most $n+1$ on Γ. Moreover, if $z_1, \ldots, z_m$ are the distinct zeros of odd multiplicity of $G = Af_n + Bf_{n+1}$ on Γ, then m is even, since otherwise the total amount of zeroes of G on Γ would be odd. We take $y \in \Omega$ satisfying (4.8) and construct Φ as above, except that now $y \in \gamma_1$. Thus, $z_1, \ldots, z_m$ are the only (simple) zeros of Φ on Γ. Reasoning as above, we conclude that $m \geq n$, and once again, all zeros of G on Γ are simple.

In both cases *(i)* and *(ii)*, we have shown that G has either n or $n+1$ zeros on Γ, all simple. This implies that the linear system

$$\begin{pmatrix} f_n(z) & f_{n+1}(z) \\ f_n'(z) & f_{n+1}'(z) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

admits only the trivial solution for $A = B = 0$. Thus, the determinant of the matrix is non-vanishing:

$$f_{n+1}(z)f_n'(z) - f_n(z)f_{n+1}'(z) \neq 0, \quad z \in \Gamma. \tag{4.9}$$

In particular, the left-hand side does not change sign on Γ.

Finally, we can proceed with the proof of interlacing. Let x and y be two consecutive zeros of f_n on Γ. All the zeros of f_n are simple (Proposition 4.5), and thus $f_n'(x)f_n'(y) < 0$. From (4.9), $f_{n+1}(x)f_{n+1}(y) < 0$, which implies that there is a zero of f_{n+1} on (x, y). This proves the interlacing result: $f_n \prec f_{n+1}$ or $f_{n+1} \prec f_n$.

In the case *(i)*, when $0 \in \Gamma$, we can take $z \to 0$ to conclude that

$$f_{n+1}(z)f_n'(z) - f_n(z)f_{n+1}'(z) > 0, \quad z \in \Gamma \setminus \{0\}.$$

Let x be the left-most zero of f_n on γ_1; since the leading coefficient of f_n is strictly positive, $f_n'(x) < 0$, and equation (4.9) shows that $f_{n+1}(x) < 0$. As $\lim_{z\to 0^+} f_{n+1}(z) = +\infty$, there exists $x' < x$ such that $f_{n+1}(x) = 0$. A similar argument shows that if y is the right-most zero of f_n then there exists $y' > y$ such that $f_{n+1}(y') = 0$. This finishes the proof for *(i)*. □

5. A correspondence between OPRL and EOP

In this section, we explore the possible correspondence between the orthogonal a-polynomials on $\mathcal{T}$, and families of OPRL.

5.1. A totally symmetric case

Starting with a positive weight function w on a compact interval of the real-line, we discuss next how to lift its corresponding family of orthogonal polynomials to a family of orthogonal elliptic a-polynomials, corresponding to a specific torus $\mathcal{T}$, for which the orthogonality weight W is an even function and a is a half-period. This corresponds to the maximum amount of symmetry possible over W and a. We do it with the goal of illustrating the procedure in the simplest situation.

Given two real numbers e_2, e_3 satisfying

$$e_3 < 0, \quad e_3 < e_2 < |e_3|/2, \tag{5.1}$$

define $e_1 = -e_2 - e_3$; notice that with our assumptions, the relation

$$e_3 < e_2 < e_1$$

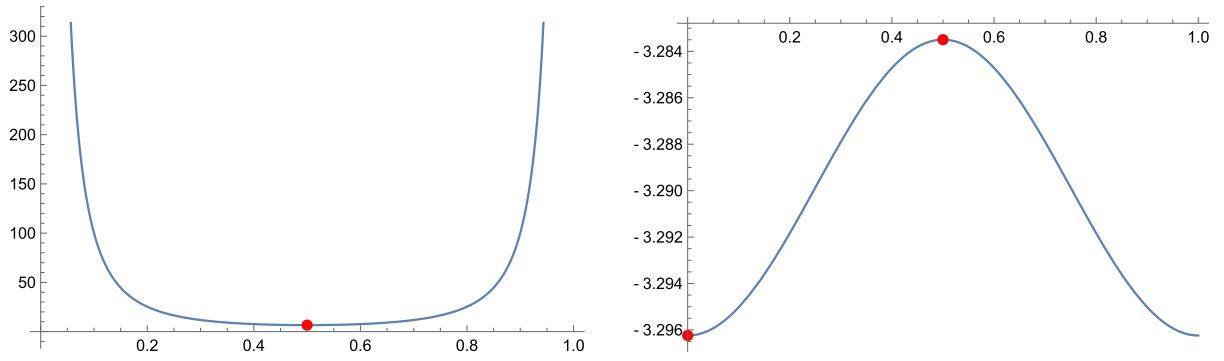
holds. Let w be a weight that is > 0 almost everywhere on the real interval $[e_3, e_2]$, and let

$$\widetilde{w}(x) := \frac{(x - e_3)(e_2 - x)}{e_1 - x}\, w(x), \quad x \in [e_3, e_2] \tag{5.2}$$

be its rational modification. We denote by $P_n(\cdot; w)$ and $P_n(\cdot; \widetilde{w})$ the algebraic monic polynomial of degree n, orthogonal on $[e_3, e_2]$ with respect to w and $\widetilde{w}$, respectively.

By formula (2.4), the triplet (e_1, e_2, e_3) determines uniquely the Weierstrass function $\wp = \wp(\cdot; \Lambda)$, where the lattice Λ is rectangular, meaning that we can take the half-periods $\omega_1 > 0$ and $\omega_3 = i\tau$, $\tau > 0$ (see (2.7)).

Theorem 5.1. *Under the assumptions above, let $a = \omega_1 \in \gamma_1$. Then the function $F_n \in \mathcal{L}(n \cdot 0 + a)$, $n \in \mathbb{Z}_{\geq 0}$, defined by*



a $\wp(t\omega_1)$, $0 \leqslant t \leqslant 2$. b $\wp(\omega_3 + t\omega_1)$, $0 \leqslant t \leqslant 2$.

Fig. 2. Plots of $\wp(z) \in \mathbb{R}$ for $2\omega_1 = 1$ and $2\omega_3 = 3i/2$ on the contours γ_1 (left) and γ_2 (right). Here $e_1 \approx 6.57974$, $e_2 \approx -3.29624$, and $e_3 \approx -3.2835$.

$$F_n(z) := \begin{cases} P_j(\wp(z); w), & n = 2j,\ j \geq 0, \\ -\dfrac{1}{2}\dfrac{\wp'(z)}{\wp(z) - e_1} P_j(\wp(z); \widetilde{w}), & n = 2j+1,\ j \geq 0, \end{cases}$$

is the (monic) a-polynomial of the polynomial degree n on the torus $\mathcal{T} = \mathbb{C}/\Lambda$, orthogonal with respect to the even weight W on γ_2, uniquely defined by

$$W(z) := \begin{cases} \dfrac{1}{2}\, w(\wp(z))\wp'(z), & z = \omega_3 + t\omega_1, \quad t \in [0,1], \\ W(2\omega_2 - z), & z = \omega_3 + t\omega_1, \quad t \in [1,2]. \end{cases} \tag{5.3}$$

Notice that F_n has a pole at a if, and only if, n is odd. When n is even, F_n coincide with the even elliptic polynomials with symmetric orthogonality weight constructed in [4].

Proof. Under our assumptions on the periods, function $\wp(\omega_3 + t\omega_1)$ is strictly increasing on $t \in (0,1)$, taking all values in the interval (e_3, e_2), see e.g., [12, §22]. By symmetry, $\wp(\omega_3 + t\omega_1)$ is strictly decreasing on $t \in (1,2)$; see Fig. 2, right.

In a similar fashion, $\wp(t\omega_1)$ is strictly decreasing on $t \in (0,1)$, taking all values in the interval $(e_1, +\infty)$. By symmetry, $\wp(t\omega_1)$ is strictly increasing on $t \in (1,2)$ (see Fig. 2, left).

Additionally, since $\wp(\omega_j) = e_j$ and $\wp'(\omega_j) = 0$ (see Equations (2.4) and (2.5)), in this case b_1 from (2.9) is an odd function that takes the form

$$b_1(z) = -\frac{1}{2}\frac{\wp'(z)}{\wp(z) - e_1}. \tag{5.4}$$

It is straightforward to check that F_n is a monic elliptic a-polynomial of degree n. We only need to verify the orthogonality conditions. Note that F_n is an even (resp. odd) function if, and only if, n is even (resp. odd).

Since W is an even weight, for $n, m \in \mathbb{Z}_{\geq 0}$,

$$\int_\Gamma F_{2n}(z) F_{2m+1}(z) W(z)\, \mathrm{d}z = 0.$$

For $n = 2j \in \mathbb{Z}_{\geq 0}$, $m = 2l \in \mathbb{Z}_{\geq 0}$, $m \neq n$,

$$\begin{aligned}\int_{\Gamma} F_n(z)F_m(z)W(z)\,\mathrm{d}z &= 2\int_{\omega_3}^{\omega_3+\omega_1} P_j(\wp(z);w)P_l(\wp(z);w)W(z)\,\mathrm{d}z\\ &= \int_{\omega_3}^{\omega_3+\omega_1} P_j(\wp(z);w)P_l(\wp(z);w)w(\wp(z))\wp'(z)\,\mathrm{d}z\\ &= \int_{e_3}^{e_2} P_j(x;w)P_l(x;w)w(x)\,\mathrm{d}x = 0.\end{aligned}$$

Analogously, using (2.4) we get that for $n = 2j+1 \in \mathbb{N}$, $m = 2l+1 \in \mathbb{N}$, $m \neq n$,

$$\int_{\Gamma} F_n(z)F_m(z)W(z)\,\mathrm{d}z = \frac{1}{4}\int_{e_3}^{e_2} P_j(x;\widetilde{w})P_l(x;\widetilde{w})\frac{(x-e_2)(x-e_3)}{(x-e_1)}w(x)\,\mathrm{d}x = 0. \quad \square$$

Theorem 5.1 shows how to lift the orthogonal polynomials from the real line to a torus $\mathcal{T}$, corresponding to a rectangular lattice. The obtained construction had the feature that

$$W \text{ is an even function on } \Gamma = \gamma_2, \qquad a = \omega_1 \text{ (half period)}. \tag{5.5}$$

Note that for every positive even weight function W defined on γ_2, equation (5.3) holds with

$$w(x) = \frac{W(\wp^{-1}(x))}{\sqrt{(e_1-x)(e_2-x)(x-e_3)}} > 0, \quad e_3 < x < e_2,$$

therefore Theorem 5.1 can be applied to w, characterizing the sequence of a-EOP with respect to W in terms of $P_n(x,w)$ and $P_n(x,\tilde{w})$, where

$$\tilde{w}(x) = \sqrt{\frac{(x-e_3)(e_2-x)}{(e_1-x)^3}}W(\wp^{-1}(x)).$$

The following identities also hold true for the monic orthogonal polynomials associated to w and $\tilde{w}$ defined by (5.2),

$$P_j(x,w) = F_{2j}(\wp^{-1}(x)), \quad P_j(x,\tilde{w}) = \sqrt{\frac{e_1-x}{(e_2-x)(x-e_3)}}F_{2j+1}(\wp^{-1}(x)), \quad x \in [e_3,e_2],$$

where F_n is the n-th monic a-EOP associated to W, with $a = \omega_1$.

In other words, the construction of Theorem 5.1 is reversible, and establishes a bijection between a-EOPs with respect to the weight W satisfying (5.5), and two related sequences of OPRL.

Example 5.2. Consider the torus $\mathcal{T}$ defined by the square lattice

$$\Lambda = 2\omega_1\mathbb{Z} + 2i\omega_1\mathbb{Z}, \quad \omega_1 = \frac{32\pi}{\Gamma\left(\frac{1}{4}\right)^4},$$

for which the constants (2.5) can be found explicitly, see [6, §23.5(iii)]:

$$e_2 = 0, \quad e_1 = 1 = -e_3,$$

as well as the Weierstrass invariants,

$$g_2 = 4\omega_1^{-6}, \quad g_3 = 0.$$

We illustrate the application of Theorem 5.1 by an explicit construction of elliptic orthogonal a-polynomials with respect to a weight on $\mathcal{T}$ in terms of the Jacobi family of OPRL.

Let $\wp(z) = \wp(z;\Lambda)$, and consider the weight

$$|\wp(z)|^{\alpha+\frac{1}{2}}\,(\wp(z)+1)^{\beta+\frac{1}{2}}(1-\wp(z))^{\frac{1}{2}} > 0, \quad z \in \gamma_2, \tag{5.6}$$

where $\alpha, \beta > -1$.

If $P_j^{(\alpha,\beta)}$ denotes the j-th monic Jacobi polynomial, then

$$P_j(x;w) = 2^{-j}P_j^{(\alpha,\beta)}(2x+1),$$

is the monic polynomial of degree j, orthogonal on $[-1,0]$ with respect to the weight

$$w(x) = |x|^{\alpha}(1+x)^{\beta}.$$

According to Theorem 5.1, for even values of n,

$$F_n(z) = \frac{1}{2^j}P_j^{(\alpha,\beta)}(2\wp(z)+1), \quad n = 2j,$$

is the n-th elliptic orthogonal a-polynomial, $a = \omega_1$, with respect to the weight (5.6).

On the other hand, the corresponding $\widetilde{w}$ function defined in (5.2) is

$$\widetilde{w}(x) := \frac{|x|^{\alpha+1}(x+1)^{\beta+1}}{1-x}, \quad x \in [-1,0],$$

which is a rational deformation of a Jacobi weight on $[-1,0]$. Define

$$\lambda_n = \lambda_n(\alpha,\beta) := \int_{-1}^{1} \frac{P_n^{(\alpha+1,\beta+1)}(s)}{3-s}(1-s)^{\alpha+1}(1+s)^{\beta+1}\,\mathrm{d}s. \tag{5.7}$$

It is easy to directly check (see e.g. [10, Theorem 2.7.2]) that in this case,

$$P_j\,(x,\widetilde{w}) = \frac{1}{2^j\lambda_{j-1}}\left[\lambda_{j-1}P_j^{(\alpha+1,\beta+1)}(2x+1) - \lambda_j P_{j-1}^{(\alpha+1,\beta+1)}(2x+1)\right].$$

Therefore, by Theorem 5.1, for odd values of n,

$$\begin{aligned} F_n(z) =& \frac{1}{2^{j+1}\lambda_{j-1}}\frac{\wp'(z)}{1-\wp(z)} \\ &\times \left[\lambda_{j-1}P_j^{(\alpha+1,\beta+1)}(2\wp(z)+1) - \lambda_j P_{j-1}^{(\alpha+1,\beta+1)}(2\wp(z)+1)\right], \quad \text{if } n = 2j+1 \end{aligned}$$

is the n-th elliptic a-orthogonal polynomial with respect to W in (5.6).

Example 5.3. Let $\mathcal{T}$ be the torus of Example 5.2, but instead of the weight W as in (5.6), we consider

$$V(x) = |\wp(z)|^{\alpha-\frac{1}{2}}\,(\wp(z)+1)^{\beta-\frac{1}{2}}(1-\wp(z))^{\frac{3}{2}}, \quad z \in \gamma_2. \tag{5.8}$$

Take

$$v(x) = (1-x)(1+x)^{\beta-1}|x|^{\alpha-1}, \quad \alpha,\beta > 0, \quad x \in [-1,0].$$

and the corresponding weight $\widetilde{v}$ (see (5.2)),

$$\widetilde{v}(x) = |x|^{\alpha}(1+x)^{\beta}, \quad x \in [-1,0].$$

Note that v is a polynomial (Christoffel) deformation of the Jacobi weight on $[-1,0]$ with parameters $\beta-1$ and $\alpha-1$.

Then, by Theorem 5.1, for odd values of n,

$$G_n(z) = \frac{1}{2^{j+1}}\frac{\wp'(z)}{1-\wp(z)}P_j^{(\alpha,\beta)}(2\wp(z)+1), \quad n = 2j+1,$$

is the n-th elliptic orthogonal a-polynomial, $a = \omega_1$, with respect to the weight (5.8).

On the other hand, [10, Theorem 2.7.1] and direct computation shows that

$$P_j(x,v) = \frac{K_j^{(\alpha-1,\beta-1)}(2x+1,3)}{2^j P_j^{(\alpha-1,\beta-1)}(3)},$$

where $K_j^{(\alpha,\beta)}$ is the (scaled) Christoffel-Darboux kernel,

$$K_j^{(\alpha,\beta)}(x,y) := \frac{P_{j+1}^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(y) - P_{j+1}^{(\alpha,\beta)}(y)P_j^{(\alpha,\beta)}(x)}{x-y}. \tag{5.9}$$

By Theorem 5.1, for even values of n,

$$G_n(z) = \frac{K_j^{(\alpha-1,\beta-1)}(2\wp(z)+1,3)}{2^j P_j^{(\alpha-1,\beta-1)}(3)}, \quad n = 2j,$$

is the n-th elliptic orthogonal a-polynomial associated to the weight V.

Interestingly, the interlacing of the a-EOPs obtained in Theorem 4.2 has a consequence for the OPRL sequence with respect to w and $\tilde{w}$. This is not expected, as no such result holds for general rational deformations of measures.

Corollary 5.4. *Set $\widehat{\gamma}_2 = \{\omega_3 + t\omega_1; t \in [0,1]\}$. Then, with the notations of Definition 4.1 and Theorem 5.1, $F_n \prec F_{n+2}$ in $\widehat{\gamma}_2$. Additionally, for the orthogonal polynomials with respect to w and its rational deformation $\widetilde{w}$ as in (5.2), the following interlacing property holds:*

$$P_n(\cdot,w) \prec P_{n-1}(\cdot,\widetilde{w}) \text{ in } [e_3,e_2].$$

Proof. Assume that $n = 2m$ and let $z_1^{(n)},\dots,z_m^{(n)}$ be the zeros of F_n on $\widehat{\gamma}_2$. By Theorem 5.1, $\wp\left(z_1^{(n)}\right),\dots,\wp\left(z_k^{(n)}\right)$ are the zeros of $P_m(\cdot,w)$. Since $P_{m+1}(\cdot,w) \prec P_m(\cdot,w)$ in $[e_2,e_3]$ and $\wp$ is monotone on $\widehat{\gamma}_2$, we see that

$$z_1^{(n+2)} < z_1^{(n)} < z_2^{(n+2)} < \cdots < z_m^{(n+2)} < z_m^{(n)} < z_{m+1}^{(n+2)},$$

which establishes that $F_n \prec F_{n+2}$ in $\widehat{\gamma}_2$ in this case. The proof for $n = 2m+1$ is analogous, by identifying the zeros of f_n with those of $P_m(\cdot, \widetilde{w})$.

Similarly, the interlacing of $P_n(\cdot, w)$ and $P_{n-1}(\cdot, \widetilde{w})$ in $[e_3, e_2]$ is consequence of the identification of their zeros with the zeros of f_{2n-1} and f_{2n}, and their corresponding interlacing given by Proposition 4.7. □

Applying Corollary 5.4 to the construction of Examples 5.2 and 5.3 we obtain some interlacing properties for Jacobi polynomials:

Corollary 5.5. *Let $P_n^{(\alpha,\beta)}$ be the n-th monic Jacobi polynomial, $K_j^{(\alpha,\beta)}$ defined by (5.9) and $\lambda_n = \lambda_n(\alpha,\beta)$ defined by (5.7). The polynomials*

$$R_{n-1}(x) := P_{n-1}^{(\alpha+1,\beta+1)}(x)\lambda_{n-2} - P_{n-2}^{(\alpha+1,\beta+1)}(x)\lambda_{n-1},$$
$$S_n(x) := K_{n+1}^{(\alpha-1,\beta-1)}(x,3),$$

satisfy the following interlacing properties:

$$S_n \prec P_n^{(\alpha,\beta)} \text{ in } [-1,1], \quad \alpha,\beta > 0,$$
$$P_n^{(\alpha,\beta)} \prec R_{n-1} \text{ in } [-1,1], \quad \alpha,\beta > -1.$$

Remark 5.6. Let us briefly put Corollary 5.5 into the context of the literature on classical orthogonal polynomials.

Note that R_{n-1} and S_n are polynomials of degree $n-1$ and n, respectively, orthogonal with respect to the weights

$$\omega(x) = \frac{(x-1)^{\alpha+1}(x+1)^{\beta+1}}{3-x}, \quad \widetilde{\omega}(x) = (3-x)(1+x)^{\beta-1}(x-1)^{\alpha-1}, \quad x \in [-1,1].$$

Theorem 2.3 in [7] establishes the following interlacing between Jacobi polynomials of different degrees,

$$P_n^{(\alpha,\beta)} \prec P_{n-1}^{(\alpha+t,\beta+t')}, \quad t,t' \in (0,2], \quad \alpha,\beta > -1.$$

We can see the Jacobi weight with parameters $(\alpha+1, \beta+1)$ as a Christoffel deformation of $\omega(x)$ and therefore Theorem 2 of [5] gives the interlacing

$$P_{n-1}^{(\alpha+1,\beta+1)} \prec R_{n-1}.$$

Corollary 5.5 establishes that the chain of interlacing

$$P_n^{(\alpha,\beta)} \prec P_{n-1}^{(\alpha+1,\beta+1)} \prec R_{n-1}$$

is transitive, fact that apparently is new.

Similarly, we can see $\widetilde{\omega}$ as a Christoffel deformation of the Jacobi weight with parameters $(\alpha-1, \beta-1)$ and, by Theorem 2 from [5] to $\widetilde{\omega}$,

$$S_n(x) \prec P_n^{(\alpha-1,\beta-1)}(x).$$

The simultaneous increase or decrease of parameters α and β can break the interlacing between Jacobi polynomials of the same degree, so there is no expected interlacing between $P_n^{(\alpha-1,\beta-1)}$ and $P_n^{(\alpha,\beta)}$. Corollary 5.5 establishes that S_n is a common interlacing polynomial to both $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha-1,\beta-1)}$, something that we could not find explicitly stated in the literature.

5.2. Decomposition of EOPs and orthogonality

In Subsection 5.1, we proved that when W is an even weight and a is a half-period, the sequence of a-EOPs splits into two sequences of orthogonal polynomials on the real-line given in terms of the weight $w(x) = \frac{W(z)}{|\wp'(z)|}$ where $x = \wp(z)$. We now drop the symmetries assumed on W and a.

We start with a general decomposition lemma holding for any a-polynomial and show that, in the case of F_n, the polynomials involved satisfy certain orthogonality conditions.

Lemma 5.7. *Let $f \in \mathcal{L}(n\cdot 0+a)$ be an elliptic a-polynomial of polynomial degree exactly n. There exist unique algebraic polynomials $p_{n,1}$ and $p_{n,2}$, with $\deg p_{n,1} \le \lfloor \frac{n}{2} \rfloor$, $\deg p_{n,2} \le \lfloor \frac{n-1}{2} \rfloor$, such that f can be written in the form*

$$f(z) = p_{n,1}(\wp(z)) + b_1(z)\, p_{n,2}(\wp(z)) + \frac{\wp'(a)}{2}\, p_{n,3}(\wp(z)), \quad p_{n,3}(x) := \frac{p_{n,2}(x) - p_{n,2}(\wp(a))}{x - \wp(a)}, \tag{5.10}$$

where $b_1(z) = -\frac{1}{2}\frac{\wp'(z)+\wp'(a)}{\wp(z)-\wp(a)}$, as defined in (2.9).

Proof. Let $n = 2m$ and write $f(z) = \sum_{j=0}^{n} \lambda_j b_j(z)$, $\lambda_n \neq 0$. Using that the second expression for the function b_1 in (2.9) is equivalent to the identity

$$\wp'(z) = -2\, b_1(z)[\wp(z) - \wp(a)] - \wp'(a)$$

and the definition of b_j in (2.10), one gets (5.10) with

$$p_{n,1}(x) = \sum_{j=0}^{m} \lambda_{2j} x^j, \quad p_{n,2}(x) = \lambda_1 + \sum_{j=0}^{m-2} \lambda_{2j+3}[x - \wp(a)]x^j, \quad p_{n,3}(x) = \sum_{j=0}^{m-2} \lambda_{2j+3} x^j.$$

The proof for $n = 2m+1$ is analogous. □

Recall that the Weierstrass $\wp$ function maps γ_2 twice on the interval $[e_3, e_2]$; more precisely, $\wp(\omega_3 + 2t)$ is a strictly increasing function on $[0,1]$, mapping $[0,1]$ onto $[e_3, e_2]$. It is strictly decreasing on $[1,2]$, taking all values from e_2 to e_3 as t transitions from 1 to 2. In this subsection, we denote by $z = \wp^{-1}(x)$ the inverse of $\wp^{-1}$ on the first half of the interval γ_2. In particular, $-z$ equals the inverse of $\wp$ restricted to the second half of γ_2. Moreover,

$$b_1(z) = -\frac{\sqrt{\prod_{i=1}^{3}(x-e_i)} + \frac{\wp'(a)}{2}}{x - \wp(a)}, \quad b_1(-z) = \frac{\sqrt{\prod_{i=1}^{3}(x-e_i)} - \frac{\wp'(a)}{2}}{x - \wp(a)}, \tag{5.11}$$

$$b_{2l}(z) = x^l, \qquad b_{2l+3}(z) = -x^l \sqrt{\prod_{i=1}^{3}(x-e_i)}. \tag{5.12}$$

Let F_n be the n-th monic elliptic orthogonal a-polynomial with respect to a general weight W on γ_2 and $a \in \gamma_1$. We preserve the notation of $p_{n,1}$ and $p_{n,2}$ when applying Lemma 5.7 to F_n. Additionally, define

$$q_n(x) := p_{n,1}(x)(\wp(a) - x) + \frac{\wp'(a)}{2} p_{n,2}(\wp(a)), \tag{5.13}$$

as well as, for $j \in \{-1, 0, 1\}$, the following real valued weight functions,

$$w_j^{\pm}(x) := \left(\sqrt{\prod_{l=1}^{3} (x - e_l)} \right)^j \frac{1}{\wp(a) - x} \frac{W(\wp^{-1}(x)) \pm W(-\wp^{-1}(x))}{2}, \quad x \in [e_3, e_2]. \tag{5.14}$$

Theorem 5.8. *For $n \in \mathbb{Z}_{\geq 0}$, let $m = \left\lfloor \frac{n}{2} \right\rfloor$ and $k = n - 2m \in \{0, 1\}$. With the notation above, the polynomials $p_{n,2}$ and q_n satisfy the orthogonality conditions*

$$\int_{e_3}^{e_2} \left[q_n(x) w_{-1}^+(x) + p_{n,2}(x) w_0^-(x) \right] x^l \, \mathrm{d}x = 0, \quad 0 \leq l < m + \frac{k}{2}; \tag{5.15}$$

$$\int_{e_3}^{e_2} \left[q_n(x) w_0^-(x) + p_{n,2}(x) w_1^+(x) \right] x^l \, \mathrm{d}x = 0, \quad 0 \leq l < m - 2; \tag{5.16}$$

$$\int_{e_3}^{e_2} \left[q_n(x) \left(w_0^-(x) + \frac{\wp'(a)}{2} w_{-1}^+(x) \right) + p_{n,2}(x) \left(w_1^+(x) + \frac{\wp'(a)}{2} w_0^-(x) \right) \right] \frac{\mathrm{d}x}{\wp(a) - x} = 0. \tag{5.17}$$

Remark 5.9. The results of Theorem 5.8 hold even in the general case of $a \in \mathcal{T} \setminus \{\gamma_1 \cup \gamma_2\}$, when the weights $w_j^{\pm}$ are generally complex-valued.

Equations (5.15)-(5.17) define *type II multiple orthogonality* conditions satisfied by the vector of polynomials $(q_n, p_{n,2})$, see, e.g., [10, §23].

Proof. The change of variable $x = \wp(z)$ and the definitions (5.11) give us

$$\begin{aligned} 0 = \int_{\gamma} f_n(z) b_j(z) W(z) \, \mathrm{d}z &= \int_{\omega_3}^{\omega_3 + \omega_1} f_n(z) b_j(z) W(z) \, \mathrm{d}z + \int_{\omega_3 + \omega_1}^{\omega_3 + 2\omega_1} f_n(z) b_j(z) W(z) \, \mathrm{d}z \\ &= \int_{e_3}^{e_2} \left[p_{n,1}(x) + \frac{\wp'(a)}{2} p_{3,n}(x) \right] \left[\frac{b_j(z) W(z) + b_j(-z) W(-z)}{2\sqrt{\prod_{i=1}^{3} (x - e_i)}} \right] \mathrm{d}x \\ &\quad + \int_{e_3}^{e_2} p_{n,2}(x) \left[\frac{b_1(z) b_j(z) W(z) + b_1(-z) b_j(-z) W(-z)}{2\sqrt{\prod_{i=1}^{3} (x - e_i)}} \right] \mathrm{d}x. \end{aligned} \tag{5.18}$$

Using again (5.11) with $j = 2l < n$, we get

$$\int_{e_3}^{e_2} \left[p_{n,1}(x) + \frac{\wp'(a)}{2} p_{3,n}(x) \right] x^l \left[\wp(a) - x \right] w_{-1}^+(x) \, \mathrm{d}x = - \int_{e_3}^{e_2} p_{n,2}(x) x^l \left[w_0^-(x) + \frac{\wp'(a)}{2} w_{-1}^+(x) \right] \mathrm{d}x.$$

By the definition of q_n (5.13) and the fact that $p_{3,n}(x) = \frac{p_{n,2}(\wp(a)) - p_{n,2}(x)}{\wp(a) - x}$, we can rewrite the above as

$$\int_{e_3}^{e_2} q_n(x) x^l w_{-1}^+(x) = - \int_{e_3}^{e_2} p_{n,2}(x) x^l w_0^-(x), \quad 0 \leq 2l < n,$$

which is exactly (5.15).

On the other hand, (5.11) with $j = 2l + 3 < n$ shows that (5.18) takes the form

$$-\int_{e_3}^{e_2}\left[p_{n,1}(x)+\frac{\wp'(a)}{2}p_{3,n}(x)\right]x^l[\wp(a)-x]w_0^-(x)\,\mathrm{d}x=\int_{e_3}^{e_2}p_{n,2}(x)x^l\left[w_1^+(x)+\frac{\wp'(a)}{2}w_0^-(x)\right]\,\mathrm{d}x.$$

Similarly to the previous case, we rewrite the identity above as

$$\int_{e_3}^{e_2}q_n(x)x^lw_0^-(x)\,\mathrm{d}x=-\int_{e_3}^{e_2}p_{n,2}(x)x^lw_1^+(x)\,\mathrm{d}x,\quad 0\le 2l+3<n,$$

which is (5.16).

Finally, for $j = 1$, (5.18) becomes

$$\begin{aligned}&\int_{e_3}^{e_2}\left(p_{n,1}(x)+\frac{\wp'(a)}{2}p_{3,n}(x)\right)\left[w_0^-(x)+\frac{\wp'(a)}{2}w_{-1}^+(x)\right]\,\mathrm{d}x\\&=-\int_{e_3}^{e_2}\frac{p_{n,2}(x)}{\wp(a)-x}\left[w_1^+(x)+\wp'(a)w_0^-(x)+\left(\frac{\wp'(a)}{2}\right)^2w_{-1}^+(x)\right]\,\mathrm{d}x.\end{aligned}$$

This is equivalent to

$$\int_{e_3}^{e_2}q_n(x)\left[\frac{w_0^-(x)}{\wp(a)-x}+\frac{\wp'(a)}{2}\frac{w_{-1}^+(x)}{\wp(a)-x}\right]=-\int p_{n,2}(x)\left[\frac{w_1^+(x)}{\wp(a)-x}+\frac{\wp'(a)}{2}\frac{w_0^-(x)}{\wp(a)-x}\right]\,\mathrm{d}x$$

that is, equation (5.17). This finishes the proof of Theorem 5.8. □

Notice that when W is even, functions $w_j^- \equiv 0$, and equations (5.15)-(5.17) specialize to

$$\int_{e_3}^{e_2}q_n(x)x^lw_{-1}^+(x)\,\mathrm{d}x=0,\quad 0\le l<m+\frac{k}{2},\tag{5.19}$$

$$\int_{e_3}^{e_2}p_{n,2}(x)x^lw_1^+(x)\,\mathrm{d}x=0,\quad 0\le l\le m-2,\tag{5.20}$$

$$\int_{e_3}^{e_2}\left[q_n(x)\frac{\wp'(a)}{2}\frac{w_{-1}^+(x)}{\wp(a)-x}+p_{n,2}(x)\frac{w_1^+(x)}{\wp(a)-x}\right]\,\mathrm{d}x=0.\tag{5.21}$$

Equation (5.19) defines orthogonality of q_n with respect to $w_{-1}^+(x)$. By Lemma 5.7,

- if $n = 2m$ then q_n has degree $m + 1$ and orthogonality conditions in (5.19) are up to degree $m - 1$. In other words, q_n is quasi-orthogonal with respect to $w_{-1}^+(x)$.
- if $n = 2m + 1$, then q_n has degree $\le m + 1$ and (5.19) corresponds to orthogonality up to degree m. Now, if the degree is exactly $m + 1$, then q_n is orthogonal with respect to $w_{-1}^+(x)$. Otherwise, $q_n \equiv 0$.

Similarly, equation (5.20) means that,

- when $n = 2m+1$, $p_{n,2}$ has degree m. Thus, $p_{n,2}$ is quasi-orthogonal with respect to $w_1^{(a)}$;
- when $n = 2m$, $p_{n,2}$ has degree $\leq m-1$. If it is exactly $m-1$, then $p_{n,2}$ is orthogonal with respect to $w_1^{(a)}$, otherwise it must be identically equal to 0.

In summary,

Corollary 5.10. *Let W be an even weight on γ_2. For each $n \in \mathbb{Z}_{\geq 0}$, $m = \left\lfloor \frac{n}{2} \right\rfloor$ and $k = n - 2m \in \{0,1\}$, there exist constants $\kappa_n, \nu_n \in \mathbb{C}$ such that*

$$q_n(x) = -\delta_{0,k} P_{m+1}(x, w_{-1}^+) + \kappa_n P_{m+k}(x, w_{-1}^+), \quad p_{n,2}(x) = \delta_{1,k} P_m(x, w_1^+) + \nu_n P_{m-1}(x, w_1^+),$$

where q_n and $p_{n,2}$ are given by Lemma 5.8 and equation (5.13), and $\delta_{i,k}$ is the Kronecker delta.

Remark 5.11. Further assuming that $a = \omega_1$, one can show that $\kappa_n = \nu_n = 0$, and the decomposition from Lemma 5.7 for F_n reduces to the one given by Theorem 5.1.

5.3. General decomposition of EOPs

Now we generalize the construction of Section 5.1. Our goal here is to lift the family of orthogonal polynomial with respect to a generic weight w defined on an interval $[e_3, e_2]$ to a family of a-EOP with respect to an even weight W on γ_2 for a suitable torus $\mathcal{T}$, for a general $a \in \gamma_1$.

We assume that e_2, e_3 satisfy the conditions (5.1) and define $e_1 = -e_2 - e_3$, so that

$$e_3 < e_2 < e_1.$$

As in Section 5.1, the triplet e_3, e_2, e_1 determines uniquely the Weierstrass elliptic function $\wp = \wp(\cdot, \Lambda)$ for a rectangular lattice Λ, with corresponding half-periods $\omega_1 > 0$ and $\omega_3 = i\tau$, $\tau > 0$.

Given a weight function $w(x)$ supported on $[e_3, e_2]$ with finite moments and $a \in \gamma_1$, consider the weight function

$$W(z) := (\wp(a) - \wp(z))\, w\left(\wp(z)\right) \sqrt{\prod_{l=1}^{3} (\wp(z) - e_l)}, \quad z \in \gamma_2.$$

With the notation (5.14),

$$w_{-1}^+(x) = w(x), \quad w_1^+(x) = \prod_{j=1}^{3} (x - e_i) w(x) =: \widehat{w}(x). \tag{5.22}$$

Set $n = 2m$. By Lemma 5.7, the n-th a-EOP with respect to W is

$$F_n(z) = p_{n,1}(\wp(z)) + b_1(z) p_{n,2}(\wp(z)) + \frac{\wp'(a)}{2} p_{3,n}(\wp(z)),$$

where $p_{n,1}$ has degree m and $p_{n,2}$ has degree $\leq m-1$ and, for q_n as in (5.13),

$$q_n(x) = -P_{m+1}(x, w) + \kappa_n P_m(x, w), \quad p_{n,2}(x) = \nu_n P_{m-1}(x, \widehat{w}), \quad \kappa_n, \nu_n \in \mathbb{C}.$$

Equation (5.13) implies that

$$q_n(\wp(a)) = \frac{\wp'(a)}{2} p_{n,2}(\wp(a)) = -P_{m+1}(\wp(a), w) + \kappa_n P_m(\wp(a), w),$$

which gives the following expression for κ_n,

$$\kappa_n = \frac{\wp'(a) p_{n,2}(\wp(a)) + 2P_{m+1}(\wp(a), w)}{2P_m(\wp(a), w)}.$$

Similarly, we find ν_n by evaluating $p_{2,n}$ at $\wp(a)$:

$$\nu_n = \frac{p_{n,2}(\wp(a))}{P_{m-1}(\wp(a), \widehat{w})}.$$

Equation (5.17) becomes

$$\frac{\wp'(a)}{2} \left(\left[\frac{\wp'(a) p_{n,2}(\wp(a)) + 2P_{m+1}(\wp(a), w)}{2P_m(\wp(a), w)} \right] \int_{e_3}^{e_2} \frac{P_m(x, w) w(x)}{\wp(a) - x} \, \mathrm{d}x \right.$$
$$\left. - \int_{e_3}^{e_2} \frac{P_{m+1}(x, w) w(x)}{\wp(a) - x} \, \mathrm{d}x \right) + \frac{p_{n,2}(\wp(a))}{P_{m-1}(\wp(a), \widehat{w})} \int_{e_3}^{e_2} \frac{P_{m-1}(x, \widehat{w}) \widehat{w}(x)}{\wp(a) - x} \, \mathrm{d}x = 0.$$

Solving for $p_{n,2}(\wp(a))$ in the expression above we get

$$p_{n,2}(\wp(a)) = \frac{\wp'(a)}{2} \left[T_{m-1}(a) - \frac{P_{m+1}(\wp(a), w) T_m(a)}{P_m(\wp(a), w)} \right] \left[\frac{\wp'(a)^2 T_m(a)}{4P_m(\wp(a), w)} + \frac{\widehat{T}_{m-1}(a)}{P_{m-1}(\wp(a), \widehat{w})} \right]^{-1},$$

where

$$T_n(a) := \int_{e_3}^{e_2} \frac{P_n(x, w) w(x)}{\wp(a) - x} \, \mathrm{d}x, \quad \widehat{T}_n(a) := \int_{e_3}^{e_2} \frac{P_n(x, \widehat{w}) \widehat{w}(x)}{\wp(a) - x} \, \mathrm{d}x. \tag{5.23}$$

This formula, although cumbersome, is an explicit expression for $p_{n,2}(\wp(a))$ given only in terms of the sequences of monic OPRL with respect to w and $\widehat{w}$, and their Cauchy transforms evaluated at $\wp(a)$.

The case $n = 2m + 1$ is similar:

$$q_n(x) = \kappa_n P_{m+1}(x, w), \quad p_{n,2}(x) = P_m(x, \widehat{w}) + \nu_n P_{m-1}(x, \widehat{w}),$$

with

$$\kappa_n = \frac{\wp'(a) p_{n,2}(\wp(a))}{4P_{m+1}(\wp(a), w)}, \quad \nu_n = \frac{p_{n,2}(\wp(a)) - P_m(\wp(a), \widehat{w})}{P_{m-1}(\wp(a), \widehat{w})},$$

where

$$p_{n,2}(\wp(a)) = \left[\frac{P_m(\wp(a), \widehat{w})}{P_{m-1}(\wp(a), \widehat{w})} \widehat{T}_{m-1}(a) - \widehat{T}_m(a) \right] \left[\frac{\wp'(a)^2}{4} \frac{T_{m+1}(a)}{P_{m+1}(\wp(a), w)} + \frac{\widehat{T}_{m-1}(a)}{P_{m-1}(\wp(a), \widehat{w})} \right]^{-1}.$$

The following theorem summarizes the results of this subsection; it is convenient to remind the reader that $P_n(\cdot, w)$ denotes the n-th monic orthogonal polynomial with respect to the weight $w(x)\,\mathrm{d}x$.

Theorem 5.12. *Let $\mathcal{T}$ be a torus with Weierstrass invariants $e_3 < e_2 < e_1$, $\wp(z)$ its corresponding Weierstrass elliptic function, w a positive weight on $[e_3, e_2]$ with finite moments. The function*

$$W(z) = (\wp(a) - \wp(z))w(\wp(z))\sqrt{\prod_{l=1}^{3}(\wp(z) - e_l)}, \quad z \in \gamma_2,$$

defines a positive weight on γ_2.

Let $n \in \mathbb{Z}_{\geq 0}$, $m = \left\lfloor \frac{n}{2} \right\rfloor$, and $k = n - 2m \in \{0, 1\}$. Then the corresponding n-th monic orthogonal a-polynomial is

$$F_n(z) = p_{n,1}(\wp(z)) + b_1(z)p_{n,2}(\wp(z)) + \frac{\wp'(a)}{2}p_{n,3}(\wp(z)),$$

where

$$\begin{aligned} p_{n,1}(x) &:= \frac{-\delta_{0,k}P_{m+1}(x, w) + \kappa_n P_{m+k}(x, w) - \frac{\wp'(a)}{2}c_n}{\wp(a) - x}, \\ p_{n,2}(x) &:= \delta_{1,k}P_m(x, \widehat{w}) + \nu_n P_{m-1}(x, \widehat{w}), \\ p_{n,3}(x) &:= \frac{p_{n,2}(x) - p_{n,2}(\wp(a))}{x - \wp(a)}, \end{aligned}$$

$\widehat{w}$ is given by (5.22), and $T_m(a)$ and $\widehat{T}_m(a)$ as in (5.23).

The constants above are

$$c_n = \begin{cases} \dfrac{\wp'(a)}{2}\left[T_{m-1}(a) - \dfrac{P_{m+1}(\wp(a), w)T_m(a)}{P_m(\wp(a), w)}\right]\left[\dfrac{\wp'(a)^2 T_m(a)}{4P_m(\wp(a), w)} + \dfrac{\widehat{T}_{m-1}(a)}{P_{m-1}(\wp(a), \widehat{w})}\right]^{-1}, & k = 0, \\ \left[\dfrac{P_m(\wp(a), \widehat{w})}{P_{m-1}(\wp(a), \widehat{w})}\widehat{T}_{m-1}(a) - \widehat{T}_m(a)\right]\left[\dfrac{\wp'(a)^2}{4}\dfrac{T_{m+1}(a)}{P_{m+1}(\wp(a), w)} + \dfrac{\widehat{T}_{m-1}(a)}{P_{m-1}(\wp(a), \widehat{w})}\right]^{-1}, & k = 1; \end{cases}$$

$$\kappa_n = \frac{\wp'(a)c_n + 2\delta_{k,1}P_{m+1}(\wp(a), w)}{P_{m+k}(\wp(a), w)}; \quad \nu_n = \frac{c_n - \delta_{0,k}P_m(\wp(a), \widehat{w})}{P_{m-1}(\wp(a), \widehat{w})}.$$

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