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Marshall's Quotient and the Arason–Pfister Hauptsatz for Reduced Special Groups

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Abstract

We provide a new proof of the Arason–Pfister Hauptsatz (APH) in the setting of reduced special groups, as developed by Dickmann and Miraglia. Our approach avoids the use of Boolean invariants and instead relies on a construction inspired by Marshall's quotient, suitably adapted to the context of special groups. We establish structural properties of this quotient and show that it generalizes the Pfister quotient by a Pfister subgroup. Using this framework, we define iterated quadratic extensions of special groups and develop a theory of Arason–Pfister sequences. These tools allow us to prove that any anisotropic form $\varphi \in I^n(G)$ over a reduced special group G satisfies the inequality $\dim(\varphi) \geq 2^n$, where $I^n(G)$ denotes the n-th power of the fundamental ideal of the Witt ring of G. Our methods are purely algebraic and internal to the theory of special groups, contributing with novel tools to the categorical study of abstract theories of quadratic forms.

Keywords: Quadratic forms; special groups; Arason–Pfister Hauptsatz; Marshall's Quotient

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1. Introduction

It can be said that the Algebraic Theory of Quadratic Forms was founded in 1937 by E. Witt, with the introduction of the concept of the Witt ring of a given field, constructed from the quadratic forms with coefficients in the field: given F, an arbitrary field of characteristic $\neq 2$, $W(F)$, the Witt ring of F, classifies the quadratic forms over F that are regular and anisotropic, being in one-to-one correspondence with them; thus, the focus of the theory is the quadratic forms defined on the ground field where all their coefficients are invertible. In this way, the set of orders in F is in one-to-one correspondence with the set of minimal prime ideals of the Witt ring of F, and more, the set of orders in F provided with the Harrison's topology is a Boolean topological space that, by the bijection above, is identified with a subspace of the Zariski spectrum of the Witt ring of F.

Questions about the structure of Witt rings $W(F)$ could only be solved about three decades after Witt's original idea, through the introduction and analysis of the concept of Pfister form. The Pfister forms of degree $n \in \mathbb{N}$, in turn, are generators of the power $I^n(F)$ of the fundamental ideal $I(F) \subseteq W(F)$ (the ideal determined by the anisotropic forms of even dimension).

One of the most emblematic results in the algebraic and abstract theories of quadratic forms is the so-called Arason–Pfister Hauptsatz (APH), whose historical context is as given below.

In his seminal 1970 paper [1], John Milnor posed two central problems related to fields of characteristics different from 2—both of which were positively resolved in many cases within the same work. One of the questions was related to the so-called Milnor conjectures for the graded cohomology ring and the graded Witt ring, which were eventually resolved by Voevodsky and collaborators around 2000. The other question asked whether for every such field F, the intersection $\bigcap_{n\in\mathbb{N}} I^n(F)$ contains only $0 \in W(F)$, where $I^n(F)$ is the n-th power of the fundamental ideal $I(F)$ of the Witt ring of F ($I(F) = \{$even-dimensional anisotropic forms over the field $F\}$).

In the subsequent year, J. Arason and A. Pfister solved this question as an immediate Corollary of what is now known as "Arason–Pfister Hauptsatz" (APH), as stated in [2]:

Let $\phi \neq \emptyset$ be an anisotropic form. If $\phi \in I^n(F)$, then $dim(\phi) \geq 2^n$.

The theory of special groups, an abstract (first-order) theory of quadratic forms developed by Dickmann and Miraglia in the mid-1990s, allows a functorial encoding of the algebraic theory of quadratic forms of fields (with char $\neq 2$). This theory has proven effective in attacking and resolving various open problems involving the theory of quadratic forms, see, e.g., [3,4].

In [5], Dickmann–Miraglia restated the APH to the setting of special groups and, employing Boolean-theoretic methods to define and calculate the Stiefel–Whitney and the Horn–Tarski invariants of a special group, established a generalization of the APH to the setting of **reduced** special groups, in particular providing an alternative proof of the APH for formally real Pythagorean fields.

The difficulty in attacking APH for general special groups lies in the fact that the methods available for reduced special groups (the invariants) and for special groups arising from fields (quadratic and transcendental extensions, valuations and so on) do not admit a clear generalization for the class of all special groups.

In the present work, we provide a new proof of the Arason–Pfister Hauptsatz for reduced special groups. More specifically, we prove the content of Theorem 12:

Theorem 1 (Arason–Pfister Hauptsatz)**.** *Let G be a **reduced** special group, then $AP_G(n)$ holds for all $n \geq 0$. In more detail: for each $n \geq 0$ and each non-zero ($k \geq 1$), regular and anisotropic form $\varphi = \langle a_1, \ldots, a_k \rangle$, if $\varphi \in I^n(G)$, then $\dim(\varphi) = k \geq 2^n$.*

Our proof completely avoids the use of the Horn–Tarski and Stiefel–Whitney invariants developed in [3]. Instead, we use Marshall's quotient, inspired by the techniques developed by Murray Marshall in [6] for multirings. We finish this short paper by pointing out the difficulties in dealing with the general case and providing some perspectives on how to attack the general problem.

2. Preliminaries

For the benefit of the reader, we provide some basic definitions and results concerning the theory of special groups.

Definition 1 (Extension of a Relation)**.** *Let A be a set and $\equiv$ a binary relation on $A \times A$. We extend $\equiv$ to a binary relation $\equiv_n$ on A^n, by induction on $n \geq 1$, as follows:*

(i) $\equiv_1$ *is the diagonal relation* $\Delta_A \subseteq A \times A$

(ii) $\equiv_2 = \equiv$.

(iii) *if* $n \geq 3$, $\langle a_1, \ldots, a_n \rangle \equiv_n \langle b_1, \ldots, b_n \rangle$ *if and only there are* $x, y, z_3, \ldots, z_n \in A$ *such that*

$$\begin{aligned} &\langle a_1, x \rangle \equiv \langle b_1, y \rangle \\ &\langle a_2, \ldots, a_n \rangle \equiv_{n-1} \langle x, z_3, \ldots, z_n \rangle \text{ and} \\ &\langle b_2, \ldots, b_n \rangle \equiv_{n-1} \langle y, z_3, \ldots, z_n \rangle \end{aligned}$$

Whenever clear from the context, we frequently abuse notation and indicate the aforementioned extension $\equiv$ by the same symbol.

Definition 2 (Special Group, 1.2 of [5])**.** *A* ***special group*** *is a tuple* $(G, -1, \equiv)$, *where* G *is a group of exponent 2, i.e.,* $g^2 = 1$ *for all* $g \in G$; -1 *is a distinguished element of* G, *and* $\equiv \subseteq G \times G \times G \times G$ *is a relation (the special relation), satisfying the following axioms for all* $a, b, c, d, x \in G$:

SG0 $\equiv$ *is an equivalence relation on* G^2;

SG1 $\langle a, b \rangle \equiv \langle b, a \rangle$;

SG2 $\langle a, -a \rangle \equiv \langle 1, -1 \rangle$;

SG3 $\langle a, b \rangle \equiv \langle c, d \rangle$ *imply* $ab = cd$;

SG4 $\langle a, b \rangle \equiv \langle c, d \rangle$ *imply* $\langle a, -c \rangle \equiv \langle -b, d \rangle$;

SG5 *For all* $g \in G$, $\langle a, b \rangle \equiv \langle c, d \rangle$ *imply* $\langle ga, gb \rangle \equiv \langle gc, gd \rangle$.

SG6 (3-transitivity) *the extension of* $\equiv$ *for a binary relation on* G^3 *(as in 1) is a transitive relation.*

Definition 3 (1.1 of [5])**.** *A map* $(G, \equiv_G, -1) \xrightarrow{f} (H, \equiv_H, -1)$ *between pre-special groups is a* ***morphism*** *of pre-special groups or PSG-morphism if* $f : G \to H$ *is a homomorphism of groups,* $f(-1) = -1$ *and for all* $a, b, c, d \in G$

$$\langle a, b \rangle \equiv_G \langle c, d \rangle \Rightarrow \langle f(a), f(b) \rangle \equiv_H \langle f(c), f(d) \rangle$$

A ***morphism*** *of special groups or SG-morphism is a PSG-morphism between the corresponding pre- special groups.* f *will be an isomorphism if the function* f *is bijective and both* f, f^{-1} *are PSG-morphisms.*

A special group, G, is formally acknowledged if it admits some SG-morphism $f : G \to 2$. The category of special groups, respectively, reduced special groups, and their morphisms will be denoted by SG and RSG, respectively.

Example 1 (The trivial special relation, 1.9 of [5])**.** *Let* G *be a group of exponent 2 and take* -1 *as any element of* G *different of 1. For* $a, b, c, d \in G$, *define* $\langle a, b \rangle \equiv_t \langle c, d \rangle$ *if and only if* $ab = cd$. *Then,* $G_t = (G, \equiv_t, -1)$ *is an SG [5]. In particular,* $2 = \{-1, 1\}$ *is a reduced special group.*

Example 2 (Special group of a field, Theorem 1.32 of [5])**.** *Let* F *be a field. We denote* $\dot{F} = F \setminus \{0\}$, $\dot{F}^2 = \{x^2 : x \in \dot{F}\}$ *and* $\Sigma\dot{F}^2 = \{\sum_{i \in I} x_i^2 : I \text{ is finite and } x_i \in \dot{F}^2\}$. *Let* $G(F) = \dot{F}/\dot{F}^2$. *In the case of* F, *take* $\Sigma\dot{F}^2$ *as a subgroup of* $\dot{F}$, *then* $G_{red}(F) = \dot{F}/\Sigma\dot{F}^2$. *Note that* $G(F)$ *and* $G_{red}(F)$ *are groups of exponent 2. In [5] they prove that* $G(F)$ *and* $G_{red}(F)$ *are special groups with the special relation given by usual notion of isometry, and* $G_{red}(F)$ *is always reduced.*

A group of exponent 2, with a distinguished element -1, satisfying the axioms SG0–SG3 and SG5 is called a **proto-special group**; a **pre-special group** is a proto-special group that also satisfies SG4. Thus, a **special group** is a pre-special group that satisfies SG6 (or, equivalently, for each $n \geq 1$, $\equiv_n$ is an equivalent relation of G^n).

A n**-form** (or form of dimension $n \geq 1$) is an n-tuple of elements of a pre-special G. If $\varphi = \langle a_1, a_2, \ldots, a_k\rangle$ and $\psi = \langle b_1, b_2, \ldots, b_n\rangle$, then $\varphi \oplus \psi := \langle a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n\rangle$ and $\varphi \otimes \psi := \langle a_1b_1, \ldots, a_1b_n, a_2b_1, \ldots, a_2b_n, \ldots, a_kb_1, \ldots, a_kb_n\rangle$.

An element $b \in G$ is **represented** on G by the form $\varphi = \langle a_1, \ldots, a_n\rangle$, in symbols $b \in D_G(\varphi)$, if there exists $b_2, \ldots, b_n \in G$ such that $\langle b, b_2, \ldots, b_n\rangle \equiv \varphi$. A pre-special group (or special group) $(G, -1, \equiv)$ is:

- **formally real** if $-1 \notin \bigcup_{n\in\mathbb{N}} D_G(n\langle 1\rangle)$ (Here, the notation $n\langle 1\rangle$ means the form $\langle a_1, \ldots, a_n\rangle$ where $a_j = 1$ for all $j = 1, \ldots, n$. In other words, $n\langle 1\rangle$ is the form $\langle 1, \ldots, 1\rangle$ with n entries equal to 1);
- **reduced** if it is formally real and, for each $a \in G$, $a \in D_G(\langle 1, 1\rangle)$ if $a = 1$.

Definition 4 (2.1 of [5])**.** *Let G be a special group. A Pfister form over G is a quadratic form φ of the type $\otimes_{i=1}^{n}\langle 1, a\rangle_i = \langle\langle a_1, \ldots, a_n\rangle\rangle$, where $n \geq 1$ and $a_1, \ldots, a_n \in G$, or the form $\langle 1\rangle$, if $n = 0$. The integer n is called the degree of φ and written* $\deg(\varphi)$*. If the coefficients of φ happen to belong to a subgroup Δ of G, we say that φ is Pfister over Δ.*

Since the Pfister form φ contains 1 as a coefficient, we may write φ as $\langle 1\rangle \oplus \varphi'$. The subform φ' is called the pure subform of φ.

Proposition 1 (Basic properties of Pfister forms (2.2 of [5]))**.** *Let G be a special group, $\varphi = \langle\langle a_1, \ldots, a_n\rangle\rangle$ a Pfister form over G of degree $n \geq 1$ and $b \in G$. Then,*

(i) $b \in D_G(1, a_1) \Rightarrow \langle\langle a_1, a_2\rangle\rangle \equiv_G \langle\langle a_1, a_2b\rangle\rangle$.
(ii) $b \in D_G(a_1, a_2) \Rightarrow \langle\langle a_1, a_2\rangle\rangle \equiv_G \langle\langle b, a_1a_2\rangle\rangle$.
(iii) $\langle\langle a_1b, \ldots, a_nb\rangle\rangle \equiv_G \langle\langle 1, a_1b\rangle\rangle \otimes \langle\langle a_1a_2, \ldots, a_1a_n\rangle\rangle$.
(iv) If $b \in D_G(\varphi')$, then $\varphi \equiv_G \langle\langle b, b_2, \ldots, b_n\rangle\rangle$, with $b_2, \ldots, b_n \in G$.
(v) An isotropic Pfister form is hyperbolic.
(vi) $D_G(\varphi) = \{x \in G : x\varphi \equiv_G \varphi\}$. Hence, $D_G(\varphi)$ is a subgroup of G. If ψ is a Pfister form over G, then $D_G(\varphi)D_G(\psi) \subseteq D_G(\varphi \otimes \psi)$.
(vii) If $a \in D_G(\varphi)$, then $\langle\langle a_1, \ldots, a_n, b\rangle\rangle \equiv_G \langle\langle a_1, \ldots, a_n, ab\rangle\rangle$.
(viii) $a \in D_G(\varphi) \Rightarrow \langle 1, a\rangle \otimes \varphi \equiv_G 2 \otimes \varphi$ and $\langle 1, -a\rangle \otimes \varphi$ is hyperbolic.
(ix) $a \in D_G(\varphi)$ and $b \in D_G(1, a) \Rightarrow b \in D_G(2 \otimes \varphi)$.
(x) $\langle 1, a\rangle \otimes \varphi \equiv_G 2 \otimes \varphi \Rightarrow a \in D_G(\varphi)$.
(xi) $\langle 1, -a\rangle \otimes \varphi$ hyperbolic $\Rightarrow a \in D_G(\varphi)$.
(xii) The following are equivalent:

(a) G is a reduced special group.
(b) $1 \neq -1$ and for every Pfister form φ over G of degree ≥ 1 and $a \in G$:

$$a, -a \in D_G(\varphi) \Rightarrow \varphi \text{ hyperbolic}.$$

(c) $1 \neq -1$ and for every Pfister form φ over G and $a \in G$

$$a \in D_G(\langle 1, -a\rangle \otimes \varphi) \Rightarrow a \in D_G(\varphi).$$

Remark 1 (Notations and Facts)**.** *Let G be a fixed special group. Here, we summarize some notations and results about Witt equivalence, Witt ring, and the powers of fundamental ideal of the Witt ring. For more details, see for instance [5,7–9].*

- *Let φ, ψ be forms on G. We can say that φ and ψ are **Witt equivalent**, denoted $\varphi \approx_{W,G} \psi$, if there exist non negative integers k, l such that*

$$k\langle 1, -1\rangle \oplus \varphi \equiv_G l\langle 1, -1\rangle \oplus \psi.$$

*By **Witt's Decomposition Theorem**, if φ is a form on G, there are unique forms $\varphi_{an}, \varphi_{hip}, \varphi_0$ (up to isometry) with $\varphi \equiv \varphi_{an} \oplus \varphi_{hip} \oplus \varphi_0$, φ_{an} anisotropic, φ_{hip} hyperbolic and φ_0 totally isotropic. We define $\dim_{W,G}(\varphi) := \dim(\varphi_{an})$.*

- *The **Witt ring** of G, $W(G)$, is the set of equivalence classes of forms modulo $\approx_{W,G}$, with sum and product endowed form $\oplus$ and $\otimes$, respectively. The **fundamental ideal**, $I(G) \subseteq W(G)$, is the ideal determined by even dimensional anisotropic forms.*
- *For each $n \in \mathbb{N}$ consider the statement:*
  $AP_G(n)$: For each $\varphi = \langle a_1, \cdots, a_k \rangle$, a non-empty ($k \geq 1$), regular ($a_i \in G$) and anisotropic form, if $\varphi \in I^n(G)$, then $\dim(\varphi) \geq 2^n$.
- *Let ψ be a Pfister form. Then, ψ is hyperbolic if it is isotropic. Moreover, if G is reduced and $-1 \in D_G(\psi)$, then ψ is hyperbolic.*
- *$I^n(G) \subseteq W(G)$ is additively generated by the Pfister forms of degree n.*
- *If $\varphi \in I^n(G) \setminus \{\emptyset\}$, then $\varphi = \varepsilon_1\varphi_1 + \ldots + \varepsilon_r\varphi_r$, where $r \geq 1$ and $\varepsilon_j = \pm 1$ for all $j = 1, \ldots, r$. Moreover, if φ is anisotropic, we will suppose without loss of generality that $\varepsilon_j = 1$ for all $j = 1, \ldots, r$.*
- *Let $\varphi \in I^nG$ with $\varphi = \varphi_1 + \ldots + \varphi_r$ for suitable Pfister forms $\varphi_1, \ldots, \varphi_r$. If φ is anisotropic, then for all positive integers m with $1 \leq m \leq r$ and all $\sigma \in S_m$ the form $\varphi_{\sigma(1)} + \ldots + \varphi_{\sigma(m)}$ is anisotropic.*

3. Marshall's Quotient of a Special Group

Here, we present the main ingredient of the paper, that are inspired by the techniques developed by Murray Marshall in [6] in the setting of multirings.

Let G be a special group and let $\emptyset \neq S \subseteq G$ be such that $S \cdot S \subseteq S$. Denote

$$a \sim b \text{ if } ar = bs \text{ for some } r, s \in S.$$

Note that, since every special group is an exponent 2 group, the condition $S \neq \emptyset$ and $S \cdot S \subseteq S$ implies that S is a subgroup of G.

Lemma 1. *Under the above circumstances, the relation $\sim$ is an equivalence relation.*

Proof. Let $a, b, c \in G$. Of course $\sim$ is reflexive and symmetric as a consequence of reflexivity and symmetry of equality. Now, suppose $a \sim b$ and $b \sim c$, saying $ar = bs$ and $bv = cw$ with $r, s, v, w \in S$. Then, $ars = b$ and $b = cwv$ imply $a(rs) = c(vw)$ with $rs, vw \in S$. Then, $\sim$ is transitive. □

Elements in $G/_mS$ will be denoted by $[a] \in G/_mS$, $a \in S$. Then, $[a] = [b]$ means $a \sim b$ and we denote

$$G/_mS := \{[a] : a \in G\}.$$

We call $G/_mS$ **the Marshall quotient of** G **by** S.

Lemma 2. *The set $G/_mS = \{[a] : a \in G\}$ is a group with the operation inherited from G.*

Proof. Let $a, a', b, b' \in S$ with $[a] = [a']$ and $[b] = [b']$, saying $ar_1 = a's_1$ and $br_2 = b's_2$. Then

$$(ab)(r_1r_2) = (ar_1)(br_2) = (a's_1)(b's_2) = (a'b')(r_2s_2).$$

Then, $[ab] = [a'b']$. Moreover, we have an operation in $G/_mS$ given by the rule $[a][b] := [ab]$. The fact that this operation provides a group structure on $G/_mS$ is an immediate consequence of the group properties holding in G. □

Our next step is to define a structure of special group over $G/_mS$. Note that if $-1 \in S$, then $G/_mS \cong \{[1]\}$. Thus, we may always suppose that $-1 \notin S$.

We proceed by steps. First, for $[a], [b] \in G/_mS$ define

$$D_{G/_mS}([a],[b]) := \{[d] : d \in D_G(as, bt) \text{ for some } s, t \in S\}$$

and

$$\langle [a],[b]\rangle \equiv_{G/_mS} \langle [c],[d]\rangle \text{ if } [ab] = [cd] \text{ and } [ac] \in D_{G/_mS}([1],[cd]).$$

Theorem 2. *With the above notations, $(G/_mS, \cdot, \equiv_{G/_mS}, -[1])$ satisfies the axioms [SG1]–[SG5] of Definition 2.*

Proof. We verify axioms [SG1]–[SG5].

SG1 $\langle [a],[b]\rangle \equiv_{G/_mS} \langle [b],[a]\rangle$ follows directly from the definition of $\equiv_{G/_mS}$.

SG2 Since $a \in D_G(1,-1)$ for all $a \in G$ and $1 \in S$, it follows that $[a] \in D_{G/_mS}([1],[-1])$, hence $\langle [a], -[a]\rangle \equiv \langle [1], -[1]\rangle$.

SG3 $\langle [a],[b]\rangle \equiv \langle [c],[d]\rangle \Rightarrow [ab] = [cd]$ follows directly from the definition of $\equiv_{G/_mS}$.

SG4 Let $\langle [a],[b]\rangle \equiv_{G/_mS} \langle [c],[d]\rangle$. Then

$$ab = cdt \text{ and } ac \in D_G(r, cds) \text{ for some } r, s, t \in S.$$

In particular, $a(-c) = (-b)dt$. Moreover, $ac \in D_G(r, cds) = D_G(r, abst)$ imply, by axiom SG4 for G, that $-abst \in D_G(r, -ac) = D_G(r, -bst)$, with $st, r \in S$, which proves that $\langle [a], -[c]\rangle \equiv_{G/_mS} \langle -[b], [d]\rangle$.

SG5 Since

$$ab = cdt \text{ and } ac \in D_G(r, cds) \text{ for some } r, s, t \in S$$

provide

$$(ag)(bg) = (cg)(dg)t \text{ and } (ag)(cg) \in D_G(r, (cg)(dg)s) \text{ for some } r, s, t \in S.$$

we obtain that $\langle [a],[b]\rangle \equiv_{G/_mS} \langle [c],[d]\rangle$ imply $\langle [ag],[bg]\rangle \equiv_{G/_mS} \langle [cg],[dg]\rangle$ for all $g \in G$.

□

Of course, the relation $\equiv_{G/_mS}$ is transitive and symmetric: Let $[a],[b],[c],[d],[e],[f] \in G/_mS$. Since $1 \in D_G(1, ab)$, we have

$$[aa] = [1] \in D_{G/_mS}([1],[ab]),$$

which means that $\langle [a],[b]\rangle \equiv_{G/_mS} \langle [a],[b]\rangle$ ($\equiv_{G/_mS}$ is reflexive). Now, let $\langle [a],[b]\rangle \equiv_{G/_mS} \langle [c],[d]\rangle$. This means that $[ab] = [cd]$ and $[ac] \in D_{G/_mS}([1],[cd])$. Then, $ab = cdt$ and $ac \in D_G(r, cds)$ for suitable $r, s, t \in S$. Hence, we get

$$cd = abt \text{ and } ca \in D_G(r, abts) \text{ with } r, s, ts \in S.$$

This means that $\langle [c],[d]\rangle \equiv_{G/_mS} \langle [a],[b]\rangle$ ($\equiv_{G/_mS}$ is symmetric).

Unfortunately, axioms [SG0] (the transitive condition) and [SG6] do not hold for a general S.

Since our main goal here is to discuss what should be a quadratic extension for special groups (and not deal with the most general quotient available), we christen the following Definition .

Definition 5. *Let G be a special group and $S \subseteq G$. We say that S is a* ***Dickmann–Miraglia subset*** *of G (or DM-subset for short) if $S \cdot S \subseteq S$ and the above described structure on $(G/_m S, \equiv_{G/_m S}, [-1])$ provides a special group.*

The terminology "Dickmann–Miraglia" subset suits two purposes: (1) it pays homage to professors Maximo Dickmann and Francisco Miraglia, the creators of the special group theory; (2) it makes the notation coherent with other papers in the area, for example, [10]. Moreover, the Axioms [DM 0]–[DM 3] in [10] (in the language of hyperfields) provides a general description of DM-subsets.

The next step is to investigate the relations between the Pfister quotient defined and developed in Chapter 2 of [5] and the Marshall's quotient. We recover some terminology and results from Chapter 2 of [5].

Definition 6 (2.15 of [5])**.** *Let G be a special group. A collection $\mathcal{S}$ of Pfister forms is said to be (upward) directed if for every $\varphi, \psi \in \mathcal{S}$, there exists $\theta \in \mathcal{S}$ such that $D_G(\varphi), D_G(\psi) \subseteq D_G(\theta)$.*

A subgroup Δ of G is a ***Pfister subgroup*** *if there is a directed family $\mathcal{S}$ of Pfister forms over G such that $\Delta = \bigcup\{D_G(\varphi) : \varphi \in \mathcal{S}\}$.*

Note that if φ is a Pfister form, then $D_G(\varphi)$ is a Pfister subgroup, as $\mathcal{S} = \{\varphi\}$ is directed.

Proposition 2 (2.18 of [5])**.** *Let G be a special group and Δ a Pfister subgroup of G, $\Delta = \bigcup\{D_G(\varphi) : \varphi \in \mathcal{S}\}$, $\mathcal{S}$ a directed family of Pfister forms. For $a, b, c, d \in G$, the following are equivalent:*

(a) $\langle a/\Delta, b/\Delta\rangle \equiv^*_{G/\Delta} \langle c/\Delta, d/\Delta\rangle$.

(b) *There is a $\varphi \in \mathcal{S}$ such that $\langle a, b\rangle \otimes \varphi \equiv_G \langle c, d\rangle \otimes \varphi$.*

Lemma 3 (2.19 of [5])**.** *Let G be a special group and φ a Pfister form over G.*

(a) *For $a, b, c, d \in G$, the following are equivalent:*
   (i) $\langle a, b\rangle \otimes \varphi \equiv_G \langle c, d\rangle \otimes \varphi$.
   (ii) *There are $a', b', c', d' \in G$ such that $aa', bb', cc', dd' \in D_G(\varphi)$ and $\langle a', b'\rangle \equiv_G \langle c', d'\rangle$.*

(b) *Conditions (i) or (ii) imply $abcd \in D_G(\varphi)$.*

Lemma 4 (2.20 of [5])**.** *Let G be a special group and let φ_1, φ_2 be anisotropic Pfister forms over G, such that $D_G(\varphi_1) \subseteq D_G(\varphi_2)$. Then, for all forms ψ, θ over G,*

$$\psi \otimes \varphi_1 \equiv_G \theta \otimes \varphi_1 \Rightarrow \psi \otimes \varphi_2 \equiv_G \theta \otimes \varphi_2.$$

Proposition 2 with Lemmas 3 and 4, yield

Proposition 3 (2.21 of [5])**.** *Let G be a special group and Δ a Pfister subgroup of G, determined by the directed family $\mathcal{S}$ of Pfister forms over G. Then, $(G/\Delta, \equiv^*_{G/\Delta}, -1/\Delta)$ is a special group, and the quotient map $\pi : G \to G/\Delta$ is a morphism of special groups. Further, $1 \neq -1$ in G/Δ if $-1 \notin \Delta$. Moreover, in this situation we have*

(a) *If φ, ψ are n-forms in G, then $\pi \star \varphi \equiv^*_{G/\Delta} \pi \star \psi$ if there is a Pfister form $\mathcal{P}$ in $\mathcal{S}$ such that $\varphi \otimes \mathcal{P} \equiv_G \psi \otimes \mathcal{P}$.*

(b) *If $f : G \to H$ is a morphism of special groups satisfying $\Delta \subseteq Ker(f)$, then there is a unique SG-morphism $\hat{f} : G/\Delta \to H$ such that $f = \hat{f} \circ \pi$.*

Concerning the Marshall's quotient under Pfister subgroups, we have a similar result to the Pfister quotients in [5].

Theorem 3. *Let G be a special group and $\Delta \subseteq G$ be a Pfister subgroup.*

$$\Delta = \bigcup \{D_G(\varphi) : \varphi \in \mathcal{S}\}$$

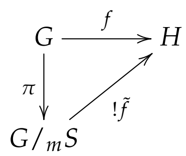
where $\mathcal{S}$ is a directed family of Pfister forms. Then,

$$[\varphi] \equiv_{G/_m S} [\psi] \text{ if there is a Pfister form } \theta \in \mathcal{S} \text{ such that } \theta \otimes \varphi \equiv_G \theta \otimes \psi. \quad (1)$$

In particular $(G/_m S, \equiv_{G/_m S}, -[1])$ is a special group.

Proof. What remains is to check the Axioms [SG0] and [SG6]. Transitivity of the relation $\equiv_{G/_m S}$ follows from Lemma 4, using Proposition 2. Likewise, Axiom [SG6] is an immediate consequence of Equation (1), which is proven by induction on n, using Lemma 4 and Proposition 2. □

Theorem 4 (Universal Property of Marshall's Quotient)**.** *Let G, H be special groups and $S \subseteq A$ a DM-subset of G. Then, for every SG-morphism $f : G \to H$ such that $f[S] = \{1\}$, there exists a unique morphism $\tilde{f} : G/_m S \to H$ such that the following diagram commute:*



where $\pi : G \to G/_m S$ is the canonical projection $\pi(a) = [a]$.

Proof. Note that if $[a] = [b]$, saying $ar = bs$, we have (under the hypothesis that $f[S] = \{1\}$) that

$$f(a) = f(a)f(r) = f(ar) = f(bs) = f(b)f(s) = f(b).$$

Then we are able to define $\tilde{f}([a]) := f(a)$. It is straightforward to prove that $\tilde{f}$ is the unique morphism such that $f = \tilde{f} \circ \pi$. □

Using the Universal Properties of Marshall's Quotient (Theorem 4) and Pfister's Quotient (Proposition 2.21, Chapter 2 of [5]) we obtain the following.

Theorem 5. *Let G be a special group and $\Delta \subseteq G$ be a Pfister subgroup of G. Then,*

$$G/\Delta \cong G/_m \Delta.$$

We have an interesting (and non obvious) consequence of the Universal Property.

Proposition 4. *Let G be a special group and $S_1, S_2 \subseteq G$ be Pfister subgroups of G with $S_1 \subseteq S_2$. Then, there exists a unique surjective morphism $G/_m S_1 \to G/_m S_2$ in the sense of Universal Property.*

Proof. Let $\pi_1 : G \to G/_m S_1$ and $\pi_2 : G \to G/_m S_2$ be the quotient morphisms. Since

$$\pi_2[S_1] \subseteq \pi_2[S_2] = \{[1]\},$$

the Universal property Theorem 4 provides a unique morphism $\varphi : G/_m S_1 \to G/_m S_2$ such that $\pi_2 = \varphi \circ \pi_1$. Since both π_1 and π_2 are surjective, it follows that φ is also surjective. □

The following was unexpected: for $S_1 \subseteq S_2$, it was desired to obtain an injective morphism $G/_m S_2 \to G/_m S_1$, but Proposition 4 provides a morphism in the reverse direction.

4. Quadratic Extensions of Special Groups

Let G be a special group and let $\delta \in G$. Note that $D_G(1,\delta) = D_G(\langle\langle\delta\rangle\rangle)$ is a Pfister subgroup of G. We define

$$G(\sqrt{\delta}) := G/_m D_G(1,\delta).$$

Remark 2. *By Theorem 3, we have*

$$[\varphi] \equiv_{G(\sqrt{\delta})} [\psi] \text{ if } \langle 1,\delta\rangle \otimes \varphi \equiv_G \langle 1,\delta\rangle \otimes \psi. \tag{2}$$

Theorem 6. *Let G be a special group and $\alpha, \beta \in G$. Then,*

$$[G(\sqrt{\alpha})](\sqrt{\beta}) \cong [G(\sqrt{\beta})](\sqrt{\alpha}).$$

Proof. Here, we will deal with many quotients. In order to simplify the proof, let us denote the following:

$$\begin{aligned}
G(\sqrt{\alpha}) &:= G/_m D_G(1,\alpha) = \{[g]_\alpha : g \in G\} \\
G(\sqrt{\beta}) &:= G/_m D_G(1,\beta) = \{[g]_\beta : \beta \in G\} \\
[G(\sqrt{\alpha})](\sqrt{\beta}) &= G(\sqrt{\alpha})/_m D_{G(\sqrt{\alpha})}([1]_\alpha, [\beta]_\alpha) = \{[g]_{\alpha,\beta} : [g]_\alpha \in G(\sqrt{\alpha})\} \\
[G(\sqrt{\beta})](\sqrt{\alpha}) &= G(\sqrt{\beta})/_m D_{G(\sqrt{\beta})}([1]_\beta, [\alpha]_\beta) = \{[g]_{\beta,\alpha} : [g]_\beta \in G(\sqrt{\beta})\}.
\end{aligned} \tag{3}$$

In this sense, the quotient morphisms $\pi_1 : G(\sqrt{\alpha}) \to [G(\sqrt{\alpha})](\sqrt{\beta})$ and $\pi_2 : G(\sqrt{\beta}) \to [G(\sqrt{\beta})](\sqrt{\alpha})$ are given, respectively, by $\pi_1([g]_\alpha) = [g]_{\alpha,\beta}$ and $\pi_2([g]_\beta) = [g]_{\beta,\alpha}$.

We have morphisms $q_1 : G(\sqrt{\alpha}) \to [G(\sqrt{\beta})](\sqrt{\alpha})$ and $q_2 : G(\sqrt{\beta}) \to [G(\sqrt{\alpha})](\sqrt{\beta})$ given, respectively, by the rules $q_1([g]_\alpha) = [g]_{\beta,\alpha}$ and $q_2([g]_\beta) = [g]_{\alpha,\beta}$. We check for q_1; the case for q_2 is analogous. Let $g, h \in G$ with $[g]_\alpha = [h]_\alpha$. Then, $gr = hs$ for some $r, s \in D_G(1,\alpha)$ which imply (after application of the morphism π_2) that $[g]_\beta[r]_\beta = [h]_\beta[s]_\beta$ with $[r]_\beta, [s]_\beta \in D_{G(\sqrt{\beta})}([1]_\beta, [\alpha]_\beta)$. In particular, $[r]_{\beta,\alpha} = [s]_{\beta,\alpha} = [1]_{\beta,\alpha}$. Then

$$\begin{aligned}
[g]_{\beta,\alpha} &= [g]_{\beta,\alpha}[1]_{\beta,\alpha} = [g]_{\beta,\alpha}[r]_{\beta,\alpha} = [gr]_{\beta,\alpha} \\
&= [hs]_{\beta,\alpha} = [h]_{\beta,\alpha}[s]_{\beta,\alpha} = [h]_{\beta,\alpha}[1]_{\beta,\alpha} = [h]_{\beta,\alpha}.
\end{aligned}$$

Then, the rule $[g]_\alpha \mapsto [g]_{\beta,\alpha}$ in fact defines a function $q_1 : G(\sqrt{\alpha}) \to [G(\sqrt{\beta})](\sqrt{\alpha})$. The fact that q_1 is a morphism follows directly then.

Hence, we have a morphism $q_1 : G(\sqrt{\alpha}) \to [G(\sqrt{\beta})](\sqrt{\alpha})$ with $q_1[D_{G(\sqrt{\alpha})}([1]_\alpha, [\beta]_\alpha)] = \{[1]_{\beta,\alpha}\}$. The Universal Property Theorem 4 provides a unique morphism $q_{\alpha,\beta} : [G(\sqrt{\alpha})](\sqrt{\beta}) \to [G(\sqrt{\beta})](\sqrt{\alpha})$ such that $q_1 = q_{\alpha,\beta} \circ \pi_1$. Similarly, there exists a unique morphism $q_{\beta,\alpha} : [G(\sqrt{\beta})](\sqrt{\alpha}) \to [G(\sqrt{\alpha})](\sqrt{\beta})$ such that $q_2 = q_{\beta,\alpha} \circ \pi_2$. The universal property Theorem 4 forces $q_{\alpha\beta} \circ q_{\beta\alpha} = id$ and $q_{\beta\alpha} \circ q_{\alpha\beta} = id$. □

By Theorem 6, we may now define iterated quadratic extensions generalizing $G(\sqrt{\delta})$.

Definition 7. *For $\delta_1, \ldots, \delta_n \in G$, we define recursively:*

$$\begin{aligned}
G(\sqrt{\delta_1}, \sqrt{\delta_2}) &:= [G(\sqrt{\delta_1})](\sqrt{\delta_2}); \\
G(\sqrt{\delta_1}, \ldots, \sqrt{\delta_{n+1}}) &:= [G(\sqrt{\delta_1}, \ldots, \sqrt{\delta_n})](\sqrt{\delta_{n+1}}).
\end{aligned}$$

Later in this paper, we will show that this definition is independent (up to isomorphism) of the order of the "roots" $\sqrt{\delta_1}, \ldots, \sqrt{\delta_n}$.

Theorem 7. *Let G be a special group, $\delta_1, \ldots, \delta_n \in G$ and $\sigma \in S_n$. Then*

$$G(\sqrt{\delta_1}, \ldots, \sqrt{\delta_n}) \cong G(\sqrt{\delta_{\sigma(1)}}, \ldots, \sqrt{\delta_{\sigma(n)}}).$$

Furthermore, we have another (and more significant) description for $G(\sqrt{\delta_1}, \ldots, \sqrt{\delta_n})$.

Theorem 8. *Let $\alpha, \beta \in G$. Then,*

$$G(\sqrt{\alpha}, \sqrt{\beta}) \cong G/_m D_G(\langle\langle \alpha, \beta \rangle\rangle).$$

Proof. We import the terminology of Equation (3) of the proof of Theorem 6. Using Equation (1) (Theorem 3) and Equation (2), we have

$$\begin{aligned} [\varphi]_\alpha &\equiv_{G(\sqrt{\alpha})} [\psi]_\alpha \text{ if } \varphi \otimes \langle 1, \alpha \rangle \equiv_G \psi \otimes \langle 1, \alpha \rangle \\ [\varphi]_\beta &\equiv_{G(\sqrt{\beta})} [\psi]_\beta \text{ if } \varphi \otimes \langle 1, \beta \rangle \equiv_G \psi \otimes \langle 1, \beta \rangle \\ [\varphi] &\equiv_{G/_m D_G(\langle\langle \alpha, \beta \rangle\rangle)} [\psi] \text{ if } \varphi \otimes \langle\langle \alpha, \beta \rangle\rangle \equiv_G \psi \otimes \langle\langle \alpha, \beta \rangle\rangle \end{aligned} \tag{4}$$

Since $\langle 1, \alpha \rangle = \langle\langle \alpha \rangle\rangle$, $\langle 1, \beta \rangle = \langle\langle \beta \rangle\rangle$ and $\langle\langle \alpha, \beta \rangle\rangle$ are Pfister forms and both $D_G(\langle 1, \alpha \rangle)$ and $D_G(\langle 1, \beta \rangle)$ are subsets of $D_G(\langle\langle \alpha, \beta \rangle\rangle)$, using Equation (1) (Theorem 3), Equation (2), Lemma 4 and Equation (4) (above), we get

$$[\varphi] \equiv_{G/_m D_G(\langle\langle \alpha, \beta \rangle\rangle)} [\psi] \text{ imply that } [\varphi]_\alpha \equiv_{G(\sqrt{\alpha})} [\psi]_\alpha \text{ and } [\varphi]_\beta \equiv_{G(\sqrt{\beta})} [\psi]_\beta. \tag{5}$$

In particular, by the Universal Property Theorem 4 we have a unique morphism $q_1 : G/_m D_G(\langle\langle \alpha, \beta \rangle\rangle) \to G(\sqrt{\alpha}, \sqrt{\beta})$ given by the rule $q_1([g]) := [g]_{\alpha\beta}$. On the other hand, Equations (4) and (5) provide a morphism $\pi : G(\sqrt{\alpha}) \to G/_m D_G(\langle\langle \alpha, \beta \rangle\rangle)$ given by $\pi([g]_\alpha) := [g]$, for which $\pi(D_{G(\sqrt{\alpha})}([1]_\alpha, [\beta]_\alpha) = \{[1]\}$. Then, there exists a unique morphism $q_2 : G(\sqrt{\alpha}, \sqrt{\beta}) \to G/_m D_G(\langle\langle \alpha, \beta \rangle\rangle)$, given by the rule $q_2([g]_{\alpha\beta}) := [g]$ and such that $\pi = q_2 \circ \pi_{G(\sqrt{\alpha}, \sqrt{\beta})}$ (here $\pi_{G(\sqrt{\alpha}, \sqrt{\beta})} : G(\sqrt{\alpha}) \to G(\sqrt{\alpha}, \sqrt{\beta})$ is the quotient morphism).

The Universal Property Theorem 4 forces $q_1 \circ q_2 = id$ and $q_2 \circ q_1 = id$. □

Using Theorems 6–8 with an inductive argument we get the following description.

Theorem 9. *Let $\delta_1, \ldots, \delta_n \in G$. Then*

$$G(\sqrt{\delta_1}, \ldots, \sqrt{\delta_n}) \cong G/_m D_G(\langle\langle \delta_1, \ldots, \delta_n \rangle\rangle).$$

This leads to the following interesting Corollary of Theorem 9 (remember Equation (1) in Theorem 3!):

Corollary 1. *Let G be a special group and $\delta_1, \ldots, \delta_n \in G$. Then,*

$$\langle [a_1], \ldots, [a_m] \rangle \equiv_{G(\sqrt{\delta_1}, \ldots, \sqrt{\delta_n})} \langle [b_1], \ldots, [b_m] \rangle$$

if

$$\langle a_1, \ldots, a_m \rangle \otimes \langle\langle \delta_1, \ldots, \delta_n \rangle\rangle \equiv_G \langle b_1, \ldots, b_m \rangle \otimes \langle\langle \delta_1, \ldots, \delta_n \rangle\rangle.$$

Based on these results, we introduce the following.

Definition 8. *Let G be a special group and $\varphi = \langle\langle \delta_1, \dots, \delta_n \rangle\rangle$ be a Pfister form. We define:*

$$G(\varphi) := G/_m D_G(\langle\langle\langle \delta_1, \dots, \delta_n \rangle\rangle\rangle) \cong G(\sqrt{\delta_1}, \dots, \sqrt{\delta_n}).$$

5. A New Proof of Arason–Pfister Hauptsatz

In the sequel, our objective is to characterize for a special group G, whether $[a] = [b]$ in $G(\sqrt{a_1}, \dots, \sqrt{a_n})$ in terms of the equations in G.

Theorem 10. *Let G be a special group and $a, b, c \in G$. Then, $[a] = [b]$ in $G(\sqrt{a})$ if there is $s, t \in D_G(1, a)$ with $as = bt$ (or $a = bst$ or even $b = ats$).*

Proof. Using the Definition of $G(\sqrt{a})$ and the Marshall quotient we have $[a] = [b]$ (in $G(\sqrt{a})$) if $as = bt$ for some $s, t \in D_G(1, a)$ (and since $D_G(1, a) \cdot D_G(1, a) \subseteq D_G(1, a)$, we have assumed $st \in D_G(1, a)$ and $ast = b$, or $a = bst$). $\square$

Theorem 11. *Let G be a special group and $a_1, \dots, a_n, b, c \in G$. Then, $[a] = [b]$ in $G(\sqrt{a_1}, \dots, \sqrt{a_n})$ if there is $s, t \in D_G(\langle\langle\langle a_1, a_2, \dots, a_n \rangle\rangle\rangle)$ such that $as = bt$ (or $a = bst$ or even $ab \in D_G(\langle\langle\langle a_1, a_2, \dots, a_n \rangle\rangle\rangle)$).*

Proof. We proceed by induction. The case $n = 1$ is just Theorem 10 (since $\langle\langle a \rangle\rangle = \langle 1, a \rangle$). Now, suppose the result is valid for n. Here, we denote elements in $G(\sqrt{a_1}, \dots, \sqrt{a_{n+1}})$ by $[[a]] \in G(\sqrt{a_1}, \dots, \sqrt{a_{n+1}})$; and elements in $G(\sqrt{a_1}, \dots, \sqrt{a_n})$ by $[a] \in G(\sqrt{a_1}, \dots, \sqrt{a_n})$ ($a \in G$).

Let $[[a]] = [[b]]$ in $G(\sqrt{a_1}, \dots, \sqrt{a_{n+1}})$. Since

$$G(\sqrt{a_1}, \dots, \sqrt{a_{n+1}}) \cong G(\sqrt{a_1}, \dots, \sqrt{a_n})(\sqrt{a_{n+1}}),$$

by Theorem 10 we have $[ar] = [bs]$ for some $[r], [s] \in D_{G(\sqrt{a_1}, \dots, \sqrt{a_n})}([1], [a_{n+1}])$. By induction hypothesis, we get $art = bsw$ with $[r], [s] \in D_{G(\sqrt{a_1}, \dots, \sqrt{a_n})}([1], [\sqrt{a_{n+1}}])$ and $t, w \in D_G(\langle\langle\langle a_1, \dots, a_n \rangle\rangle\rangle)$.

Now, since $[r] \in D_{G(a_1, \dots, a_n)}([1], [a_{n+1}])$, we have

$$\langle [r], [ra_{n+1}] \rangle \equiv_{G(\sqrt{a_1}, \dots, \sqrt{a_n})} \langle [1], [a_{n+1}] \rangle.$$

Using Corollary 1, we have

$$\langle r, ra_{n+1} \rangle \otimes \langle\langle a_1, \dots, a_n \rangle\rangle \equiv_G \langle 1, a_{n+1} \rangle \otimes \langle\langle a_1, \dots, a_n \rangle\rangle,$$

which implies that $r \in D_G(\langle\langle\langle a_1, \dots, a_n, a_{n+1} \rangle\rangle\rangle)$. Similarly, $s \in D_G(\langle\langle\langle a_1, \dots, a_n, a_{n+1} \rangle\rangle\rangle)$. Since Pfister forms are multiplicative, we conclude that $a(rt) = b(sw)$ with $rt, sw \in D_G(\langle\langle\langle a_1, \dots, a_n, a_{n+1} \rangle\rangle\rangle)$, completing the proof. $\square$

Corollary 2. *Let G be a **reduced** special group and φ be a Pfister form, say $\varphi = \langle\langle a_1, a_2, \dots, a_n \rangle\rangle$ with $a_1, a_2, \dots, a_n \in G$. If φ is anisotropic ($-1 \notin D_G(\varphi)$) then $G(\varphi)$ is formally real.*

Proof. Since $-1 \notin D_G(\varphi)$, by Separation Theorem (Theorem 2.11, Chapter 2 of [5]), there is a maximal saturated subgroup Δ such that $D_G(\varphi) \subseteq \Delta$ and $-1 \notin \Delta$. Then, $G/_m\Delta \cong \mathbb{Z}_2$ and by the Universal Property of Pfister quotient Theorem 4, there exists a morphism $\sigma\colon G(\varphi) \to \mathbb{Z}_2$. Thus, $\sigma \in X_{G(\varphi)}$ and $G(\varphi)$ is formally real. $\square$

Before we proceed with the proof of the main Theorem 12, let us establish some notation. Let G be a special group and $\varphi \in I^n(G)$ be an anisotropic form. As we already have seen in Remark 1, for an anisotropic form $\varphi \in I^n G$, we can suppose without loss of generality that $\varphi = \varphi_1 + \ldots + \varphi_r$ with φ_j an anisotropic Pfister form ($j = 1, \ldots, r$). Recall (Definition 8): If $\varphi_1 = \langle\langle a_1, \ldots, a_n \rangle\rangle$, we have

$$G(\varphi_1) = G/_m D_G(\langle\langle a_1, \ldots, a_n \rangle\rangle) \cong G(\sqrt{a_1}, \ldots, \sqrt{a_n})$$

If $\psi = \langle b_1, \ldots, b_m \rangle$ is a form over G, we denote $[\psi] = \langle [b_1], \ldots, [b_m] \rangle$ the quotient form over $G(\varphi_1)$. In this sense, for $\varphi = \varphi_1 + \ldots + \varphi_r$ we have

$$[\varphi] = 2^n \cdot \langle [1] \rangle + [\varphi_2] + \ldots + [\varphi_r].$$

where $2^n = \dim(\varphi_1)$. Note that $\dim_{W,G}(\varphi) \geq \dim_{W,G(\varphi_1)}([\varphi])$.

We can suppose without loss of generality that for some positive integer $r \geq 1$, we have $\varphi = \varphi_1 + \ldots + \varphi_r$, with $\varphi_1, \ldots, \varphi_r$ anisotropic Pfister forms (see Remark 1).

Definition 9. *Let G be a special group, $\varphi_1, \ldots, \varphi_r$ anisotropic Pfister forms and $\varphi = \varphi_1 + \ldots + \varphi_r$. The **Arason–Pfister sequence** of φ over G, $AP(G, \varphi) := (G_0, \ldots, G_r)$, is defined recursively by:*

$$\begin{aligned} G_0 &:= G, \\ G_1 &:= G(\varphi_1) \\ G_{m+1} &:= G_m([\varphi_{m+1}]_m) \text{ for } m = 2, \ldots, r-1. \end{aligned}$$

where $[\varphi_{m+1}]_m$ means the class of the form φ_{m+1} in G_m, with $[\varphi]_m$ hyperbolic over G_m and $G_{m+1} \cong \{1\}$, or (reindexing if necessary) $[\varphi_{m+1}]_m$ anisotropic (over G_{m+1}) (The idea behind Arason–Pfister sequences is making iterate quotients by all the anisotropic forms in $\{\varphi_1, \ldots, \varphi_r\}$ until we achieve r steps or are left only the isotropic ones). We also denote elements (and forms) in G_m by $[a]_m \in G_m$, $a \in G$ ($[\psi]_m \in G_m$, ψ form over G).

Note that for all $m = 1, \ldots, r$ we have

$$[\varphi]_m = m \cdot 2^n \cdot \langle [1]_m \rangle + [\varphi_{m+1}]_m + \ldots + [\varphi_r]_m \in G_m.$$

Lemma 5. *Let G be a formally real special group, $\varphi_1, \ldots, \varphi_r$ anisotropic Pfister forms and*

$$\varphi = \varphi_1 + \ldots + \varphi_r$$

with Arason–Pfister sequence $AP(G, \varphi) := (G_0, \ldots, G_r)$. Then:

(i) *there exists the minimum $p \in \{1, \ldots, r-1\}$ such that $2^n \langle [1]_m \rangle$ is anisotropic for all $m \leq p$ and $G_m \cong \{1\}$ for all $m > p$ (which means $[\varphi]_p$ hyperbolic over G_p);*

(ii) *There exists a maximum $p \in \{1, \ldots, r\}$ such that $[\varphi_m]_m + \ldots + [\varphi_r]_m$ is anisotropic over G_m for all $m \leq p$ (Observe that if condition (ii) holds, then $[\varphi_{m+1}]_m + \ldots + [\varphi]_r$ is also anisotropic over G_m for all $m \leq p$).*

Definition. *This unique $p \geq 1$ is called **the Arason–Pfister index of AP**(G, φ).*

Proof. We proceed by induction on r. If $r = 1$, then condition (ii) holds (just take $p = 1$).

Let $r = 2$, with $\varphi = \varphi_1 + \varphi_2$. We have two cases:

(I) $[\varphi_2]_1$ isotropic over G_1. Then, condition (i) and (ii) holds with $p = 1$.

(II) $[\varphi_2]_1$ anisotropic over G_1. Then, condition (ii) holds with $p = 2$.

Now suppose the result valid for $r - 1$ ($r \geq 2$) and let $\varphi_1, \ldots, \varphi_r$ and $\varphi = \varphi_1 + \ldots + \varphi_r$ be anisotropic Pfister forms over G. Note that

$$\mathrm{AP}(G_1, [\varphi]_1) = (G_2, \ldots, G_r).$$

In other words, $(G_2, \dots, G_r)$ is the Arason–Pfister sequence of $[\varphi]_1$ over G_1. By the induction hypotheses, there exist an index p' from which one condition (i) or condition (ii) holds over G_1. This implies that condition (i) or condition (ii) holds over G with $p = p' + 1$. □

Theorem 12 (Arason–Pfister Hauptsatz)**.** *Let G be a* ***reduced*** *special group, then $AP_G(n)$ holds, for all $n \geq 0$. In more detail: for each $n \geq 0$ and each non-zero ($k \geq 1$), regular and anisotropic form $\varphi = \langle a_1, \dots, a_k \rangle$, if $\varphi \in I^n(G)$, then $\dim(\varphi) = k \geq 2^n$.*

Proof. If $1 = -1$ in G the result is trivially valid. Then, we can suppose $1 \neq -1$. Additionally, since φ is a non-zero regular anisotropic form in $\varphi \in I^n G$ we can suppose without loss of generality that for some positive integer $r \geq 1$, we have $\varphi = \varphi_1 + \dots + \varphi_r$, with $\varphi_1, \dots, \varphi_r$ anisotropic Pfister forms (see Remark 1).

We proceed by induction on r. If $r = 1$, then $\varphi = \varphi_1$, with $\dim(\varphi) = \dim(\varphi_1) = 2^n$.

If $r = 2$, we have some cases to consider:

I $[\varphi]_1$ is hyperbolic. Then, $[\varphi]_1 \equiv_{G_1} \cdot 2^{n-1} \cdot \langle [1]_1, -[1]_1 \rangle$, which means

$$2^n \cdot \langle [1]_1 \rangle + [\varphi_2]_1 \equiv_{G_1} 2^n \cdot \langle [1]_1, -[1]_1 \rangle.$$

By Witt's cancellation, we get

$$[\varphi_2]_1 \equiv_{G_1} 2^n \cdot \langle -[1]_1 \rangle$$

Since hyperbolic forms are zero in the Witt ring, we get $\dim_{W,G}(\varphi) \geq \dim_{W,G}(\varphi_2)$ (see Remark 1) and $\dim_{W,G}(\varphi_2) \geq \dim_{W,G_1}([\varphi_2]_1)$ by the quotient morphism $\pi : G \to G_1$. Putting this together, we arrive at

$$\dim_{W,G}(\varphi) \geq \dim_{W,G}(\varphi_2) \geq \dim_{W,G_1}([\varphi_2]_1) = \dim_{W,G_1} 2^n \cdot \langle -[1]_1 \rangle = 2^n.$$

II $[\varphi]_1$ is anisotropic. Then, φ_2 is an anisotropic Pfister form (see Remark 1) and $\dim_{W,G}(\varphi_2) \geq 2^n$. Moreover,

$$\dim_{W,G}(\varphi) \geq \dim_{W,G_1}([\varphi]_1) \geq \dim_{W,G_1}([\varphi_2]_1) \geq 2^n.$$

III $[\varphi]_1$ is isotropic and not hyperbolic. Observe that this means that

$$[\varphi]_1 = 2^n \langle [1]_1 \rangle + [\varphi_2]_1$$

is isotropic over G_1. If $[\varphi_2]_1$ is anisotropic over G_1, then $G_2 = G_1([\varphi_2]_1)$ is a formally real reduced special group and $[\varphi]_2 = 2 \cdot 2^n \langle [1]_2 \rangle$ is isotropic, contradiction. Then, $[\varphi_2]_1$ is isotropic over G_1, which means

$$\dim_{W,G}(\varphi) \geq \dim_{W,G_1}([\varphi]_1) \geq \dim_{W,G_1} 2^n \cdot \langle [1]_1 \rangle \geq 2^n.$$

Let $r \geq 2$ and suppose the result valid for $r - 1$. Then,

$$[\varphi]_1 = [\varphi_1 + \dots + \varphi_r]_1 = [\varphi_1]_1 + \dots + [\varphi_r]_1 = 2^n \cdot \langle [1]_1 \rangle + [\varphi_2]_1 + \dots + [\varphi_r]_1.$$

We already know that $\dim_{W,G}(\varphi) \geq \dim_{W,G}(\varphi_2 + \dots + \varphi_r) \geq \dim_{W,G(\varphi_1)}([\varphi_2]_1 + \dots + [\varphi_r]_1)$. Then we have three cases:

I $[\varphi]_1$ is hyperbolic. Then, $[\varphi]_1 \equiv_{G_1} r \cdot 2^n \cdot \langle [1]_1, -[1]_1 \rangle$, which means

$$2^n \cdot \langle [1]_1 \rangle + [\varphi_2]_1 + \dots + [\varphi_r]_1 \equiv_{G_1} r \cdot 2^n \cdot \langle [1]_1, -[1]_1 \rangle.$$

By Witt's cancellation we get

$$[\varphi_2]_1 + \ldots + [\varphi_r]_1 \equiv_{G_1} 2^n \cdot \langle -[1]_1 \rangle + (r-1) \cdot 2^n \cdot \langle [1]_1, -[1]_1 \rangle$$

Since hyperbolic forms are zero in the Witt ring, we get

$$\dim_{W,G}(\varphi) \geq \dim_{W,G}(\varphi_2 + \ldots + \varphi_r) \geq \dim_{W,G_1}([\varphi_2]_1 + \ldots + [\varphi_r]_1) = \dim_{W,G_1} 2^n \cdot \langle -[1]_1 \rangle = 2^n.$$

II $[\varphi]_1$ is anisotropic. Since $\varphi_2 + \ldots + \varphi_r$ is also anisotropic (see Remark 1), by induction hypothesis we have $\dim_{W,G}(\varphi_2 + \ldots + \varphi_r) \geq 2^n$.

Moreover,

$$\dim_{W,G}(\varphi) \geq \dim_{W,G}(\varphi_2 + \ldots + \varphi_r) \geq \dim_{W,G_1}([\varphi_2]_1 + \ldots + [\varphi_r]_1) \geq 2^n.$$

III $[\varphi]_1$ is isotropic and not hyperbolic. Let $\mathrm{AP}(G,\varphi) = (G_0, \ldots, G_r)$ be the Arason–Pfister sequence of φ over G (see Definition 5) and p its Arason–Pfister index (see Lemma 5). We have two subcases to consider:

1. Condition (i) of Lemma 5 holds for p. Then, means that G_m is formally real for all $m \leq p$ and $[\varphi]_p$ hyperbolic over G_p. Then,
$$[\varphi]_p = p \cdot 2^n \cdot \langle [1]_p \rangle + [\varphi_{p+1}]_p + \ldots + [\varphi_r]_p$$
is hyperbolic, which means
$$p \cdot 2^n \cdot \langle [1]_p \rangle + [\varphi_{p+1}]_p + \ldots + [\varphi_r]_p \equiv_{G_p} p \cdot 2^n \langle [1]_p, -[1]_p \rangle.$$
By Witt's cancellation we get
$$[\varphi_{p+1}]_p + \ldots + [\varphi_r]_p \equiv_{G_p} 2^n \cdot \langle -[1]_1 \rangle + (r-p) \cdot 2^n \cdot \langle [1]_1, -[1]_1 \rangle$$
Since hyperbolic forms are zero in the Witt ring, we get
$$\dim_{W,G}(\varphi) \geq \dim_{W,G}(\varphi_{p+1} + \ldots + \varphi_r) \geq$$
$$\dim_{W,G_p}([\varphi_{p+1}]_p + \ldots + [\varphi_r]_p) = \dim_{W,G_p} 2^n \cdot \langle -[1]_1 \rangle = 2^n.$$
2. Condition (ii) of Lemma 4.6 holds for p. In particular, $[\varphi_{p+1}]_p + \ldots + [\varphi]_p$ is anisotropic over G_p. By induction hypothesis $\dim_{W,G_p}([\varphi_{p+1}]_p + \ldots + [\varphi]_p) \geq 2^n$. Then
$$\dim_{W,G}(\varphi) \geq \dim_{W,G}(\varphi_p + \ldots + \varphi_r) \geq \dim_{W,G_p}([\varphi_{p+1}]_p + \ldots + [\varphi]_p) \geq 2^n.$$

□

The classical (and equivalent) way to state Theorem 12 is the following: if a form φ belongs to I^nG and $\dim \varphi < 2^n$ then φ must be a hyperbolic form (see for instance [9]).

6. Final Remarks and Further Research

We have obtained a new proof of Arason–Pfister Hauptsatz in the same abstract setting of the proof presented in [5]—the class of reduced special groups–, but avoiding completely the use of Horn–Tarski and Stiefel–Whitney invariants for n-ary isometry.

On the other hand, extending the Arason–Pfister Hauptsatz to arbitrary (or, at least, formally real) special groups remains a central challenge, specially because we do not have an analogous of Corollary 2 for general special groups. Possibly approaches require either a generalization of the quotient techniques or the development of some notion of "transcendental extensions" for special groups.

Since Marshall's original notion of quotient arose in the context of real reduced hyperfields [6], it would be fruitful to explore whether our constructions admit natural reformulations in terms of hyperstructures or real spectra, possibly leading to a unified approach to abstract quadratic form theories.

The theory of special groups, their quotients, and associated invariants might benefit from a reinterpretation in categorical terms—e.g., through the lens of topos theory, or model categories. These perspectives may yield further structural results and pave the way for homotopical generalizations.

The explicit structure of Marshall's quotient and quadratic extensions suggests algorithmic possibilities for detecting isotropy and computing dimensions in abstract settings, which could be valuable in calculations involving special groups.

The categorical tools introduced here may allow for AP-type results in settings beyond classical quadratic forms—for example, in hermitian or sesquilinear form theories, especially defined over (non-commutative) hyperfields and multirings endowed with involution [11].

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