CENTRAL LIMIT THEOREM FOR THE MODULUS OF CONTINUITY OF AVERAGES OF OBSERVABLES ON TRANSVERSAL FAMILIES OF PIECEWISE EXPANDING UNIMODAL MAPS

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Abstract. Consider a C^2 family of good mixing C^4 piecewise expanding unimodal maps $t \in [a,b] \mapsto f_t$, with a critical point c, that is transversal to the topological classes of such maps. Given a Lipchitz observable ϕ consider the function

$$\mathcal{R}_\phi(t) = \int \phi \, d\mu_t,$$

where μ_t is the unique absolutely continuous invariant probability of f_t. Suppose that $\sigma_t > 0$ for every $t \in [a,b]$, where

$$\sigma_t^2 = \sigma_t^2(\phi) = \lim_{n\to\infty} \int \left(\frac{\sum_{j=0}^{n-1} \left(\phi \circ f_t^j - \int \phi d\mu_t \right)}{\sqrt{n}} \right)^2 d\mu_t.$$

We show that

$$m\left\{ t \in [a,b] \colon t+h \in [a,b] \text{ and } \frac{1}{\Psi(t)\sqrt{-\log|h|}} \left(\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds,$$

where $\Psi(t)$ is a dynamically defined function and m is the Lebesgue measure on $[a,b]$, normalized in such way that $m([a,b]) = 1$. As a consequence we show that $\mathcal{R}_\phi$ is not a Lipchitz function on any subset of $[a,b]$ with positive Lebesgue measure.

Contents

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1. Introduction and statement of the main results

Let f_t be a smooth family of (piecewise) smooth maps on a manifold M, and let us suppose that for each f_t there is a physical (or SBR) probability μ_t on M. Given an observable $\phi : M \to \mathbb{R}$, we can ask if the function

$$\begin{array}{rccc} \mathcal{R}_\phi & : & [0,1] & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \int \phi \, d\mu_t \end{array}$$

is differentiable and if we can find an explicit formula for its derivative. The study of this question is the so called *linear response problem.*

D. Ruelle showed that $\mathcal{R}_\phi$ is differentiable and also gave the formula for ${\mathcal{R}_\phi}'$, in the case of smooth uniformly hyperbolic dynamical systems (see Ruelle in [16] and [17], and Baladi and Smania in [4] for more details).

In the setting of smooth families of piecewise expanding unimodal maps, Baladi and Smania (see [2]) proved that if we have a C^2 family of piecewise expanding unimodal maps of class C^3, then $\mathcal{R}_\phi$ is differentiable in t_0, with $\phi \in C^{1+Lip}$, *provided* that the family f_t is tangent to the topological class of f_{t_0} at $t = t_0$. It turns out that the family $s \mapsto f_s$ is tangent to the topological class of f_t at the parameter t if and only if

$$J(f_t, v_t) = \sum_{k=0}^{M_t-1} \frac{v_t(f_t^k(c))}{Df_t^k(f_t(c))} = 0,$$

where $v_t = \partial_s f_s|_{s=t}$ and M_t is either the period of the critical point c if c is periodic, or ∞, otherwise (see [3]). Now, let us consider a C^2 family of piecewise expanding unimodal maps of class C^4 that is *transversal* to the topological classes of piecewise unimodal maps, that is

$$J(f_t, v_t) = \sum_{k=0}^{M_t-1} \frac{v_t(f_t^k(c))}{Df_t^k(f_t(c))} \neq 0 \tag{1}$$

for every t.

Baladi and Smania, [2] and [5], proved that $\mathcal{R}_\phi$ is not differentiable, for most of the parameters t, even if ϕ is quite regular. One can ask what is the regularity of the function $\mathcal{R}_\phi$ in this case. We know from Keller [9] (see also Mazzolena [14]) that $\mathcal{R}_\phi$ has modulus of continuity $|h|(\log(1/|h|) + 1)$.

We will show the Central Limit Theorem for the modulus of continuity of the function $\mathcal{R}_\phi$ where ϕ is a lipschitzian observable. Let

$$\sigma_t^2 = \sigma_t^2(\phi) = \lim_{n\to\infty} \int \left(\frac{\sum_{j=0}^{n-1} \left(\phi \circ f_t^j - \int \phi d\mu_t \right)}{\sqrt{n}} \right)^2 d\mu_t \neq 0.$$

Let $t \mapsto f_t$ be a C^2 family of C^4 piecewise expanding unimodal maps. Note that each f_t has a unique absolutely continuous invariant probability $\mu_t = \rho_t m$, where its density ρ_t has bounded variation. Let

$$L_t = \int \log|Df_t| \, d\mu_t > 0, \; \ell_t = \frac{1}{\sqrt{L_t}}. \tag{2}$$

Indeed ρ_t is continuous except on the forward orbit $f_t^j(c)$ of the critical point (see Baladi [1]). Let S_t be the jump of ρ_t at the critical value, that is

$$S_t = \lim_{x\to f_t(c)^-} \rho_t(x) - \lim_{x\to f_t(c)^+} \rho_t(x) = \lim_{x\to f_t(c)^-} \rho_t(x) > 0. \tag{3}$$

A piecewise expanding C^r unimodal map f is *good* if either c is not a periodic point of f or

$$\liminf_{x\to c} |Df^p(x)| \geq 2.$$

where $p \geq 2$ is the prime period of c (see [2] and [3]for more details).

Theorem 1.1. *Let*

$$t \in [a,b] \mapsto f_t,$$

be a transversal C^2 family of good and mixing C^4 piecewise expanding unimodal maps

$$f_t\colon [0,1] \to [0,1].$$

If ϕ is a lipschitzian observable satisfying $\sigma_t \neq 0$ for every $t \in [a,b]$, then for every $y \in \mathbb{R}$

(4)

$$\lim_{h\to 0} m\left\{ t \in [a,b]\colon t+h \in [a,b] \text{ and } \frac{1}{\Psi(t)\sqrt{-\log|h|}}\left(\frac{\mathcal{R}_\phi(t+h)-\mathcal{R}_\phi(t)}{h}\right) \leq y \right\}$$

converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds,$$

where

$$\Psi(t) = \sigma_t S_t J_t \ell_t.$$

and m is the Lebesgue measure normalized in such way that $m([a,b]) = 1$.

Corollary 1.2. *Under the same assumptions above, the function $\mathcal{R}_\phi$ is not a lipschitzian function on any subset of $[a,b]$ with positive Lebesgue measure.*

The proof of Corollary 1.2 will be given in the last section as a consequence of a stronger result (Corollary 9.1).

2. Families of piecewise expanding unimodal maps

We begin this section by setting the one-parameter family of piecewise expanding unimodal maps.

Definition 2.1. A piecewise expanding C^r unimodal map $f : [0,1] \to [0,1]$ is a continuous map with a critical point $c \in (0,1)$, $f(0) = f(1) = 0$ and such that $f|_{[0,c]}$ and $f|_{[c,1]}$ are C^r and

$$\left|\frac{1}{Df}\right|_\infty < 1.$$

We say that f is mixing if f is topologically mixing on the interval $[f^2(c), f(c)]$. For instance, if

$$\inf_x |Df(x)| > \sqrt{2}$$

then f is not renormalisable. In particular f is topologically mixing on $[f^2(c), f(c)]$.

We can see the set of all C^r piecewise expanding unimodal maps that share the same critical point $c \in (0,1)$ as a convex subset of the affine subspace $\{f \in B^r\colon\ f(0) = f(1)\}$ of the Banach space B^r of all continuous functions $f\colon [0,1] \to \mathbb{R}$ that are C^r on the intervals $[0,c]$ and $[c,1]$, with the norm

$$|f|_r = |f|_\infty + |f|_{[0,c]}|_{C^r} + |f|_{[c,1]}|_{C^r}.$$

Let $f_t : [0,1] \to [0,1]$, $t \in [a,b]$ be a one-parameter family of piecewise expanding C^4 unimodal maps. We assume

(1) For all $t \in [a,b]$ the critical point of f_t is c.
(2) The maps f_t are uniformly expanding, that is, there exist constants $1 < \lambda \leq \Lambda < \infty$ such that for all $t \in [a,b]$,
$$\left|\frac{1}{Df_t}\right|_\infty < \frac{1}{\lambda} \quad \text{and} \quad |Df_t|_\infty < \Lambda.$$
(3) The map
$$t \in [a,b] \mapsto f_t \in B^4$$
is of class C^2.

Each f_t admits a unique absolutely continuous invariant probability measure μ_t and its density ρ_t has bounded variation (see [12]). By Keller (see [9]),

$$|\rho_{t+h} - \rho_t|_{L^1} \leq C|h|(\log \frac{1}{|h|} + 1). \tag{5}$$

3. Good transversal families

It turns out that we can cut the parameter interval of a transversal family f_t in smaller intervals in such way that the family, when restricted to each one of those intervals satisfies stronger assumptions. Here, we introduce the notation of partitions following Schnellmann in [19]. Let us denote by $K(t) = [f_t^2(c), f_t(c)]$ the support of ρ_t.

Let $\mathcal{P}_j(t)$, $j \geq 1$ be the partition on the dynamical interval composed by the maximal open intervals of smooth monotonicity for the map $f_t^j : K(t) \to K(t)$, where t is a fixed parameter value. Therefore, $\mathcal{P}_j(t)$ is the set of open intervals $\omega \subset K(t)$ such that $f_t^j : \omega \to K(t)$ is C^4 and ω is maximal.

We can also define analogous partitions on the parameter interval $[a,b]$. Let

$$\begin{array}{rccc} x_0 : & [a,b] & \longrightarrow & [0,1] \\ & t & \longmapsto & f_t(c) \end{array}$$

be a C^2 map from the parameter interval into the dynamical interval. We will denote by

$$x_j(t) := f_t^j(x_0(t)),$$

$j \geq 0$, the orbit of the point $x_0(t)$ under the map f_t.

Consider a interval $J \subset [a,b]$. Let us denote by $\mathcal{P}_j|J$, $j \geq 1$, the partition on the parameter interval composed by all open intervals ω in J such that $x_i(t) \neq c$, for all i satisfying $0 \leq i < j$, that is

$$f_t^i(x_0(t)) = f_t^{i+1}(c) \neq c,$$

for all $t \in \omega$, and such that ω is maximal, that is, if $s \in \partial\omega$, then there exists $0 \leq i < j$ such that $x_i(s) = c$.

The intervals $\omega \in \mathcal{P}_j$ are also called cylinders.

We quote almost verbatim the definition of the Banach spaces V_α given in [19]. The spaces V_α were introduced by Keller [10]. Let m be the Lesbegue measure on the interval $[0,1]$

Definition 3.1. (Banach space V_α) For every $\psi\colon [0,1] \to \mathbb{R}$ be a function in $L^1(m)$ and $\gamma > 0$, we can define

$$\text{osc}\ (\psi, \gamma, x) = \text{ess sup}\ \psi|_{(x-\gamma, x+\gamma)} - \ \text{ess inf}\ \psi|_{(x-\gamma,x+\gamma)}.$$

Given $A > 0$ and $0 < \alpha \leq 1$ denote

$$|\psi|_\alpha = \sup_{0<\gamma\leq A} \frac{1}{\gamma^\alpha} \int_0^1 \text{osc}\ (\psi, \gamma, x) dx.$$

The Banach space V_α is the set of all $\psi \in L^1(m)$ such that $|\psi|_\alpha < \infty$, endowed with the norm

$$\|\psi\|_\alpha = |\psi|_\alpha + |\psi|_{L^1}.$$

We quote almost verbatim the definition of the almost sure invariant principle given in [19].

Definition 3.2. Given a sequence of functions ξ_i on a probability space, we say that it satisfies the **almost sure invariance principle (ASIP)**, with exponent $\kappa < 1/2$ if one can construct a new probability space that has a sequence of functions σ_i, $i \geq 1$ and a representation of the Weiner process W satisfying

- We have
$$\left| W(n) - \sum_{i=1}^{n} \sigma_i \right| = O(n^\kappa),$$
almost surely as $n \to \infty$.
- The sequences $\{\sigma_i\}_{i\geq 1}$ and $\{\xi_i\}_{i\geq 1}$ have identical distributions.

Definition 3.3. A C^2 transversal (see equation (1)) family of good mixing C^4 piecewise expanding unimodal maps f_t, $t \in [c,d]$ is a **good transversal family** if we can extend this family to a C^2 transversal family of good mixing C^4 piecewise expanding unimodal maps f_t, $t \in [c-\delta, d+\delta]$, for some $\delta > 0$, with the following properties

(I) There exists $j_0 > 0$ with the following property. For every $t \in [c,d]$ and for each $j \geq j_0$ there exists a neighborhood V of t such that for all $t' \in V \setminus \{t\}$ and all $0 < i < j$, we have $f_{t'}^i(c) \neq c$. In particular the one-sided limits

$$\lim_{t' \to t^+} \frac{\partial_{t'} f_{t'}^j(c)}{D f_{t'}^{j-1}(f_{t'}(c))} \quad \text{and} \quad \lim_{t' \to t^-} \frac{\partial_{t'} f_{t'}^j(c)}{D f_{t'}^{j-1}(f_{t'}(c))}$$

exist for every $j \geq j_0$, and there is $C \geq 1$ so that

$$\frac{1}{C} \leq \left| \lim_{t' \to t^+} \frac{\partial_{t'} f^j_{t'}(c)}{Df^{j-1}_{t'}(f_{t'}(c))} \right| \leq C, \tag{6}$$

and

$$\frac{1}{C} \leq \left| \lim_{t' \to t^-} \frac{\partial_{t'} f^j_{t'}(c)}{Df^{j-1}_{t'}(f_{t'}(c))} \right| \leq C, \tag{7}$$

for all $j \geq j_0$ and $t \in [c-\delta, d+\delta]$.

(II) The map f_t is mixing and there are constants $\delta > 0$, $L \geq 1$ and $0 < \tilde{\beta} < 1$ such that for all $\psi \in V_\alpha$

$$\|\mathcal{L}^n_t \psi\|_\alpha \leq L \tilde{\beta}^n \, |\psi|_\alpha + L \, |\psi|_{L^1}, \tag{8}$$

for all $t \in [c-\delta, d+\delta]$. Here $\mathcal{L}_t$ is the Ruelle-Perron-Frobenious operator of f_t given by

$$(\mathcal{L}_t \psi)(x) = \sum_{f_t(y)=x} \frac{1}{|Df_t(y)|} \psi(y).$$

(III) There is $\delta > 0$ such that for every $\zeta > 0$ there is a constant $\tilde{C}$ satisfying

$$\sum_{\omega \in \mathcal{P}_n | [a-\delta, b+\delta]} \frac{1}{|x'_n|_\omega|_\infty} \leq \tilde{C} e^{n^\zeta}$$

for all $n \geq 1$.

(IV) For all $\varphi \in V_\alpha$ such that $\sigma_t(\varphi) > 0$ the functions $\xi_i : [c-\delta, d+\delta] \to \mathbb{R}$ $i \geq 1$, defined by

$$\xi_i(t) = \frac{1}{\sigma_t(\varphi)} \left(\varphi(f^{i+1}_t(c)) - \int \varphi \, d\mu_t \right)$$

satisfy the ASIP for every exponent $\gamma > 2/5$.

(V) There are positive constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4, \tilde{C}_5, \tilde{C}_6$ and $\beta \in (0,1)$ such that for every $t \in [c-\delta, d+\delta]$ and its respective density ρ_t of the unique absolutely continuous invariant probability of f_t

A_1. The Perron-Frobenious operator $\mathcal{L}_t$ satisfies the Lasota-Yorke inequality in the space of bounded variation functions

$$|\mathcal{L}^k_t \phi|_{BV} \leq \tilde{C}_6 \beta^k |\phi|_{BV} + \tilde{C}_5 |\phi|_{L^1(m)}.$$

A_2. We have $\rho_t \in BV$ and $|\rho_t|_{BV} < \tilde{C}_1$.

A_3. We have $\rho'_t \in BV$ and $|\rho'_t|_{BV} < \tilde{C}_2$. Moreover

$$\rho_t(x) = \int_0^x \rho'_t(u) \, du + \sum_{k=1}^{M_t - 1} s_k(t) H_{f^k_t(c)}(x),$$

where $H_a(x) = 0$ if $x < a$ and $H_a(x) = -1$ if $x \geq a$,

$$s_1(t) = \frac{\rho_t(c)}{|Df_t(c-)|} + \frac{\rho_t(c)}{|Df_t(c+)|} \tag{9}$$

and

$$s_k(t) = \frac{s_1(t)}{Df^{k-1}_t(f_t(c))}.$$

A$_4$. We have $\rho''_t \in BV$ and $|\rho''_t|_{BV} < \tilde{C}_3$. Moreover

$$\rho'_t(x) = \int_0^x \rho''_t(u)\ du + \sum_{k=1}^{M_t-1} s'_k(t) H_{f_t^k(c)}(x),$$

where

$$|s'_k(t)| \leq \frac{\tilde{C}_4}{|Df_t^{k-1}(f_t(c))|}.$$

(VI) Let $j_0 > 0$ be the constant given by condition (I). For all i, j satisfying $0 \leq i, j \leq j_0$ and $t \in [c, d]$, such that $t + h \in [c - \delta, d + \delta]$ we have

$$c \notin I_{i,j}(t, h),$$

where $I_{i,j}(t, h)$ is the smallest interval that contains the set

$$\{f_{t+h}^{i+j+1}(c), f_t^{i+j+1}(c), f_{t+h}^i \circ f_t^{j+1}(c), f_{t+h}^{i+1} \circ f_t^j(c)\}.$$

Remark 3.4. Conditions (I), (II) and (III) are exactly those that appears in Schnellmann [19], with obvious cosmetic modifications.

Remark 3.5. If f_t is a good transversal family then of course Eq. (4) converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds$$

if and only if

$$\lim_{h \to 0} m \left\{ t \in [a, b] \colon \frac{1}{\Psi(t)\sqrt{-\log |h|}} \left(\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\} \tag{10}$$

converges to it as well.

Proposition 3.6. *Let f_t, $t \in [a, b]$, be a transversal C^2 family of good mixing C^4 piecewise expanding unimodal maps. Then there is a countable family of intervals $[c_i, d_i] \subset [a, b]$, $i \in \Delta \subset \mathbb{N}$, with pairwise disjoint interior and*

$$m([a, b] \setminus \bigcup_{i \in \Delta} [c_i, d_i]) = 0,$$

such that f_t is a good transversal family on each $[c_i, d_i]$, $i \in \Delta$.

Proof. Since f_t is transversal, there is just a countable number Q of parameters where f_t has a periodic critical point. Consider $\Omega = [a, b] \setminus (Q \cup \{a, b\})$. It follows from the analysis in the proof of [4, Theorem 4.1] and [1, Proposition 3.3] that for every $t' \in \Omega$ there exists $\epsilon_1 = \epsilon_1(t')$ such that if $[c, d] \subset (t' - \epsilon_1, t' + \epsilon_1)$ then the family f_t restricted to $[c, d]$ satisfies condition (V). By Schnellmann [19] for every $t' \in \Omega$ there exists $\epsilon_2 = \epsilon_2(t')$ such that if $[c, d] \subset (t' - \epsilon_2, t' + \epsilon_2)$ then the family f_t restricted to $[c, d]$ satisfies conditions (I), (II), (III) and (IV).
We claim that for every $t' \in \Omega$ there is $\epsilon_3 = \epsilon_3(t')$ such that if $[c, d] \subset (t' - \epsilon_3, t' + \epsilon_3)$ and $\delta > 0$ is small enough then the family f_t, with $t \in [c, d]$, satisfies condition (VI). Indeed, since c is not a periodic point of $f_{t'}$, there is $\epsilon_3(t') > 0$ such that

$$\eta := \min \{|f_t^{i+j+1}(c) - c| : 0 \leq j \leq j_0 \ \text{ and } \ 0 < i \leq j_0, t \in (t' - \epsilon_3, t' + \epsilon_3)\} > 0, \tag{11}$$

Since $t \in [t' - \epsilon_3/2, t' + \epsilon_3/2] \mapsto f_t$ is a C^2 family the map

$$(t, h) \mapsto f_{t+h}^i(f_t^j(c))$$

is continuous for every $0 < i \leq j_0$ and every j satisfying $0 \leq j \leq j_0$. Therefore there is $\gamma_1 := \gamma_1(i,j) < \epsilon_3/2$ such that, if $|h| \leq \gamma_1$ and $t \in [t' - \epsilon_3/2, t' + \epsilon_3/2]$, then

$$|f_{t+h}^{i+1}(f_t^j(c)) - f_t^{i+1}(f_t^j(c))| \leq \eta,$$

and

$$|f_{t+h}^{i}(f_t^{j+1}(c)) - f_t^{i}(f_t^{j+1}(c))| \leq \eta,$$

for all $0 \leq j \leq j_0$ and $0 < i \leq j_0$. Let $\gamma := \min\{\gamma_1(i,j) : 0 \leq j \leq j_0 \text{ and } 0 < i \leq j_0\}$. In particular if $|h| \leq \gamma_1$ and $t \in [t' - \epsilon_3/2, t' + \epsilon_3/2]$ then $c \notin I_{i,j}(t,h)$ for all $0 \leq j \leq j_0$, $0 < i \leq j_0$.
Let $\epsilon_4(t') = \min\{\epsilon_1(t'), \epsilon_2(t'), \gamma\}$. Consider the family $\mathcal{F}$ of intervals $[c,d] \subset [a,b]$ such that $[c,d] \subset (t' - \epsilon_4(t'), t' + \epsilon_4(t'))$ for some $t' \in \Omega$. By the Vitali's covering theorem there exists a countable family of intervals $[c_i, d_i] \subset [a,b]$, $[c_i,d_i] \in \mathcal{F}$, $i \in \Delta \subset \mathbb{N}$, with pairwise disjoint interior and

$$m([a,b] \setminus \bigcup_{i\in\Delta} [c_i,d_i]) = m(\Omega \setminus \bigcup_{i\in\Delta} [c_i,d_i]) = 0.$$

□

We will also need

Lemma 3.7. *Let*

$$t \in [a,b] \mapsto f_t$$

be a good transversal C^2 family of good and mixing C^4 piecewise expanding unimodal maps

$$f_t\colon [0,1] \to [0,1].$$

If ϕ is a lipschitzian observable satisfying $\sigma_t \neq 0$ for every $t \in [a,b]$ then

$$\underline{J} = \inf_{t\in[a,b]} |J(f_t, v_t)|, \ \ \underline{\sigma} = \inf_{t\in[a,b]} \sigma_t(\phi), \ \ \underline{s} = \inf_{t\in[a,b]} |S_t|, \ \ \underline{\ell} = \inf_{t\in[a,b]} |\ell_t|,$$

are positive, where S_t and ℓ_t are as defined in Eqs. (3) and (2) respectively.

Proof. The function

$$t \mapsto J(f_t, v_t)$$

is *not* continuous in a transversal family (see [3]). Indeed, its points of discontinuity lie on the parameters t where the critical point c is periodic for f_t, where this function have one-sided limits. However, in [3], Baladi and Smania showed that if v_n converges to v and f_n converges to f, and $J(f_n, v_n) \to 0$ when $n \to \infty$ then $J(f,v) = 0$. From this follows that $\underline{J} > 0$. In [19], Schnellmann proved that $t \mapsto \sigma_t$ is Hölder continuous. Therefore, $\underline{\sigma} > 0$. Suppose that $\lim_n s_1(t_n) = 0$, where s_1 is as defined in Eq. (9). Remember that (see [2] and [1]),

$$\rho_{t_n} = \rho_{abs,t_n} + \rho_{sal,t_n} = \rho_{abs,t_n} + \sum_{k=1}^{M_{t_n}-1} \frac{s_1(t_n)}{Df_{t_n}^{k-1}(f_{t_n}(c))} H_{f_{t_n}^k(c)} \tag{12}$$

where ρ_{abs,t_n} is absolutely continuous, $(\rho_{abs,t_n})'$ has bounded variation and

$$|(\rho_{abs,t_n})'|_{BV} \leq C. \tag{13}$$

Taking a subsequence, if necessary, we can assume that $\lim_n t_n = t$ and that ρ_{t_n} converges in $L^1(m)$ to ρ_t. But if $\lim_n s_1(t_n) = 0$ then by Eqs. (12) and (13) we conclude that ρ_t is a continuous function. But this is absurd since $s_1(t) \neq 0$ for every t. □

Remark 3.8. As an example, we have the family of tent maps defined by

$$f_t(x) = \begin{cases} tx, & \text{if } x < 1/2, \\ t - tx, & \text{if } x \geq 1/2, \end{cases}$$

$t \in (1,2)$. Tsujii [20] showed that the family of tent maps is a transversal family. We can observe that, since f_t is a piecewise linear map for all t, the density ρ_t is purely a saltus function.

4. Decomposition of the Newton quotient for good families

In this section we will assume that f_t is a good family. In order to prove Theorem 1.1 we will decompose the quotient

$$\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h}$$

in two parts which will be called the *Wild part* and the *Tame part* of the decomposition.

Definition 4.1. Let $g : [0,1] \to \mathbb{R}$ be a function of bounded variation and $t \in [a,b]$. We define the projection

$$\begin{array}{rcccl} \Pi_t & : & BV & \longrightarrow & BV \\ & & g & \longmapsto & g - \rho_t \int g dm. \end{array}$$

Indeed Π_t is also a well defined operator in $L^1(m)$ and

$$\sup_t |\Pi_t|_{BV} < \infty \text{ and } \sup_t |\Pi_t|_{L^1(m)} < \infty.$$

A function $g \in L^1(m)$ belongs to $\Pi_t(BV)$ if and only if $\int g \; dm = 0$. In particular the operator $(I - \mathcal{L}_t)^{-1}$ is well defined on $\Pi_t(BV)$.We are going to use the following observation quite often. If $\int g \; dm = 0$, and

$$g = \sum_{i=0}^{\infty} g_i,$$

with $g_i \in BV$ and the convergence of the series is in the BV norm, then

$$(I - \mathcal{L}_t)^{-1} g = \sum_{i=0}^{\infty} (I - \mathcal{L}_t)^{-1} \Pi_t(g_i).$$

Note also that

$$\Pi_t \circ \mathcal{L}_t = \mathcal{L}_t \circ \Pi_t.$$

Proposition 4.2. *Assume that f_t is a family of piecewise expanding unimodal maps as defined in section 2 and let $\mathcal{L}_t$ be the Perron-Frobenius operator. Then*

$$\frac{\rho_{t+h} - \rho_t}{h} = (I - \mathcal{L}_{t+h})^{-1} \left(\frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} \right).$$

Proof. Note that $(I - \mathcal{L}_t)^{-1}$ is well defined in $\Pi_t(BV)$ and is given by

$$(I - \mathcal{L}_t)^{-1}(\rho) = \sum_{i=0}^{\infty} \mathcal{L}_t^i(\rho),$$

for every $\rho \in \Pi_t(BV)$. Therefore, the result follows as an immediate consequence of the identity

$$(I - \mathcal{L}_{t+h})(\rho_{t+h} - \rho_t) = (I - \mathcal{L}_t)(\rho_t) - (I - \mathcal{L}_{t+h})(\rho_t).$$

□

Proposition 4.3. *Let f_t be a C^2 family of good mixing C^4 piecewise expanding unimodal maps that satisfies property (V) in Definition 3.3. There exists $C > 0$ with the following property. For every $t \in [a, b]$ such that the critical point of f_t is not periodic, we can decompose*

$$\frac{\mathcal{L}_{t+h}(\rho_t) - \mathcal{L}_t(\rho_t)}{h} = \Phi_h + r_h$$

where

$$\Phi_h = \frac{1}{h}\sum_{k=0}^{\infty} s_{k+1}(t)\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)$$

and r_h satifies

$$\int r_h dm = 0 \quad \text{and} \quad \sup_{h\neq 0} |r_h|_{BV} < C.$$

We will prove Proposition 4.3 in Section 8. We will call $\mathcal{W}(t,h) = (I - \mathcal{L}_{t+h})^{-1}\Phi_h$ the *Wild part* and $(I - \mathcal{L}_{t+h})^{-1} r_h$ will be called the *Tame part* of the decomposition. Note that

$$\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} = \int \phi\mathcal{W}(t,h)\, dm + \int \phi(I - \mathcal{L}_{t+h})^{-1} r_h\, dm.$$

Definition 4.4. Given $h \neq 0$ and $t \in [0, 1]$, let $N := N(t, h)$ be the unique integer such that

$$\frac{1}{|Df_t^{N+1}(f_t(c))|} \leq |h| < \frac{1}{|Df_t^{N}(f_t(c))|}. \tag{14}$$

There is some ambiguity in the definition of $N(t, h)$ when $f_t^k(c) = c$ for some $k > 0$. But since the family is transversal, there exists just a countable number of such parameters (see [3]).

Given $a \in \mathbb{R}$ define

$$\lfloor a \rfloor = \max\{k \in \mathbb{Z}\colon\ a \geq k\}.$$

The following proposition gives us a control on the orbit of the critical point.

Proposition 4.5. *For large $\mathcal{K} > 0$ and every $\gamma > 0$ there exists $\delta > 0$ such that for every small h_0 there are sets $\Gamma^\delta_{h',h_0}, \Gamma^\delta_{h_0} \subset I = [a, b]$, with $\Gamma^\delta_{h',h_0} \subset \Gamma^\delta_{h_0}$, for every h' satisfying $0 < h' < h_0$, with the following properties*

A. $\lim_{h'\to 0} m(\Gamma^\delta_{h',h_0}) = m(\Gamma^\delta_{h_0}) > 1 - \gamma$.

B. *If $t \in \Gamma^\delta_{h',h_0}$ and $|h| \leq h'$ then there exists $N_3(t, h)$ such that*

$$\lfloor \frac{\mathcal{K}}{2} \log N(t,h) \rfloor \leq N(t,h) - N_3(t,h) \leq C_5\mathcal{K} \log N(t,h) \tag{15}$$

and

$$c \notin I_{i,j} \tag{16}$$

for all $0 \leq j < N_3(t, h)$ and $0 \leq i < N_3(t, h) - j$, where $I_{i,j}$ is the smallest interval that contains the set

$$\{f^{i+j+1}_{t+h}(c), f^{i+j+1}_t(c), f^i_{t+h} \circ f^{j+1}_t(c), f^{i+1}_{t+h} \circ f^j_t(c)\}$$

C. *For every $t \in \Gamma^\delta_{h',h_0}$ the critical point of f_t is not periodic.*

D. *If* $0 < \hat{h} < h' \leq h_0$ *then* $\Gamma^{\delta}_{h',h_0} \subset \Gamma^{\delta}_{\hat{h},h_0}$,

where m *is the normalized Lebesgue measure on* $I = [a, b]$.

We will prove Proposition 4.5 in Section 6.

The following proposition is one of the most important result because it relates the Birkhoff sum to the Wild part. That is, the integral of the observable ϕ with respect to the Wild part is related to the Birkhoff sum of ϕ and this fact will allow us to use Lemma 5.1.

Proposition 4.6. *Let* f_t *be a good transversal family. Let* $\phi : [0, 1] \to \mathbb{R}$ *be a lipschitzian observable. If* $t \in \Gamma^{\delta}_{h,h_0}$, *where* Γ^{δ}_{h,h_0} *is the set given by Proposition 4.5, then*

$$\int \phi\, \mathcal{W}(t, h) dm = s_1(t) J(f_t, v_t) \sum_{j=0}^{N_3(t,h)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + O\left(\log\log \frac{1}{|h|} \right).$$

We will prove Proposition 4.6 in Section 7.

Proposition 4.7. *Let* f_t *be a good transversal family. Let* $\phi : [0, 1] \to \mathbb{R}$ *be a lipschitzian observable. If* $t \in \Gamma^{\delta}_{h,h_0}$, *where* Γ^{δ}_{h,h_0} *is the set given by Proposition 4.5, then*

$$\begin{aligned}\frac{\mathcal{R}_\phi(t + h_n) - \mathcal{R}_\phi(t)}{S_t J_t h_n} &= \sum_{j=0}^{N_3(t,h_n)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t \right) + O\left(\log\log \frac{1}{|h_n|} \right) \\ &\quad + \frac{1}{S_t J_t} \int \phi (I - \mathcal{L}_{t+h})^{-1} r_h \; dm.\end{aligned}$$

The proof follows directly from Propositions 4.3 and 4.6

5. Proof of the Central Limit Theorem for the modulus of continuity of $\mathcal{R}_\phi$

To simplify the notation in this section, given a transversal family $t \mapsto f_t$ we will denote $S_t^f = s_1^f(t)$, $J_t^f = J(f_t, \partial_s f_s|_{s=t})$, $\sigma_t^f = \sigma_t^f(\phi)$. Moreover

$$L_t^f = \int \log |Df_t| d\mu_t^f,$$

where μ_t^f is the unique absolutely continuous invariant probability of f_t, and

$$\ell_t^f = \frac{1}{\sqrt{L_t^f}}.$$

When there are not confusion with respect to which family we are dealing with, we will omit f in the notation.

Lemma 5.1 (Functional Central Limit Theorem)**.** *Let* f_t *be a good transversal* C^2 *family of* C^4 *unimodal maps and* $\sigma_t(\phi) \neq 0$ *for every* t. *For each* $t \in [a, b]$ *let us consider the continuous function* $\theta \mapsto X_N(\theta, t)$, *where*

$$X_N(\theta, t)$$

$$= \frac{1}{\sigma_t \sqrt{N}} \sum_{k=0}^{\lfloor N\theta \rfloor - 1} \big(\phi(f_t^k(c)) - \int \phi \; d\mu_t \big) + \frac{(N\theta - \lfloor N\theta \rfloor)}{\sigma_t \sqrt{N}} \big(\phi(f_t^{\lfloor N\theta \rfloor}(c)) - \int \phi \; d\mu_t \big).$$

These functions and the normalized Lebesgue measure on $[a,b]$ define a distribution on the continuous functions space and this distribution converges to the Wiener measure. We denote $X_N \xrightarrow{D}_N W$.

Proof. By Schnellmann [19] we know that the sequence of functions

$$\xi_i(t) = \frac{1}{\sigma_t}\left(\phi(f_t^{i+1}(c)) - \int_0^1 \phi d\mu_t\right)$$

satisfies the ASIP for every exponent error larger than 2/5. By [15, Theorem E], the ASIP implies the Functional Central Limit Theorem for $X_N(\theta,t)$. □

We are going to need the following

Proposition 5.2 ([6])**.** *If*

$$\frac{\nu_n}{a_n} \xrightarrow{P}_n L, \tag{17}$$

where L is a positive constant and $(a_n)_n$ is a sequence such that $a_n \to \infty$ when $n \to \infty$, then

$$X_N \xrightarrow{D}_N W$$

implies

$$Y_n \xrightarrow{D}_n W,$$

where Y_n is

$$\frac{1}{\sigma_t\sqrt{\nu_n(t)}}\sum_{k=0}^{\lfloor \nu_n\theta\rfloor - 1}\left(\phi(f_t^k(c)) - \int \phi \, d\mu_t\right) + \frac{(\nu_n\theta - \lfloor \nu_n\theta\rfloor)}{\sigma_t\sqrt{\nu_n(t)}}\left(\phi(f_t^{\lfloor \nu_n\theta\rfloor}(c)) - \int \phi \, d\mu_t\right),$$

Proof. See [6], page 152.

From now on we will denote

$$\mathcal{D}_{\mathcal{N}}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds.$$

The following lemma will be used many times

Lemma 5.3 (A variation of Slutsky's Theorem)**.** *Let $A_n\colon [0,1] \to \mathbb{R}$ be functions and $\Omega_n \subset [0,1]$ be such that*

$$\liminf_n m(\Omega_n) > 1 - \gamma,$$

and for every $y \in \mathbb{R}$ the sequence

$$a_n(y) = m(t \in \Omega_n \colon A_n(t) \le y)$$

eventually belongs to

$$O(y,\epsilon) = (\mathcal{D}_{\mathcal{N}}(y) - \epsilon, \mathcal{D}_{\mathcal{N}}(y) + \epsilon),$$

that is, there is $n_0 = n_0(y)$ such that $a_n(y) \in O(y,\epsilon)$ for every $n \ge n_0$. Then

A. *There exists $\delta > 0$ such that if $B_n\colon [0,1] \to \mathbb{R}$ is a function such that*

$$\liminf_n m(t \in [0,1] \colon |B_n(t) - 1| < \delta) > 1 - \gamma,$$

then the sequence

$$b_n(y) = m(t \in [0,1] \colon A_n(t)B_n(t) \le y)$$

eventually belong to $O(y, \epsilon + 3\gamma)$.

B. *There exists $\delta > 0$ such that if $B_n\colon [0,1] \to \mathbb{R}$ is a function such that*

$$\liminf_n m(t \in [0,1]\colon |B_n(t)| < \delta) > 1 - \gamma,$$

then the sequence

$$b_n(y) = m(t \in [0,1]\colon A_n(t) + B_n(t) \le y)$$

eventually belong to $O(y, \epsilon + 3\gamma)$.

Proof of A. Define

$$D_A^n(y) = \{t \in \Omega_n\colon A_n(t) \le y\}$$
$$D_B^n = \{t \in [0,1]\colon |B_n(t) - 1| < \delta\}$$
$$D_{AB}^n(y) = \{t \in [0,1]\colon A_n(t)B_n(t) \le y\}$$

Choose $\delta > 0$ such that

$$\sup_{y\in\mathbb{R}} \sup_{|\delta'|<\delta} |\mathcal{D}_{\mathcal{N}}(y) - \mathcal{D}_{\mathcal{N}}(y(1-\delta'))| < \gamma,$$

and

$$\sup_{y\in\mathbb{R}} \sup_{|\delta'|<\delta} |\mathcal{D}_{\mathcal{N}}(y) - \mathcal{D}_{\mathcal{N}}(y(1-\delta')^{-1})| < \gamma.$$

If $y \ge 0$

$$D_A^n((1-\delta)y) \cap D_B^n \subset D_{AB}^n(y) \text{ and } D_{AB}^n(y) \cap D_B^n \cap \Omega_n \subset D_A^n((1-\delta)^{-1}y),$$

Thus, if n is large

$$\begin{aligned} m(D_{AB}^n(y)) &\ge m(D_A^n((1-\delta)y) \cap D_B^n) \\ &\ge m(D_A^n((1-\delta)y)) - \gamma \ge \mathcal{D}_{\mathcal{N}}((1-\delta)y) - \epsilon - \gamma \\ &\ge \mathcal{D}_{\mathcal{N}}(y) - \epsilon - 2\gamma, \end{aligned} \tag{18}$$

and

$$\begin{aligned} m(D_{AB}^n(y)) &\le m(D_{AB}^n(y) \cap D_B^n) + \gamma \\ &\le m(D_{AB}^n(y) \cap D_B^n \cap \Omega_n) + 2\gamma \\ &\le m(D_A^n((1-\delta)^{-1}y)) + 2\gamma \le \mathcal{D}_{\mathcal{N}}((1-\delta)^{-1}y) + \epsilon + 2\gamma \\ &\le \mathcal{D}_{\mathcal{N}}(y) + \epsilon + 3\gamma, \end{aligned} \tag{19}$$

and if $y < 0$ we have

$$D_A^n((1-\delta)^{-1}y) \cap D_B^n \subset D_{AB}^n(y) \text{ and } D_{AB}^n(y) \cap D_B^n \cap \Omega_n \subset D_A^n((1-\delta)y),$$

and an analogous analysis as above gives

$$m(D_{AB}^n(y)) \in O(y, \epsilon + 3\gamma).$$

□

Proof of B. Since the proof is is quite similar to the proof of A, we will skip it.

□

Lemma 5.4. *Let $t \mapsto f_t$, $t \in [a,b]$ be a good transversal C^2 family of C^4 unimodal maps. Let $\psi\colon [c,d] \to [a,b]$ be an affine map, $\psi(c) = a$ and $\psi(d) = b$ and $g_\theta = f_{\psi(\theta)}$. For every small enough $h \ne 0$ we can define*

$$\Omega_g(h,y) = \left\{ \theta \in [c,d]\colon \frac{1}{\sigma_\theta^g \ell_\theta^g S_\theta^g J_\theta^g \sqrt{-\log|h|}} \left(\frac{\mathcal{R}_{\phi_g}(\theta+h) - \mathcal{R}_{\phi_g}(\theta)}{h} \right) \le y \right\}$$

and

$$\Omega_f(w,y) = \left\{ t \in [a,b]\colon \frac{1}{\sigma_t^f \ell_t^f S_t^f J_t^f \sqrt{-\log|w|}} \left(\frac{\mathcal{R}_{\phi_f}(t+w) - \mathcal{R}_{\phi_f}(t)}{w} \right) \leq y \right\}.$$

If

$$\frac{m(\Omega_g(h,y))}{m([c,d])}$$

eventually belong to $O(y,\gamma)$ *when* h *converges to* 0 *then*

$$\frac{m(\Omega_f(rh,y))}{m([a,b])}$$

eventually belong to $O(y,\gamma')$ *when* h *converges to* 0*, for every* $\gamma' > \gamma$*. Here* $r = \psi'$*.*

Proof. It follows easily from Lemma 5.3.A. □

Remark 5.5. Lemma 5.4 implies that it is enough to show our main theorem for families parametrized by $[0,1]$.

Proposition 5.6. *For every* $\gamma > 0$ *there exists* Q_1 *with the following property. Let* f_t *be a good transversal* C^2 *family of* C^4 *piecewise expanding unimodal maps with* $\sigma_t(\phi) \neq 0$ *for every* t *and*

$$Q = \sup_{t,t' \in [c,d]} \left| 1 - \frac{L_{t'}}{L_t} \right| < Q_1.$$

Then for every h *small enough we have*

$$\frac{1}{m([c,d])} m \left\{ t \in [c,d]\colon \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log|h|}} \left(\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \right) \leq y \right\}$$

belongs to $O(y, 13\gamma)$*.*

Proof. Without loss of generality we assume that $[c,d] = [0,1]$. It is enough to prove the following claim: For every sequence

$$h_n \to_n 0$$

and every $\gamma > 0$, the sequence

$$s_n = m \left\{ t \in [0,1] : \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log|h_n|}} \left(\frac{\mathcal{R}_\phi(t+h_n) - \mathcal{R}_\phi(t)}{h_n} \right) \leq y \right\}$$

eventually belong to the interval $O(y, 12\gamma)$.
Fix a large $\mathcal{K} > 0$. By Proposition 4.5, for every $\gamma > 0$ there exist $\delta > 0$, $h_0 > 0$ and sets $\Gamma^\delta_{h,h_0}, \Gamma^\delta_{h_0} \subset I$, with $\Gamma^\delta_{h,h_0} \subset \Gamma^\delta_{h_0}$, for every $h \neq 0$ satisfying $|h| < h_0$, such that

A. $\lim_{h \to 0} m(\Gamma^\delta_{h,h_0}) = m(\Gamma^\delta_{h_0}) > 1 - \gamma$.
B. If $t \in \Gamma^\delta_{h,h_0}$ then there exists $N_3(t,h)$ such that

$$\lfloor \frac{\epsilon}{2} \log N(t,h) \rfloor \leq N(t,h) - N_3(t,h) \leq C_5 \mathcal{K} \log N(t,h)$$

and

$$c \notin [f^i_{t+h} \circ f^{j+1}_t(c), f^{i+1}_{t+h} \circ f^j_t(c)]$$

for all $1 \leq j < N_3(t,h)$ and $0 \leq i < N_3(t,h) - j$.

For all $h \neq 0$ and $t \in [0,1]$, define $N_4(t,h) = N_3(t,h)$ if $t \in \Gamma^{\delta}_{h,h_0}$ and $|h| < h_0$, and $N_4(t,h) = N(t,h)$, otherwise. Therefore, for B we have

$$N(t,h) - N_4(t,h) \leq C_5 \mathcal{K} \log N(t,h) \tag{20}$$

for every (t,h). Since

$$\frac{1}{|Df_t^{N(t,h)+1}(f_t(c))|} \leq |h| < \frac{1}{|Df_t^{N(t,h)}(f_t(c))|},$$

we have

$$\frac{1}{N(t,h)} \sum_{k=1}^{N(t,h)} \log |Df_t(f_t^k(c))| < \frac{-\log |h|}{N(t,h)} \leq \frac{1}{N(t,h)} \sum_{k=1}^{N(t,h)+1} \log |Df_t(f_t^k(c))|.$$

By Schnellmann[18], we have for almost every t

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} \log |Df_t(f_t^k(c))| = L_t = \int \log |Df_t| \, d\mu_t,$$

which implies that for almost every t

$$\lim_{h \to 0} \frac{-\log |h|}{N(t,h)} = \int \log |Df_t| \, d\mu_t.$$

And by Eq. (20)

$$\frac{-\log |h|}{N(t,h)} \leq \frac{-\log |h|}{N_4(t,h)} \leq \frac{-\log |h|}{N(t,h) - C_5 \mathcal{K} \log N(t,h)},$$

we also have

$$\lim_{h \to 0} \frac{L_t N_4(t,h)}{-\log |h|} = 1. \tag{21}$$

for almost every $t \in [0,1]$. Fix $t_0 \in [0,1]$ such that $L_{t_0} = min_{t \in [0,1]} L_t$. Then

$$\frac{L_t / L_{t_0} N_4(t,h_n)}{-\log |h_n|} \xrightarrow{P} \frac{1}{L_{t_0}}. \tag{22}$$

By Lemma 5.1 and Propostion 5.2,

$$Y_n(\theta, t) \xrightarrow{D}_n W, \tag{23}$$

where Y_n is given in Propostion 5.2 and W is the Wiener measure, with

$$\nu_n(t) = N_4(t,h_n) \frac{L_t}{L_{t_0}}.$$

Hence, taking $\theta = 1$ we conclude that

$$Y_n(1,t) \xrightarrow{D}_n \mathcal{N}(0,1), \tag{24}$$

where $\mathcal{N}(0,1)$ denotes the Normal distribution with average zero and variance one. Let

$$Q = \sup_{t \in [0,1]} \left| 1 - \frac{L_{t_0}}{L_t} \right|.$$

Fix $\alpha \in (0, 1/2)$. The Lévy's modulus of continuity theorem (see for instance Karatzas and Shreve [8]) implies that for almost every function f with respect to the Wiener measure there exists C_f such that

$$|f(\theta') - f(\theta)| \leq C_f |\theta' - \theta|^{\alpha}$$

for all $\theta', \theta \in [0,1]$. In particular there exist $H = H(\gamma)$ and a set Ω_γ of α-Hölder continuous functions in $C([0,1], \mathbb{R})$, whose measure with respect to the Wiener measure is larger than $1-\gamma$, such that

$$|f(\theta') - f(\theta)| \leq H|\theta' - \theta|^\alpha.$$

In particular for $f \in \Omega_\gamma$ we have

$$\max_{\theta \in [1-Q,1]} |f(1) - f(\theta)| \leq HQ^\alpha, \tag{25}$$

Due to Eq. (23),

$$\liminf_n m\{t \in [0,1]\colon \max_{\theta \in [1-Q,1]} |Y_n(1,t) - Y_n(\theta,t)| \leq HQ^\alpha\} > 1-\gamma.$$

In particular if

$$D_n = \{t \in [0,1]\colon\ |Y_n(1,t) - Y_n(\frac{L_{t_0}}{L_t}, t)| \leq 2HQ^\alpha\}$$

then $\liminf_n m(D_n) > 1-\gamma$. Let us apply Lemma 5.3.B with $\Omega_n = D_n$, $A_n(t) = Y_n(1,t)$ and $B_n(t) = Y_n(\frac{L_{t_0}}{L_t}, t) - Y_n(1,t)$. Observe that by Eq.(24) the sequence $a_n(y)$ defined in Lemma 5.3 eventually belongs to $O(y,\epsilon)$ for all $\epsilon > 0$. Hence, taking $\epsilon = \gamma$, there exists $\delta_1 = \delta_1(\gamma) > 0$ such that if $2HQ^\alpha < \delta$ we have

$$m(t \in [0,1]\colon Y_n(\frac{L_{t_0}}{L_t}, t) \leq y) \tag{26}$$

eventually belongs to $O(y, 4\gamma)$. Choose $Q_0 = Q_0\gamma > 0$ such that if $Q < Q_0$ then $2HQ^\alpha < \delta_1$. Note that

$$Y_n(\frac{L_{t_0}}{L_t}, t) = \sqrt{\frac{L_{t_0}}{L_t}} \frac{1}{\sigma_t \sqrt{N_4(t,h_n)}} \sum_{k=0}^{\lfloor N_4(t,h_n) \rfloor - 1} \big(\phi(f_t^k(c)) - \int \phi \, d\mu_t\big). \tag{27}$$

By Eq. (21) and Lemma 5.3.A, the sequence

$$m(t \in [0,1]\colon \frac{\sqrt{L_{t_0}}}{\sigma_t \sqrt{-\log|h_n|}} \sum_{k=0}^{\lfloor N_4(t,h_n) \rfloor - 1} \big(\phi(f_t^k(c)) - \int \phi \, d\mu_t\big) \leq y)$$

eventually belongs to $O(y, 7\gamma)$. Applying again Lemma 5.3.A, with

$$A_n(t) = \frac{\sqrt{L_{t_0}}}{\sigma_t \sqrt{-\log|h_n|}} \sum_{k=0}^{\lfloor N_4(t,h_n) \rfloor - 1} \big(\phi(f_t^k(c)) - \int \phi \, d\mu_t\big),$$

$\Omega_n = [0,1]$ and

$$B_n(t) = \sqrt{\frac{L_t}{L_{t_0}}},$$

there exists $\delta_2 = \delta_2(\gamma) > 0$ such that if

$$\left| \sqrt{\frac{L_t}{L_{t_0}}} - 1 \right| < \delta_2 \tag{28}$$

for every t then

$$m(t \in [0,1]\colon \frac{\sqrt{L_t}}{\sigma_t \sqrt{-\log|h_n|}} \sum_{k=0}^{\lfloor N_4(t,h_n) \rfloor - 1} \big(\phi(f_t^k(c)) - \int \phi \, d\mu_t\big) \leq y)$$

eventually belong to $O(y, 10\gamma)$. Choose $Q_1 = \min\{Q_0, \delta_2\}$ such that $Q < Q_1$ implies Eq. (28). Finally by Propositions 4.6 and 4.7, if $0 < |h_n| \leq h_0$ and $t \in \Gamma^{\delta}_{h_n,h_0}$ we have

$$\frac{\mathcal{R}_\phi(t+h_n) - \mathcal{R}_\phi(t)}{S_t J_t h_n} = \sum_{j=0}^{N_3(t,h_n)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t\right) + O\left(\log\log\frac{1}{|h_n|}\right) + \frac{1}{S_t J_t}\int \phi(I - \mathcal{L}_{t+h})^{-1} r_h \, dm.$$

Since

$$\frac{\log\log\frac{1}{|h_n|}}{\sqrt{\log\frac{1}{|h_n|}}} \to_n 0$$

and

$$\sup |(I - \mathcal{L}_{t+h})^{-1} r_h|_{L^1} < \infty,$$

we have

$$\frac{\mathcal{R}_\phi(t+h_n) - \mathcal{R}_\phi(t)}{S_t \sigma_t J_t h_n \sqrt{-\log|h_n|}} = \frac{1}{\sigma_t\sqrt{-\log|h_n|}} \sum_{j=1}^{N_3(t,h_n)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t\right) + r(t,h_n),$$

where

$$\lim_n \sup_{t\in\Gamma^{\delta}_{h_n,h_0}} |r(t,h_n)| = 0.$$

Hence, it is easy to conclude that

$$\frac{\mathcal{R}_\phi(t+h_n) - \mathcal{R}_\phi(t)}{S_t \sigma_t \ell_t J_t h_n \sqrt{-\log|h_n|}} = \frac{1}{\ell_t\sigma_t\sqrt{-\log|h_n|}} \sum_{j=1}^{N_3(t,h_n)} \left(\phi(f_t^j(c)) - \int \phi d\mu_t\right) + r'(t,h_n), \tag{29}$$

for every $t \in \Gamma^{\delta}_{h_n,h_0}$, where

$$\ell_t = \frac{1}{\sqrt{L_t}}$$

and

$$\lim_n \sup_{t\in\Gamma^{\delta}_{h_n,h_0}} |r'(t,h_n)| = 0.$$

Since $m(\Gamma^{\delta}_{h,h_0}) > 1-\gamma$, we can apply Lemma 5.3 (remember that $N_4(t,h) = N_3(t,h)$ for $t \in \Gamma^{\delta}_{h,h_0}$) to conclude that the sequence

$$m(t \in [0,1] \colon \frac{\mathcal{R}_\phi(t+h_n) - \mathcal{R}_\phi(t)}{S_t \sigma_t \ell_t J_t h_n \sqrt{-\log|h_n|}} \leq y)$$

eventually belong to the interval $O(y, 13\gamma)$. □

Lemma 5.7. *Let $[c_i, d_i] \subset [a,b]$, $i \in \Delta \subset \mathbb{N}$, be intervals with pairwise disjoint interior and such that*

$$m([a,b] \setminus \bigcup_{i\in\Delta} [c_i, d_i]) = 0.$$

If $t \mapsto f_t$, with $t \in [c_i, d_i]$, are good transversal families such that for all $i \in \Delta$ and $y \in \mathbb{R}$ we have

$$\frac{1}{m([c_i,d_i])} m\left\{t \in [c_i,d_i] \colon \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log|h|}} \left(\frac{\mathcal{R}_\phi(t+h)-\mathcal{R}_\phi(t)}{h}\right) \leq y\right\}$$

eventually belongs to $O(y,\gamma)$, then

$$\frac{1}{m([a,b])} m\left\{t \in [a,b] \colon t+h \in [a,b] \text{ and } \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log|h|}} \left(\frac{\mathcal{R}_\phi(t+h)-\mathcal{R}_\phi(t)}{h}\right) \leq y\right\}$$

eventually belongs to $O(y,\gamma+\epsilon)$, for every $\epsilon > 0$.

Proof. Define

$$\Omega(h,y) = \left\{t \in [a,b] \colon t+h \in [a,b] \text{ and } \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log|h|}} \left(\frac{\mathcal{R}_\phi(t+h)-\mathcal{R}_\phi(t)}{h}\right) \leq y\right\}$$

and

$$\Omega_i(h,y) = \left\{t \in [c_i,d_i] \colon t+h \in [a,b] \text{ and } \frac{1}{\sigma_t \ell_t S_t J_t \sqrt{-\log|h|}} \left(\frac{\mathcal{R}_\phi(t+h)-\mathcal{R}_\phi(t)}{h}\right) \leq y\right\}.$$

Of course $\Omega_i(h,y)$ are pairwise disjoint up to a countable set, $\Omega_i(h,y) \subset \Omega(h,y)$ and

$$m(\Omega(h,y) \setminus \cup_i \Omega_i(h,y)) = 0.$$

Then

$$m(\Omega(h,y)) = \sum_{i\in\Delta} m(\Omega_i(h,y)).$$

Given $\epsilon \in (0,1)$, choose i_0 such that

$$m(\cup_{i>i_0}[c_i,d_i]) < \epsilon\, m([a,b]).$$

For every $i \leq i_0$ there exists $h_i > 0$ such that for every $|h| < h_i$ we have

$$\frac{m(\Omega_i(h,y))}{m([c_i,d_i])}$$

belongs to $O(y,\gamma+\epsilon)$. Let $\hat{h} = \min_{i\leq i_0} h_i$. Let

$$U_{i_0}(h,y) = \cup_{i\leq i_0}\Omega_i(h,y),$$

and

$$W_{i_0}(h,y) = \cup_{i\leq i_0}[c_i,d_i].$$

Then for $|h| < \hat{h}$ we have

$$\frac{m(U_{i_0}(h,y))}{m(W_{i_0}(h,y))} = \sum_{i\leq i_0} \frac{m([c_i,d_i])}{m(W_{i_0}(h,y))} \frac{m(\Omega_i(h,y))}{m([c_i,d_i])}$$

is a convex combination of elements of $O(y,\gamma+\epsilon)$, then it belongs to $O(y,\gamma+\epsilon)$. We conclude that

$$
\begin{aligned}
&(\mathcal{D}_{\mathcal{N}}(y)-\gamma-2\epsilon)m([a,b])\\
&\leq(\mathcal{D}_{\mathcal{N}}(y)-\gamma-\epsilon)(m([a,b])-\epsilon\, m([a,b]))\\
&\leq(\mathcal{D}_{\mathcal{N}}(y)-\gamma-\epsilon)m(W_{i_0}(h,y))\\
&\leq m(U_{i_0}(h,y))\\
&\leq m(\Omega(h,y))\\
&\leq m(U_{i_0}(h,y))+\epsilon\, m([a,b])\\
&\leq(\mathcal{D}_{\mathcal{N}}(y)+\gamma+\epsilon)m(W_{i_0}(h,y))+\epsilon\, m([a,b])\\
&\leq(\mathcal{D}_{\mathcal{N}}(y)+\gamma+2\epsilon)m([a,b]).
\end{aligned}
\tag{30}
$$

□

Proof of Theorem 1.1. Remember that

$$t\mapsto L_t$$

is a continuous and positive function on $[a,b]$. Given $\gamma>0$, let $Q_1>0$ be as in Proposition 5.6. Then there are $k>0$ and intervals $[c_i,d_i]$, $i\leq k=k(\gamma)$, which forms a partition $\mathcal{F}$ of $[a,b]$ and

$$\sup_{t,t'\in[c_i,d_i]}\Big|1-\frac{L_{t'}}{L_t}\Big|<Q_1$$

for every $i\leq k$. Then the restrictions of the family f_t to each one of the intervals $[c_i,d_i]$ satisfy the assumptions of Proposition 5.6. Now it remains to apply Lemma 5.7 to the full family and the partition $\mathcal{F}$. Since $\gamma>0$ is arbitrary we completed the proof of Theorem 1.1. □

6. Controlling how the orbit of the critical point moves

The aim of this section is to prove Proposition 4.5. Let us denote by $I=[0,1]$ the interval of parameters.

Remark 6.1. In Schnellmann [19, Lemma 4.4] it is proven that there is $C_1>0$ such that if $N\geq 1$, $|t_1-t_2|<1/N$ and if $\omega_1\in\mathcal{P}_N(t_1)$ and $\omega_2\in\mathcal{P}_N(t_2)$ have the same combinatorics up to the $(N-1)$-th iteration then

$$\left|\frac{Df_{t_1}^N(x_1)}{Df_{t_2}^N(x_2)}\right|\leq C_1,$$

for all $x_1\in\omega_1$ and $x_2\in\omega_2$.

We also observe that if $x,y\in\omega\in\mathcal{P}_N(t)$, then by the bounded distortion lemma, there is $C_2>0$ such that

$$\left|\frac{Df_t^j(x)}{Df_t^j(y)}\right|\leq C_2,$$

for every $j\leq N$. Let

$$\tilde{M}=\sup_{0\leq j\leq j_0}\sup_{t\in[a,b]}|\partial_t f_t^j(c)|,$$

and let us define

$$C_3=\max\{C,\tilde{M}\}, \tag{31}$$

where C is the constant given by the transversality condition (see Eqs (6) and (7)) and

$$C_4 = \sup_{t\in[0,1]} \sup_{x\in[0,1]} |\partial_t f_t(x)|.$$

To prove Proposition 4.5 we will need

Lemma 6.2. *Let $N_3 \in \mathbb{N}$ and $\omega \in \mathcal{P}_{N_3}$ be such that*

$$|\omega| \leq \frac{1}{N_3}.$$

If $t \in \omega$ and

$$dist(t, \partial\omega) > (M+1)|h|, \tag{32}$$

where

$$M > \max\{C_1C_3C_4, C_1^2C_2C_3^2\} \tag{33}$$

Then

$$c \notin I_{i,j}(t,h) \tag{34}$$

for all $0 \leq j < N_3$ and $0 \leq i < N_3 - j$, where $I_{i,j}(t,h)$ is the smallest interval that contains the set

$$\{f_{t+h}^{i+j+1}(c), f_t^{i+j+1}(c), f_{t+h}^i \circ f_t^{j+1}(c), f_{t+h}^{i+1} \circ f_t^j(c)\}.$$

Proof. Let j_0 be as defined in condition (I) (see Definition 3.3). If $j > j_0$ define $i_1 = 0$ and if $0 \leq j < j_0$ define $i_1 = j_0$. First of all, we observe that if $0 \leq j \leq j_0$ and $0 \leq i \leq j_0$ then Eq. (34) follows from condition (VI). In particular

$$c \notin I_{i,j}(t,h) \text{ for every } i < i_1. \tag{35}$$

Hence, it is left to consider the cases when

$$j_0 < j < N_3 \text{ and } 0 < i \leq N_3 - j$$

and

$$0 < j < j_0 \text{ and } j_0 < i \leq N_3 - j.$$

We claim that

$$c \notin I_{i_1,j}. \tag{36}$$

Indeed, if $0 \leq j \leq j_0$, it follows from condition (VI). Now, if $j_0 < j < N_3$, due to condition (I), Eqs. (6) and (7) the maps

$$\theta \in \omega \to f_\theta^k(c) \in [0,1]$$

are diffeomorphisms on their images for every $j_0 < k \leq N_3$ and they do not contain the critical point in its image, for all $j_0 < k < N_3$, $\theta \in \omega$. In particular if $\omega = (s_1, s_2)$ then

$$c \notin \{f_\theta^k(c) \colon \theta \in \omega\} = (f_{s_1}^k(c), f_{s_2}^k(c)) \tag{37}$$

for every $j_0 < k < N_3$. Therefore,

$$c \notin [f_t^k(c), f_{t+h}^k(c)].$$

By the Mean Value Theorem and Remark 6.1, for every $j < N_3$

$$|f_t^{j+1}(c) - f_{t+h}^{j+1}(c)| = |\partial_\theta f_\theta^{j+1}(c)|_{\theta=\theta_1}||h| \leq C_3|Df_{\theta_1}^j(f_{\theta_1}(c))||h| \leq C_3C_1|Df_t^j(f_t(c))||h|.$$

Moreover,

$$|f_{t+h}(f_t^j(c)) - f_t(f_t^j(c))| \leq |\partial_\theta f_\theta(f_t^j(c))|_{\theta=\theta_2}||h| \leq C_4|h|. \tag{38}$$

By assumption, $d([t,t+h],\partial\omega) > M|h|$. Thus,

$$|\omega| \geq (2M+1)|h|. \tag{39}$$

If $\partial\omega = \{s_1, s_2\}$ and $s \in [t,t+h]$ then

$$\begin{aligned} |f_{s_i}^{k+1}(c) - f_s^{k+1}(c)| &= |\partial_\theta f_\theta^{k+1}(c)|_{\theta=\theta_3}||s_i - s| \\ &\geq \frac{1}{C_3}|Df_{\theta_3}^k(f_{\theta_3}(c))|M|h| \\ &\geq \frac{1}{C_1C_3}|Df_t^k(f_t(c))|M|h| \end{aligned} \tag{40}$$

for every $k < N_3$. Taking $k = j$ we obtain

$$\begin{aligned} |f_{t+h}(f_t^j(c)) - f_t(f_t^j(c))| &\leq C_4|h| \leq \frac{M}{C_1C_3}|h| \\ &\leq \frac{M}{C_1C_3}|Df_t^j(f_t(c))||h| \leq |f_{s_i}^{j+1}(c) - f_t^{j+1}(c)|. \end{aligned} \tag{41}$$

Hence,

$$[f_{t+h}(f_t^j(c)), f_t(f_t^j(c))] \subset (f_{s_1}^{j+1}(c), f_{s_2}^{j+1}(c)). \tag{42}$$

In particular

$$c \notin I_{0,j}(t,h) = I_{i_1,j}(t,h).$$

We concluded the proof of our claim. Now fix $0 \leq j < N_3$. We are going to prove by induction on i that, for every $i_1 \leq i < N_3 - j$,

$$c \notin I_{i,j}(t,h), \tag{43}$$

The case $i = i_1$ follows from Eq. (36). Now suppose that Eq. (43) holds up to i. Provided that $i \geq i_1$, we have $i + j + 1 > j_0$. Therefore, by Eq. (37), with $k = i + j + 2$, we obtain

$$f_{t+h}^{(i+1)+j}(f_{t+h}(c)) \in (f_{s_1}^{(i+1)+j+1}(c), f_{s_2}^{(i+1)+j+1}(c)).$$

And as in Eq. (40)

$$|f_{s_i}^{(i+1)+j}(f_{s_i}(c)) - f_{t+h}^{(i+1)+j}(f_{t+h}(c))| \geq \frac{1}{C_3C_1}|Df_{t+h}^{(i+1)+j}(f_{t+h}(c))|M|h| \tag{44}$$

Moreover by induction assumption and Eq. (35) , we have for every $0 \leq k \leq i$

$$c \notin I_{k,j}(t,h)$$

Thus the points

$$f_{t+h}^{j+1}(c) \text{ and } f_t^{j+1}(c)$$

have the same combinatorics up to i iterations of the map f_{t+h}. Then by Remark 6.1

$$\begin{aligned}
|f_{t+h}^{i+1}(f_t^{j+1}(c)) - f_{t+h}^{i+1}(f_{t+h}^{j+1}(c))| &\leq C_2|Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))||f_t^{j+1}(c) - f_{t+h}^{j+1}(c)| \\
&\leq C_2|Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))||\partial_\theta f_\theta^{j+1}(c)|_{\theta=\theta_4}||h| \\
&\leq C_3C_2|Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))||Df_{\theta_4}^{j}(f_{\theta_4}(c))||h| \\
&\leq C_1C_2C_3|Df_{t+h}^{i+1}(f_{t+h}^{j+1}(c))||Df_{t+h}^{j}(f_{t+h}(c))||h| \\
&\leq C_1C_2C_3|Df_{t+h}^{(i+1)+j}(f_{t+h}(c))||h| \qquad (45)
\end{aligned}$$

and

$$f_t^j(c) \text{ and } f_{t+h}^j(c)$$

have the same combinatorics up to $i+1$ iterations of the map f_{t+h}. Then by Remark 6.1

$$\begin{aligned}
|f_{t+h}^{(i+1)+1}(f_t^j(c)) - f_{t+h}^{(i+1)+1}(f_{t+h}^j(c))| &\leq C_2|Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))||f_t^j(c) - f_{t+h}^j(c)| \\
&\leq C_2|Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))||\partial_\theta f_\theta^j(c)|_{\theta=\theta_5}||h| \\
&\leq C_2C_3|Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))||Df_{\theta_5}^{j-1}(f_{\theta_5}(c))|||h| \\
&\leq C_1C_2C_3|Df_{t+h}^{(i+1)+1}(f_{t+h}^j(c))||Df_{t+h}^{j-1}(f_{t+h}(c))||h| \\
&\leq C_1C_2C_3|Df_{t+h}^{(i+1)+j}(f_{t+h}(c))||h|. \qquad (46)
\end{aligned}$$

Since

$$C_1C_2C_3 < \frac{M}{C_1C_3},$$

Eqs. (44), (45) and (46) imply that

$$\{f_{t+h}^{(i+1)+1}(f_t^j(c)), f_{t+h}^{i+1}(f_t^{j+1}(c))\} \subset (f_{s_1}^{(i+1)+j+1}(c), f_{s_2}^{(i+1)+j+1}(c)).$$

In particular, $c \notin I_{i,j}(t,h)$ for all $0 \leq j < N_3$ and $i_1 < i < N_3 - j$. □

To prove Proposition 4.5 we need to show that, for each given $h \neq 0$, for most of the parameters $t \in [0,1]$ we can find a cylinder $\omega \in \mathcal{P}_{N_3(t,h)}$ where $[t, t+h]$ is deep inside ω (see Eq. (32)) and moreover $N_3(t,h)$ satisfies Eq. (15). To this end, for most t we will find ω, with $t \in \omega$, in such way that $|\omega|$ is quite large with respect to $|h|$ and $N_3(t,h)$ satisfies Eq. (15), but not necessarily the whole interval $[t, t+h]$ is deep inside ω. Then we will use a simple argument to conclude that for most of the parameters t this indeed occurs.

Let $\mathcal{P}_j$ be the partition of level $j > j_0$. Observe that for each cylinder $\omega \in \mathcal{P}_j$

$$|\omega| \leq C_3 \left(\frac{1}{\lambda}\right)^j,$$

where C_3 is the constant given by Eq. (31).

Let $N > 1$ and define $j = j(N)$ as

$$j = \left\lfloor \frac{\log(C_3N)}{\log \lambda} \right\rfloor + 1.$$

Note that the cylinders of $\mathcal{P}_j$ divide the interval of parameters I in subintervals of length shorter than $1/N$. Let J be one of these intervals in $\mathcal{P}_j$. And we will denote by t_R the right boundary point of J.

Observe that, by defintion, there is an integer i, $0 \leq i < j$ such that

$$x_i(t_R) = f_{t_R}^{i+1}(c) = c.$$

Fix an integer τ such that $2^{1/\tau} \leq \sqrt{\lambda}$.

Definition 6.3 (The sets $E_{N,J}$)**.** Let $J \in \mathcal{P}_j$, $j = j(N)$. Let $E_{N,J}$ be the family of all intervals $\omega \in \mathcal{P}_N$ such that for every k satisfying

$$0 \leq k \leq \left\lfloor \frac{\mathcal{K} \log N}{\tau} \right\rfloor$$

there is not

$$\tilde{\omega} = (a,b) \in \mathcal{P}_{N - \lfloor \mathcal{K} \log N \rfloor + q}, \text{ with } \omega \subset \tilde{\omega} \subset J,$$

where

$$q = \min\{(k+1)\tau, \lfloor \mathcal{K} \log N \rfloor\}$$

and for every i satisfying

$$0 \leq i < N - \lfloor \mathcal{K} \log N \rfloor + k\tau$$

we have

$$x_i(a) \neq c \text{ and } x_i(b) \neq c.$$

Define

$$E_N = \bigcup_{J \in \mathcal{P}_j} E_{N,J}.$$

Let us denote by $|E_N|$ the sum of the lengths of the intervals in families
Given $n \in \mathbb{N}$ and $\tilde{\omega} \in \mathcal{P}_n$ define

$$\delta_t := \min\{|f_t^i(c) - f_t^j(c)| \colon \ f_t^i(c) \neq f_t^j(c) \ i,j \leq \tau.\}$$

$$\delta_\omega := \frac{\min_{t \in \overline{\omega}} \delta_t}{2}.$$

Notice that if $\tilde{\omega} \supset \omega$ then $\delta_{\tilde{\omega}} \leq \delta_\omega$.

Let C_L be such that

$$|f_t^i(c) - f_s^i(c)| \leq C_L |t - s|$$

for all $i \leq \tau$, $s, t \in [0,1]$.

Lemma 6.4. *There is $C > 0$ such that the following holds. If $\tilde{\omega} \in \mathcal{P}_i$, $i > j_0$, with $|\tilde{\omega}| < 1/i$ and $t \in \tilde{\omega}$ then*

$$\frac{1}{C} \frac{|x_i(\tilde{\omega})|}{|Df_t^i(f_t(c))|} \leq |\tilde{\omega}| \leq C \frac{|x_i(\tilde{\omega})|}{|Df_t^i(f_t(c))|}. \tag{47}$$

Moreover, if $\omega \in \mathcal{P}_N \setminus E_N$ then there exists i satisfying

$$N - \lfloor \mathcal{K} \log N \rfloor \leq i \leq N$$

such that $\omega \subset \tilde{\omega} \in \mathcal{P}_i$ and if

$$C_L |\tilde{\omega}| < \delta_{\tilde{\omega}}$$

then

$$|x_i(\tilde{\omega})| \geq \delta_{\tilde{\omega}}$$

and

$$\frac{1}{C} \frac{\delta_{\tilde{\omega}}}{|Df_t^i(f_t(c))|} \leq |\tilde{\omega}| \leq C \frac{1}{|Df_t^i(f_t(c))|} \tag{48}$$

for every $t \in \tilde{\omega}$.

Proof. If $t \in \tilde{\omega} \in \mathcal{P}_k$ then by the Mean Value Theorem for some $\theta_1 \in \tilde{\omega}$

$$|x_k(\tilde{\omega})| = |\partial_\theta f_\theta^{k+1}(c)|_{\theta=\theta_1}||\tilde{\omega}|,$$

then

$$\frac{|Df_t^k(f_t(c))||\tilde{\omega}|}{C_1C_3} \le \frac{|Df_{\theta_1}^k(f_{\theta_1}(c))||\tilde{\omega}|}{C_3} \le |x_k(\tilde{\omega})|$$

and

$$|x_k(\tilde{\omega})| \le C_3|Df_{\theta_1}^k(f_{\theta_1}(c))||\tilde{\omega}| \le C_1C_3|Df_t^k(f_t(c))||\tilde{\omega}|,$$

therefore, Eq. (47) holds. Now assume $\omega \in \mathcal{P}_N \setminus E_N$. Then there exists k satisfying

$$1 \le k \le \left\lfloor \frac{\mathcal{K}\log N}{\tau} \right\rfloor$$

and

$$\tilde{\omega} = (a,b) \in \mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor + k\tau}$$

such that $x_{i_a}(a) = c = x_{i_b}(b)$, where

$$N - \lfloor \mathcal{K}\log N\rfloor + (k-1)\tau \le i_a, i_b < N - \lfloor \mathcal{K}\log N\rfloor + k\tau,$$

in particular

$$x_{N-\lfloor \mathcal{K}\log N\rfloor + k\tau}(\tilde{\omega}) = (f_a^{n_a}(c), f_b^{n_b}(c)),$$

where

$$0 \le n_a, n_b < \tau, \text{ with } n_a \ne n_b.$$

Thus,

$$\begin{aligned} |x_{N-\lfloor \mathcal{K}\log N\rfloor + k\tau}(\tilde{\omega})| &= |f_a^{n_a}(c) - f_b^{n_b}(c)| \\ &\ge |f_a^{n_a}(c) - f_a^{n_b}(c)| - |f_a^{n_b}(c) - f_b^{n_b}(c)| \\ &\ge 2\delta_{\tilde{\omega}} - C_L|a-b| \ge \delta_{\tilde{\omega}}. \end{aligned} \tag{49}$$

□

Since $\delta_{\tilde{\omega}} \ge 0$ depends only on a fixed finite number of iterations of the family f_t, it will be easy to give positive lower bounds to it that hold for most of the intervals $\tilde{\omega}$. Indeed define

$$\Lambda_{N_0}^{\delta} = \{t \in [0,1]\colon \text{ for every } N \ge N_0 \text{ if } t \in \omega \in \mathcal{P}_{N-2\lfloor \mathcal{K}\log N\rfloor} \text{ then } \delta_\omega > \delta\}.$$

Note that $\Lambda_{N_0}^{\delta} \subset \Lambda_{N_0+1}^{\delta}$. Moreover $\delta' < \delta$ implies $\Lambda_{N_0}^{\delta'} \supset \Lambda_{N_0}^{\delta}$.

Lemma 6.5. *Given $\gamma > 0$ there exists $\delta > 0$ such that*

$$\lim_{N_0\to\infty} |\Lambda_{N_0}^{\delta}| \ge 1-\gamma.$$

Proof. Since f_t is a transversal family, the set of parameters t such that $f_t^i(c) = f_t^j(c)$ for some $i \ne j$, with $i, j \le \tau + 1$ is finite. Let $t_1, \dots, t_m$ be those parameters. The function $t \to \delta_t$ is positive and continuous on

$$O = [0,1] \setminus \{t_1, \dots, t_m\}.$$

Choose N_0 large enough such that

$$\#\{\omega \in \mathcal{P}_{N_0-2\lfloor \mathcal{K}\log N_0\rfloor}\colon \overline{\omega} \cap \{t_1,\dots,t_m\} \ne \emptyset\} \le 2m.$$

Thus,

$$|\{\omega \in \mathcal{P}_{N_0-2\lfloor \mathcal{K}\log N_0\rfloor}\colon \overline{\omega} \subset O\}| \ge 1 - \frac{2Cm}{\lambda^{N_0-2\lfloor \mathcal{K}\log N_0\rfloor}} > 1-\gamma,$$

provided N_0 is large enough. Let

$$\delta := \frac{1}{2}\min\{\delta_\omega\colon\ \omega \in \mathcal{P}_{N-2\lfloor \mathcal{K}\log N\rfloor},\ \overline{\omega} \subset O\}.$$

Note that $\delta > 0$ and

$$\Lambda_N^\delta \supset \bigcup\{\omega \in \mathcal{P}_{N-2\lfloor \mathcal{K}\log N\rfloor} : \overline{\omega} \subset O\}$$

for every $N \geq N_0$, provided that N_0 is large. □

Proposition 6.6. *There exist $\hat{C}_1, \hat{C}_2 > 0$, that do not depend on $\mathcal{K}$, such that for every $\mathcal{K}' < \mathcal{K}$ there exists $K = K(\mathcal{K}') > 0$ such that*

$$|E_N| \leq K N^{\hat{C}_2 - \hat{C}_1 \mathcal{K}'}. \tag{50}$$

The proof of this proposition follows from

Lemma 6.7. *There exists $\hat{C}_1 > 0$, that does not depend on $\mathcal{K}$, such that for every $\mathcal{K}' < \mathcal{K}$ there exists $K = K(\mathcal{K}') > 0$ such that if $J \in \mathcal{P}_j$, j =j(N), and $E_{N,J}$ is as defined before, then*

$$|E_{N,J}| \leq K N^{-\hat{C}_1 \mathcal{K}'}. \tag{51}$$

We will prove Lemma 6.7 later in this section.

Proof of Proposition 6.6. We have

$$E_N = \bigcup_{J \in \mathcal{P}_j} E_{N,J}.$$

Since there are at most 2^j cylinders of level j, we have by Lemma 6.7 that there exist $\hat{C}_1 > 0$ and $K = K(\mathcal{K}')$ such that

$$|E_N| \leq 2^{\left(\frac{\log(C_3 N)}{\log \lambda}\right)} K N^{-\hat{C}_1 \mathcal{K}'} = K C_3^{\frac{\log 2}{\log \lambda}} N^{\frac{\log 2}{\log \lambda} - \hat{C}_1 \mathcal{K}'}. \tag{52}$$

□

Define

$$\Omega_{N_0} = \left\{t \in [0,1] \colon \forall N \geq N_0\, \exists\, \omega \in \mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor} \backslash E_{N-\lfloor \mathcal{K}\log N\rfloor} \ \text{ and } \ \ t \in \omega\right\}. \tag{53}$$

Note that $\Omega_{N_0} \subset \Omega_{N_0+1}$.

Corollary 6.8. *If $\hat{C}_2 - \hat{C}_1\mathcal{K} < -1$ we have*

$$\lim_{N_0 \to \infty} |\Omega_{N_0}| = 1. \tag{54}$$

Proof. Notice that

$$\Omega_{N_0} = \bigcap_{N \geq N_0} \bigcup_{\omega \in \mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor} \backslash E_{N-\lfloor \mathcal{K}\log N\rfloor}} \omega.$$

If we choose $\mathcal{K}' < \mathcal{K}$ such that $\hat{C}_2 - \hat{C}_1\mathcal{K}' < -1$ we have

$$|\Omega_{N_0}^c| = \Big|\bigcup_{N \geq N_0} \bigcup_{\omega \in E_{N-\lfloor \mathcal{K}\log N\rfloor}} \omega\Big| \leq \sum_{N \geq N_0} K(N - \lfloor \mathcal{K}\log N\rfloor)^{\hat{C}_2 - \hat{C}_1\mathcal{K}'} \overset{N_0 \to \infty}{\longrightarrow} 0.$$

□

From now on we choose and fix $\mathcal{K} > 0$ satisfying $\hat{C}_2 - \hat{C}_1\mathcal{K} < -1$.

Corollary 6.9. *For every $\gamma > 0$ there exists $\delta > 0$ such that*

$$\lim_{N_0\to\infty} m(\Lambda^\delta_{N_0} \cap \Omega_{N_0}) > 1-\gamma.$$

Definition 6.10. Given $\delta > 0$ and $h_0 > 0$, define

$$\Gamma^\delta_{h_0}$$

as the set of all parameters $t \in [0,1]$ such that for every h, $0 < |h| \leq h_0$, there exists k satisfying

$$N(t,h) - 2\lfloor \epsilon \log N(t,h)\rfloor \leq k \leq N(t,h) - \lfloor \epsilon \log N(t,h)\rfloor$$

such that if $t \in \hat{\omega} \in \mathcal{P}_k$ then $|x_k(\hat{\omega})| > \delta$.

Given $t \in \Gamma^\delta_{h_0}$ and $h \neq 0$, let $N_2(t,h)$ be the largest k with this property.

Definition 6.11. Given h and $t \in [0,1]$, define

$$N_1(t,h) := N(t,h) - \lfloor \mathcal{K} \log N(t,h)\rfloor, \tag{55}$$

and for $h_0 > 0$ define

$$\hat{N}_1(h_0) := \min_{t\in I, |h|\leq h_0} N_1(t,h).$$

Since

$$\lim_{N\to\infty} \max_{t\in[0,1]} \frac{1}{|Df^N_t(f_t(c))|} = 0,$$

we have

$$\lim_{h_0\to 0} \hat{N}_1(h_0) = +\infty.$$

Lemma 6.12. *For every $\gamma > 0$ there exists $\delta > 0$ such that*

$$\lim_{h_0\to 0} m(\Gamma^\delta_{h_0}) > 1-\gamma.$$

Proof. By Corollary 6.9 there exist $\delta > 0$ and N_0 such that

$$m(\Lambda^\delta_{N_0} \cap \Omega_{N_0}) > 1-\gamma.$$

Choose h_0 such

$$\hat{N}_1(h_0) > N_0.$$

Let $|h| \leq h_0$. Then

$$N(t,h) - \lfloor \mathcal{K} \log N(t,h)\rfloor \geq N_0.$$

If $t \in \Lambda^\delta_{N_0} \cap \Omega_{N_0}$, choosing $\tilde{\omega}$ such that $t \in \tilde{\omega} \in \mathcal{P}_{N(t,h)-\lfloor \mathcal{K}\log N(t,h)\rfloor}$ then

$$\tilde{\omega} \notin E_{N(t,h)-\lfloor \mathcal{K}\log N(t,h)\rfloor}.$$

Hence, by Lemma 6.4 there exists k satisfying (here $N = N(t,h)$)

$$N - \lfloor \mathcal{K}\log N\rfloor - \lfloor \mathcal{K}\log(N - \lfloor \mathcal{K}\log N\rfloor)\rfloor \leq k \leq N - \lfloor \mathcal{K}\log N\rfloor$$

such that if $t \in \tilde{\omega} \subset \hat{\omega} \in \mathcal{P}_k$ then

$$|x_k(\hat{\omega})| \geq \delta_{\hat{\omega}} > \delta$$

since $t \in \Lambda^\delta_{N_0}$, so that $C_L|\hat{\omega}| < \delta < \delta_{\tilde{\omega}}$. Therefore, $\Gamma^\delta_{h_0} \supset \Lambda^\delta_{N_0} \cap \Omega_{N_0}$. □

Definition 6.13. Given $h_0 > 0$ and $\delta > 0$, for every h such that $|h| \leq h_0$ let $\mathcal{A}^\delta_{h,h_0}$ be a covering of $\Gamma^\delta_{h_0}$ by intervals ω with the following properties

P_1. There exists $t \in \Gamma^\delta_{h_0}$ such that $t \in \omega \in \mathcal{P}_{N_2(t,h)}$.

P_2. If $t' \in \Gamma^\delta_{h_0}$ and $t' \in \omega$ then $\omega' \subset \omega$, where $t' \in \omega' \in \mathcal{P}_{N_2(t',h)}$.

P_3. There does not exist $t'' \in \Gamma^\delta_{h_0}$ such that $t'' \in \omega'' \in \mathcal{P}_{N_2(t'',h)}$ and $\omega \subsetneq \omega''$.

One can check that one such collection $\mathcal{A}^\delta_{h,h_0}$ does exist. Indeed, consider the covering of $\Gamma^\delta_{h_0}$ given by

$$\{\omega\colon \text{ there exists } t\in\Gamma^\delta_{h_0} \text{ such that } t\in\omega\in\mathcal{P}_{N_2(t,h)}\}.$$

Of course this covering satisfies property P_1. Remove from this covering all intervals ω that do not satisfy property P_3. Then the remaining collection is a covering of $\Gamma^\delta_{h_0}$ satisfying properties P_1, P_2 and P_3. Note also that the distinct intervals in $\mathcal{A}^\delta_{h,h_0}$ are pairwise disjoint. Indeed, if $\omega,\omega'\in\mathcal{A}^\delta_{h,h_0}$, with $\omega\neq\omega'$ and $\omega\cap\omega'\neq\emptyset$ then either $\omega\subsetneq\omega'$ or $\omega'\subsetneq\omega$, which is in contradiction with property P_3.

We note that $|\mathcal{A}^\delta_{h,h_0}|\geq m(\Gamma^\delta_{h_0})$, since $\mathcal{A}^\delta_{h,h_0}$ covers $\Gamma^\delta_{h_0}$. Here $|\mathcal{A}^\delta_{h,h_0}|$ denotes the Lebesgue measure of the union of the intervals in the family $\mathcal{A}^\delta_{h,h_0}$.

Lemma 6.14. *If h_0 is small enough there are $C_5>0$ and $C_6>0$ such that the following holds. Given $t'\in\Gamma^\delta_{h_0}$, let ω be the unique interval in $\mathcal{A}^\delta_{h,h_0}$ such that $t'\in\omega$. Let $t\in\Gamma^\delta_{h_0}$ be such that $t\in\omega\in\mathcal{P}_{N_2(t,h)}$. Then*

$$\lfloor\frac{\epsilon}{2}\log N(t',h)\rfloor\leq N(t',h)-N_2(t,h)\leq C_5\mathcal{K}\log N(t',h) \tag{56}$$

and

$$|\omega|\geq C_6\delta N(t',h)^{\mathcal{K}\frac{\log\lambda}{2}}|h|. \tag{57}$$

Proof. Consider ω' such that

$$t'\in\omega'\in\mathcal{P}_{N_2(t',h)}.$$

Then by property P_2 we have $\omega'\subset\omega$. Since

$$\delta\leq|x_{N_2(t',h)}(\omega')|=|\partial_\theta f^{N_2(t',h)}(c)||\omega'|\leq C_1C_3|Df_{t'}^{N_2(t',h)}(f_{t'}(c))|,$$

it follows that

$$\frac{\delta}{C_1C_3}\frac{1}{|Df_{t'}^{N_2(t',h)}(f_{t'}(c))|}\leq|\omega'|\leq|\omega|\leq\frac{C_1C_3}{|Df_t^{N_2(t,h)}(f_t(c))|}. \tag{58}$$

Since $t,t'\in\omega$, there is $C_1>1$ such that

$$\frac{1}{C_1}\frac{1}{|Df^i_{t'}(f_{t'}(c))|}\leq\frac{1}{|Df^i_t(f_t(c))|}\leq C_1\frac{1}{|Df^i_{t'}(f_{t'}(c))|}.$$

for every $i\leq N_2(t,h)$. Choose $\bar{C}$ such that

$$\frac{\delta}{C_3^2C_1^3}>\frac{1}{\lambda^{\bar{C}}}. \tag{59}$$

Then

$$N_2(t',h)\geq N_2(t,h)-\bar{C},$$

otherwise

$$\begin{aligned}\frac{\delta}{C_1C_3}\frac{1}{|Df_{t'}^{N_2(t',h)}(f_{t'}(c))|}&\leq\frac{C_1C_3}{|Df_t^{N_2(t,h)}(f_t(c))|}\\&\leq\frac{C_1C_3}{|Df_t^{N_2(t,h)-N_2(t',h)}(f_t^{N_2(t',h)+1}(c))|}\frac{1}{|Df_t^{N_2(t',h)}(f_t(c))|}\\&\leq\frac{C_1C_3}{\lambda^{\bar{C}}}\frac{C_1}{|Df_{t'}^{N_2(t',h)}(f_{t'}(c))|},\end{aligned}$$

which contradicts Eq. (59). In particular

$$\begin{aligned} N(t',h) - N_2(t,h) &\geq N(t',h) - N_2(t',h) - \bar{C} \\ &\geq \lfloor \epsilon \log N(t',h) \rfloor - \bar{C} \\ &\geq \lfloor \frac{\epsilon}{2} \log N(t',h) \rfloor. \end{aligned}$$

Note that the lower bound holds if h_0 is small enough. Thus,

$$N(t',h) > N_2(t,h).$$

Moreover,

$$\begin{aligned} |h| &\leq \frac{1}{|Df_{t'}^{N(t',h)}(f_{t'}(c))|} \\ &\leq \frac{1}{|Df_{t'}^{N(t',h)-N_2(t,h)}(f_{t'}^{N_2(t,h)+1}(c))|} \frac{1}{|Df_{t'}^{N_2(t,h)}(f_{t'}(c))|} \\ &\leq \frac{1}{|Df_{t'}^{N(t',h)-N_2(t,h)}(f_{t'}^{N_2(t,h)+1}(c))|} \frac{C_1}{|Df_{t}^{N_2(t,h)}(f_{t}(c))|}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |h| &\geq \frac{1}{|Df_{t}^{N(t,h)+1}(f_{t}(c))|} \\ &\geq \frac{1}{|Df_{t}^{N(t,h)-N_2(t,h)}(f_{t}^{N_2(t,h)+1}(c))|} \frac{1}{|Df_{t}^{N_2(t,h)}(f_{t}(c))|} \frac{1}{\Lambda}. \end{aligned} \tag{60}$$

Then

$$\begin{aligned} &\log |Df_{t'}^{N(t',h)-N_2(t,h)}(f_{t'}^{N_2(t,h)+1}(c))| - \log C_1 \\ &\leq \log |Df_{t}^{N(t,h)-N_2(t,h)}(f_{t}^{N_2(t,h)+1}(c))| + \log \Lambda \end{aligned}$$

and consequently

$$N(t',h) - N_2(t,h) \leq \hat{C}_3(N(t,h) - N_2(t,h)) + \hat{C}_4.$$

In a similar way, we can obtain

$$N(t,h) - N_2(t,h) \leq \hat{C}_3(N(t',h) - N_2(t,h)) + \hat{C}_4,$$

where

$$\hat{C}_3 = \frac{\log \Lambda}{\log \lambda}$$

and

$$\hat{C}_4 = \frac{\log C_1}{\log \lambda}.$$

$$\begin{aligned} N(t,h) &= N(t,h) - N_2(t,h) + N_2(t,h) \\ &\leq 2\epsilon \log N(t,h) + N(t',h) \\ &\leq \frac{N(t,h)}{N(t,h) - 2\epsilon \log N(t,h)} N(t',h) \\ &\leq 2N(t',h), \end{aligned}$$

provided that h_0 is small. Consequently

$$\begin{aligned} N(t',h)-N_2(t,h) &\leq \hat{C}_3(N(t,h)-N_2(t,h))+\hat{C}_4 \\ &\leq \hat{C}_3 2\lfloor \mathcal{K}\log N(t,h)\rfloor+\hat{C}_4 \\ &\leq \hat{C}_3 2\mathcal{K}\log[2N(t',h)]+\hat{C}_4 \\ &\leq C_5\mathcal{K}\log N(t',h). \end{aligned} \tag{61}$$

Here the last inequality holds if h_0 is small enough. Moreover, by Eq. (58)

$$\begin{aligned} |\omega| &\geq \frac{1}{C_1C_3}\frac{\delta}{|Df_{t'}^{N_2(t',h)}(f_{t'}(c))|}=\frac{\delta}{C_1C_3}\frac{|Df_{t'}^{N(t',h)-N_2(t',h)}(f_{t'}^{N_2(t',h)+1}(c))|}{|Df_{t'}^{N(t',h)}(f_{t'}(c))|} \\ &\geq \frac{\delta}{C_1C_3}\frac{\lambda^{N(t',h)-N_2(t',h)}}{|Df_{t'}^{N(t',h)}(f_{t'}(c))|}\geq\frac{\delta}{C_1C_3}\lambda^{\frac{\mathcal{K}\log N(t',h)}{2}-1}|h|=\frac{\delta}{C_1C_3\lambda}N(t',h)^{\mathcal{K}\frac{\log\lambda}{2}}|h|. \end{aligned} \tag{62}$$

Hence, we obtain Eq. (57). □

Choose $\epsilon>0$ such that

$$\frac{1}{\sqrt{\lambda}}<1-\epsilon.$$

Lemma 6.15. *Given $M>0$, define*

$$B^{\delta}_{h,h_0,M}=\{t: t\in\omega\in\mathcal{A}^{\delta}_{h,h_0} \text{ and } dist(t,\partial\omega)>\frac{M+1}{1-\epsilon}|h|\}.$$

Let $h_i=(1-\epsilon)^i h_0$. Given h satisfying $0<|h|\leq h_0$, let

$$i(h)=max\{i\in\mathbb{N}\colon |h|<(1-\epsilon)^{i-1}h_0\}.$$

For every $h>0$ define

$$\hat{\Gamma}^{\delta}_{h,h_0}=\Gamma^{\delta}_{h_0}\cap\Big(\bigcap_{i\geq i(h)}B^{\delta}_{h_i,h_0,M}\Big).$$

Then

A. *If $0<\hat{h}<h$ then $\hat{\Gamma}^{\delta}_{h,h_0}\subset\hat{\Gamma}^{\delta}_{\hat{h},h_0}$,*
B. *We have*

$$\lim_{h\to 0}m(\hat{\Gamma}^{\delta}_{h,h_0})=m(\Gamma^{\delta}_{h_0}).$$

Proof. Note that

$$\min_{t\in[0,1]}N(t,h_i)\geq-\frac{\log h_0}{\log\Lambda}-1-\frac{i\log(1-\epsilon)}{\log\Lambda},$$

where

$$-\frac{\log h_0}{\log\Lambda}>0 \quad\text{and}\quad -\frac{i\log(1-\epsilon)}{\log\Lambda}>0.$$

Therefore, if h_0 is samall enough, there are $K_1,K_2>0$, such that

$$\min_{t\in[0,1]}N(t,h_i)\geq K_1+iK_2.$$

Define

$$A_h=\bigcup_{\omega\in\mathcal{A}^{\delta}_{h,h_0}}\omega.$$

If $\omega \in \mathcal{A}^\delta_{h,h_0}$ then there is $t \in \Gamma^\delta_{h_0}$ such that $t \in \omega \in \mathcal{P}_{N_2(t,h)}$. By Lemma 6.14

$$
\begin{aligned}
m(\omega \cap (B^\delta_{h,h_0,M})^c) &= m\{t' \in \omega \colon dist(t', \partial\omega) \leq \frac{M+1}{1-\epsilon}|h|\} \\
&\leq 2\frac{M+1}{1-\epsilon}|h| \\
&\leq \frac{2(M+1)|h|}{(1-\epsilon)|\omega|}|\omega| \\
&\leq \frac{2C_6(M+1)}{\delta(1-\epsilon)N(t,h)^{\mathcal{K}\frac{\log\lambda}{2}}}|\omega|. \qquad (63)
\end{aligned}
$$

Choose $\mathcal{K}$ large enough such that $\mathcal{K}\log\lambda > 2$. Then

$$
\sum_{i=0}^{\infty} m(A_{h_i} \cap (B^\delta_{h_i,h_0,M})^c) \leq \sum_{i=0}^{\infty} \frac{2C_6(M+1)\sqrt{\lambda}}{\delta(K_1+iK_2)^{\mathcal{K}\frac{\log\lambda}{2}}} < \infty. \qquad (64)
$$

In particular

$$
\begin{aligned}
m\Big(\Gamma^\delta_{h_0} \cap \big(\bigcap_{i\geq i(h)} B^\delta_{h_i,h_0,M}\big)\Big) &= m(\Gamma^\delta_{h_0}) - m(\Gamma^\delta_{h_0} \cap (\bigcap_{i\geq i(h)} B^\delta_{h_i,h_0,M})^c) \\
&\geq m(\Gamma^\delta_{h_0}) - \sum_{i\geq i(h)} m(\Gamma^\delta_{h_0} \cap (B^\delta_{h_i,h_0,M})^c) \\
&\geq m(\Gamma^\delta_{h_0}) - \sum_{i\geq i(h)} m(A_{h_i} \cap (B^\delta_{h_i,h_0,M})^c). \qquad (65)
\end{aligned}
$$

Eq. (64) implies that

$$
\lim_{h\to 0} \sum_{i\geq i(h)} m(A_{h_i} \cap (B^\delta_{h_i,h_0,M})^c) = 0.
$$

□

Proof of Proposition 4.5. By Lemma 6.12 for every $\gamma > 0$ there exists $\delta > 0$ such that for every small h_0 we have

$$
m(\Gamma^\delta_{h_0}) > 1 - \gamma.
$$

Choose M satisfying Eq. (33). Define

$$
\Gamma^\delta_{h,h_0} = \hat{\Gamma}^\delta_{h,h_0} \setminus Q,
$$

where $\hat{\Gamma}^\delta_{h,h_0}$ is the set defined in Lemma 6.15 and Q is the countable set of parameters where f_t has a periodic critical point. By Lemma 6.15 Property A. holds. Let $t' \in \Gamma^\delta_{h,h_0}$, with $|h| < h_0$. There exists $i \geq i(h)$ such that

$$
h_{i+1} \leq |h| \leq h_i,
$$

where $h_i = (1-\epsilon)^i h_0$. Thus, $N(t',h) = N(t',h_j)$, for some $j \in \{i, i+1\}$, and consequently $N_2(t',h) = N_2(t',h_j)$. Then there exists a unique $\omega \in \mathcal{A}^\delta_{h_j,h_0}$ and $t \in \Gamma^\delta_{h_0}$ such that $t, t' \in \omega \in \mathcal{P}_{N_2(t,h)}$. Moreover, since $t' \in B^\delta_{h_j,h_0,M}$ we have

$$
dist(t', \partial\omega) \geq \frac{M+1}{1-\epsilon}h_j \geq (M+1)|h|.
$$

Define $N_3(t',h) = N_2(t,h)$. By Lemma 6.14 we have Eq. (15) holds. By Lemma 6.2, Eq. (16) holds.

□

6.1. **Proof of Lemma 6.7.** Let J be the interval as in the statement of Lemma 6.7. The sets $E_{N,J}$ 'live' in the parameter space. To estimate its measures we will compare them, following [18], with the measures of similarly defined sets in the phase space of the map f_{t_R}.

Definition 6.16 (The sets $\hat{E}_{N,t_R}$)**.** Let $J = [t_L, t_R]$. Denote by $\hat{E}_{N,t_R}$ the set of all

$$\eta \in \mathcal{P}_N(t_R)$$

such that for all k satisfying

$$0 \leq k \leq \left\lfloor \frac{\mathcal{K} \log N}{\tau} \right\rfloor$$

there is not

$$\tilde{\eta} \in \mathcal{P}_{N - \lfloor \mathcal{K} \log N \rfloor + j}(t_R), \ \ \eta \subset \tilde{\eta},$$

where

$$j = \min\{(k+1)\tau, \lfloor \mathcal{K} \log N \rfloor\},$$

such that

$$f_{t_R}^{N - \lfloor \mathcal{K} \log N \rfloor + k\tau}(\tilde{\eta}) \in \mathcal{P}_{j - k\tau}(t_R).$$

Using a strategy similar to the one applied in [18], we estimate the measure $|E_{N,J}|$ in terms of the measure $|\hat{E}_{N,t_R}|$. To this end we need to define the map $\mathcal{U}_J$. Recall that if $\mathcal{F}$ is a family of disjoint intervals then $|\mathcal{F}|$ denotes the sum of the measures of the intervals.

Definition 6.17 (The map $\mathcal{U}_J$)**.** Let $J = (t_L, t_R)$. Consider the map $\mathcal{U}_J$

$$\mathcal{U}_J : \mathcal{P}_N|_J \to \mathcal{P}_N(t_R)$$

defined by Schnellmann [18, proof of Lemma 3.2] in the following way. Let $\omega \in \mathcal{P}_N|_J$ and choose $t \in \omega$. Since ω is a cylinder, it follows that $x_j(t) \neq c$ for all $0 \leq j < N$. Therefore, there is a cylinder $\omega(x_0(t))$ in the partition $\mathcal{P}_N(t)$ such that $x_0(t) \in \omega(x_0(t))$.

Let

$$\mathcal{U}_J(\omega) = \mathcal{U}_{t,t_R,N}(\omega(x_0(t))),$$

where $\mathcal{U}_{t,t_R,N} : \mathcal{P}_N(t) \to \mathcal{P}_N(t_R)$ is such that for all $\eta \in \mathcal{P}_N(t)$, the elements η and $\mathcal{U}_J(\eta)$ have the same combinatorics.

$$\mathrm{symb}_t(f_t^i(\eta)) = \mathrm{symb}_{t_R}(f_{t_R}^i(\mathcal{U}_{t,t_R,N}(\eta)),$$

for $0 \leq i < N$. Schnellmann [18] proved that $\mathcal{U}_{t,t_R,N}$ is well defined when f_t is a family of piecewise expanding unimodal maps satisfying our assumptions. In particular, if $t < t'$ and a certain symbolic dynamic appears in the dynamic of f_t, then it also appears in the dynamic of f_t'.

Therefore, the cylinder $\omega' = \mathcal{U}_J(\omega) = \mathcal{U}_{t,t_R,N}(\omega(x_0(t)))$ has the same combinatorics as ω, that is,

$$\mathrm{symb}(x_j(\omega)) = \mathrm{symb}_{t_R}(f_{t_R}^j(\omega')),$$

when $0 \leq j < N$. Since there are not two cylinders in $\mathcal{P}_N(t_R)$ with the same combinatorics, the element ω' does not depend on the choice of $t \in \omega$. Therefore, $\mathcal{U}_J$ is well defined.

Lemma 6.18. *If $\omega \in E_{N,J}$, then $\mathcal{U}_J(\omega) \in \hat{E}_{N,t_R}$. Moreover, there exists $C' \geq 1$ such that*

$$|\omega| \leq C'|\mathcal{U}_J(\omega)|. \tag{66}$$

In particular

$$|E_{N,J}| \leq C'|\hat{E}_{N,t_R}|. \tag{67}$$

Proof. Note that $\mathcal{U}_J(\omega) \in \hat{E}_{N,t_R}$ follows from the fact that ω and $\mathcal{U}_J(\omega)$ have the same combinatorics [18]. By [18, Lemma 3.2], there exists a constant $C' \geq 1$ such that

$$|\omega| \leq C'|\mathcal{U}_J(\omega)|.$$

Thus,

$$|E_{N,J}| \leq \sum_{\omega \in E_{N,J}} |\omega| \leq \sum_{\omega \in E_{N,J}} C'|\mathcal{U}_J(\omega)| \leq C'|\hat{E}_{N,t_R}|. \tag{68}$$

□

Definition 6.19. For each $\eta' \in \mathcal{P}_{N-\lfloor \mathcal{K} \log N \rfloor}(t_R)$, define the set

$$\hat{E}_{N,t_R,\eta'} = \left\{ \eta \in \mathcal{P}_N(t_R) : \eta \in \hat{E}_{N,t_R} \text{ and } \eta \subset \eta' \right\}.$$

Lemma 6.20. *Let $\eta' \in \mathcal{P}_{N-\lfloor \mathcal{K} \log N \rfloor}(t_R)$. Then*

$$\#\hat{E}_{N,t_R,\eta'} \leq 2^{\lfloor \frac{\mathcal{K} \log N}{\tau} \rfloor + 1}. \tag{69}$$

Proof. Define

$$k_0 = \left\lfloor \frac{\mathcal{K} \log N}{\tau} \right\rfloor.$$

Notice that

$$N \geq N - \lfloor \mathcal{K} \log N \rfloor + k_0 \tau > N - \tau.$$

If $N = N - \lfloor \mathcal{K} \log N \rfloor + k_0 \tau$ define $k_1 = k_0$. Otherwise define $k_1 = k_0 + 1$. For every k satisfying

$$0 \leq k \leq k_1,$$

define families of intervals $\mathcal{F}_k$ in the following way. If $k \leq k_0$ define

$$\mathcal{F}_k = \{\hat{\eta} \subset \eta' : \ \hat{\eta} \in \mathcal{P}_{N-\lfloor \mathcal{K} \log N \rfloor + k\tau}(t_R) \text{ and there is } \eta \in \hat{E}_{N,t_R,\eta'} \text{with } \eta \subset \hat{\eta}\} \tag{70}$$

otherwise $k = k_1 = k_0 + 1$ and

$$\mathcal{F}_{k_1} = \hat{E}_{N,t_R,\eta'}. \tag{71}$$

Note that if $k_1 = k_0$ then we also have $\mathcal{F}_{k_1} = \hat{E}_{N,t_R,\eta'}$. We claim that

$$\#\mathcal{F}_k \leq 2^k. \tag{72}$$

We observe that, taking $k = k_1$ in Eq. (72) we obtain Eq. (69). Note that either $\mathcal{F}_0$ is the empty set or $\mathcal{F}_0 = \{\eta'\}$. Then $\#\mathcal{F}_0 \leq 1$. Moreover, it is easy to see that if $\hat{\eta}_{k+1} \in \mathcal{F}_k$, with $k < k_1$, then there exists a unique $\hat{\eta}_k \in \mathcal{F}_k$ such that $\hat{\eta}_{k+1} \subset \hat{\eta}_k$. Therefore, it is enough to show that for each $\hat{\eta}_k \in \mathcal{F}_k$, with $k < k_1$, there are at most two intervals $\hat{\eta}_{k+1} \in \mathcal{F}_{k+1}$ such that $\hat{\eta}_{k+1} \subset \hat{\eta}_k$. Indeed, given $k < k_1$, for every $\hat{\eta}_k \in \mathcal{F}_k$ we have $\hat{\eta}_k \in \mathcal{P}_{N-\lfloor \mathcal{K} \log N \rfloor + k\tau}(t_R)$. Moreover, there is j such that for every $\hat{\eta}_{k+1} \in \mathcal{F}_{k+1}$ we have $\hat{\eta}_{k+1} \in \mathcal{P}_{N-\lfloor \mathcal{K} \log N \rfloor + j}(t_R)$, with $k\tau < j \leq \lfloor \mathcal{K} \log N \rfloor$, and $j \leq k\tau + \tau$. Note that if the closure of $\hat{\eta}_{k+1} = (a,b)$ is contained in the interior of $\hat{\eta}_k$, then for every $x \in \overline{\hat{\eta}_{k+1}}$ we have $f^p_{t_R}(x) \neq c$, for every $p < N - \lfloor \mathcal{K} \log N \rfloor + k\tau$. Furthermore, there are n_a, n_b such that

$$f^{n_a}_{t_R}(a) = c = f^{n_b}_{t_R}(b),$$

where

$$N - \lfloor \mathcal{K} \log N \rfloor + k\tau \leq n_a,\ \ n_b < N - \lfloor \mathcal{K} \log N \rfloor + j.$$

We conclude that

$$f^{N-\lfloor \mathcal{K} \log N \rfloor + k\tau}_{t_R}(\hat{\eta}_{k+1}) \in \mathcal{P}_{j-k\tau}(t_R).$$

where $j - k\tau \leq \tau$. Therefore, if $\eta \subset \hat{\eta}_{k+1}$, with $\eta \in \mathcal{P}_N(t_R)$, then $\eta \notin \hat{E}_{N,t_R,\eta'}$ and consequently $\hat{\eta}_{k+1} \notin \mathcal{F}_{k+1}$. Since there are at most two intervals $\mathcal{P}_{N-\lfloor \mathcal{K} \log N \rfloor + j}(t_R)$ whose closure is not contained in the interior of $\hat{\eta}_k$, we conclude that there are at most two intervals in $\mathcal{F}_{k+1}$ that are contained in $\hat{\eta}_k$.

□

Lemma 6.21. *Let $\eta', \eta'' \in \mathcal{P}_{N-\lfloor \mathcal{K} \log N \rfloor}(t_R)$ such that*

$$f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\eta') = f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\eta'').$$

Then

$$f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\hat{E}_{N,t_R,\eta'}) = f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\hat{E}_{N,t_R,\eta''}).$$

Proof. Let $\omega' = (y'_1, y'_2) \in \mathcal{P}_N(t_R)$, with $\omega' \subset \eta'$, be a cylinder in $\hat{E}_{N,t_R,\eta'}$. Then

$$f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\omega') \subset f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\eta') = f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\eta''). \tag{73}$$

Remember that since $\omega' \in \mathcal{P}_N(t_R)$, it follows that for all $x \in \omega'$

$$f^i_{t_R}(x) \neq c \quad \text{for all} \quad 0 \leq i < N, \tag{74}$$

and if $y \in \partial\omega'$, then there exists j, $0 \leq j < N$ such that $f^j_{t_R}(y) = c$. Define

$$a_i = f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(y'_i).$$

Then $f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\omega') = (a_1, a_2)$ is an open interval and, by Eq. (73), we have $(a_1, a_2) \subset f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\eta'')$. Therefore, there is an open interval $\omega'' = (y''_1, y''_2) \subset \eta''$ such that $f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\omega'') = (a_1, a_2)$ with

$$a_i = f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(y''_i).$$

We claim that ω'' is also a cylinder. Indeed, let $x \in \omega''$. Then, since $\omega'' \subset \eta''$ and η'' is a cylinder of level $N - \lfloor \mathcal{K} \log N \rfloor$, it follows that

$$f^i_{t_R}(x) \neq c,$$

for all $1 \leq i < N - \lfloor \mathcal{K} \log N \rfloor$. On the other hand,

$$f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\omega'') = f^{N-\lfloor \mathcal{K} \log N \rfloor}_{t_R}(\omega'),$$

and by Eq. (74), we can conclude that $f^i_{t_R}(x) \neq c$ for all i satisfying $N - \lfloor \mathcal{K} \log N \rfloor \leq i < N$. Therefore, for all $x \in \omega''$, we have $f^i_{t_R}(x) \neq c$ for all $0 \leq i < N$. Now, let $y''_i \in \partial\omega_2$. Since $\omega'' \subset \eta''$, we have two cases.

Case 1: $y''_i \in \partial\eta''$. In this case, there is an integer j, $0 \leq j < N - \lfloor \mathcal{K} \log N \rfloor$, such that $f^j_{t_R}(y''_i) = c$.

Case 2: $y_i'' \notin \partial\eta''$. In this case, $f^j_{t_R}(y_i'') \neq c$ for all $0 \leq j < N - \lfloor \mathcal{K} \log N \rfloor$. Then $f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(y_i'') = a_i = f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(y_i')$ belongs to the interior of $f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\eta'') = f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\eta')$. Thus, y_i' belongs to the interior of η', which implies that there exists j such that $N - \lfloor \mathcal{K}\log N\rfloor \leq j < N$ such that $f^j_{t_R}(y_i') = f^j_{t_R}(y_i'') = c$.

Therefore $\omega'' \in \mathcal{P}_N(t_R)$.

By assumption, $\omega' \in \hat{E}_{N,t_R,\eta'}$. Then for all $0 \leq k \leq \lfloor \frac{\mathcal{K}\log N}{\tau} \rfloor$, if

$$\tilde{\omega}_k \in \mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor + j(k)}(t_R),$$

where $\omega' \subset \tilde{\omega}_k \subset \eta'$ and

$$j(k) = \min\{(k+1)\tau, \lfloor \mathcal{K}\log N\rfloor\},$$

then there is $z_k' \in \partial\tilde{\omega}$ satisfying

$$f_{t_R}^{q_k'}(z_k') = c, \text{ for some } q_k',\ 0 \leq q_k' < N - \lfloor \mathcal{K}\log N\rfloor + k\tau. \tag{75}$$

In the same manner as for ω', there exists a unique cylinder $\hat{\omega}_k \in \mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor + j(k)}$, $\hat{\omega}_k \subset \eta''$, such $f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\tilde{\omega}_k) = f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{\omega}_k)$. Note that $\omega'' \subset \hat{\omega}_k$. Let $z_k'' \in \partial\hat{\omega}_k$ such that

$$f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(z_k') = f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(z_k'').$$

If $z_k'' \in \partial\eta''$ then there exists $i < N - \lfloor \mathcal{K}\log N\rfloor$ such that $f^i_{t_R}(z_k'') = c$. Define $q_k'' = i$.
If $z_k'' \notin \partial\eta''$ then $z_k' \notin \partial\eta'$. Thus, $f^q_{t_R}(z_k') \neq c$ for every $q < N - \lfloor \mathcal{K}\log N\rfloor$, which implies that

$$N - \lfloor \mathcal{K}\log N\rfloor \leq q_k' < N - \lfloor \mathcal{K}\log N\rfloor + k\tau.$$

Then $f_{t_R}^{q_k'}(z_k'') = f_{t_R}^{q_k'}(z_k'') = c$. Define $q_k'' = q_k'$.
In both cases we have $0 \leq q_k'' < N - \lfloor \mathcal{K}\log N\rfloor + k\tau$, then $\omega'' \in \hat{E}_{N,t_R,\eta''}$.

$$f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{E}_{N,t_R,\eta''}) \subset f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{E}_{N,t_R,\eta'}).$$

A similar argument shows that

$$f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{E}_{N,t_R,\eta'}) \subset f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{E}_{N,t_R,\eta''}).$$

□

Proof of Lemma 6.7. Due to Lemma 6.18 it is enough to show that for every $\mathcal{K}' < \mathcal{K}$ there exists $C > 0$ and $K = K(\mathcal{K}') > 0$ such that if $J \in \mathcal{P}_j$, $j = j(N)$ then

$$|\hat{E}_{N,t_R}| \leq K N^{-C\mathcal{K}'}. \tag{76}$$

By Lemma 6.20 we have

$$\#\hat{E}_{N,t_R,\eta'} \leq 2^{\lfloor \frac{\mathcal{K}\log N}{\tau} \rfloor + 1}.$$

Let us define the set

$$\Omega = \bigcup_{\eta' \in \mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor}(t_R)} f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{E}_{N,t_R,\eta'}).$$

Note that

$$\hat{E}_{N,t_R} \subset f_{t_R}^{-(N-\lfloor \mathcal{K}\log N\rfloor)}(\Omega).$$

Therefore, if μ_{t_R} is the acip for f_{t_R} we have

$$\mu_{t_R}(\hat{E}_{N,t_R}) \leq \mu_{t_R}(\Omega). \tag{77}$$

In [18, Section 6.2] it is shown that there is $C_1' \geq 1$ such that for every density ρ_t of the unique acip of f_t

$$\frac{1}{C_1'} \leq \rho_t(x) \leq C_1',$$

for μ_t-almost every $x \in [0,1]$, then

$$|\hat{E}_{N,t_R}| \leq {C_1'}^2 |\Omega|.$$

Since $J \in \mathcal{P}_j$, $j = j(N)$, there exists an integer p, $0 \leq p < j$ such that $x_p(t_R) = f_{t_R}^p(f_{t_R}(c)) = c$. In particular

$$\#\{f_{t_R}^i(c)\}_{i\geq 0} = p+1.$$

Thus,

$$\#\{f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\eta'),\ \eta' \in \mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor}(t_R)\} \leq (p+1)^2.$$

Therefore, by Lemma 6.21,

$$\begin{aligned}
|\hat{E}_{N,t_R}| &\leq {C_1'}^2|\Omega| = {C_1'}^2 |\cup_{\eta'\in\mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor}(t_R)} f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{E}_{\eta'}))| \\
&\leq {C_1'}^2(p+1)^2 \max_{\eta'\in\mathcal{P}_{N-\lfloor \mathcal{K}\log N\rfloor}(t_R)} |f_{t_R}^{N-\lfloor \mathcal{K}\log N\rfloor}(\hat{E}_{\eta'})| \\
&\leq {C_1'}^2(p+1)^2 \left(\frac{1}{\lambda}\right)^{\lfloor \mathcal{K}\log N\rfloor} \#\left\{\eta \in \mathcal{P}_N(t_R)|_{\hat{E}_{\eta'}}\right\} \\
&\leq {C_1'}^2(p+1)^2 \left(\frac{1}{\lambda}\right)^{\lfloor \mathcal{K}\log N\rfloor} 2^{\lfloor \frac{\mathcal{K}\log N}{\tau}\rfloor+1} \leq {C_1'}^2(p+1)^2 \left(\frac{1}{\lambda}\right)^{\frac{\lfloor \mathcal{K}\log N\rfloor}{2}} \\
&\leq {C_1'}^2 j^2 \left(\frac{1}{\lambda}\right)^{\frac{\lfloor \mathcal{K}\log N\rfloor}{2}} \leq {C_1'}^2 \left(\left\lfloor\frac{\log(C_3N)}{\log\lambda}\right\rfloor\right)^2 \left(\frac{1}{\lambda}\right)^{\frac{\lfloor \mathcal{K}\log N\rfloor}{2}} \\
&\leq KN^{-\frac{\log\lambda}{2}\epsilon'}
\end{aligned}$$

where $K = K(\epsilon')$. ∎

7. Estimates for the Wild part

We start this section with a technical lemma.

Lemma 7.1. *Given a good transversal family f_t there are constants L_1 and L_2 such that the following holds. Let $\varphi : [0,1] \to \mathbb{R}$, $|\varphi|_{L^1(m)} > 0$, be a function of bounded variation such that*

$$\int \varphi dm = 0.$$

Then

$$\left|(I-\mathcal{L}_t)^{-1}(\varphi)\right|_{L^1} \leq \left(L_1 \log\frac{|\varphi|_{BV}}{|\varphi|_{L^1}} + L_2\right)|\varphi|_{L^1}.$$

Proof. Let $\tilde{j} > 0$ such that

$$L\tilde{\beta}^{\tilde{j}}|\varphi|_{BV} = |\varphi|_{L^1}.$$

And let j_0 the smallest integer such that $j_0 - 1 \leq \tilde{j} \leq j_0$. Hence, we have

$$(I-\mathcal{L}_t)^{-1}(\varphi) = \sum_{i=0}^{j_0} \mathcal{L}_t^i(\varphi) + \sum_{l=1}^{\infty} \mathcal{L}_t^l(\mathcal{L}_t^{j_0}(\varphi)).$$

Observing Assumption $(V)A_1$, the fact that $|\mathcal{L}_t^l \varphi|_{BV} \leq \tilde{L}\theta^l |\varphi|_{BV}$, when $\int \varphi \, dm = 0$ with constants $\tilde{L}$ and θ uniform in t, as well as the elementary facts that $|\mathcal{L}_t|_{L^1} = 1$ and $|\cdot|_{L^1} \leq |\cdot|_{BV}$, we see that

$$\begin{aligned} |(I-\mathcal{L}_t)^{-1}(\varphi)|_{L^1} &\leq (j_0+1)|\varphi|_{L^1} + \frac{\tilde{L}}{1-\theta}|\mathcal{L}_t^{j_0}\varphi|_{BV} \\ &\leq (j_0+1)|\varphi|_{L^1} + \frac{\tilde{L}}{1-\theta}\left(\tilde{C}_6\beta^{j_0}|\varphi|_{BV} + \tilde{C}_5|\varphi|_{L^1}\right) \\ &\leq \left(c_1\beta^{j_0}\frac{|\varphi|_{BV}}{|\varphi|_{L^1}} + (j_0+c_2)\right)|\varphi|_{L^1}. \end{aligned}$$

By the choice of j_0, we have the desired estimate.

□

The following proposition will be quite important to study the Wild part of the decomposition. Denote

$$\mathrm{supp}(\psi) = \overline{\{x \in [0,1]\colon\ \psi(x) \neq 0\}}.$$

Proposition 7.2. *There exist $K, K_1', K_2' > 0$ such that the following holds. For all $i, k \geq 0$, $t \in [0,1]$ and $h \neq 0$, let*

$$\varphi_{k,i,h} = \frac{1}{h}\mathcal{L}_{t+h}^i\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right).$$

Then

$$\left|\varphi_{k,i,h}\right|_{L^1} \leq K, \tag{78}$$

and

$$\left|\varphi_{k,i,h}\right|_{BV} \leq \frac{K}{|h|}. \tag{79}$$

Furthermore,

$$\left|(I-\mathcal{L}_{t+h})^{-1}\Pi_{t+h}(\varphi_{k,i,h})\right|_{L^1} \leq K_1' \max\{0, \log|\varphi_{k,i,h}|_{BV}\} + K_2'. \tag{80}$$

Proof. Note that

$$\begin{aligned} &\left|\mathcal{L}_{t+h}^i\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\right|_{L^1} \\ &\leq \left|H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right|_{L^1} \\ &\leq (\sup_t |v_t|)|h|, \end{aligned} \tag{81}$$

and, by Assumption (V) in Definition 3.3

$$\begin{aligned} &\left|\mathcal{L}_{t+h}^i\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\right|_{BV} \\ &\leq 2\tilde{C}_6\beta^i + \tilde{C}_5(\sup_t |v_t|)|h| \leq \hat{C}. \end{aligned} \tag{82}$$

Thus, we have Eqs. (78) and (79). In particular

$$|\Pi_{t+h}(\varphi_{k,i,h})|_{L^1(m)} \leq 2|\varphi_{k,i,h}|_{L^1(m)} \leq 2K,$$

and if h is small

$$|\Pi_{t+h}(\varphi_{k,i,h})|_{BV} \leq |\varphi_{k,i,h}|_{BV} + |\varphi_{k,i,h}|_{BV} \sup_{t\in[0,1]} |\rho_t|_{BV} \leq C|\varphi_{k,i,h}|_{BV},$$

where $C \geq 1$.
Now we can easily obtain Eq. (80) applying Lemma 7.1.

□

Proposition 7.3. *Let ϕ be a Lipchitz function. There exists $K > 0$ such that the following holds. Let $t \in \Gamma^{\delta}_{h',h_0}$ and $0 < |h| \leq h'$. Then*

$$\text{var}\big(\frac{1}{h}\mathcal{L}^i_{t+h}\left(H_{f_{t+h}(f^k_t(c))} - H_{f_t(f^k_t(c))}\right)\big) \leq \frac{K}{|h||Df^i_t(f^{k+1}_t(c))|}, \tag{83}$$

and

$$\begin{aligned}&\int \phi(x)\mathcal{L}^i_{t+h}\left(\frac{H_{f_{t+h}(f^k_t(c))} - H_{f_t(f^k_t(c))}}{h}\right)(x)\ dx\\ &= \phi(f^{i+k+1}_t(c))v_t(f^k_t(c)) + O(|Df^i_t(f^{k+1}_t(c))||h|),\end{aligned} \tag{84}$$

where $0 \leq k \leq N_3(t,h)$ and $i < N_3(t,h) - k$.

Proof. By Eq. (16), the points $f^{k+1}_{t+h}(c), f_{t+h}(f^k_t(c)), f_t(f^k_t(c))$ belong to the same interval of monotonicity of f^i_{t+h}. Let

$$\zeta\colon Dom(\zeta) \to Im(\zeta)$$

be an inverse branch associated to such interval of monotonicity, that is, ζ is a diffeomorphism such that $f^i_{t+h}(\zeta(y)) = y$ for every $y \in Dom(\zeta)$ and

$$\{f^{k+1}_{t+h}(c), f_{t+h}(f^k_t(c)), f_t(f^k_t(c))\} \subset Im(\zeta).$$

Hence,

$$\begin{aligned}&\mathcal{L}^i_{t+h}\left(H_{f_{t+h}(f^k_t(c))} - H_{f_t(f^k_t(c))}\right)(x)\\ &= \frac{1}{Df^i_{t+h}(\zeta(x))}\mathbb{1}_{Dom(\zeta)}(x)\left(H_{f_{t+h}(f^k_t(c))}(\zeta(x)) - H_{f_t(f^k_t(c))}(\zeta(x))\right).\end{aligned} \tag{85}$$

There is a constant $K \geq 1$ such that for all $t \in [0,1]$, h, and i, and every interval of monotonicity Q of f^i_{t+h} we have

$$\frac{1}{K} \leq \left|\frac{Df^i_{t+h}(y_1)}{Df^i_{t+h}(y_2)}\right| \leq K$$

for all $y_1, y_2 \in Q$. Now we can estimate the variation of the function in Eq. (83) using familiar properties of the variation of functions (see Chapter 3 from Viana

[21], for instance).

$$
\begin{aligned}
&\mathrm{var}_{[0,1]}\left(\mathcal{L}^i_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\right)\\
&= \mathrm{var}_{[0,1]}\left(\frac{1}{Df^i_{t+h}(\zeta(x))}\mathbb{1}_{Dom(\zeta)}(x)\left(H_{f_{t+h}(f_t^k(c))}(\zeta(x)) - H_{f_t(f_t^k(c))}(\zeta(x))\right)\right)\\
&= \mathrm{var}_{Dom(\zeta)}\left(\frac{1}{Df^i_{t+h}(\zeta(x))}\right)\sup_{[0,1]}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\\
&+ 2\sup_{[0,1]}\left(\frac{1}{Df^i_{t+h}(\zeta(x))}\mathbb{1}_{Dom(\zeta)}(x)\right)\sup_{[0,1]}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\\
&+ \sup_{[0,1]}\left(\frac{1}{Df^i_{t+h}(\zeta(x))}\mathbb{1}_{Dom(\zeta)}(x)\right)\mathrm{var}_{[0,1]}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\\
&\le 2\mathrm{var}_{Dom(\zeta)}\left(\frac{1}{Df^i_{t+h}(\zeta(x))}\right) + \frac{6K}{|Df^i_{t+h}(f^{k+1}_{t+h}(c))|}.
\end{aligned}
$$

Now, note that since ζ is a diffeomorphism, it follows that

$$
\begin{aligned}
&\mathrm{var}_{Dom(\zeta)}\left(\frac{1}{Df^i_{t+h}(\zeta(x))}\right) = \mathrm{var}_{Im(\zeta)}\left(\frac{1}{Df^i_{t+h}(y)}\right)\\
&= \int_{Im(\zeta)}\left|D\left(\frac{1}{Df^i_{t+h}(y)}\right)\right|dy\\
&= \int_{Im(\zeta)}\left|\sum_{j=1}^{i} -\frac{D^2 f_{t+h}(f^{j-1}_{t+h}(y))}{Df^{i-j}_{t+h}(f^j_{t+h}(y))\left(Df_{t+h}(f^{j-1}_{t+h}(y))\right)^2}\right|dy\\
&\le K_1|Im(\zeta)| \le K_1\frac{|Dom(\zeta)|}{|Df^i_{t+h}(f^{k+1}_{t+h}(c))|}\\
&\le \frac{CK_2}{|Df^i_{t+h}(f^{k+1}_{t+h}(c))|}.
\end{aligned}
$$

Here we used that

$$
\begin{aligned}
&\left|\sum_{j=1}^{i} -\frac{D^2 f_{t+h}(f^{j-1}_{t+h}(y))}{Df^{i-j}_{t+h}(f^j_{t+h}(y))\left(Df_{t+h}(f^{j-1}_{t+h}(y))\right)^2}\right|\\
&\le \sum_{j=1}^{i}\frac{C}{\lambda^{i-j}}\\
&\le K_1, \qquad (86)
\end{aligned}
$$

and that

$$
|Im(\zeta)| \le K\frac{|Dom(\zeta)|}{|Df^i_{t+h}(f^{k+1}_{t+h}(c))|} \le K\frac{1}{|Df^i_{t+h}(f^{k+1}_{t+h}(c))|}.
$$

Therefore,

$$\operatorname{var}_{[0,1]}\left(\mathcal{L}^i_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\right) \leq \frac{K_3}{|Df^i_{t+h}(f^{k+1}_{t+h}(c))|}. \tag{87}$$

Finally, by Eq. (16) note that the combinatorics up to i iterations of $f^{k+1}_{t+h}(c)$ by the map f_{t+h} is the same as the combinatorics up to i iterations of $f_t^{k+1}(c)$ by the map f_t. By Remark 6.1 we obtain

$$\frac{1}{|Df^i_{t+h}(f^{k+1}_{t+h}(c))|} \leq C_1 \frac{1}{|Df^i_t(f^{k+1}_t(c))|}. \tag{88}$$

Eqs. (88) and (87) give us Eq. (83). Since

$$\operatorname{supp}\frac{1}{h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right) = [f_{t+h}(f_t^k(c)), f_t(f_t^k(c))],$$

by Eq. (85) we conclude that

$$Z_{i,k} = \operatorname{supp}\frac{1}{h}\mathcal{L}^i_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right) = [f^{i+1}_{t+h}(f_t^k(c)), f^i_{t+h}(f_t^{k+1}(c))].$$

By Eq. (16), the points $f^{k+1}_{t+h}(c), f_{t+h}(f_t^k(c)), f_t(f_t^k(c))$ belong to the same interval of monotonicity of f^i_{t+h}. Hence,

$$\begin{aligned}
&\mathit{diam}\ \operatorname{supp}\frac{1}{h}\mathcal{L}^i_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\\
&= \mathit{diam}\ [f^{i+1}_{t+h}(f_t^k(c)), f^i_{t+h}(f_t^{k+1}(c))]\\
&= |f^{i+1}_{t+h}(f_t^k(c)) - f^i_{t+h}(f_t^{k+1}(c))|\\
&\leq K|Df^i_{t+h}(f_t^{k+1}(c))||f_{t+h}(f_t^k(c)) - f_t(f_t^k(c))|\\
&\leq K|Df^i_{t+h}(f_t^{k+1}(c))||\sup_t v_t||h|\\
&\leq C_1K|Df^i_t(f_t^{k+1}(c))||\sup_t v_t||h|.
\end{aligned} \tag{89}$$

Therefore,

$$\begin{aligned}
&\int \phi(x)\mathcal{L}^i_{t+h}\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\ dx\\
&= \phi(f_t^{i+k+1}(c))\int \mathcal{L}^i_{t+h}\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\ dx\\
&+ \int \left(\phi(x) - \phi(f_t^{i+k+1}(c))\right)\mathcal{L}^i_{t+h}\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\ dx.
\end{aligned} \tag{90}$$

Note that

$$\begin{aligned}
&\int \mathcal{L}^i_{t+h}\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\ dx\\
&= \int \frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}(x)dx = v_t(f_t^k(c)) + O(|h|).
\end{aligned} \tag{91}$$

Due to Eq (89) and the fact that ϕ is a lipschitzian function with Lipschitz constant L, and that $f_t^{i+k+1}(c) \in Z_{i,k}$

$$
\begin{aligned}
&|\int \left(\phi(x) - \phi(f_t^{i+k+1}(c))\right) \mathcal{L}^i_{t+h} \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\ dx| \\
&\leq \int_{Z_{i,k}} |\phi(x) - \phi(f_t^{i+k+1}(c))| \left|\mathcal{L}^i_{t+h} \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\right|\ dx \\
&\leq LC_1K|Df_t^i(f_t^{k+1}(c))||\sup_t v_t||h|\left|\mathcal{L}^i_{t+h} \left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)\right|_{L^1} \\
&\leq LC_1K|Df_t^i(f_t^{k+1}(c))||\sup_t v_t|^2|h|. \qquad (92)
\end{aligned}
$$

□

Proof of Proposition 4.6. Let Φ_h be as in Proposition 4.3, that is

$$\Phi_h = \frac{1}{h}\sum_{k=0}^{\infty} s_{k+1}(t)\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right).$$

Given $t \in \Gamma^{\delta}_{h,h_0}$. Let $N_3(t,h)$ be as in Proposition 4.5. Since t and h are fixed throughout this proof, we will write N_3 instead of $N_3(t,h)$ and N instead of $N(t,h)$. Let us divide Φ_h as follows

$$\Phi_h = S_1 + S_2.$$

Where

$$S_1 = \frac{1}{h}\sum_{k=0}^{N_3} s_{k+1}(t)\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)$$

and

$$S_2 = \frac{1}{h}\sum_{k=N_3+1}^{\infty} s_{k+1}(t)\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right).$$

Let us first estimate S_2.

$$(I-\mathcal{L}_{t+h})^{-1}S_2 = \frac{1}{h}\sum_{k=N_3+1}^{\infty} s_{k+1}(t)(I-\mathcal{L}_{t+h})^{-1}\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right).$$

Thus,

$$
\begin{aligned}
&\left|(I-\mathcal{L}_{t+h})^{-1}S_2\right|_{L^1} \\
&\leq \sum_{k=N_3+1}^{\infty} |s_{k+1}(t)| \left|\frac{1}{h}(I-\mathcal{L}_{t+h})^{-1}\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)\right|_{L^1}.
\end{aligned}
$$

By Proposition 7.2 and Lemma 7.1, taking

$$\varphi = \frac{1}{h}\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right),$$

we have,

$$\begin{aligned}&\left|(I-\mathcal{L}_{t+h})^{-1}\frac{1}{h}\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right)\right|_{L^1}\\&\le K_1\log\frac{1}{|h|}+K_2\le K_1\log\Lambda^{N+1}+K_2\\&\le K_1(N+1)\log\Lambda+K_2\le K_3N+K_4.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|(I-\mathcal{L}_{t+h})^{-1}S_2\right|_{L^1}&\le\sum_{k=N_3+1}^{\infty}\frac{1}{\lambda^k}(K_3N+K_4)\le\frac{K_5N}{\lambda^{N_3}}+K_6\\&\le\frac{K_5N}{\lambda^{N-C_5\mathcal{K}\log N}}+K_6\le K_7h^{K_8\log\lambda}\left(\log\frac{1}{|h|}\right)^{1+C_5\mathcal{K}\log\lambda}+K_6.\end{aligned}$$

It is left to analyze S_1. Applying the operator $(I-\mathcal{L}_{t+h})^{-1}$,

$$(I-\mathcal{L}_{t+h})^{-1}(S_1)=\frac{1}{h}\sum_{i=0}^{\infty}\mathcal{L}_{t+h}^i\sum_{k=0}^{N_3}s_{k+1}(t)\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right).$$

Then

$$\begin{aligned}(I-\mathcal{L}_{t+h})^{-1}(S_1)=&\frac{1}{h}\sum_{k=0}^{N_3}s_{k+1}(t)\sum_{i=0}^{\infty}\mathcal{L}_{t+h}^i\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right)\\=&S_{11}+S_{12}.\end{aligned}$$

Where

$$S_{11}=\sum_{k=0}^{N_3}s_{k+1}(t)\sum_{i=0}^{N_3-k}\frac{1}{h}\mathcal{L}_{t+h}^i\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right)$$

and

$$\begin{aligned}S_{12}&=\sum_{k=0}^{N_3}s_{k+1}(t)\sum_{i=N_3-k+1}^{\infty}\frac{1}{h}\mathcal{L}_{t+h}^i\Pi_{t+h}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right)\\&=\sum_{k=0}^{N_3}s_{k+1}(t)\frac{1}{h}\mathcal{L}_{t+h}\circ(I-\mathcal{L}_{t+h})^{-1}\circ\Pi_{t+h}\circ\mathcal{L}_{t+h}^{N_3-k}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right).\end{aligned}$$

We observe that

$$|S_{12}|_{L^1}\le C\sum_{k=0}^{N_3}|s_{k+1}(t)||(I-\mathcal{L}_{t+h})^{-1}\circ\Pi_{t+h}\circ\mathcal{L}_{t+h}^{N_3-k}\frac{1}{h}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right)|_{L^1}.$$

Let

$$\varphi_k=\frac{1}{h}\mathcal{L}_{t+h}^{N_3-k}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right),$$

By Proposition 7.3 it follows that

$$|\varphi_k|_{BV} = var(\varphi_k) + |\varphi_k|_{L^1} \tag{93}$$

$$\leq \frac{C}{|h||Df_t^{N_3(t,h)-k}(f_t^{k+1}(c))|} + K_1 \tag{94}$$

$$\leq C\frac{|Df_t^{N(t,h)+1}(f_t(c))|}{|Df_t^{N_3(t,h)-k}(f_t^{k+1}(c))|} + K_1$$

$$\leq C|Df_t^{N(t,h)+1-N_3(t,h)}(f_t^{N_3(t,h)+1}(c))||Df_t^k(f_t(c)))| + K_1$$

$$\leq C\Lambda^{N(t,h)+1-N_3(t,h)+k} + K_1. \tag{95}$$

By Lemma 7.1 we have

$$\begin{aligned}
&|(I-\mathcal{L}_{t+h})^{-1}\circ\Pi_{t+h}(\varphi_k)|_{L^1}\\
&= |(I-\mathcal{L}_{t+h})^{-1}\circ\Pi_{t+h}\circ\mathcal{L}_{t+h}^{N_3-k}\frac{1}{h}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)|_{L^1}\\
&\leq K_1'\log(C\Lambda^{N(t,h)+1-N_3(t,h)+k} + K_1) + K_2'\\
&\leq K_1'\log(K_2\Lambda^{N(t,h)+1-N_3(t,h)+k}) + K_2'\\
&\leq K_3(N-N_3+k+1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|S_{12}|_{L^1} &\leq K_3\sum_{k=0}^{N_3}|s_{k+1}(t)|(N-N_3+k+1)\\
&\leq K_3(N-N_3)\sum_{k=0}^{N_3}\frac{1}{\lambda^k} + K_3\left(\sum_{k=0}^{N_3}\frac{k}{\lambda^k} + \sum_{k=0}^{N_3}\frac{1}{\lambda^k}\right)\\
&\leq K_4\mathcal{K}\log N + K_5 \leq K\left(\log\log\frac{1}{|h|} + 1\right).
\end{aligned}$$

We proceed to examine S_{11}.

$$\begin{aligned}
S_{11} = &\underbrace{\sum_{k=0}^{N_3}s_{k+1}(t)\sum_{i=0}^{N_3-k}\mathcal{L}_{t+h}^i\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)}_{S_{111}}\\
&\underbrace{-\sum_{k=0}^{N_3}s_{k+1}(t)\sum_{i=0}^{N_3-k}\mathcal{L}_{t+h}^i\left(\rho_{t+h}\int\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\,dm\right)}_{S_{112}}.
\end{aligned}$$

Note that

$$\begin{aligned}
S_{112} &= -\sum_{k=0}^{N_3}s_{k+1}(t)\sum_{i=0}^{N_3-k}\rho_{t+h}\int\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)dm\\
&= -\sum_{k=0}^{N_3}s_{k+1}(t)\sum_{i=0}^{N_3-k}\left(v_t(f_t^k(c)) + O(h)\right)\rho_{t+h}.
\end{aligned}$$

Adding and subtracting the sum

$$\sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} v_t(f_t^k(c))\rho_t,$$

we obtain

$$S_{112} = S_{1121} + S_{1122},$$

where

$$S_{1121} = -\sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} v_t(f_t^k(c))\rho_t$$

and

$$S_{1122} = -(\rho_{t+h} - \rho_t)\sum_{k=0}^{N_3} s_{k+1}(t)(N_3-k)v_t(f_t^k(c)) - O(h)\sum_{k=0}^{N_3} s_{k+1}(t)(N_3-k)\rho_{t+h}.$$

By Eq. (5)

$$\begin{aligned}|S_{1122}|_{L^1} &\leq K_1 \sup_t |v_t||h|\log\big(\frac{1}{|h|}\big)\sum_{k=0}^{N_3}|s_{k+1}(t)|(N_3-k)\\ &+ |\rho_{t+h}|_{L^1}\,|O(h)|\sum_{k=0}^{N_3}|s_{k+1}(t)|(N_3-k)\\ &\leq \left(K_2|h|\log\frac{1}{|h|} + K_3|O(h)|\right) N_3 \sum_{k=0}^{N_3}\frac{1}{\lambda^k}\\ &\leq K_4 N\left(|h|\log\frac{1}{|h|} + |O(h)|\right) \leq K\log\frac{1}{|h|}\left(|h|\log\big(\frac{1}{|h|}\big) + |O(h)|\right).\end{aligned}$$

Therefore, taking $\phi : [0,1] \to \mathbb{R}$ a lipschitzian observable,

$$\begin{aligned}&\int \phi(x)\mathcal{W}(x)\ dx = \int \phi(x)(I - \mathcal{L}_{t+h})^{-1}\Phi_h(x)\ dx\\ &= \int \phi(x)(S_{111} + S_{1121})(x)\ dx + O\left(\log\log\frac{1}{|h|}\right)\\ &= \sum_{k=0}^{N_3} s_{k+1}(t)\sum_{i=0}^{N_3-k}\int \phi(x)\mathcal{L}_{t+h}^i\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\ dx\\ &- \sum_{k=0}^{N_3} s_{k+1}(t)\sum_{i=0}^{N_3-k} v_t(f_t^k(c))\int \phi(x)\rho_t(x)\ dx + O\left(\log\log\frac{1}{|h|}\right).\end{aligned}$$

By Eq. (84) we have

$$\begin{aligned}&\int \phi(x)\mathcal{L}_{t+h}^i\left(\frac{H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}}{h}\right)(x)\ dx\\ &= \phi(f_t^{i+k+1}(c))v_t(f_t^k(c)) + O(|Df_t^i(f_t^{k+1}(c))||h|)\\ &= \phi(f_t^{i+k+1}(c))v_t(f_t^k(c)) + O(\frac{|Df_t^i(f_t^{k+1}(c))|}{|Df_t^N(f_t(c))|}).\end{aligned}$$

Since

$$
\begin{aligned}
&|\sum_{k=0}^{N_3} s_{k+1}(t) \sum_{i=0}^{N_3-k} O(\frac{|Df_t^i(f_t^{k+1}(c))|}{|Df_t^N(f_t(c))|})| \\
&\leq K_1 \sum_{k=0}^{N_3} \left(\frac{1}{\lambda}\right)^k \sum_{i=0}^{N} \left(\frac{1}{\lambda}\right)^{N-i} < K, \qquad (96)
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\int \phi(x)\mathcal{W}(x)dx \\
&= \sum_{k=0}^{N_3} s_{k+1}(t)v_t(f_t^k(c)) \sum_{i=0}^{N_3-k} \left(\phi(f_t^{i+k+1}(c)) - \int \phi \; d\mu_t\right) + O\left(\log\log\frac{1}{|h|}\right) \\
&= \sum_{k=0}^{N_3} s_{k+1}(t)v_t(f_t^k(c)) \sum_{j=k+1}^{N_3+1} \left(\phi(f_t^{j}(c)) - \int \phi \; d\mu_t\right) + O\left(\log\log\frac{1}{|h|}\right) \\
&= \sum_{j=1}^{N_3+1} \left(\phi(f_t^{j}(c)) - \int \phi \; d\mu_t\right) \sum_{k=0}^{j-1} s_{k+1}(t)v_t(f_t^k(c)) + O\left(\log\log\frac{1}{|h|}\right).
\end{aligned}
$$

Adding and subtracting the series

$$
\sum_{j=1}^{N_3+1} \left(\phi(f_t^{j}(c)) - \int \phi \; d\mu_t\right) \sum_{k=j}^{\infty} s_{k+1}(t)v_t(f_t^k(c)),
$$

we obtain

$$
\begin{aligned}
\int \phi(x)\mathcal{W}(x)dx = &\sum_{j=1}^{N_3+1} \left(\phi(f_t^{j}(c)) - \int \phi \; d\mu_t\right) \sum_{k=0}^{\infty} \frac{s_1(t)}{Df_t^k(f_t(c))} v_t(f_t^k(c)) \\
&\underbrace{- \sum_{j=1}^{N_3+1} \left(\phi(f_t^{j}(c)) - \int \phi \; d\mu_t\right) \sum_{k=j}^{\infty} \frac{s_1(t)}{Df_t^k(f_t(c))} v_t(f_t^k(c))}_{I_1} \\
&+ O\left(\log\log\frac{1}{|h|}\right).
\end{aligned}
$$

Note that $|I_1| < \infty$. Indeed,

$$
\begin{aligned}
|I_1| &\leq K_1 \sum_{j=1}^{N_3+1} \left|\phi(f_t^{j}(c)) - \int \phi \; d\mu_t\right| \sum_{k=j}^{\infty} \left(\frac{1}{\lambda}\right)^k \\
&\leq K_2 \sum_{j=1}^{N_3+1} \left(\frac{1}{\lambda}\right)^j \leq K.
\end{aligned}
$$

Therefore,

$$\int \phi(x)\mathcal{W}(x)dx = s_1(t)J(f_t, v_t)\sum_{j=1}^{N_3+1}\left(\phi(f_t^j(c)) - \int \phi\ d\mu_t\right) + O\left(\log\log\frac{1}{|h|}\right)$$

$$= s_1(t)J(f_t, v_t)\sum_{j=0}^{N_3}\left(\phi(f_t^j(c)) - \int \phi\ d\mu_t\right) + O\left(\log\log\frac{1}{|h|}\right). \tag{97}$$

□

8. Estimates for the Tame part

Let ν be a signed, finite and borelian measure on $[0,1]$. Denote by $|\nu|$ the variation measure of ν and by $||\nu||$ the total variation of ν. Define the push-forward of ν by f_t as the borelian measure

$$(f_t^\star \nu)(A) = \nu(f_t^{-1}(A)).$$

Note that for every bounded borelian function $g\colon [0,1] \to \mathbb{R}$

$$\int g\ d(f_t^\star \nu) = \int g \circ f_t\ d\nu.$$

It is also easy to see that

$$|f_t^\star \nu| \leq f_t^\star |\nu|.$$

Suppose that ν has the form

$$\nu = \pi m + \sum_{x\in\hat{\Delta}} q_x \delta_x, \tag{98}$$

where $\pi \in L^\infty(m)$ with support on $[0,1]$, m is the Lebesgue measure, $\hat{\Delta} \subset [0,1]$ is a countable subset, $q_x \in \mathbb{R}$, with

$$\sum_{x\in\hat{\Delta}} |q_x| < \infty,$$

and δ_x is the Dirac measure supported on $\{x\}$. Then

$$|\nu| = |\pi| m + \sum_{x\in\hat{\Delta}} |q_x|\delta_x,$$

$$||\nu|| = |\pi|_{L^1(m)} + \sum_{x\in\hat{\Delta}} |q_x|.$$

Furthermore, $f_t^\star \nu$ has the form

$$f_t^\star \nu = \mathcal{L}_t(\pi) m + \sum_{x\in\hat{\Delta}} q_x \delta_{f_t(x)}.$$

Proposition 8.1. *Let f_t be a C^1 family of C^1 piecewise expanding unimodal maps. Let ν be a signed, finite and borelian measure. Let $\psi_t : [0,1] \mapsto \mathbb{R}$, $t \in [0,1]$ be such that $\psi_t \in L^\infty(\nu)$ and $t \to \psi_t$ is a lipschitzian function with respect to the $L^\infty(|\nu|)$ norm, that is, there exists L such that for all t, h we have*

$$|\psi_{t+h} - \psi_t|_{L^\infty(\nu)} \leq L|h|.$$

Define

$$\Delta_{t,h}(x) = \int_0^x\ df_{t+h}^\star(\psi_{t+h}\nu) - \int_0^x\ df_t^\star(\psi_t \nu).$$

Then there exist positive constants K_1, K_2 such that

$$|\Delta_{t,h}|_{L^1(m)} \leq (L + K_1K_2)||\nu|||h|$$

for all $t \in [0,1]$, h, where

$$K_1 = \sup_t |\psi_t|_{L^\infty(\nu)} \text{ and } K_2 = \sup_{t,x} |\partial_t f_t(x)|.$$

Proof. Observe that

$$\begin{aligned}\Delta_{t,h}(x) &= \int_0^x \ df^\star_{t+h}(\psi_{t+h}\nu) - \int_0^x \ df^\star_t(\psi_t\nu) \\ &= \underbrace{\int_0^x \ df^\star_{t+h}(\psi_{t+h}\nu) - \int_0^x \ df^\star_{t+h}(\psi_t\nu)}_{\Delta_1} \\ &+ \underbrace{\int_0^x \ df^\star_{t+h}(\psi_t\nu) - \int_0^x \ df^\star_t(\psi_t\nu)}_{\Delta_2}.\end{aligned}$$

Therefore,

$$|\Delta_{t,h}(x)| \leq |\Delta_1(x)| + |\Delta_2(x)|.$$

We first estimate Δ_1.

$$\begin{aligned}|\Delta_1(x)| &\leq \int \mathbb{1}_{[0,x]} \ d|f^\star_{t+h}(\psi_{t+h}\nu - \psi_t\nu)| \leq \int \mathbb{1}_{[0,x]} \ d(f^\star_{t+h}(|\psi_{t+h} - \psi_t||\nu|)) \\ &\leq \int \mathbb{1}_{[0,x]} \circ f_{t+h}|\psi_{t+h} - \psi_t| \ d|\nu| \leq |\psi_{t+h} - \psi_t|_{L^\infty(\nu)}||\nu|| \leq L||\nu|||h|.\end{aligned}$$

In particular

$$|\Delta_1|_{L^1(m)} \leq L||\nu|||h|.$$

We now estimate Δ_2.

$$\begin{aligned}\Delta_2(x) &= \int \mathbb{1}_{[0,x]} \ df^\star_{t+h}(\psi_t\nu) - \int \mathbb{1}_{[0,x]} \ df^\star_t(\psi_t\nu) \\ &= \int \mathbb{1}_{[0,x]} \circ f_{t+h} \ d(\psi_t\nu) - \int \mathbb{1}_{[0,x]} \circ f_t d(\psi_t\nu) \\ &= \int (\mathbb{1}_{f^{-1}_{t+h}([0,x])} - \mathbb{1}_{f^{-1}_t([0,x])}) \ d(\psi_t\nu).\end{aligned}$$

Therefore,

$$|\Delta_2(x)| \leq \int |\mathbb{1}_{f^{-1}_{t+h}([0,x])} - \mathbb{1}_{f^{-1}_t([0,x])}||\psi_t| \ d|\nu| \leq K_1 \int |\mathbb{1}_{f^{-1}_{t+h}([0,x])} - \mathbb{1}_{f^{-1}_t([0,x])}| \ d|\nu|$$

where

$$K_1 = \sup_t |\psi_t|_{L^\infty(\nu)}.$$

By the Fubini's Theorem

$$\begin{aligned}|\Delta_2|_{L^1(m)} &\leq K_1 \int\int |\mathbb{1}_{f^{-1}_{t+h}([0,x])}(y) - \mathbb{1}_{f^{-1}_t([0,x])}(y)| \ d|\nu|(y) \ dm(x) \\ &\leq K_1 \int\int |\mathbb{1}_{f^{-1}_{t+h}([0,x])}(y) - \mathbb{1}_{f^{-1}_t([0,x])}(y)| \ dm(x) \ d|\nu|(y) \qquad (99)\end{aligned}$$

Note that

$$|\mathbb{1}_{f^{-1}_{t+h}([0,x])}(y) - \mathbb{1}_{f^{-1}_t([0,x])}(y)| = \mathbb{1}_{U_y}(x),$$

where

$$U_y = \{x \in [0,1] \colon f_{t+h}(y) < x \le f_t(y) \text{ or } f_t(y) < x \le f_{t+h}(y)\}.$$

Observe that

$$m(U_y) = |f_{t+h}(y) - f_t(y)| \le K_2|h|.$$

Thus,

$$\begin{aligned} |\Delta_2|_{L^1(m)} &\le K_1 \int\int \mathbb{1}_{U_y}(x)\ dm(x)\ d|\nu|(y) \\ &\le K_1K_2||\nu|||h|. \end{aligned} \tag{100}$$

□

Remark 8.2. To avoid a cumbersome notation, in the Proof of Proposition 4.3 we will use the following notation. Whenever we take the supremum over all $t \in [0,1]$ we actually take the supremum over all $t \in [0,1]$ such that f_t do not have a periodic critical point. And whenever we take the supremum over all $h \neq 0$ we indeed mean taking the supremum over all $h \neq 0$ such that $0 < |h| < \delta$, where $\delta > 0$ is given by Definition 3.3.

Proof of Proposition 4.3. We first examine

$$\frac{1}{h}(\mathcal{L}_{t+h}\rho_t - \mathcal{L}_t\rho_t).$$

As we have seen, the density ρ_t can be decomposed as

$$\rho_t = (\rho_t)_{abs} + (\rho_t)_{sal}.$$

We also have $\mathcal{L}_{t+h}\rho_t \in BV$ and

$$\mathcal{L}_{t+h}\rho_t = (\mathcal{L}_{t+h}\rho_t)_{abs} + (\mathcal{L}_{t+h}\rho_t)_{sal}.$$

Therefore,

$$(\mathcal{L}_{t+h}\rho_t - \mathcal{L}_t\rho_t) = ((\mathcal{L}_{t+h}\rho_t)_{abs} - (\mathcal{L}_t\rho_t)_{abs}) + ((\mathcal{L}_{t+h}\rho_t)_{sal} - (\mathcal{L}_t\rho_t)_{sal}).$$

Let us examine the absolutely continuous term

$$\frac{1}{h}((\mathcal{L}_{t+h}\rho_t)_{abs} - (\mathcal{L}_t\rho_t)_{abs}).$$

Observe that for every t

$$(\mathcal{L}_t\rho_t)(x) = (\mathcal{L}_t\rho_t)_{abs}(x) + (\mathcal{L}_t\rho_t)_{sal}(x).$$

Differentiating with respect to x,

$$\begin{aligned} ((\mathcal{L}_t\rho_t)_{abs})'(x) &= (\mathcal{L}_t\rho_t)'(x) \\ &= ((\mathcal{L}_t\rho_t)')_{abs}(x) + ((\mathcal{L}_t\rho_t)')_{sal}(x). \end{aligned}$$

Then

$$(\mathcal{L}_t\rho_t)_{abs}(x) = \int_0^x (\mathcal{L}_t\rho_t)'\ dm.$$

Similarly

$$(\mathcal{L}_t\rho_{t+h})_{abs}(x) = \int_0^x (\mathcal{L}_{t+h}\rho_t)'(y)\ dm.$$

Therefore,

$$
\begin{aligned}
(\mathcal{L}_{t+h}\rho_t)_{abs}(x) - (\mathcal{L}_t\rho_t)_{abs}(x) &= \int_0^x (\mathcal{L}_{t+h}\rho_t)' - (\mathcal{L}_t\rho_t)'\ dm \\
&= \int_0^x ((\mathcal{L}_{t+h}\rho_t)')_{abs} - ((\mathcal{L}_t\rho_t)')_{abs}\ dm \\
&+ \int_0^x ((\mathcal{L}_{t+h}\rho_t)')_{sal} - ((\mathcal{L}_t\rho_t)')_{sal}\ dm.
\end{aligned}
$$

We define

$$A_{t,h}(x) = \int_0^x ((\mathcal{L}_{t+h}\rho_t)')_{abs} - ((\mathcal{L}_t\rho_t)')_{abs}\ dm, \tag{101}$$

and

$$B_{t,h}(x) = \int_0^x ((\mathcal{L}_{t+h}\rho_t)')_{sal} - ((\mathcal{L}_t\rho_t)')_{sal}\ dm. \tag{102}$$

Our goal is to prove that

$$\sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{A_{t,h}}{h}\right|_{BV} < \infty \quad \text{and} \quad \sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{B_{t,h}}{h}\right|_{BV} < \infty.$$

Since $A_{t,h}$ is absolutely continuous, it follows that

$$\mathrm{var}(A_{t,h}) = \int |A'_{t,h}| dm.$$

Hence, to prove that

$$\sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{A_{t,h}}{h}\right|_{BV} < \infty,$$

it is enough to prove that

$$\sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{A'_{t,h}}{h}\right|_{L^1(m)} dm < \infty \quad \text{and} \quad \sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{A_{t,h}}{h}\right|_{L^1(m)} < \infty. \tag{103}$$

According to Eq. (101),

$$A'_{t,h}(x) = (\mathcal{L}_{t+h}\rho_t)'_{abs}(x) - (\mathcal{L}_t\rho_t)'_{abs}(x).$$

Differentiating $(\mathcal{L}_{t+h}\rho_t)'$, we have, for every h,

$$(((\mathcal{L}_{t+h}\rho_t)')_{abs})'(x) = (\mathcal{L}_{t+h}\rho_t)''(x).$$

for m-almost every x. In particular

$$A''_{t,h}(y) = (\mathcal{L}_{t+h}\rho_t)''(y) - (\mathcal{L}_t\rho_t)''(y),$$

for m-almost every y. and

$$A'_{t,h}(x) = \int_0^x (\mathcal{L}_{t+h}\rho_t)'' - (\mathcal{L}_t\rho_t)''\ dm. \tag{104}$$

As we have seen the Ruelle-Perron-Frobenius operator for f_{t+h} is given by

$$(\mathcal{L}_{t+h}\rho_t)(x) = \sum_{f_{t+h}(y)=x} \frac{\rho_t(y)}{|Df_{t+h}(y)|}. \tag{105}$$

Differentiating the equation (105) with respect to x we obtain

$$(\mathcal{L}_{t+h}\rho_t)'(x) = \sum_{f_{t+h}(y)=x} \frac{\rho_t'(y)}{Df_{t+h}(y)|Df_{t+h}(y)|} - \frac{\rho_t(y)D^2f_{t+h}(y)}{|Df_{t+h}(y)|^3}. \tag{106}$$

Now, differentiating the equation (106) with respect to x we obtain

$$\begin{aligned}(\mathcal{L}_{t+h}\rho_t)''(x) = &\sum_{f_{t+h}(y)=x} \left(\frac{\rho_t''(y)}{|Df_{t+h}(y)||Df_{t+h}(y)|^2} - 3\frac{\rho_t'(y)D^2f_{t+h}(y)}{|Df_{t+h}(y)|Df_{t+h}(y)^3}\right)\\ &+ \sum_{f_{t+h}(y)=x}\left(-\frac{\rho_t(y)D^3f_{t+h}(y)}{|Df_{t+h}(y)|Df_{t+h}(y)^3} + 3\frac{\rho_t(y)(D^2f_{t+h}(y))^2}{|Df_{t+h}(y)||Df_{t+h}(y)|^4}\right).\end{aligned}$$

Observe that we can rewrite $(\mathcal{L}_{t+h}\rho_t)''$ as follows

$$\begin{aligned}(\mathcal{L}_{t+h}\rho_t)'' = \mathcal{L}_{t+h}\left(\frac{\rho_t''}{|Df_{t+h}|^2}\right) - 3\mathcal{L}_{t+h}\left(\frac{\rho_t' D^2f_{t+h}}{(Df_{t+h})^3}\right)\\ - \mathcal{L}_{t+h}\left(\frac{\rho_t D^3f_{t+h}}{(Df_{t+h})^3}\right) + 3\mathcal{L}_{t+h}\left(\frac{\rho_t(D^2f_{t+h})^2}{|Df_{t+h}|^4}\right).\end{aligned} \tag{107}$$

We obtain a similar expression for $(\mathcal{L}_t\rho_t)''$.
Substituting Eq. (107) into Eq. (104) we obtain

$$\begin{aligned}A'_{t,h}(x) = &\underbrace{\int_0^x df^\star_{t+h}\left(\frac{\rho_t''}{|Df_{t+h}|^2}m\right) - \int_0^x df^\star_t\left(\frac{\rho_t''}{|Df_t|^2}m\right)}_{A_1}\\ &+ \underbrace{\int_0^x df^\star_{t+h}\left(\frac{-3\rho_t' D^2f_{t+h}}{(Df_{t+h})^3}m\right) - \int_0^x df^\star_t\left(\frac{-3\rho_t' D^2f_t}{(Df_t)^3}m\right)}_{A_2}\\ &+ \underbrace{\int_0^x df^\star_{t+h}\left(\frac{-\rho_t D^3f_{t+h}}{(Df_{t+h})^3}m\right) - \int_0^x df^\star_t\left(\frac{-\rho_t D^3f_t}{(Df_t)^3}m\right)}_{A_3}\\ &+ \underbrace{\int_0^x df^\star_{t+h}\left(\frac{3\rho_t(D^2f_{t+h})^2}{|Df_{t+h}|^4}m\right) - \int_0^x df^\star_t\left(\frac{3\rho_t(D^2f_t)^2}{|Df_t|^4}m\right)}_{A_4}.\end{aligned}$$

Observe that A_i, $1 \le i \le 4$, satisfy the assumptions of Proposition 8.1 and the total variation of each one of the measures that appears above has a upper bound that depends on the constants in Assumption (V) of Definition 3.3. Therefore,

$$\sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{A'_{t,h}}{h}\right|_{L^1(m)} < \infty$$

and, consequently

$$\sup_{t\in[0,1]}\sup_{h\neq 0}\operatorname{var}\left(\frac{A_{t,h}}{h}\right) = \sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{A'_{t,h}}{h}\right|_{L^1(m)} dm < \infty. \tag{108}$$

It remains to verify that the second part of Eq. (103). Note that

$$\left|\frac{A_{t,h}}{h}\right|_{L^1} = \int\left|\frac{A_{t,h}}{h}\right| dm = \int\left|\int_0^x \frac{A'_{t,h}(y)}{h}dy\right| dm \le \left|\frac{A'_{t,h}}{h}\right|_{L^1(m)}.$$

Hence, by Eq. (108), Eq. (103) holds. Hence, we need to show that

$$\sup_{t\in[0,1]}\sup_{h\neq 0}\left|\frac{B_{t,h}}{h}\right|_{BV}<\infty.$$

By Eq. (106) and Property (V) in Definition 3.3 we have

$$\begin{aligned}
&(\mathcal{L}_{t+h}\rho_t)'_{sal}(x)\\
&=\sum_{k=1}^{\infty}\left(\frac{s'_k(t)H_{f_{t+h}(f_t^k(c))}(x)}{Df_{t+h}(f_t^k(c))|Df_{t+h}(f_t^k(c))|}-\frac{s_k(t)H_{f_{t+h}(f_t^k(c))}(x)}{|Df_{t+h}(f_t^k(c))|^3}D^2f_{t+h}(f_t^k(c))\right)\\
&+\left(\frac{\rho'_t(c)}{Df_{t+h}(c-)|Df_{t+h}(c-)|}+\frac{\rho'_t(c)}{Df_{t+h}(c+)|Df_{t+h}(c+)|}\right)H_{f_{t+h}(c)}(x)\\
&-\left(\frac{\rho_t(c)D^2f_{t+h}(c-)}{|Df_{t+h}(c-)|^3}+\frac{\rho_t(c)D^2f_{t+h}(c+)}{|Df_{t+h}(c+)|^3}\right)H_{f_{t+h}(c)}(x).
\end{aligned}$$

Since for every $a\in[0,1]$ we have

$$H_a(x)=\int_0^x\, d(-\delta_a),$$

we can write

$$B_{t,h}(x)=\int_0^x\sum_{i=1}^4 B_i(y)\, dm(y),$$

with functions B_i given by

$$B_1(x)=\int_0^x\, df^\star_{t+h}\left(\frac{1}{Df_{t+h}|Df_{t+h}|}\nu_1\right)-\int_0^x\, df^\star_t\left(\frac{1}{Df_t|Df_t|}\nu_1\right)$$

where

$$\nu_1=\sum_{k=1}^{\infty}s'_k(t)(-\delta_{f_t^k(c)}),$$

$$B_2(x)=-\int_0^x\, df^\star_{t+h}\left(\frac{D^2f_{t+h}}{|Df_{t+h}|^3}\nu_2\right)+\int_0^x\, df^\star_t\left(\frac{D^2f_t}{|Df_t|^3}\nu_2\right),$$

where

$$\nu_2=\sum_{k=1}^{\infty}s_k(t)(-\delta_{f_t^k(c)}).$$

Let $\hat{\psi}$ be the constant borelian function $\hat{\psi}\colon[0,1]\to\mathbb{R}$ given by

$$\hat{\psi}_t(y)=\frac{1}{Df_t(c-)|Df_t(c-)|}+\frac{1}{Df_t(c+)|Df_t(c+)|}.$$

Then

$$B_3(x)=\int_0^x\, df^\star_{t+h}(\hat{\psi}_{t+h}\nu_3)-\int_0^x\, df^\star_t(\hat{\psi}_t\nu_3).$$

where

$$\nu_3=-\rho'_t(c)\delta_c.$$

Let $\tilde{\psi}$ be the constant borelian function $\tilde{\psi}\colon[0,1]\to\mathbb{R}$ given by

$$\tilde{\psi}_t(y)=\frac{D^2f_t(c-)}{|Df_t(c-)|^3}+\frac{D^2f_t(c+)}{|Df_t(c+)|^3}.$$

then

$$B_4(x) = -\int_0^x df^\star_{t+h}(\tilde{\psi}_{t+h}\nu_4) + \int_0^x df^\star_t(\tilde{\psi}_t\nu_4).$$

Here

$$\nu_4 = -\rho_t(c)\delta_c.$$

We can apply Proposition 8.1 on each one of the pairs (B_i, ν_i). Moreover, by property (V) of Definition 3.3 there is a upper bound for the total variation of the measures ν_i, $i = 1, 2, 3, 4$, that holds for every $t \in [0, 1]$. Hence,

$$\sup_{t\in[0,1]} \sup_{h\neq 0} \left|\frac{B_i}{h}\right|_{L^1(m)} < \infty,$$

and consequently

$$\sup_{t\in[0,1]} \sup_{h\neq 0} \operatorname{var}\left(\frac{B_{t,h}}{h}\right) < \infty.$$

Since

$$\left|\frac{B_{t,h}}{h}\right|_{L^1} = \int \left|\frac{B_{t,h}}{h}\right| dm = \int \left|\int_0^x \sum_{i=1}^4 \frac{B_i(y)}{h}\ dy\right| dm \leq \sum_{i=1}^4 |\frac{B_i}{h}|_{L^1(m)},$$

we obtain

$$\sup_{t\in[0,1]} \sup_{h\neq 0} \left|\frac{B_{t,h}}{h}\right|_{BV} < \infty.$$

Therefore,

$$\sup_{t\in[0,1]} \sup_{h\neq 0} \left|\frac{(\mathcal{L}_{t+h}\rho_t)_{abs} - (\mathcal{L}_t\rho_t)_{abs}}{h}\right|_{BV} < \infty.$$

It remains to examine the saltus.

$$\frac{(\mathcal{L}_{t+h}\rho_t)_{sal} - (\mathcal{L}_t\rho_t)_{sal}}{h}$$
$$= \underbrace{\frac{1}{h}\sum_{k=1}^\infty \left(\frac{s_k(t)}{Df_{t+h}(f_t^k(c))} H_{f_{t+h}(f_t^k(c))} - \frac{s_k(t)}{Df_t(f_t^k(c))} H_{f_t(f_t^k(c))}\right)}_{\tilde{S}_1}$$
$$+ \frac{1}{h}\left(\left(\frac{\rho_t(c)}{|Df_{t+h}(c-)|} + \frac{\rho_t(c)}{|Df_{t+h}(c+)|}\right) H_{f_{t+h}(c)} - \left(\frac{\rho_t(c)}{|Df_t(c-)|} + \frac{\rho_t(c)}{|Df_t(c+)|}\right) H_{f_t(c)}\right).$$

Let us analyze $\tilde{S}_1$. Notice that

$$\tilde{S}_1 = \frac{1}{h}\sum_{k=1}^\infty \frac{s_k(t)}{Df_t(f_t^k(c))}\left(H_{f_{t+h}(f_t^k(c))} - H_{f_t(f_t^k(c))}\right)$$
$$+ \underbrace{\frac{1}{h}\sum_{k=1}^\infty \left(\frac{s_k(t)}{Df_{t+h}(f_t^k(c))} - \frac{s_k(t)}{Df_t(f_t^k(c))}\right) H_{f_{t+h}(f_t^k(c))}}_{\tilde{S}_{11}}.$$

Note that

$$\begin{aligned}\left|\tilde{S}_{11}\right|_{BV} &\leq \frac{1}{|h|}\sum_{k=1}^{\infty}|s_k(t)|\left|\frac{1}{Df_{t+h}(f_t^k(c))}-\frac{1}{Df_t(f_t^k(c))}\right|\left|H_{f_{t+h}(f_t^k(c))}\right|_{BV}\\ &\leq \frac{2}{|h|}\sum_{k=1}^{\infty}|s_k(t)|\frac{|Df_{t+h}(f_t^k(c))-Df_t(f_t^k(c))|}{|Df_{t+h}(f_t^k(c))Df_t(f_t^k(c))|}\\ &\leq \frac{K_1}{|h|}\sum_{k=1}^{\infty}\frac{1}{|Df_t^{k-1}(f_t(c))|}|(\partial_s Df_s(f_t^k(c)))|_{s=\theta_{t,h,k}}||h| \leq K.\end{aligned}$$

Hence, $\sup_h \left|\tilde{S}_{11}\right|_{BV} < \infty$. Therefore,

$$\begin{aligned}&\frac{(\mathcal{L}_{t+h}\rho_t)_{sal}-(\mathcal{L}_t\rho_t)_{sal}}{h}\\ &=\frac{1}{h}\sum_{k=1}^{\infty}\frac{s_k(t)}{Df_t(f_t^k(c))}\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right)+\tilde{S}_{11}\\ &+\frac{1}{h}\left(\left(\frac{\rho_t(c)}{|Df_{t+h}(c-)|}+\frac{\rho_t(c)}{|Df_{t+h}(c+)|}\right)H_{f_{t+h}(c)}-\left(\frac{\rho_t(c)}{|Df_t(c-)|}+\frac{\rho_t(c)}{|Df_t(c+)|}\right)H_{f_t(c)}\right)\\ &=\underbrace{\frac{1}{h}\sum_{k=0}^{\infty}s_{k+1}(t)\left(H_{f_{t+h}(f_t^k(c))}-H_{f_t(f_t^k(c))}\right)}_{\tilde{S}}+\tilde{S}_{11}\\ &+\underbrace{\frac{1}{h}\left(\frac{\rho_t(c)}{|Df_{t+h}(c-)|}-\frac{\rho_t(c)}{|Df_t(c-)|}\right)H_{f_{t+h}(c)}}_{\tilde{S}_2}\\ &+\underbrace{\frac{1}{h}\left(\frac{\rho_t(c)}{|Df_{t+h}(c+)|}-\frac{\rho_t(c)}{|Df_t(c+)|}\right)H_{f_{t+h}(c)}}_{\tilde{S}_3}.\end{aligned}$$

We will analize only $\tilde{S}_2$, the term $\tilde{S}_3$ is analogous.

$$\left|\tilde{S}_2\right|_{BV} \leq K_1\frac{1}{|h|}\left|\frac{1}{|Df_{t+h}(c-)|}-\frac{1}{|Df_t(c-)|}\right| \leq \frac{K_2}{|h|}|h| \leq K.$$

Hence,

$$\sup_{h\neq 0}\left|\tilde{S}_2\right|_{BV} < \infty \text{ and } \sup_{h\neq 0}\left|\tilde{S}_3\right|_{BV} < \infty.$$

We can write

$$\begin{aligned}\frac{\mathcal{L}_{t+h}(\rho_t)-\mathcal{L}_t(\rho_t)}{h} &= \Pi_{t+h}\left(\frac{\mathcal{L}_{t+h}(\rho_t)-\mathcal{L}_t(\rho_t)}{h}\right)\\ &= \underbrace{\Pi_{t+h}(\tilde{S})}_{\Phi_h}+\underbrace{\Pi_{t+h}\left(\frac{A}{h}+\frac{B}{h}+\tilde{S}_{11}+\tilde{S}_2+\tilde{S}_3\right)}_{r_h}.\end{aligned}$$

Therefore

$$\int r_h\,dm = 0 \text{ and } \sup_{t\in[0,1]}\sup_{h\neq 0}|r_h|_{BV} < \infty.$$

This finishes the proof. □

9. The function $\mathcal{R}_\phi$ is not Lipschitz on any subset of positive measure

We give two interesting and simple consequences of our main result. They tell us that, under the assumptions of our main result, the function $\mathcal{R}_\phi$ is *not* very regular in *any* subset of the parameter space with positive Lebesgue measure. This show that there is not way to make $\mathcal{R}_\phi$ more regular using some "parameter exclusion" strategy.

Corollary 9.1. *Under the same assumptions of our main result, for every set $\Omega \subset [a,b]$, with $m(\Omega) > 0$, we have for almost every $t \in \Omega$*

$$\limsup_{h\to 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) = +\infty \tag{109}$$

and

$$\liminf_{h\to 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) = -\infty, \tag{110}$$

where $\mathbb{1}_\Omega$ denotes the indicator function of Ω.

Proof. Due Propostition 3.6, it is enough to prove Corollary 9.1 for good transversal families. We are going to prove that Eq. (109) holds for almost every $t \in \Omega$. The proof that Eq. (110) holds for almost every $t \in \Omega$ is similar.

If Eq. (109) fails for t in a subset of Ω with positive Lebesgue measure, then there exist $\hat\Omega \subset \Omega$, with $m(\hat\Omega) > 0$ and $K_1 > 0$ such that for every $t \in \hat\Omega$ we have

$$\limsup_{h\to 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) \le K_1.$$

Since f_t is a good transversal family, without loss of generality we can assume $\inf_t \Psi(t) > 0$, there exists $K_2 > 0$ such that

$$\limsup_{h\to 0+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{\Psi(t)h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) \le K_2$$

for every $t \in \hat\Omega$. Then there exists $h_0 > 0$ and a set $S \subset \hat\Omega$ with $m(S) > 0$ such that for every $t \in S$ we have

$$\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{\Psi(t)h\sqrt{-\log|h|}} \mathbb{1}_\Omega(t+h) \le K_2 + 1$$

for every h satisfying $0 < h \le h_0$. Let $t_0 \in (a,b)$ be a Lebesgue density point of S. Choose $\delta > 0$ such that

$$\mathcal{D}_\mathcal{N}(K_2+1) + \delta < 1.$$

Then for every $\epsilon > 0$ small enough,

$$\frac{m(S\cap I_\epsilon)}{m(I_\epsilon)} > \mathcal{D}_\mathcal{N}(K_2+1) + \delta,$$

where $I_\epsilon = [t_0 - \epsilon, t_0 + \epsilon]$. Let $S_\epsilon = S \cap I_\epsilon$. It is a well-known fact that if

$$S_\epsilon - h = \{t - h \colon t \in S_\epsilon\}$$

then

$$\lim_{h\to 0} m(S_\epsilon \cap (S_\epsilon - h)) = m(S_\epsilon) > 0.$$

Note that for every $t \in S_\epsilon \cap (S_\epsilon - h)$, we have $t, t+h \in S_\epsilon \subset S \subset \Omega$, then

$$\frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{\Psi(t) h \sqrt{-\log |h|}} \leq K_2 + 1$$

for every $0 < h \leq h_0$. In particular

$$\limsup_{h \to 0+} \frac{1}{m(I_\epsilon)}\, m\left(t \in I_\epsilon\colon\ \frac{1}{\Psi(t) h \sqrt{-\log |h|}} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \leq K_2 + 1 \right)$$

$$\geq \frac{m(S_\epsilon)}{m(I_\epsilon)} \geq \mathcal{D}_\mathcal{N}(K_2+1) + \delta. \tag{111}$$

On the other hand the restriction of f_t to the interval I_ϵ is a transversal family, then by Theorem 1.1 we obtain

$$\lim_{h \to 0+} \frac{1}{m(I_\epsilon)}\, m\left(t \in I_\epsilon\colon\ \frac{1}{\Psi(t) h \sqrt{-\log |h|}} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h} \leq K_2 + 1 \right)$$
$$= \mathcal{D}_\mathcal{N}(K_2+1),$$

which contradicts Eq.(111). □

Proof of Corollary 1.2. It follows from Corollary 9.1. □

Remark 9.2. In Baladi and Smania [2][5] it is proven that for almost every $t \in [a,b]$ there exists a sequence $h_n \to 0$ such that

$$\frac{\mathcal{R}_\phi(t+h_n) - \mathcal{R}_\phi(t)}{h_n}$$

is not bounded. In particular $\mathcal{R}_\phi$ is not a lipschitzian function on the whole interval $[a,b]$. Naturally Corollaries 9.1 and 1.2 do not follow from this when Ω is not an interval.

Remark 9.3. Two weeks before this work be completed, Fabián Contreras sent us his Ph. D. Thesis [7] where he proves a result sharper than Corollary 9.1 when $\Omega = [a,b]$ and ϕ is a C^1 generic observable. He proves that for almost every $t \in [a,b]$ the limit

$$\lim_{h \to 0^+} \frac{\mathcal{R}_\phi(t+h) - \mathcal{R}_\phi(t)}{h\sqrt{|\log h \log\log|\log h||}} \tag{112}$$

exists and it is non zero. Note again that Corollaries 9.1 and 1.2 do not seem to follow from his result when Ω is not an interval.

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