Approach to Teaching the Finite Element Method Applied to Electromagnetic Problems with Axial Symmetry to Electrical Engineering Students

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ABSTRACT: An approach to the finite element method applied to the solution of stationary electromagnetic problems with axial symmetry is presented. This method is suitable for teaching electrical engineering students at the undergraduate level. The problem formulation is based solely on the direct integration of Maxwell's equations, and the approach is valid only for first-order elements, thereby avoiding the use of an excessively complex mathematical treatment. Numerical examples illustrate the application of the proposed methodology to academic problems. Its validation is verified both by classical expressions of electromagnetic theory and by an educational finite-element computer package. © 1999 John Wiley & Sons, Inc. Comput Appl Eng Educ 7: 133–145, 1999

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INTRODUCTION

In previous work [1–3], the authors described an approach to the finite element method (FEM), which is especially suited for introducing this subject to electrical engineering students at the undergraduate level. This approach is being used at Escola Politécnica da Universidade de São Paulo, São Paulo, Brazil. In that work, only bidimensional problems where the cross section regularly repeats itself in planes parallel and perpendicular to the coordinate axes were treated.

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Magnetostatic, electrostatic, and electrokinetic problems were considered.

In this work, stationary electromagnetic problems with cylindrical (axisymmetric) symmetry are dealt with. In this case, the geometry of the device under consideration repeats itself in planes that contain the device's axis. Since this work is mainly devoted to undergraduate students, the problem formulation is based solely on the direct integration of Maxwell's equations, as in the previous works. This approach is valid only for first-order finite elements and avoids the use of an excessively complex mathematical treatment, such as the variational approach. In the authors' opinion, this complexity has prevented the widespread use of the FEM in Brazilian engineering schools. At the end of this article, some examples are presented to illustrate the application of all steps required in the use of the FEM. The results are compared to classical expressions of electromagnetic theory, as well as solutions obtained from a finite element–based computer package, LMAG-2D [2].

STEPS REQUIRED IN APPLICATION OF THE FEM

Application of the FEM requires the execution of well-defined steps to guarantee the precision of results. These steps are described next.

Open or Closed Domain?

The first step is related to the definition of the domain to be studied. In a closed-domain problem, the field (electric or magnetic) is totally contained within the device under consideration. This is the case in some types of cylindrical inductors, as can be seen in Figure 1.

On the other hand, in an open-domain problem, the field distribution includes the space surrounding the device. Some types of actuators fall in this category (Fig. 2). In this case, the original domain has to be transformed into a closed domain by imposing a virtual boundary beyond which the field can be safely neglected.

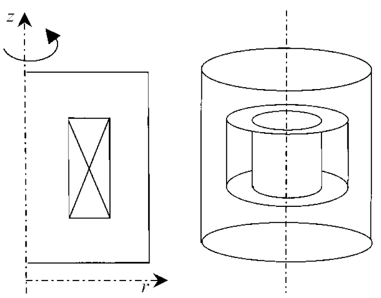

Figure 1 Example of a closed-domain problem.

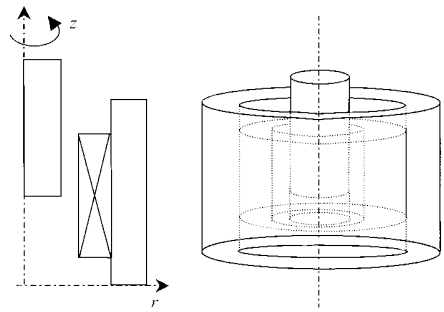

Figure 2 Example of a open-domain problem.

Domain Subdivision

In the second step, the domain resulting from the previous step is divided into a number of subdomains, the finite elements, which in their simplest form are triangles. Special care is required in this task since an inadequate subdivision can adversely affect the obtained solution. As a general rule, the following conditions should be satisfied:

1. The region inside any given element should possess constant physical properties.
2. The element size is not relevant, but a higher element density should be used in regions where significant field variations are expected to occur.

Figure 3 shows a triangular mesh for a closed domain originated from the domain of Figure 2.

Attribution of Physical Properties

In this step, physical properties have to be assigned to each element within the domain. Depending on the type of study, a value of either magnetic permeability, electric permitivity, or electric conductivity is assigned to each element.

Specification of Field Sources

In magnetostatics, the field sources are given by current density values, whereas in electrostatics they are given by values of electric charge density. In this step, these values have to be assigned to those elements that belong to field sources.

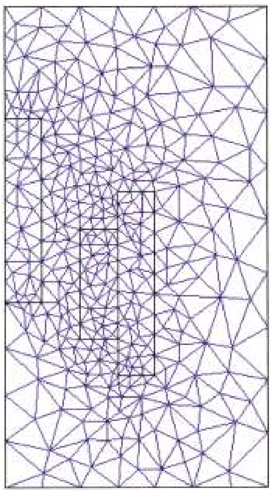

Figure 3 Triangular mesh of domain in Figure 2.

Specification of Boundary Conditions

Boundary conditions are to be imposed on the domain boundaries. They can be of one of the following types:

1. Fixed potential (or Dirichlet condition): In this case, the electric or magnetic potential is partially or totally known alongside the boundary. The known potential values are then assigned to the element vertices that belong to this boundary line.
2. Normal field known (or homogeneous Neumann condition): In electrostatics and electrokinetics, the field perpendicular to the boundary line is often null. In these cases, a null field value has to be assigned to all vertices belonging to the boundary.
3. Tangential field known (or homogeneous Neumann condition): In magnetostatics, the field tangential to the boundary line is often null. In this case, a null field value also has to be assigned to all vertices belonging to the boundary.

Having obtained all the above information, a linear system of equations is assembled and solved. The order of the system equals the number of vertices of the subdivided domain; the unknowns are either electric scalar potential values (in electrostatics and electrokinetics) or magnetic vector potential values (in magnetostatics). The assembly of the system of equations has already been treated in greater detail in [1] and [3].

Exploitation of Results

In this step, the solution of the system of equations is presented in graphic form. Equipotential lines and color shades are common tools for presenting the solution. Some global quantities can also be obtained from the system solution, such as circuit parameters, forces, and torque, stored energy.

MAXWELL'S EQUATIONS IN MAGNETOSTATICS

The differential form of Maxwell's equations, which are used to the study of magnetostatic phenomena are as follows:

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} \text{ (Ampere's law)} \tag{1}$$

$$\nabla \cdot \vec{\mathbf{B}} = \mathbf{0} \tag{2}$$

If we recall that $\nabla \cdot \nabla \times \vec{\mathbf{P}} = \mathbf{0}$ for any vector $\vec{\mathbf{P}}$, a magnetic vector potential, $\vec{\mathbf{A}}$, can be defined as follows:

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} \tag{3}$$

To guarantee the uniqueness of this vector, we will choose $\vec{\mathbf{A}}$ such that [4]

$$\nabla \cdot \vec{\mathbf{A}} = \mathbf{0} \tag{4}$$

It is possible to demonstrate that a vector field having these properties is continuous throughout its domain [4]. In problems presenting axial symmetry, the current density vector and the magnetic potential vectors can be reduced to only one component, φ, normal to the plane of study, as follows:

$$\vec{\mathbf{J}} = J_\varphi(r, z)\vec{\mathbf{u}}_\varphi$$

and

$$\vec{\mathbf{A}} = A_\varphi(r, z)\vec{\mathbf{u}}_\varphi$$

Thus,

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} = -\frac{\partial A_\varphi}{\partial z}\vec{\mathbf{u}}_r + \frac{1}{r}\frac{\partial (rA_\varphi)}{\partial r}\vec{\mathbf{u}}_z \tag{5}$$

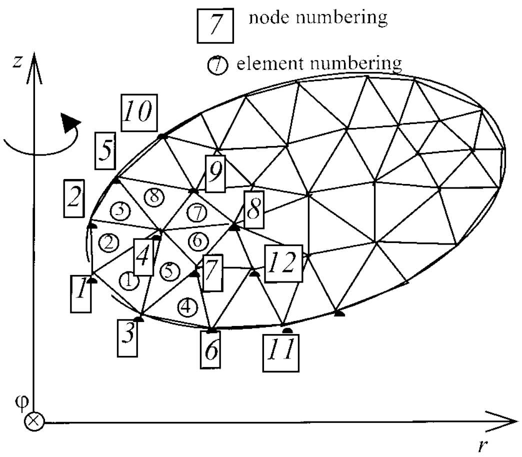

Figure 4 Domain with axial symmetry subdivided in triangles.

If we define the modified magnetic vector potential as $\vec{\phi} = r\vec{\mathbf{A}}$ (with its only component $\phi = rA_{\varphi}$), Equation (5) can be rewritten as

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} = -\frac{1}{r}\frac{\partial \phi}{\partial z}\vec{\mathbf{u}}_r + \frac{1}{r}\frac{\partial \phi}{\partial r}\vec{\mathbf{u}}_z \qquad (6)$$

Now, by applying the constitutive equation

$$\vec{\mathbf{H}} = \nu \vec{\mathbf{B}} \qquad (7)$$

with $\nu = 1/\mu$ being the magnetic reluctivity (supposed constant), we finally obtain

$$\vec{\mathbf{H}} = -\frac{\nu}{r}\frac{\partial \phi}{\partial z}\vec{\mathbf{u}}_r + \frac{\nu}{r}\frac{\partial \phi}{\partial r}\vec{\mathbf{u}}_z \qquad (8)$$

Figure 4 illustrates a generic axisymmetric domain that has been subdivided in triangular finite elements. In this figure, node and element numbers have been indicated.

The integral form of Maxwell's second equation (i.e., Ampere's law) states that the line integral of the magnetic field $\vec{\mathbf{H}}$ alongside a closed contour equals the current enclosed by this contour.

Let us apply Equation (8) to polygonal contours involving each node in the finite element mesh, in such a way that each segment of the polygonal contour consists of a segment of one median going from one side of the triangle to its centroid (Fig. 5).

Figure 6 shows a generic element where point **O** represents its centroid and segments $\overline{\mathbf{PO}}$ and $\overline{\mathbf{OS}}$ are part of the closed contour involving node i. Similarly, segments $\overline{\mathbf{PO}}$ and $\overline{\mathbf{OG}}$, and $\overline{\mathbf{SO}}$ and $\overline{\mathbf{OG}}$, are part of

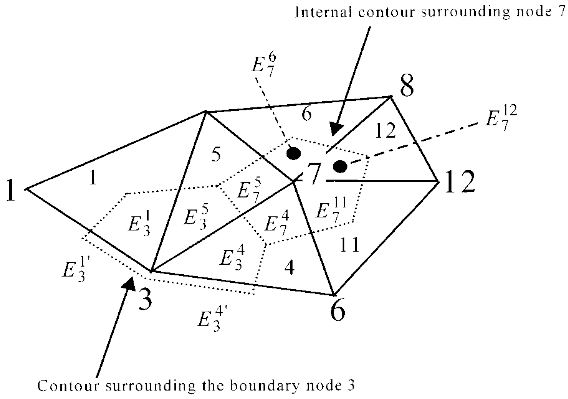

Figure 5 Typical contours (dotted lines) involving nodes from the mesh.

the closed contours involving nodes j and k, respectively.

Referring to Figure 6, the term E_i^e represents the line integral of the magnetic field $\vec{\mathbf{H}}$ along segments $\overline{\mathbf{OS}}$ and $\overline{\mathbf{OP}}$, yielding

$$E_i^e = \oint_{\overline{\mathrm{SOP}}} \vec{\mathbf{H}} \cdot \vec{\mathbf{dl}} \qquad (9)$$

Terms E_j^e and E_k^e are defined in a similar way. Then we have for the contour surrounding node 7 in Figure 5:

$$E_7^5 + E_7^6 + E_7^{12} + E_7^{11} + E_7^4 = \oint_{C_7} \vec{\mathbf{H}} \cdot \vec{\mathbf{dl}} \qquad (10)$$

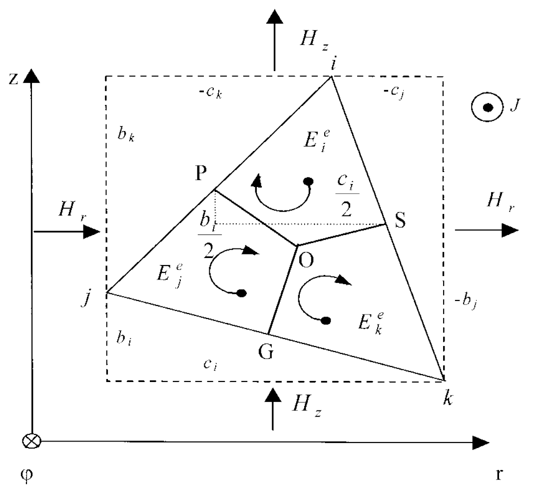

Figure 6 Triangular element showing partial contours: $\overline{\mathbf{POG}}$, $\overline{\mathbf{SOP}}$, and $\overline{\mathbf{GOS}}$ involving nodes i, j, and k, respectively.

and for the contour involving node 3:

$$E_3^1 + E_3^5 + E_3^4 + E_3^{4'} + E_3^{1'} = \oint_{C_3} \vec{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} \quad (11)$$

ANALYSIS FOR A GENERIC TRIANGULAR ELEMENT

Distribution of Modified Potential ϕ Inside the Element

Let us call i, j, and k the clockwise ordered vertices of a generic triangular element as shown in Figure 6. Also, let us define ϕ_i, ϕ_j, and ϕ_k the potentials at nodes i, j, and k, respectively.

Now we will assume that the potential ϕ inside the element can be given by the following linear interpolation:

$$\phi(r, z) = \alpha_1 + \alpha_2 r + \alpha_3 z \quad (12)$$

This is true because, similarly to $\vec{\mathbf{A}}$ (or A_φ), ϕ is continuous throughout its domain. Constants α_1, α_2, and α_3 can be determined by substituting both potential and node coordinates in Equation (12), as follows:

$$\left.\begin{aligned} \phi_i &= \alpha_1 + \alpha_2 r_i + \alpha_3 z_i \\ \phi_j &= \alpha_1 + \alpha_2 r_j + \alpha_3 z_j \\ \phi_z &= \alpha_1 + \alpha_2 r_k + \alpha_3 z_k \end{aligned}\right\} \quad (13)$$

Solving this system of equations, we obtain

$$\left.\begin{aligned} \alpha_1 &= \frac{1}{2\Delta}(a_i\phi_i + a_j\phi_j + a_k\phi_k) \\ \alpha_2 &= \frac{1}{2\Delta}(b_i\phi_i + b_j\phi_j + b_k\phi_k) \\ \alpha_3 &= \frac{1}{2\Delta}(c_i\phi_i + c_j\phi_j + c_k\phi_k) \end{aligned}\right\} \quad (14)$$

with the constants a, b, and c given by

$$\left.\begin{matrix} a_i = r_j z_k - r_k z_j & a_j = r_k z_i - r_i z_k & a_k = r_i z_j - r_j z_i \\ b_i = z_j - z_k & b_j = z_k - z_i & b_k = z_i - z_j \\ c_i = r_k - r_j & c_j = r_i - r_k & c_k = r_j - r_i \end{matrix}\right\} \quad (15)$$

The quantity Δ, given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_i \\ 1 & r_k & z_i \end{vmatrix} = \frac{1}{2}(b_i c_j - b_j c_i)$$

equals the element area.

By substituting Equations (13) and (14) into Equation (12), we obtain the distribution of the potential ϕ inside the element:

$$\phi(r, z) = N_i\phi_i + N_j\phi_j + N_k\phi_k \quad (16)$$

with functions N_i, N_j, and N_k being called the shape functions of the first-order triangular element. In this case, they are given by

$$\left.\begin{aligned} N_i &= \frac{1}{2\Delta}(a_i + b_i r + c_i z) \\ N_j &= \frac{1}{2\Delta}(a_j + b_j r + c_j z) \\ N_k &= \frac{1}{2\Delta}(a_k + b_k r + c_k z) \end{aligned}\right\} \quad (17)$$

By analyzing the shape functions above, we can conclude that the modified magnetic vector potential changes linearly inside the element. Also, this vector is continuous along the element sides. Figure 7(a,b) shows the geometrical meaning of this approximation.

Flux Density Inside the Element

The flux density inside the element is the most important piece of information that can be obtained from a computational tool based on the FEM, since it is the main quantity involved in the energy conversion process and in the evaluation of the stress imposed on the magnetic material.

From Equation (6) we obtain

$$B_r = -\frac{1}{r}\frac{\partial\phi}{\partial z}$$

and

$$B_z = \frac{1}{r}\frac{\partial\phi}{\partial r}$$

By substituting the value of ϕ obtained in Equation (16), it follows

$$B_r = -\frac{1}{2r\Delta}[c_i\phi_i + c_j\phi_j + c_k\phi_k] \quad (18)$$

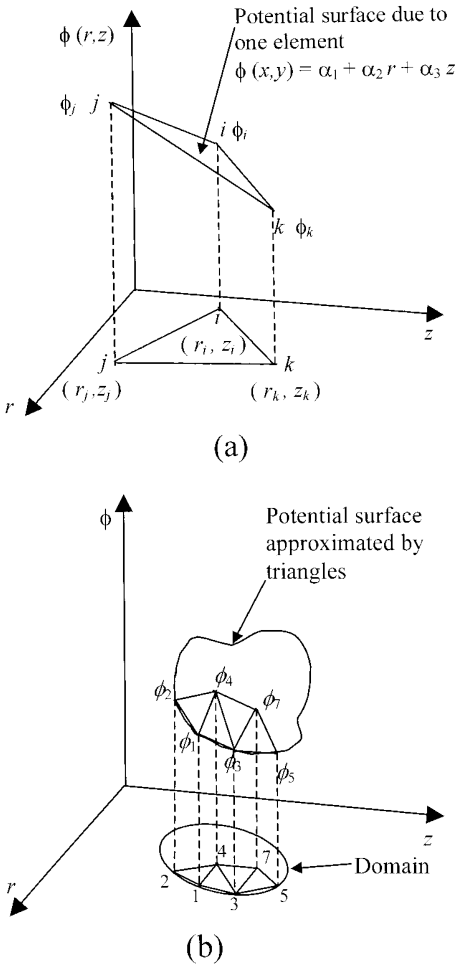

Figure 7 Geometrical meaning of potentials.

$$B_z = \frac{1}{2r\Delta}\left[b_i\phi_i + b_j\phi_j + b_k\phi_k\right] \tag{19}$$

Considering that the size of all elements is much smaller than the domain size, we can admit that the distribution of magnetic field inside each element is (nearly) constant and equal to the value obtained in the centroid of the element. Therefore, we substitute the value of r in Equations (18) and (19) by an average value r_c, given by the distance from the centroid of the element to the domain axis of symmetry.

From Equation (7) we obtain

$$H_r = \nu B_r = -\frac{\nu}{2r_c\Delta}\left[c_i\phi_i + c_j\phi_j + c_k\phi_k\right] \tag{20}$$

and

$$H_z = \nu B_z = \frac{\nu}{2r_c\Delta}\left[b_i\phi_i + b_j\phi_j + b_k\phi_k\right] \tag{21}$$

with $r_c = (r_i + r_j + r_k)/3$ being the r coordinate of the element centroid.

Partial Line Integral of Vector $\vec{\mathbf{H}}$ around a Node

In this section, we will compute the contribution of a given element to the line integral of vector $\vec{\mathbf{H}}$ around one of the element's nodes. Let E_i^e be the contribution of element e to the line integral corresponding to the contour surrounding node i (Fig. 6). Recalling that $\vec{\mathbf{H}}$ is constant inside the element, we can then write

$$E_i^e = \oint_{\overline{\mathrm{POS}}} \vec{\mathbf{H}} \cdot \vec{\mathbf{dl}} = \oint_{\overline{\mathrm{PS}}} \vec{\mathbf{H}} \cdot \vec{\mathbf{dl}}$$

and

$$E_i^e = -H_r\Delta r + H_z\Delta z \tag{22}$$

If we note that

$$\Delta r = r_P - r_S = -c_i/2$$

$$\Delta z = z_P - z_S = b_i/2$$

and replacing H_r and H_z in Equation (22) by Equations (20) and (21), we obtain

$$E_i^e = \frac{\nu}{2r_c\Delta}\left[(b_ib_i + c_ic_i)\phi_i + (b_ib_j + c_ic_j)\phi_j + (b_ib_k + c_ic_k)\phi_k\right] \tag{23}$$

E_j^e and E_k^e can be both computed in a similar manner:

$$E_j^e = \frac{\nu}{2r_c\Delta}\left[(b_jb_i + c_jc_i)\phi_i + (b_jb_j + c_jc_j)\phi_j + (b_jb_k + c_jc_k)\phi_k\right] \tag{24}$$

$$E_k^e = \frac{\nu}{2r_c\Delta}\left[(b_kb_i + c_kc_i)\phi_i + (b_kb_j + c_kc_j)\phi_j + (b_kb_k + c_kc_k)\phi_k\right] \tag{25}$$

Using matrix notation, we finally obtain

$$\begin{bmatrix} E_i^e \\ E_j^e \\ E_k^e \end{bmatrix} = \frac{\nu}{2r_c\Delta} \begin{bmatrix} b_ib_i + c_ic_i & b_ib_j + c_ic_j & b_ib_k + c_ic_k \\ & b_jb_j + c_jc_j & b_jb_k + c_jc_k \\ & & b_kb_k + c_kc_k \end{bmatrix} \times \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_k \end{bmatrix} \quad (26)$$

or

$$[E^e] = [S^e][\phi^e] \quad (27)$$

Matrix $[S^e]$ is both symmetric and singular and is referred to as the element matrix.

Current Enclosed by the Element

A generic element encloses a total current to which all three nodes contribute. The contribution of each node is due to the partial contour associated to the node and belonging to the element. If we consider the current density within the element as uniform and assign a positive value for a current "entering" the plane of study, we can then write

$$I_i^e = I_j^e = I_k^e = J^e \frac{\Delta}{3}$$

or, in matrix notation

$$[I^e] = \begin{bmatrix} I_i^e \\ I_j^e \\ I_k^e \end{bmatrix} = \begin{bmatrix} J^e \frac{\Delta}{3} \\ J^e \frac{\Delta}{3} \\ J^e \frac{\Delta}{3} \end{bmatrix} \quad (28)$$

MAXWELL'S EQUATIONS IN ELECTROSTATICS

The differential form of Maxwell's equations for electrostatic problems is as follows:

$$\nabla \times \vec{\mathbf{E}} = 0 \quad (29)$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho \quad (30)$$

We also recall that

$$\nabla \times \nabla f = 0 \quad \forall f \quad (31)$$

Thus, in this case we define the scalar electric potential function V as

$$\vec{\mathbf{E}} = -\nabla V \quad (32)$$

It should be pointed out that the negative sign is chosen just for convenience. The uniqueness of V is guaranteed by specifying a value to a particular point in the domain. In the axisymmetric problems, we have $V = V(r, z)$, so that

$$\vec{\mathbf{E}} = -\frac{\partial V}{\partial r}\vec{\mathbf{u}}_r - \frac{\partial V}{\partial z}\vec{\mathbf{u}}_z \quad (33)$$

The integral form of Maxwell's fourth equation states that the flux of the displacement vector $\vec{\mathbf{D}}$ through a closed surface equals the total electric charge inside the volume delimited by the surface, which is

$$\oiint_{\Gamma} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} = \int_V \rho dV \quad (34)$$

Let us apply this equation to closed surfaces involving each one of the nodes of the domain. These surfaces' cross sections are the typical contours that involve the nodes previously defined.

As an example, the closed surface surrounding node 7 in Figure 5 represents a surface of revolution around the z axis, whose cross section is the contour formed by the medians of the triangular elements that have node 7 as a vertex. Figure 8 shows again a generic element of a domain subdivided in triangles.

Let E_i^e be the flux of the displacement vector $\vec{\mathbf{D}}$ in the region of the closed surface that involves node i (this surface is formed by faces $\overline{\mathbf{PO}}$ and $\overline{\mathbf{OS}}$). We can then write

$$E_i^e = \oiint_{\overline{\mathbf{POS}}} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} \quad (35)$$

Also, let V_i, V_j, and V_k be the electric potential at each node of the element. Since the potential function is continuous, the potential of any point inside the element can be seen as a linear interpolation, similar to the magnetostatic case

$$V(r, z) = N_iV_i + N_jV_j + N_kV_k \quad (36)$$

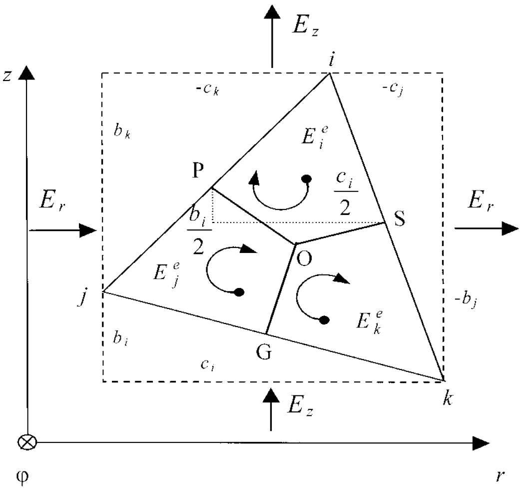

Figure 8 Surfaces surrounding nodes of a generic element.

with N_i, N_j, and N_k being the shape functions given by Equation (17).

Electric Field Inside the Element

The electric field inside devices that possess axial symmetry is given by

$$\vec{\mathbf{E}} = E_r\vec{\mathbf{u}}_r + E_z\vec{\mathbf{u}}_z \tag{37}$$

with

$$E_r = -\frac{\partial V}{\partial r} = -\frac{1}{2\Delta}(b_iV_i + b_jV_j + b_kV_k) \tag{38}$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{1}{2\Delta}(c_iV_i + c_jV_j + c_kV_k) \tag{39}$$

From the constitutive equation $\vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}}$, we can determine the components of the displacement vector, as follows:

$$D_r = \varepsilon E_r = -\frac{\varepsilon}{2\Delta}(b_iV_i + b_jV_j + b_kV_k) \tag{40}$$

$$D_z = \varepsilon E_z = -\frac{\varepsilon}{2\Delta}(c_iV_i + c_jV_j + c_kV_k) \tag{41}$$

Partial Flux of Displacement Vector

The flux of the displacement vector through the closed surface that involves node i of the triangular element is given by

$$E_i^e = \oiint_{\overline{\text{POS}}} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} = \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{\Delta s}} \tag{42}$$

since $\vec{\mathbf{D}}$ is constant inside the element. Recalling that

$$\vec{\mathbf{D}} = D_r\vec{\mathbf{u}}_r + D_z\vec{\mathbf{u}}_z \tag{43}$$

and

$$\overrightarrow{\mathbf{\Delta s}} = -\frac{b_i}{2}2\pi r_c\vec{\mathbf{u}}_r - \frac{c_i}{2}2\pi r_c\vec{\mathbf{u}}_z$$

we then obtain for node i

$$E_i^e = \frac{2\pi r_c\varepsilon}{4\Delta}[(b_ib_i + c_ic_i)V_i + (b_ib_j + c_ic_j)V_j + (b_ib_k + c_ic_k)V_k] \tag{44}$$

Following a similar procedure for nodes j and k, we obtain

$$E_j^e = \frac{2\pi r_c\varepsilon}{4\Delta}[(b_jb_i + c_jc_i)V_i + (b_jb_j + c_jc_j)V_j + (b_jb_k + c_jc_k)V_k] \tag{45}$$

$$E_k^e = \frac{2\pi r_c\varepsilon}{4\Delta}[(b_kb_i + c_kc_i)V_i + (b_kb_j + c_kc_j)V_j + (b_kb_k + c_kc_k)V_k] \tag{46}$$

Using matrix notation, we can finally write

$$\begin{bmatrix} E_i^e \\ E_j^e \\ E_k^e \end{bmatrix} = \frac{2\pi r_c\varepsilon}{4\Delta}\begin{bmatrix} b_ib_i + c_ic_i & b_ib_j + c_ic_j & b_ib_k + c_ic_k \\ & b_jb_j + c_jc_j & b_jb_k + c_jc_k \\ & & b_kb_k + c_kc_k \end{bmatrix} \times \begin{bmatrix} V_i \\ V_j \\ V_k \end{bmatrix} \tag{47}$$

Internal Charge

For each element that has node i as a vertex, there is a charge contribution to the total electric charge inside the surface surrounding the node. If we assume a constant volumetric charge density, the contribution

Table 1 Correspondence Between Electrostatics and Electrokinetics

Electrostatics	Electrokinetics
$\vec{\mathbf{E}}$	$\vec{\mathbf{E}}$
V	V
$\vec{\mathbf{D}}$	$\vec{\mathbf{J}}$
ρ	0
ε	σ

of nodes i, j, and k to the total element charge in the closed surface that surrounds the nodes is given by

$$Q_i^e = Q_j^e = Q_k^e = \rho 2\pi r_c \frac{\Delta}{3}$$

or, in matrix form,

$$\begin{bmatrix} Q_i^e \\ Q_j^e \\ Q_k^e \end{bmatrix} = \begin{bmatrix} \rho 2\pi r_c \frac{\Delta}{3} \\ \rho 2\pi r_c \frac{\Delta}{3} \\ \rho 2\pi r_c \frac{\Delta}{3} \end{bmatrix}$$

ELECTROKINETIC PHENOMENA

The equations that govern the phenomenon of electric conduction in continuous media are as follows:

$$\nabla \times \vec{\mathbf{E}} = 0 \text{ (Maxwell's 1st equation)} \quad (48)$$

$$\nabla \cdot \vec{\mathbf{J}} = 0 \text{ (continuity equation)} \quad (49)$$

$$\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} \text{ (Ohm's law)} \quad (50)$$

From Equation (48) we can define the scalar electric potential, V, as follows:

$$\vec{\mathbf{E}} = -\nabla V \quad (51)$$

in the same way as in the electrostatic case.

Looking at Equations (29), (30), and (32), and (48), (49), and (50), we can identify a complete similarity that allows us to establish the following correspondence between electrostatics and electrokinetics (Table 1).

As a consequence, the partial fluxes of vector $\vec{\mathbf{J}}$ for the generic element are obtained by following a similar procedure, which yields

$$\begin{bmatrix} E_i^e \\ E_j^e \\ E_k^e \end{bmatrix} = \frac{2\pi r_c \sigma}{4\Delta} \begin{bmatrix} b_i b_i + c_i c_i & b_i b_j + c_i c_j & b_i b_k + c_i c_k \\ & b_j b_j + c_j c_j & b_j b_k + c_j c_k \\ & & b_k b_k + c_k c_k \end{bmatrix} \times \begin{bmatrix} V_i \\ V_j \\ V_k \end{bmatrix} \quad (52)$$

In this case, the load vector is null owing to the continuity equation.

SYSTEM OF EQUATIONS

Magnetostatics

In magnetostatic problems, the term

$$E_i^e = \oint_{\overline{\text{POS}}} \vec{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} \quad (53)$$

represents the contribution of vector $\vec{\mathbf{H}}$ to the line integral along a closed contour that involves node i of element e. The total value of the integral along the same contour is given by

$$\sum_{e=1}^{NE} E_i^e = \oint_{C_i} \vec{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} \quad (54)$$

In this equation, index e denotes those elements that have node i as a vertex and NE, the total number of elements. Similarly, the total current enclosed by this contour is given by

$$\sum_{e=1}^{NE} I_i^e = \oiint_{\Gamma_i} \vec{\mathbf{J}} \cdot \overrightarrow{\mathbf{ds}} \quad (55)$$

Therefore, Maxwell's second equation applied to the closed contour C_i that surrounds node i (to which surface Γ_i is associated) is given by

$$\sum_{e=1}^{NE} E_i^e = \sum_{e=1}^{NE} I_i^e \quad (56)$$

For all nodes in the domain, we have

$$\begin{bmatrix} \sum_{e=1}^{NE} E_1^e \\ \sum_{e=1}^{NE} E_2^e \\ \vdots \\ \sum_{e=1}^{NE} E_n^e \end{bmatrix} = \begin{bmatrix} \sum_{e=1}^{NE} I_1^e \\ \sum_{e=1}^{NE} I_2^e \\ \vdots \\ \sum_{e=1}^{NE} I_n^e \end{bmatrix} \tag{57}$$

with n = the total number of nodes. Using matrix notation, we obtain

$$[S][A] = [I] \tag{58}$$

with $[S]$ being called global system matrix, and $[I]$ the load vector.

Electrostatics and Electrokinetics

In electrostatic problems, the term

$$E_i^e = \oiint_{\overline{\text{POS}}} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} \tag{59}$$

represents the flux contribution due to the displacement vector through the closed surface, which surrounds node i of element e. The total flux through the surface surrounding node i is then given by

$$\sum_{e=1}^{NE} E_i^e = \oiint_{\Gamma_i} \vec{\mathbf{D}} \cdot \overrightarrow{\mathbf{ds}} \tag{60}$$

The total charge inside the surface is

$$\sum_{e=1}^{NE} Q_i^e = \int_V \rho dV \tag{61}$$

so that Gauss' theorem applied to this surface yields

$$\sum_{e=1}^{NE} E_i^e = \sum_{e=1}^{NE} Q_i^e \tag{62}$$

For all nodes in the domain, we have

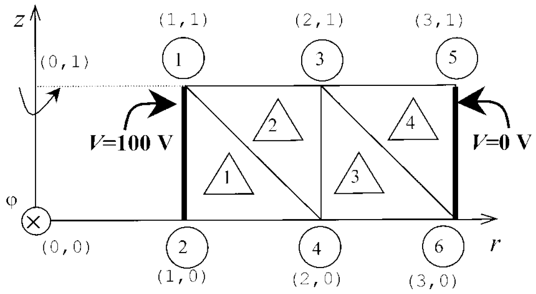

Figure 9 Domain with axial symmetry used in the numerical example: a resistor manually meshed in four triangles.

$$\begin{bmatrix} \sum_{e=1}^{NE} E_1^e \\ \sum_{e=1}^{NE} E_2^e \\ \vdots \\ \sum_{e=1}^{NE} E_n^e \end{bmatrix} = \begin{bmatrix} \sum_{e=1}^{NE} Q_1^e \\ \sum_{e=1}^{NE} Q_2^e \\ \vdots \\ \sum_{e=1}^{NE} Q_n^e \end{bmatrix} \tag{63}$$

or, in matrix notation:

$$[C][V] = [Q] \tag{64}$$

In electrokinetic problems, the continuity equation applied to the closed surface, which surrounds node i is given by

$$\sum_{e=1}^{NE} E_i^e = 0 \tag{65}$$

For all nodes in the domain, we have the following matrix equation:

$$[G][V] = 0 \tag{66}$$

NUMERICAL APPLICATION

The methodology proposed can be applied to a simple axisymmetric domain, as the one illustrated in Figure 9, which consists of an electrokinetic problem. It is an annular resistor with internal and external radii equal to 1 m and 2 m, respectively. Its conductivity is uniform and equals 2 S/m. Between the internal and external cylindrical surfaces a fixed voltage of 100 V is applied. This problem can be solved manually, as follows.

Table 2 Element 1

Local node numbering	1	2	3
Global node numbering	1	2	4
r coordinate (m)	1	1	2
z coordinate (m)	1	0	0

Calculation of Element Matrices (Table 2)

From Equation (15), it follows:

$$b_1 = z_2 - z_3 = 0 - 0 \Rightarrow b_1 = 0$$

$$b_2 = z_3 - z_1 = 0 - 1 \Rightarrow b_2 = -1$$

$$b_3 = z_1 - z_2 = 1 - 0 \Rightarrow b_3 = 1$$

$$c_1 = r_3 - r_2 = 2 - 1 \Rightarrow c_1 = 1$$

$$c_2 = r_1 - r_3 = 1 - 2 \Rightarrow c_2 = -1$$

$$c_3 = r_2 - r_1 = 1 - 1 \Rightarrow c_3 = 0$$

with the element area given by $\Delta = (b_1c_2 - b_2c_1)/2 \Rightarrow \Delta = \frac{1}{2}$, and the centroid given by $r_c^1 = (r_1 + r_2 + r_3)/3 = (1 + 1 + 2)/3 = \frac{4}{3}$.

Thus, by applying Equation (52), the element matrix for element 1 can be evaluated by

$$M^1 = \frac{2 \cdot \pi \cdot \frac{4}{3} \cdot 2}{4 \cdot \frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow$$

$$M^1 = \frac{2\pi}{3} \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix}$$

By repeating the procedure for elements 2, 3, and 4, we obtain the following element matrices:

$$M^2 = \frac{2\pi}{3} \begin{bmatrix} 5 & 0 & -5 \\ 0 & 5 & -5 \\ -5 & -5 & 10 \end{bmatrix},$$

$$M^3 = \frac{2\pi}{3} \begin{bmatrix} 7 & -7 & 0 \\ -7 & 14 & -7 \\ 0 & -7 & 7 \end{bmatrix}$$

$$M^4 = \frac{2\pi}{3} \begin{bmatrix} 8 & 0 & -8 \\ 0 & 8 & -8 \\ -8 & -8 & 16 \end{bmatrix}$$

Assembling of Global System of Equations and Assignment of Boundary Conditions

Applying the procedure described in [1] and [3] for assembling the global system of equation and the attribution of boundary conditions, we obtain the following system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & -12 & 0 & 0 \\ 0 & 0 & -12 & 23 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 500 \\ 400 \\ 0 \\ 0 \end{bmatrix}$$

which, after solution, yields the following nodal values for the electric potentials:

$$V_1 = V_2 = 100 \text{ V}, \quad V_3 = 37.819 \text{ V},$$

$$V_4 = 37.123 \text{ V} \quad V_5 = V_6 = 0 \text{ V}$$

Evaluation of Electric Field Inside the Elements

The electric field can be evaluated inside each element by Equations (38) and (39), as follows:

$$E_r^1 = -(0.V_1 - 1.V_2 + 1.V_4) = 100 - 37.123 \Rightarrow$$

$$\boxed{E_r^1 = 62.877 \text{ V/m}} \qquad (67)$$

$$E_z^1 = -(1.V_1 - 1.V_2 + 0.V_4) = -(100 - 100) \Rightarrow E_z^1 = 0 \text{ V/m}$$

$$E_r^2 = 62.181 \text{ V/m}, \quad E_z^2 = -0.696 \text{ V/m}$$

$$E_r^3 = 37.123 \text{ V/m}, \quad E_z^3 = -0.696 \text{ V/m}$$

$$E_r^4 = 37.819 \text{ V/m}, \quad E_z^4 = 0 \text{ V/m}$$

Comparison with Analytical Calculation of Electric Field

The electric field of the annular resistor shown in Figure 9 can be calculated by the following equation:

$$E = \frac{\Delta V}{r \ln \frac{b}{a}}$$

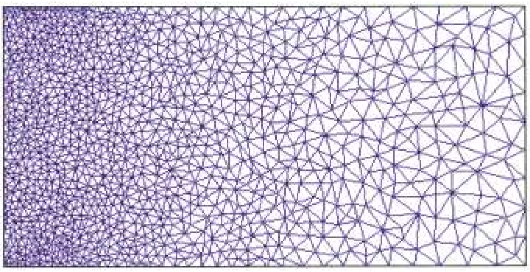

Figure 10 FE mesh of the annular resistor of Figure 9.

with b and a, respectively, the external and internal radius of the resistor, ΔV the potential drop between b and a, and r, the radial coordinate position, spanning from a to b. By applying this equation to, say, element $\underline{1}$, and by substituting r by the centroid, r_c, it follows that

$$E^1 = \frac{100}{\frac{4}{3} \cdot \ln(\frac{3}{1})} \Rightarrow E^1 = 68.27 \text{ V/m}$$

This result is different from Equation (67) by −8.9%, which can be considered acceptable in this case, since a coarse mesh has been employed to enable a calculation by hand.

Comparison with Numerical Solution Issued from the FE Simulation

The problem of Figure 9 was solved with the aid of an educational FE computer package, LMAG-2D [2]. Figures 10–12 present the results of the FE simulation. They show the triangular mesh, the lines of electric potential, and the electric field along the resistor, respectively.

According to the scale in Figure 12, it can be seen that the centroid of the element 1 (indicated in the figure by a cross) of the mesh shown in Figure 9 lies in the strip corresponding to an electric field E = 70.6 V/m (light brown strip in Fig. 12). The deviation with respect to the analytical value is 3.4%, a better

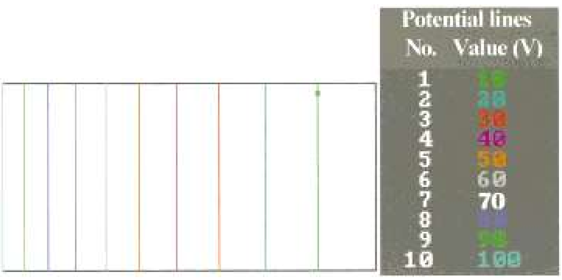

Figure 11 Visualization of potential lines in the resistor.

Figure 12 Visualization of the electric field in the resistor through color shades. The cross indicates the centroid of the element 1 of the mesh shown in Figure 9.

result when compared with the previous calculation by hand, since a finer mesh has been employed, as depicted in Figure 10.

It can be noticed that the results obtained by using the proposed methodology agree well with those obtained by the numerical solution.

CONCLUSION

An approach to the finite element method, which is suitable to teach this concept to undergraduate students of electrical engineering, as well as its application to stationary problems of electromagnetics with axial symmetry, has been presented. It is based on the direct integration of the Maxwell's equations and the use of first-order triangular elements, thereby avoiding the complex mathematical treatment of this theory that is often encountered in the literature. The methodology has been applied to the solution of a simple academic problem, and its results are equivalent to those obtained by both analytical equations and numerical solutions issued from an FE computer package for teaching purposes (LMAG-2D).

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BIOGRAPHIES

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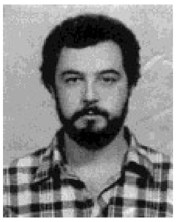

Luiz Natal Rossi was born in 1955 in Brazil. He obtained the BSc and MSc degrees in electrical engineering from Escola Politécnica da Universidade de São Paulo (EPUSP), Brazil, in 1979 and 1986, respectively. In 1982, he joined the Department of Electrical Engineering, EPUSP, as an assistant lecturer. His current research interests include electromagnetic field computation by FEM and electrical machines.