Eur. Phys. J. Plus (2021) 136:984
https://doi.org/10.1140/epjp/s13360-021-01982-z

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Pseudoclassical description of relativistic particles interacting with electromagnetic fields and weakly interacting with matter fields

D. M. Gitman[1,2,a], **D. A. Ivanov**[1,b], **A. F. de Souza**[3]

[1] Department of Physics, Tomsk State University, Tomsk, Russia 634050
[2] P. N. Lebedev Physical Institute, 53 Leninskiy Prospect, Moscow, Russia 119991
[3] Institute of Physics, University of São Paulo, Rua do Matão, 1371, São Paulo, SP 05508-090, Brazil

Received: 26 April 2021 / Accepted: 21 September 2021

Abstract Starting from an equation for causal propagator describing Dirac particles interacting with electromagnetic fields and weakly interacting with matter fields, we derive a path integral representation for the propagator. An effective gauge invariant action, which appears in the representation, is interpreted as a pseudoclassical action for the Dirac particles. Quantization of the action is nontrivial due to its gauge invariant nature as well as due to ordering problems that arise in course of a realization of commutation relations in a Hilbert space. The Dirac equation in the background under consideration appears as a quantum equation of motion in the constructed quantum mechanics, justifying, thus, the interpretation of the action. The path-integral representation allows one to calculate effectively the propagator and with its help emerging quantum currents. Studying these currents one may detect effects similar to the chiral magnetic effect. Pseudoclassical equations of motion in the nonrelativistic limit generalize nontrivially the Pauli quantum mechanics with external electromagnetic field.

1 Introduction

Construction of classical and pseudoclassical (models with Grassmann variables that describe spinning degrees of freedom) models of relativistic particles as well as their quantization (the first quantization) attracts attention already for a long time due to various reasons. First, the interest in such models was initiated by the close relationship with problems in string theory and gravity, but now it is clear that it is an important problem itself whether there exist classical model for any relativistic particle whose quantization reproduces, in a sense, the corresponding field theory, or one particle sector in the corresponding quantum field theory. One of the basic, in the above-mentioned set of models, is the pseudoclassical model of a Fermi particle with spin 1/2, proposed first in the works [1–3], investigated and quantized in many works, see for example [4–16]. A generalization of the pseudoclassical action of a spinning particle with an anomalous magnetic momentum was given in Refs. [17,18].

[a] e-mail: dmitrygitman@hotmail.com (corresponding author)
[b] e-mail: ivanofdm@phys.tsu.ru

A noncommutative version of Berezin–Marinov action is constructed in Ref. [19]. In Refs. [20,21], a pseudoclassical model whose quantization reproduces the quantum theory of Weyl particles was proposed. Here, a pseudoclassical model of relativistic spin-one particle, both massive and massless with an action, with a Chern–Simons term was considered as well. Quantum mechanics constructed for the massive case proves to be equivalent to the Proca theory and for massless case to the Maxwell theory. Pseudoclassical model for a massive Dirac particle in $2+1$ dimensions was constructed in Ref. [22] and then extended for arbitrary even- and odd-dimensional cases in Refs. [23–28]. Pseudoclassical actions of spinning particles in a non-Abelian background and in a torsion field are proposed in Refs. [29,30], respectively.

At the first glance, the construction of pseudoclassical actions and the corresponding pseudoclassical mechanics of relativistic particles has no direct physical applications, due to the difficulties of physical interpretations of the related pseudoclassical trajectories in spaces of Grassmannian variables. However, such physical quantities as propagators of relativistic particles can be represented in the form of functional integrals with kernels including such actions, see e.g., Refs. [12,27]. The path integral representations often serve as an efficient method for calculating such propagators, see e.g., [31–33]. Moreover, path integral representations of propagators underlie the so-called worldline technique in sting theory and QFT, see e.g., [34,35] and references there.

In this respect, one should mention that there exist works [36,37] in which models of relativistic particles with only Bosonic variables are constructed and a quantization of which also leads to quantum mechanics of particles with spin. On the one hand, the Bose nature of trajectories in such models makes the use of these trajectories for describing physical effects clearer, but on the other hand, the structure of such models, as a rule, is technically more complicated than the structure of pseudoclassical models, and the construction of functional integrals with the corresponding to the model actions is still an open task. Nevertheless, relatively recently [38], it was shown that Bose trajectories of a classical model describing massless charged chiral fermions in an external magnetic field can alternatively be used to calculate the chiral magnetic effect.[1]

All this once again emphasizes the importance of studying classical and pseudoclassical models of chiral particles interacting with various external background fields.

In Refs. [41,42], on the basis of the standard model, an effective relativistic-wave equation describing Dirac particles interacting with electromagnetic and matter fields was considered. The equation reads:

$$\left\{[i\partial_\nu - qA_\nu(x)]\gamma^\nu - m - \gamma_\mu\left[V_R^\mu(x)P_R + V_L^\mu(x)P_L\right]\right\}\Psi(x) = 0, \quad (1)$$

where q and m are particle charge and mass, respectively, $A_\nu(x)$ are potentials of an external electromagnetic field, $V_{L,R}^\mu$ are effective potentials of the electroweak interaction with a background matter (of left and right electrons respectively), $P_{L,R} = (1 \mp \gamma^5)/2$ are the chiral projectors, γ^μ, $\mu = 0, 1, 2, 3$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, are Dirac gamma-matrices,[2] and $\Psi(x)$ are four-component Dirac bispinor. By analogy with the behavior of chiral particles in external electromagnetic fields, one can hope for an appearance of anomalous currents due to the interaction with some specific structures of the matter. Technically, the study of such currents may be carried out by analogy either with the Vilenkin method [40] based on an analysis of specific solutions of Eq. (1), or on an approach used in work [38], where the key

[1] Chiral magnetic effect is the generation of an electric current along an external magnetic field induced by chirality imbalance, see e.g., [39]. Apparently, the first theoretical description of the effect belongs to Velenkin [40].

[2] $[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}$, $\eta_{\mu\nu}\eta^{\nu\gamma} = \delta_{\mu\gamma}$, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $[A, B]_\pm = AB \pm BA$.

role was played by a classical action and classical trajectories associated with chiral particles. In this regard, in the present study, we focus our attention on a possibility constructing a path-integral representation for the causal propagator of particles under consideration and with its help to derive a pseudoclassical action for these particles. We believe that a further analysis of both the propagator and the action may be useful in finding the above-mentioned anomalous currents, and, thus, may represent a new possibility in studying the chiral effect due a combination of electromagnetic and electroweak interactions.

In the present article, starting from an equation for causal propagator describing Dirac particles interacting with electromagnetic fields and weakly interacting with matter fields, we derive a path-integral representation for this propagator. An effective gauge-invariant action, which appears in the representation, is interpreted as a pseudoclassical action for the Dirac particles. Quantization of the action is nontrivial due to its gauge invariant nature as well as due to ordering problems that arise in course of a realization of commutation relations in a Hilbert space. The Dirac equation in the background under consideration appears as a quantum equation of motion in the constructed quantum mechanics, justifying, thus, the interpretation of the action. The path-integral representation allows one to calculate effectively the propagator and therefore emerging quantum currents, in the same manner as done in Refs. [31–33] in the case of Dirac particle in a magnetic field. Pseudoclassical equations of motion in the nonrelativistic limit generalize nontrivially the Pauli quantum mechanics with external electromagnetic field.

2 Causal propagator of a Dirac particle interacting with electromagnetic field and electroweakly interacting with matter fields

2.1 Defining equations

To begin with, let us rewrite Eq. (1) as follows:

$$\left\{\gamma^{\nu}\left[\hat{\mathscr{P}}_{\nu}-i\gamma^{5}K_{\nu}(x)\right]-m\right\}\Psi(x)=0, \tag{2}$$

where

$$\begin{aligned}
&\hat{\mathscr{P}}_{\nu}=i\partial_{\nu}-\mathbb{A}_{\nu},\ \mathbb{A}_{\nu}(x)=qA_{\nu}(x)+V_{\nu}(x),\\
&V^{\mu}(x)=\frac{1}{2}\left[V_{L}^{\mu}(x)+V_{R}^{\mu}(x)\right],\\
&K^{\mu}(x)=\frac{1}{2}\left[V_{R}^{\mu}(x)-V_{L}^{\mu}(x)\right].
\end{aligned} \tag{3}$$

Then, using the identity

$$3!\gamma^{\rho}\gamma^{5}=i\eta^{\rho\alpha}\varepsilon_{\alpha\beta\mu\nu}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}, \tag{4}$$

where $\varepsilon_{\alpha\beta\mu\nu}$ is a completely antisymmetric tensor with the normalization $\varepsilon_{0123}=1$, we can represent Eq. (2) to the following form:

$$\left[\hat{\mathscr{P}}_{\nu}\gamma^{\nu}-m-\frac{i}{3!}K^{\alpha}(x)\varepsilon_{\alpha\beta\mu\nu}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right]\Psi(x)=0. \tag{5}$$

The corresponding causal Green function $S^{c}(x,y)$ (the propagator) transformed by γ^{5} matrix, $\tilde{S}^{c}(x,y)=S^{c}(x,y)\gamma^{5}$ satisfies the inhomogeneous equation

$$\left[\hat{\mathscr{P}}_{\nu}\Gamma^{\nu}-m\Gamma^{4}-\frac{i}{3!}K^{\alpha}(x)\varepsilon_{\alpha\beta\mu\nu}\Gamma^{\beta}\Gamma^{\mu}\Gamma^{\nu}\right]\tilde{S}^{c}(x,y)=\delta^{(4)}(x-y), \tag{6}$$

where five matrices $\Gamma^n, n = (\mu, 4), \Gamma^\mu = \gamma^5\gamma^\mu,\ \Gamma^4 = \gamma^5$ are introduced. These matrices form a 5-dimensional representation of the Clifford algebra,

$$\left[\Gamma^n, \Gamma^m\right]_+ = 2\eta^{nm},\ \eta^{nm} = \mathrm{diag}(1, -1, -1, -1, -1).$$

2.2 Path integral representation for the propagator

We note that in fact, the propagator has lower spinor indices ($\tilde{S}^c(x, y) = \tilde{S}^c_{\alpha\beta}(x, y)$) that are omitted in Eq. (6). In contrast with Schwinger work [43], we represent only the spatial part of the propagator as a matrix element of an operator $\tilde{S}^c_{\alpha\beta}$:

$$\tilde{S}^c_{\alpha\beta}(x, y) = \langle x|\tilde{S}^c_{\alpha\beta}|y\rangle.$$

Here, $|x\rangle$—are eigenvectors of some coordinate operators $\hat{X}^\mu$, whereas $\hat{P}_\mu$ are the corresponding operators of momenta, all these quantities satisfy the relations:

$$\hat{X}^\mu|x\rangle = x^\mu|x\rangle,\ \langle x|y\rangle = \delta^4(x-y),\ \int |x\rangle\langle x|\,\mathrm{d}x = I,$$
$$\left[\hat{P}_\mu, \hat{X}^\nu\right]_- = -i\delta^\nu_\mu,\ \ \hat{P}_\mu|p\rangle = p_\mu|p\rangle,\ \langle p|p'\rangle = \delta^4(p-p'),$$
$$\int |p\rangle\langle p|\,\mathrm{d}p = I,\ \langle x|\hat{P}_\mu|y\rangle = -i\partial_\mu\delta^4(x-y),\ \langle x|p\rangle = (2\pi)^{-2}e^{ipx}.$$

Equation (6) implies the following representation for the operator $\tilde{S}^c_{\alpha\beta}$:

$$\tilde{S}^c = \hat{A}^{-1},\ \hat{A} = \hat{\Pi}_\nu\Gamma^\nu - m\Gamma^4 - \frac{i}{3!}K^\alpha\varepsilon_{\alpha\beta\mu\nu}\Gamma^\beta\Gamma^\mu\Gamma^\nu,$$
$$\hat{K}^\alpha = K^\alpha(\hat{X}), \hat{\Pi}_\mu = -\hat{P}_\mu - \mathbb{A}_\mu(\hat{X}),$$
$$\left[\hat{\Pi}_\mu, \hat{\Pi}_\nu\right]_- = -i\mathbb{F}_{\mu\nu}(\hat{X}),\ \mathbb{F}_{\mu\nu} = \partial_\mu\mathbb{A}_\nu - \partial_\beta\mathbb{A}_\nu\ . \tag{7}$$

Here and in what follows, the lower spinor indices are omitted again.

Through the use of identity (4) and transformation propagator with γ^5 matrix, we have ensured that the operator $\hat{A}$ is represented by terms that contain an odd number of Γ-matrices only. Now, one can consider the operator $\hat{A}$ as a pure Fermi operator, if one treats Γ-matrices as Fermi operators. In general case, an inverse operator to a Fermi operator can be presented by means of an integral over a super-proper time (λ, χ) of an exponential with an even exponent [12]. In the case under consideration, such a representation reads:

$$\hat{A}^{-1} = \int_0^\infty \mathrm{d}\lambda \int e^{i(\lambda(\hat{A}^2+i\epsilon)+\chi\hat{A})}\,\mathrm{d}\chi,$$

where the pair (λ, χ) is the super-proper time with λ being Grassmann even variable and χ being a Grassmann odd variable that anticommute with Γ matrices and therefore with $\hat{A}$ by the definition, and

$$\hat{A}^2 = \hat{\Pi}^2 - m^2 - \frac{i}{2}\mathbb{F}_{\alpha\beta}\Gamma^\alpha\Gamma^\beta - \hat{K}^2 + \partial_\mu\hat{K}^\mu\Gamma^0\Gamma^1\Gamma^2\Gamma^3$$
$$+ \frac{i}{2}\varepsilon^{\alpha\beta\mu\nu}[\hat{\Pi}_\alpha, \hat{K}_\beta]_+\Gamma_\mu\Gamma_\nu\ ,\ \partial_\mu\hat{K}^\mu = \left[\partial_\mu K^\mu(x)\right]_{x\to\hat{X}}.$$

Thus, we obtain:

$$\tilde{S}^c = \int_0^\infty d\lambda \int e^{-i\hat{\mathscr{H}}(\lambda,\chi)}\,\mathrm{d}\chi,$$

$$\hat{\mathscr{H}}(\lambda, \chi) = \lambda \Big\{ m^2 - \hat{\Pi}^2 + \frac{i}{2}\mathbb{F}_{\alpha\beta}\Gamma^\alpha\Gamma^\beta + \hat{K}^2 - \frac{i}{2}[\hat{\Pi}_\alpha, \hat{K}_\beta]_+ \varepsilon^{\alpha\beta\mu\nu}\Gamma_\mu\Gamma_\nu - \partial_\mu \hat{K}^\mu \Gamma^0\Gamma^1\Gamma^2\Gamma^3 \Big\} + \left(\hat{\Pi}_\nu \Gamma^\nu - m\Gamma^4 - \frac{i}{3!}\hat{K}^\alpha \varepsilon_{\alpha\beta\mu\nu}\Gamma^\beta\Gamma^\mu\Gamma^\nu \right)\chi,$$

and

$$\tilde{S}^c(x_{out}, x_{in}) = \int_0^\infty d\lambda \int \langle x_{out}| e^{-i\hat{\mathscr{H}}(\lambda,\chi)} |x_{in}\rangle \, d\chi. \tag{8}$$

Now, one can represent the matrix element entering in the right-hand side of Eq. (8) as a path integral following a universal (suitable for a wide class of operators, in particular, for the operator $\hat{\mathscr{H}}(\lambda, \chi)$) discretization procedure technique represented in detail in Refs. [12,27]. The result reads:

$$\begin{aligned} \tilde{S}^c(x_{out}, x_{in}) = {} & \exp\left(i\Gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty d\lambda_0 \int \mathscr{D}x \mathscr{D}p \mathscr{D}\lambda \mathscr{D}\pi \\ & \times \int d\chi_0 \int \mathscr{D}\chi \mathscr{D}\nu \int_{\psi(0)+\psi(1)=\theta} \mathscr{D}\psi \\ & \times \exp\Big\{ i \int_0^1 \lambda \left(\mathscr{P}^2 - m^2 + 2i\mathbb{F}_{\alpha\beta}\psi^\alpha\psi^\beta - K^2 \right. \\ & \left. - 4i\varepsilon^{\alpha\beta\mu\nu}\mathscr{P}_\alpha K_\beta \psi_\mu\psi_\nu + 16\partial_\mu K^\mu \psi^0\psi^1\psi^2\psi^3 \right) \\ & + 2i\left(\mathscr{P}_\alpha\psi^\alpha - m\psi^4 + \frac{2i}{3}K^\alpha \varepsilon_{\alpha\beta\mu\nu}\psi^\beta\psi^\mu\psi^\nu \right)\chi \\ & - i\psi_n\dot{\psi}^n + p\dot{x} + \pi\dot{\lambda} + \nu\dot{\chi} d\tau + \psi_n(1)\psi^n(0) \Big\}\Big|_{\theta=0}. \end{aligned} \tag{9}$$

Here, $x(\tau)$, $p(\tau)$, $\lambda(\tau)$, $\pi(\tau)$ are even and $\chi(\tau)$, $\nu(\tau)$, $\psi^n(\tau)$ are odd trajectories, obeying the boundary conditions $x(0) = x_{in}$, $x(1) = x_{out}$, $\lambda(0) = \lambda_0$, $\chi(0) = \chi_0$, and $\mathscr{P}_\nu = -p_\nu - \mathbb{A}_\nu(x)$.

The representation (9) can be considered as a Hamiltonian form of the path integral for the propagator. Integrating it over the momenta p, we obtain a Lagrangian path integral representation for the propagator:

$$\begin{aligned} \tilde{S}^c(x_{out}, x_{in}) = {} & \exp\left(i\Gamma^n \frac{\partial_l}{\partial\theta^n} \right) \int_0^\infty de_0 \int_{e_0} \mathscr{M}(e) De \int d\chi_0 \\ & \times \int_{\chi_0} D\chi \int_{x_{in}}^{x_{out}} Dx \int D\pi \int D\nu \int_{\psi(0)+\psi(1)=\theta} \mathscr{D}\psi \\ & \times \exp\Big\{ i \int_0^1 \Big[-\frac{\dot{x}^2}{2e} - \frac{e}{2}\left(M^2 - 2i\mathbb{F}_{\alpha\beta}\psi^\alpha\psi^\beta \right) \\ & - \dot{x}^\alpha (\mathbb{A}_\alpha - d_\alpha) + i\left(\frac{\dot{x}_\alpha\psi^\alpha}{e} - m\psi^4 - \frac{2}{3}\psi^\alpha d_\alpha \right)\chi \\ & - i\psi_n\dot{\psi}^n + \pi\dot{e} + \nu\dot{\chi} \Big] d\tau + \psi_n(1)\psi^n(0) \Big\}\Big|_{\theta=0}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} M^2 &= m^2 + K^2 - 16\partial_\mu K^\mu \psi^0\psi^1\psi^2\psi^3, \\ d_\mu &= -2i\varepsilon_{\mu\nu\alpha\beta}K^\nu\psi_\alpha\psi_\beta, \end{aligned}$$

and the measure $\mathscr{M}(e)$ reads:

$$\mathscr{M}(e) = \int \mathscr{D}p \exp\left[\frac{i}{2}\int_0^1 ep^2 \,\mathrm{d}\,\tau\right].$$

3 Pseudoclassical action of Dirac particle interacting with electromagnetic field and electroweakly interacting with matter fields

3.1 Derivation of the action

The integral over the time τ in the exponential expression (10) can be understood as an effective, nondegenerate Lagrangian action. It has two different parts. The first part, which consists of two terms with derivatives $\dot{e}$ and $\dot{\chi}$, can be understood as a gauge fixing term S_{GF},

$$S_{\mathrm{GF}} = \int_0^1 (\pi\dot{e} + \nu\dot{\chi})\,d\tau,$$

that corresponds to the gauge conditions $\dot{e} = \dot{\chi} = 0$. The second part may be treated as a gauge invariant action

$$S = \int_0^1 \left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}\left(M^2 - 2i\mathbb{F}_{\alpha\beta}\psi^\alpha\psi^\beta\right) - \dot{x}^\alpha\left(\mathbb{A}_\alpha - d_\alpha\right) + i\left(\frac{\dot{x}_\alpha\psi^\alpha}{e} - m\psi^4 - \frac{2}{3}\psi^\alpha d_\alpha\right)\chi - i\psi_n\dot{\psi}^n\right]\mathrm{d}\tau \tag{11}$$

of the Dirac particle interacting with electromagnetic field and electroweakly interacting with the matter fields. The derived action is a generalization of Berezin–Marinov action [1] to the background under consideration. It is easy to check that this action is invariant under the gauge transformations (a time reparameterization)

$$\delta x^\alpha = \dot{x}^\alpha\xi\,,\quad \delta e = \frac{d}{d\tau}(e\xi)\,,\quad \delta\chi = \frac{d}{d\tau}(\chi\xi)\,,\quad \delta\psi^n = \dot{\psi}^n\xi$$

with an even parameter ξ.

Lagrangian equations of motion corresponding to action (11) have the following form:

$$\begin{aligned}
\frac{\delta S}{\delta e} &= \frac{1}{e^2}\left(\frac{\dot{x}^2}{2} - i\dot{x}_\alpha\psi^\alpha\chi\right) - \frac{1}{2}\left(M^2 - 2i\mathbb{F}_{\alpha\beta}\psi^\alpha\psi^\beta\right) = 0,\\
\frac{\delta_r S}{\delta\chi} &= \frac{\dot{x}_\alpha\psi^\alpha}{e} - m\psi^4 - \frac{2}{3}\psi^\alpha d_\alpha = 0,\\
\frac{\delta_r S}{\delta\psi^4} &= -2i\dot{\psi}^4 + im\chi = 0,\\
\frac{\delta_r S}{\delta\psi^\alpha} &= 2i\dot{\psi}_\alpha + 2e\left(i\mathbb{F}_{\beta\alpha}\psi^\beta + \frac{2}{3}\partial_\sigma K^\sigma\varepsilon_{\beta\mu\nu\alpha}\psi^\beta\psi^\mu\psi^\nu\right)\\
&\quad + 2id_\alpha\chi - i\frac{\dot{x}_\alpha}{e}\chi - 4i\varepsilon_{\mu\nu\beta\alpha}\dot{x}^\mu K^\nu\psi^\beta = 0,\\
\frac{\delta S}{\delta x^\alpha} &= \frac{d}{d\tau}\left(\frac{\dot{x}_\alpha - i\psi_\alpha\chi}{e} + \mathbb{A}_\alpha - d_\alpha\right) + ie\mathbb{F}_{\mu\nu,\alpha}\psi_\mu\psi_\nu\\
&\quad + 8e(\partial_\alpha\partial_\mu K^\mu)\psi^0\psi^1\psi^2\psi^3 - \dot{x}^\mu(\partial_\alpha\mathbb{A}_\mu)\\
&\quad + \dot{x}^\mu(\partial_\alpha d_\mu) - eK^\mu(\partial_\alpha K_\mu) + \frac{2i}{3}\chi\psi^\mu(\partial_\alpha d_\mu) = 0\,.
\end{aligned} \tag{12}$$

3.2 Quantization of the action

One can quantize the action (11) to confirm its above given interpretation. For this, we first carry out a Hamiltonian analysis of the corresponding pseudoclassical theory following Refs. [16,44] . Since the action (11) is gauge invariant, there are constraint (in particular, first class constraints) in the Hamiltonian formulation. Introducing canonical momenta $p_\alpha,\ P_e,\ P_\chi,\ P_n$ to the coordinates $x^a,\ e,\ \chi,\ \psi^n$, respectively,

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = -\frac{1}{e}\left(\dot{x}_\alpha - i\psi_\alpha\chi\right) - qA_\alpha + d_\alpha\ ,$$
$$P_e = \frac{\partial L}{\partial \dot{e}} = 0,\ P_\chi = \frac{\partial_r L}{\partial \dot{\chi}} = 0,\ P_n = \frac{\partial_r L}{\partial \dot{\psi}^n} = -i\psi_n\ ,$$

we see that primary constraints $\Phi^{(1)} = 0$ are:

$$\Phi_1^{(1)} = P_\chi = 0, \quad \Phi_2^{(1)} = P_e = 0, \quad \Phi_{3n}^{(1)} = P_n + i\psi_n = 0, \tag{13}$$

and the Hamiltonian H reads:

$$H = -\frac{e}{2}\left(\mathscr{P}^2 + 2\mathscr{P}_\alpha d^\alpha + 2i\mathbb{F}_{\alpha\beta}\psi^\alpha\psi^\beta - M^2\right) + i\chi\left(\mathscr{P}_\alpha\psi^\alpha - m\psi^5 + \frac{1}{3}\psi^\alpha d_\alpha\right).$$

Conditions of the conservation of the primary constraints in time allow us to find secondary constraints $\Phi^{(2)} = 0$:

$$\Phi_1^{(2)} = \mathscr{P}_\alpha\psi^\alpha - m\psi^4 + \frac{1}{3}\psi^\alpha d_\alpha = 0, \tag{14}$$

$$\Phi_2^{(2)} = \mathscr{P}^2 + 2\mathscr{P}_\alpha d^\alpha + 2i\mathbb{F}_{\alpha\beta}\psi^\alpha\psi^\beta - M^2 = 0. \tag{15}$$

There are no other connections in the theory. One can see that the Hamiltonian H appears to be proportional to constraints, as one can expect in the case of a reparameterization-invariant theory [45].

One can go over from the initial set of constraints to the equivalent one ($\Phi^{(1)} = 0,\ T_{1,2} = 0$), where:

$$T_{1,2} = \Phi_{1,2}^{(2)} + \frac{i}{2}\left(\partial_r\Phi_{1,2}^{(2)}/\partial\psi^n\right)\Phi_{3n}^{(1)}\ . \tag{16}$$

The new set of constraints can be explicitly divided in a set of first-class constraints, which is $\left(\Phi_{1,2}^{(1)} = 0,\ T_{1,2} = 0\right)$ and in a set of second-class constraints, which is $\Phi_{3n}^{(1)} = 0$.

On this stage, we are able to perform an operator quantization. To this end, we perform only a partial gauge fixing, by imposing the supplementary gauge conditions $\Phi_{1,2}^{\mathrm{G}} = 0$ to the primary first-class constraints $\Phi_{1,2}^{(1)} = 0$,

$$\Phi_1^{\mathrm{G}} = \chi, \quad \Phi_2^{\mathrm{G}} = e - 1/m\ .$$

Thus, on this stage we reduced our Hamiltonian theory to one with the first-class constraints $T_{1,2} = 0$ and the second-class constraints $\varphi = 0$, $\varphi = \left(\Phi_{1,2}^{(1)},\ \Phi^{\mathrm{G}}\right)$. Generalizing the Dirac quantization method for systems with first-class constraints [44], we believe that commutation relations between operators are calculated according to Dirac brackets with respect to the second-class constraints $\varphi = 0$ only. At the same time second-class constraints operators must be equal zero. The operators $\hat{T}_{1,2}$ corresponding to the first-class constraints $T_{1,2} = 0$ are not zero, but define restrictions on physical state vectors, they annulate the physical states.

The second-class constraints $\varphi = 0$ have a special form [16], one can use it for direct eliminating of the variables e, P_e, χ, and P_χ, from the consideration. In this case, the Dirac brackets with respect to the constraints φ for the variables x, p and ψ^n are reduced to ones with respect to the constraints $\Phi_{3n}^{(1)}$ and can be easily calculated. The only nonvanishing Dirac brackets of this kind are:

$$\left\{x^\alpha, p_\beta\right\}_{D(\Phi_{3n}^{(1)})} = \delta_\beta^\alpha, \ \left\{\psi^n, \psi^m\right\}_{D(\Phi_{3n}^{(1)})} = \frac{i}{2}\eta^{nm} .$$

They define commutation (anticommutation) relations for the operators $\hat{x}$, $\hat{p}$ and $\hat{\psi}^n$ that correspond to the variables x, p and ψ^n, respectively,

$$\begin{aligned} \left[\hat{x}^\alpha, \hat{p}_\beta\right]_- &= i\left\{x^\alpha, p_\beta\right\}_{D(\Phi_{3n}^{(1)})} = i\delta_\beta^\alpha, \\ \left[\hat{\psi}^n, \hat{\psi}^m\right]_+ &= i\left\{\psi^n, \psi^m\right\}_{D(\Phi_{3n}^{(1)})} = -\frac{1}{2}\eta^{nm}. \end{aligned} \tag{17}$$

Besides, the following operator equations hold true:

$$\hat{\Phi}_{3n}^{(1)} = \hat{P}_n + i\hat{\psi}_n = 0. \tag{18}$$

Operators that satisfy commutation relations (17) and Eq. (18) have to be realized in a Fock space. As such a space, we chose a space of four columns $\Psi(x)$ (bispinors) that depend on x^α. At the same time, we chose $\hat{x}^\alpha$ to be operators of multiplication by x^α, and

$$\hat{p}_\alpha = -i\partial_\alpha, \ \hat{\psi}^\alpha = \frac{i}{2}\gamma^5\gamma^\alpha, \ \hat{\psi}^4 = \frac{i}{2}\gamma^5 . \tag{19}$$

The first-class constraints $\hat{T}_{1,2}$ as operators have to annihilate physical vectors $\hat{T}_{1,2}\Psi(x) = 0$. One can easily verify that in virtue of Eqs. (19), (16) and (18), these conditions are reduced to following ones:

$$\hat{T}_{1,2}\Psi(x) = 0 \Longrightarrow \begin{cases} \hat{\Phi}_1^{(2)}\Psi(x) = 0 \\ \hat{\Phi}_1^{(2)}\Psi(x) = 0 \end{cases}, \tag{20}$$

where $\hat{\Phi}_{1,2}^{(2)}$ are operators, which correspond to classical functions from Eqs. (14) and (15). There is no ambiguity in the construction of the operator $\hat{\Phi}_1^{(2)}$. Thus, taking into account the above realizations of the commutation relations, we see that the first equation $\hat{\Phi}_1^{(2)}\Psi(x) = 0$ reproduces the Dirac equation (5),

$$\hat{\Phi}_1^{(2)}\Psi(x) = 0 \Longrightarrow \left[\mathscr{P}_\nu\gamma^\nu - m - \frac{i}{3!}K^\alpha(x)\varepsilon_{\alpha\beta\mu\nu}\gamma^\beta\gamma^\mu\gamma^\nu\right]\Psi(x) = 0. \tag{21}$$

As to the construction of the operator $\hat{\Phi}_2^{(2)}$, according to the classical function $\Phi_2^{(2)}$ from Eq. (15), we meet here an ordering problem since, in the general case, the function $\Phi_2^{(2)}$ contains terms with products of the momenta and some functions of the coordinates. For such terms we choose the symmetrized (Weyl) form of the corresponding operators, which, in particular, provides the hermicity of the operator $\hat{\Phi}_2^{(2)}$. At the same time, one can see that such a correspondence rule provides the consistency of the two equations (20). Indeed, in this case, it turns out that $\hat{\Phi}_2^{(2)} = \left(\hat{\Phi}_1^{(2)}\right)^2$, such that second equation (20) appears to be a consequence of first equation (20), and therefore is equivalent to Dirac equation (21).

Thus, we see that the presented above operator quantization of action (11) reproduces the quantum theory of spinning particle interacting with electromagnetic fields and electroweakly interacting with matter fields. This fact justifies the above interpretation form of the action.

3.3 Nonrelativistic limit

Let us consider equations (12) in the nonrelativistic (low-energy) limit. To simplify the analysis, we chose the gauge conditions $\chi = 0$ and $e = 1/m$ that had appeared in the above Hamiltonian formulation. Following Berezin and Marinov [1], we realize σ-matrices as as $\sigma_k = 2i\,\varepsilon_{kjl}\psi^l\psi^j$, such that

$$\psi^j\psi^l = \frac{i}{4}\,\varepsilon^{kjl}\sigma_k\,, \quad \dot{\psi}^j\psi^l = \frac{i}{4}\,\varepsilon^{kjl}\dot{\sigma}_k\,.$$

In the limit under consideration, we have:

$$\psi^0 \approx 0\,, \quad \dot{x}^0 \approx 1\,, \quad \dot{x}^i \approx \upsilon^i = \frac{dx^i}{dx^0}\,.$$

Using standard expressions for stress tensor components, see definition (7),

$$\mathbb{F}_{0i} = -\tilde{E}_i = \partial_0\mathbb{A}_i - \partial_i\mathbb{A}_0\,, \quad \mathbb{F}_{ij} = \varepsilon_{ijk}\,\tilde{H}^k, \quad \tilde{H}^k = \frac{1}{2}\varepsilon^{kij}\mathbb{F}_{ij}$$

in Eq. (12) and disregarding terms of higher orders in external fields, we obtain:

$$\begin{aligned} m\dot{\boldsymbol{\upsilon}} &= \tilde{\mathbf{E}} + \frac{1}{c}\left[\boldsymbol{\upsilon}\times\tilde{\mathbf{H}}\right] - \nabla\left(\boldsymbol{\sigma}\mathbf{K}\right) - \frac{1}{c}\frac{d}{dt}(\boldsymbol{\sigma}K_0) \\ &\quad + \frac{1}{c}\left(\boldsymbol{\upsilon}\boldsymbol{\sigma}\right)\nabla K_0, \\ \dot{\boldsymbol{\sigma}} &= \left[\frac{2}{\hbar}\left(\frac{\hbar}{2mc}\tilde{\mathbf{H}} + \mathbf{K} - K_0\frac{\boldsymbol{\upsilon}}{c}\right)\times\boldsymbol{\sigma}\right], \end{aligned} \tag{22}$$

where

$$\tilde{E}_i = qE_i - (\partial_0 V_i - \partial_i V_0)\,, \quad \tilde{H}^k = qH^k + \frac{1}{2}\varepsilon^{kij}\left(\partial_i V_j - \partial_j V_i\right),$$

where E_i and H_i are three-dimensional components of the electric field and magnetic fields, respectively.

We note that in the absence of the potentials $V^\mu_{L,R}$ of the electroweak interaction ($V^\mu = K^\mu = 0$, see Eq. (3)) with a background matter, the low-energy limit equations coincide with ones that could be obtained from the Pauli equation. In this regard, it would be interesting to derive an analogue of the Pauli equation in the background under consideration and compare its low-energy limit with equations (22).

Acknowledgements The authors are thankful to M. Dvornikov for helpful discussions. The work is supported by Russian Science Foundation (Grant No. 19-12-00042).

Data Availability Statement This manuscript has associated data in a data repository. [Authors' comment: All data generated or analysed during this study are included in this published article.]

References

1. F.A. Berezin, M.S. Marinov, JETP Lett. **21**, 320 (1975)
2. F.A. Berezin, M.S. Marinov, Ann. Phys. (NY) **104**, 336 (1977)

3. R. Casalbuoni, Nuovo Cimento A **33**, 115 (1976)
4. A. Barducci, R. Casalbuoni, L. Lusanna, Nuovo Cimento A **35**, 377 (1976)
5. L. Brink, S. Deser, B. Zumino, P. Di Vechia, P. Howe, Phys. Lett. B **64**, 435 (1976)
6. L. Brink, P. Di Vechia, P. Howe, Nucl. Phys. B **118**, 76 (1977)
7. M. Henneaux, C. Teteilboim, Ann. Phys. (NY) **143**, 127 (1982)
8. K. Sundermeyer, *Constrained Dynamics*, Lect. Notes Phys. **169** (Springer, Berlin, 1982)
9. P.D. Mannheim, Phys. Rev. D **32**, 898 (1985)
10. P.D. Mannheim, Phys. Lett. B **166**, 191 (1986)
11. D.M. Gitman, I.V. Tyutin, Class. Quant. Grav. **7**, 2131 (1990)
12. E.S. Fradkin, D.M. Gitman, Phys. Rev. D **44**, 3230 (1991)
13. S.P. Gavrilov, D.M. Gitman, Int. J. Mod. Phys. A **15**, 4499 (2000)
14. S.P. Gavrilov, D.M. Gitman, Class. Quant. Grav. **17**, L133 (2000)
15. S.P. Gavrilov, D.M. Gitman, Class. Quant. Grav. **18**, 2989 (2001)
16. D.M. Gitman, I.V. Tyutin, *Quantization of Fields with Constraints* (Springer, Berlin, 1990)
17. D.M. Gitman, A.V. Saa, Mod. Phys. Lett. A **8**, 463 (1993)
18. D.M. Gitman, A.V. Saa, Class. Quant. Grav. **10**, 1447 (1993)
19. D.M. Gitman, V.G. Kupriyanov, Eur. Phys. J. C **54**, 325 (2008)
20. D.M. Gitman, A.E. Gonçalves, I.V. Tyutin, Phys. Rev. D **50**, 5439 (1994)
21. D.M. Gitman, A.E. Gonçalves, I.V. Tyutin, Int. J. Mod. Phys. A **10**, 701 (1995)
22. D.M. Gitman, A.E. Gonçalves, I.V. Tyutin, Phys. At. Nucl. **60**, 748 (1997)
23. D.M. Gitman, A.E. Gonçalves, J. Math. Phys. **38**, 2167 (1997)
24. D.M. Gitman, A.E. Gonçalves, Int. J. Theor. Phys. **35**, 2427 (1996)
25. D.M. Gitman, I.V. Tyutin, Mod. Phys. Lett. A **11**, 381 (1996)
26. D.M. Gitman, I.V. Tyutin, Int. J. Mod. Phys. A **12**, 535 (1997)
27. D.M. Gitman, Nucl. Phys. B **488**, 490 (1997)
28. R. Fresneda, S.P. Gavrilov, D.M. Gitman, PYu. Moshin, Class. Quant. Grav. **21**, 1419 (2004)
29. B. Geyer, D.M. Gitman, I.L. Shapiro, Int. J. Mod. Phys. A **15**, 3861 (2000)
30. R. Fresneda, D.M. Gitman, Int. J. Mod. Phys. A **23**, 835 (2008)
31. D.M. Gitman, S.I. Zlatev, Mod. Phys. Lett. A **12**, 2435 (1997)
32. D.M. Gitman, S.I. Zlatev, Phys. Rev. D **55**, 7701 (1997)
33. D.M. Gitman, S.I. Zlatev, P.B. Barros, J. Phys. A: Math. Gen. **31**, 7791 (1998)
34. C. Schubert, Phys. Rept. **355**, 73 (2001)
35. N. Ahmadiniaz et al., JHEP **08**, 018 (2020)
36. S.M. Kuzenko, J.V. Yarevskaya, Mod. Phys. Lett. A **11**, 1653 (1996)
37. S.L. Lyakhovich, A.Y. Segal, A.A. Sharapov, Phys. Rev. D **54**, 5223 (1996)
38. D.T. Son, N. Yamamoto, Phys. Rev. D **87**, 085016 (2013)
39. D.E. Kharzeev, Prog. Part. Nucl. Phys. **75**, 133 (2014)
40. A. Vilenkin, Phys. Rev. D **22**, 3080 (1980)
41. A.I. Studenikin, J. Phys. A **41**, 164047 (2008)
42. M. Dvornikov, in *Neutrinos: Properties, Sources and Detection*, ed. By J.P. Greene (Nova Science Publishers, New York, 2011), pp. 23-90
43. J. Schwinger, Phys. Rev. **82**, 664 (1951)
44. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964)
45. G. Fulop, D.M. Gitman, I.V. Tyutin, in *Topics in Theoretical Physics II, Festschrift for Abraham H. Zimerman*, ed. By H. Aratyn, L. Ferreira, J. Gomes, (IFT/UNESP, São Paulo, 1998), pp. 286–295