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THE EFFECTS OF BOUNDARY ROUGHNESS ON THE MHD DUCT FLOW WITH SLIP HYDRODYNAMIC CONDITION

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Abstract. In this paper we present the analytical study of the magnetohydrodynamic (MHD) flow through a rectangular duct driven by the pressure gradient and under the action of the transverse magnetic field. Motivated by various MHD applications in which hydrodynamic slip naturally occur, we prescribe the slipping boundary condition on the upper boundary which contains irregularities as well. Depending on the period of the boundary roughness, we derive three different limit problems by using rigorous analysis in the appropriate functional setting. This approach also enables us to determine the relative contribution of the MHD effect and the slip itself in the governing coupled system satisfied by the velocity and induced magnetic field.

1. Introduction. The magnetohydrodynamic (MHD) effects naturally appear in numerous industrial and biological applications such as MHD pumps, generators, cooling devices in nuclear fusion reactors, blood flow measurement tools, accelerators, etc. The increasing number of those devices has motivated the researchers to try to generalize the available hydrodynamic solutions to include the MHD effects for electrically conducting liquids. Due to the strong coupling of the equations of fluid dynamics and electromagnetism, the corresponding MHD equations have been mainly treated by different numerical methods, while analytical solutions are available mostly for some special (ideal) geometries of the problem region and under simple boundary conditions (see e.g. [13]).

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The increasing number of MHD applications where the duct flow under the action of a transverse magnetic field naturally appear motivated the researchers to try to generalize the available hydrodynamic solutions. The pioneering results in this field are due to Hunt [16, 17] in which the analytical solutions have been provided for laminar motion in two settings: (i) non-conducting walls parallel to the applied magnetic field and perfectly conducting walls perpendicular to the field, and (ii) perfectly conducting walls parallel to the imposed magnetic field and non-conducting walls perpendicular to the field. In view of the practical applications, the latter setting seems to be of more interest but it is more challenging from the analytical point of view due to the appearance of the boundary layers (see [17]). We would also like to mention the paper by Sezgin [30] where the analytical solution has been proposed by reducing the problem to the solution of a Fredholm integral equation of the second kind, which he solved numerically. Bluck and Wolfendale [7] proposed the analytical solution to the laminar flow in an array of partially conducting ducts of arbitrary wall thickness and indicated the limitations of the thin-wall approximations developed by Hunt. Interesting numerical results on the subject can be found in [6, 8, 14, 31, 33, 34].

In this paper we study a steady-state flow through a rectangular duct driven by the pressure gradient along a duct. The fluid is taken as viscous, incompressible and having uniform electrical conductivity. A uniform magnetic field of the constant intensity is applied perpendicular to the duct. We suppose that no secondary flow is produced, so there is no variation of the fluid movement and induced magnetic field in the duct axis z-direction, but they are subjected to change on the cross-section of the duct which is a region in the xy–plane. Since, in the practice, the boundaries almost always contain various small irregularities (rugosities, dents, etc.), we assume that the upper boundary of the cross-section of the duct is not perfectly smooth. Namely, we consider the following open set as our domain:

$$\Omega^\varepsilon = \left\{ (x,y) \in \mathbb{R}^2 \,:\, -c < x < c,\ -1 < y < 1 + \epsilon\, \psi(x)\, G\left(\frac{x}{\epsilon^\alpha}\right) \right\}, \quad 0 < \epsilon \ll 1. \tag{1.1}$$

Here we set the roughness on the upper boundary by a 1-periodic smooth function G. $\alpha \geq 0$ is a parameter which establishes the order of the roughness and ψ is a $\mathcal{C}^\infty$-function with compact support in $(-c, c)$ which places the perturbation (see Fig. 1.1 as an example). Notice that, when ϵ goes to zero, the perturbed domain Ω^ϵ uniformly converges to $\Omega = (-c, c) \times (-1, 1)$.

In MHD applications such as heat exchangers, where the fluid is interacted with the solid wall, a good wettability is to be expected at the liquid-solid interface. In this case, it is reasonable to prescribe the no-slip (zero) boundary condition for the velocity. Although many experimental studies support the no-slip condition, in some applications such as microfluidic devices or even fusion reactors with liquid metal flows in contact with ceramics, the slip in the MHD flow is likely to occur (see [11, 28] and the references therein). The slip length is defined as the distance between the liquid and the solid surface where the extrapolated fluid flow velocity vanishes, leading to a mixed (Robin-type) boundary condition for the velocity. Thus, in the present paper, our aim is to rigorously study the MHD flow in the domain Ω^ϵ by taking into account the presence of slip on the oscillating part of the boundary.

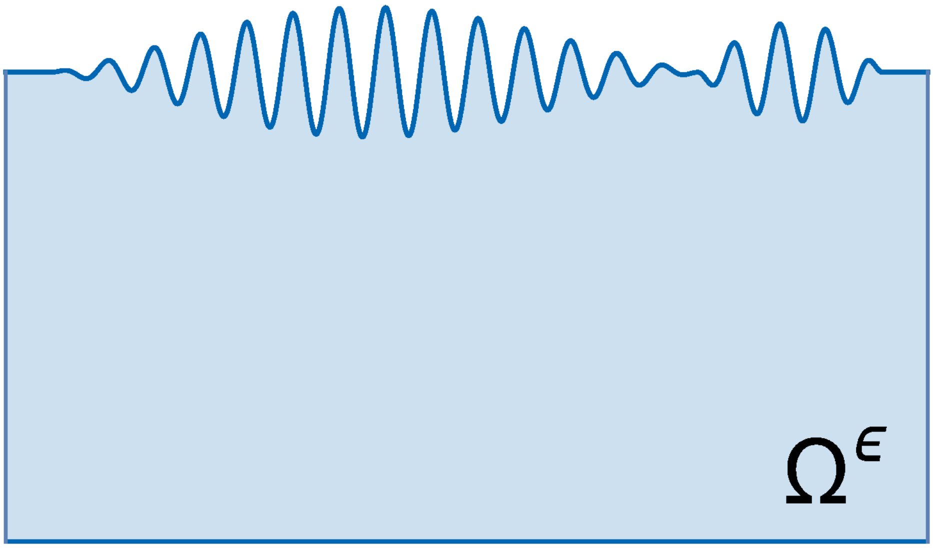

FIG. 1.1. Cross section of the duct with rough upper boundary

The paper is organized as follows. In Section 2, we formulate the problem given by the (coupled) MHD equations and the corresponding boundary conditions for the velocity and the induced magnetic field. For the sake of the reader's convenience, we also summarize the main results by providing the derived limit systems describing the effective flow. In Section 3, we first discuss the well-posedness of the governing system (see Section 3.1) and then provide some technical results focusing on the ones regarding the extension operators which are crucial for the analysis in the sequel (see Section 3.2). Section 4 represents the main part of this paper containing the proofs of the results on the asymptotic behavior of the flow, as $\epsilon \to 0$. Depending on the order of the boundary roughness α, we rigorously derive three different limit problems in Ω. Namely, for $\alpha \in [0, 1)$ (weak regime, i.e. slightly corrugated boundary), we obtain the effective system in the same form as the starting one, with no roughness-induced effects observed in the slip condition (see Theorem 4.1). In the same theorem, we prove the result for $\alpha = 1$, i.e. in the resonant case where the orders of the amplitude and the period of the boundary roughness are equal. This turns out to be the critical regime since the effective condition on the upper boundary is corrected by the reaction coefficient term confirming that the boundary roughness is affecting the flow. We also resolve the remaining case $\alpha > 1$ (large boundary roughness). Assuming for simplicity ψ nonnegative in $(-c, c)$ and satisfying $\psi(-c) = \psi(c) = 0$, we deduce the limit problem endowed with homogeneous Dirichlet boundary conditions on $y = 1$ (see Theorem 4.2). It should be emphasized that the undertaken approach based on the extension operators allows us to elegantly handle the MHD duct flow with perfectly conducting walls parallel to the imposed magnetic field and non-conducting walls perpendicular to the field. This setting cannot be treated by direct approach via Taylor series (as we did in [19, 24]) since the no-slip boundary conditions at $x = -c, c$ cannot be satisfied. Last but not least, in Section 5, using the analysis in the appropriate functional setting we also discuss how the slip itself (i.e. its order of magnitude) influence on the effective behavior of the MHD flow in a simple rectangular duct (without boundary distortion). By doing that, we can determine the relative contribution of the MHD effect and the slipping condition, motivated by the discussion from [32].

To finish the Introduction, let us provide more bibliographic remarks on the subject. Although one can find many papers on the MHD duct flows throughout the literature, the analytical results on the influence of the slip hydrodynamic condition on such flows are very sparse throughout the literature. We already mentioned the paper by Smolentsev [32] in which he considered the MHD flow in a simple rectangular channel with non-conducting walls. Rivero and Cuevas [29] investigated the effect of the slip length on the flow rate in MHD micropumps. They observed that the solution of the MHD flow cannot be derived analytically if the slip condition is prescribed on the entire boundary. The effects of small boundary perturbation ($\alpha = 0$, $\Psi \equiv 1$) on the MHD duct flow have been studied analytically in [19, 24] using the asymptotic expansion technique employed earlier for the different regimes of the fluid flow (see e.g. [20–23]). In the present paper, we allow different orders between the amplitude and the period of the boundary roughness which forces us to completely change the approach and employ mathematical tools developed in [2, 3, 12, 25, 26]. To conclude, we believe that the provided rigorous results could prove useful for the engineering practice, in particular with regards to numerical simulations of the MHD problems which are affected by the wall irregularities and allow slippage as well.

2. Setting of the problem and main result. Statement of the problem. As described in the Introduction, we study the steady-state flow of an incompressible conducting fluid governed by a pressure gradient along a duct, under an applied transverse magnetic field. We suppose that no secondary flow is produced and that there are no variations of the fluid movement and induced magnetic field along the duct (in the z-direction). As a consequence, all physical quantities (except pressure) are independent of z and the velocity and the induced magnetic field have only z-components which are subjected to change on the cross-section of the duct (in xy–plane). Finally, we assume that the induced magnetic field due to the motion of the fluid does not disturb the applied magnetic field so the latter can be taken as the constant field of flux density in y-direction. In view of that, starting from the steady-state Navier-Stokes system coupled with Maxwell's equations we arrive at the following (non-dimensional) system describing the MHD duct flow (see e.g. [7, 16] for details):

$$\frac{\partial^2 u^\epsilon}{\partial x^2} + \frac{\partial^2 u^\epsilon}{\partial y^2} + M\frac{\partial H^\epsilon}{\partial y} + 1 = f^\epsilon, \qquad \text{in } \Omega^\epsilon\,, \tag{2.1}$$

$$\frac{\partial^2 H^\epsilon}{\partial x^2} + \frac{\partial^2 H^\epsilon}{\partial y^2} + M\frac{\partial u^\epsilon}{\partial y} = g^\epsilon, \qquad \text{in } \Omega^\epsilon\,. \tag{2.2}$$

In the above system, the unknowns are the fluid velocity u^ϵ and the induced magnetic field H^ϵ. Note that we have one non-dimensional parameter M (Hartmann number [15]) appearing in the governing equations which depends on the flux density, fluid viscosity and conductivity. The functions f^ϵ and g^ϵ are given in the system (2.1)–(2.2) representing the external body and magnetic force applied to the cross-section.

Let us impose the boundary conditions following the guidelines from the Introduction:

$$u^\epsilon + \beta \frac{du^\epsilon}{d\mathbf{n}^\epsilon} = 0, \quad H^\epsilon = 0, \qquad \text{for } \ y = 1 + \epsilon\, \psi(x)\, G\left(\frac{x}{\epsilon^\alpha}\right), \tag{2.3}$$

$$u^\epsilon = 0, \ \ H^\epsilon = 0, \qquad \text{for } \ y = -1\,, \tag{2.4}$$

$$u^\epsilon = 0, \ \ \frac{\partial H^\epsilon}{\partial x} = 0, \qquad \text{for } \ x = -c, c\,. \tag{2.5}$$

We prescribe a slip boundary condition on the upper rough wall, which, in the case of a unidirectional flow, reduce to $(2.3)_1$. Here $\beta = \frac{L_s}{a}$ denotes the dimensionless slip length with a being the characteristic value related to the domain's width and L_s the (dimensional) slip coefficient. The lower wall ($y = -1$) and the side walls ($x = -c, c$) are assumed to have no-slip. The boundary conditions for the induced magnetic field describe the situation with perfectly conducting walls parallel to the imposed magnetic field and non-conducting walls perpendicular to the field. As mentioned above, this setting seems to be more interesting from the practical point of view (see [17]) and also more challenging from the mathematical point of view since we cannot use the approach from [19, 24] anymore.

Summary of the different asymptotic regimes. The main goal of this paper is to investigate the effective behavior of the flow described by (2.1)–(2.5), as $\epsilon \to 0$. In summary (see Theorems 4.1 and 4.2), we establish the following asymptotic regimes depending on the order of the boundary roughness α:

- **Critical regime** ($\alpha = 1$) and **weak regime** ($\alpha \in [0, 1)$)

  The limit systems read as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + M \frac{\partial H}{\partial y} + 1 = f, & \text{in } \ \Omega\,, \\ \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + M \frac{\partial u}{\partial y} = g, & \text{in } \ \Omega\,, \\ \gamma\, u + \beta \frac{du}{d\mathbf{n}} = 0, \quad H = 0, & \text{for } \ y = 1\,, \\ u = 0, \ \ H = 0, & \text{for } \ y = -1\,, \\ u = 0, \ \ \frac{\partial H}{\partial x} = 0, & \text{for } \ x = -c, c\,, \end{cases} \tag{2.6}$$

  where the boundary coefficient γ is given by

$$\gamma(x) = \begin{cases} \int_0^1 \sqrt{1 + \psi(x)^2 G'(y)^2}\, dy & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha \in [0, 1). \end{cases} \tag{2.7}$$

- **Large-roughness regime** ($\alpha > 1$)
  The limit system takes the following form:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + M\frac{\partial H}{\partial y} + 1 = f, & \text{in } \Omega\,, \\ \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + M\frac{\partial u}{\partial y} = g, & \text{in } \Omega\,, \\ u = 0, \quad H = 0, & \text{for } y = 1\,, \\ u = 0, \; H = 0, & \text{for } y = -1\,, \\ u = 0, \; \frac{\partial H}{\partial x} = 0, & \text{for } x = -c, c\,. \end{cases} \tag{2.8}$$

3. Preliminary results. In this section, we first discuss the well-posedness of the system (2.1)–(2.5). Next, we introduce some technical results needed for the analysis in the sequel. Among them, we focus on the existence of extension operators associated to our domain perturbation problem.

3.1. *Existence of solution.* In order to show the existence and uniqueness of the solution for each $\epsilon \geq 0$, we consider the following bilinear form $a_\epsilon : \mathcal{H}_\epsilon \times \mathcal{H}_\epsilon \mapsto \mathbb{R}$ set by

$$a_\epsilon((u, H), (\varphi, \phi)) = \int_{\Omega^\epsilon} \{\nabla u \nabla\varphi + \nabla H \nabla\phi - M\,(\varphi\partial_y H + \phi\partial_y u)\}\, dydx.$$

Here, $\mathcal{H}_\epsilon$ is the Hilbert space

$$\mathcal{H}_\epsilon = \left\{(\varphi, \phi) \in H^1(\Omega^\epsilon) \times H^1(\Omega^\epsilon) \;:\; \varphi = 0 \text{ on } \Gamma_s \cup \Gamma_l \text{ and } \phi = 0 \text{ on } \Gamma_\epsilon \cup \Gamma_l\right\},$$

where Γ_s is the side part of $\partial\Omega^\epsilon$, Γ_l its lower part and Γ^ϵ is the upper boundary. More precisely,

$$\Gamma_s = \{(-c, y) \in \mathbb{R}^2 \;:\; y \in (-1, 1)\} \;\cup\; \{(c, y) \in \mathbb{R}^2 \;:\; y \in (-1, 1)\}$$
$$\Gamma_l = \{(x, -1) \in \mathbb{R}^2 \;:\; x \in (-c, c)\} \quad \text{ and}$$
$$\Gamma_\epsilon = \left\{(x, y) \in \mathbb{R}^2 \;:\; x \in (-c, c) \;\text{ and }\; y = 1 + \epsilon\psi(x) G\left(\frac{x}{\epsilon^\alpha}\right)\right\}.$$

Also, we set

$$\Gamma = \Gamma_0 = \{(x, 1) \in \mathbb{R}^2 \;:\; x \in (-c, c)\}.$$

Remark 1. Since we are using Dirichlet boundary conditions, we can set the following norm to $\mathcal{H}_\epsilon$:

$$\|(\varphi, \phi)\|_{\mathcal{H}_\epsilon} = \sqrt{\int_{\Omega^\epsilon} \{|\nabla\varphi|^2 + |\nabla\phi|^2\}\, dxdy}$$

which is given by the inner product $(u, H) \cdot (\varphi, \phi) = \int_{\Omega^\epsilon} \{\nabla u \nabla\varphi + \nabla H \nabla\phi\} dxdy$. Notice that this norm is well defined since it holds the Poincaré's inequality in Ω^ϵ. Namely, we have

$$
\begin{aligned}
\int_{\Omega^\epsilon} u(x,y)^2 dxdy &= \int_{\Omega^\epsilon} \left(u(x,y) - u(x,-1)\right)^2 dxdy \\
&\leq \int_{\Omega^\epsilon} \left(\int_{-1}^{y} \frac{\partial u}{\partial y}(x,s) ds \right)^2 dxdy \\
&\leq 2 \int_{\Omega^\epsilon} \left(\frac{\partial u}{\partial y}(x,s) \right)^2 (2 + \epsilon\psi(x) G(x/\epsilon^\alpha))\, dsdx \\
&\leq 2 \left(2 + \epsilon \|\psi\|_{L^\infty(-c,c)} \|G\|_{L^\infty(\mathbb{R})}\right) \|\nabla u\|^2_{L^2(\Omega^\epsilon)}.
\end{aligned}
$$

Hence, there exists a constant $C_P > 0$ independent of $\epsilon \in [0,1]$ such that

$$
\|u\|_{L^2(\Omega^\epsilon)} \leq C_P \|\nabla u\|_{L^2(\Omega^\epsilon)}.
$$

REMARK 2. It is not difficult to deduce that a_ϵ set a continuous and uniformly coercive bilinear form for any ϵ varying in $[0,1]$. Indeed, we can combine Hölder and Cauchy-Schwarz inequalities to obtain

$$
|a_\epsilon((u,H),(\varphi,\phi))| \leq (1+M) \|(u,H)\|_{H^1(\Omega^\epsilon)\times H^1(\Omega^\epsilon)} \|(\varphi,\phi)\|_{H^1(\Omega^\epsilon)\times H^1(\Omega^\epsilon)}.
$$

Also, for any $(u,H) \in \mathcal{H}_\epsilon$, we may integrate by part one of the terms multiplied by M and use the given boundary conditions to get

$$
\begin{aligned}
a_\epsilon((u,H),(u,H)) &= \int_{\Omega^\epsilon} \left\{ |\nabla u|^2 + |\nabla H|^2 - M \left(u \operatorname{div} \begin{pmatrix} 0 \\ H \end{pmatrix} + H \partial_y u \right) \right\} dxdy \\
&= \int_{\Omega^\epsilon} \left\{ |\nabla u|^2 + |\nabla H|^2 + M \left(H\partial_y u - H \partial_y u \right) \right\} - M \int_{\partial\Omega^\epsilon} u\, H\, \mathbf{n}_2^\epsilon \, dS \\
&= \int_{\Omega^\epsilon} \left\{ |\nabla u|^2 + |\nabla H|^2 \right\} dxdy \\
&= \|(u,H)\|^2_{\mathcal{H}_\epsilon}.
\end{aligned}
$$

REMARK 3. As a consequence of the previous remarks, from Lax-Milgram Theorem we obtain that, for any M non-negative and β positive, and for any f^ϵ and $g^\epsilon \in L^2(\Omega^\epsilon)$, there exists a unique solution $(u^\epsilon, H^\epsilon) \in \mathcal{H}_\epsilon$ of the system (2.1)–(2.2) satisfying the boundary conditions (2.3)–(2.5). More precisely,

$$
a_\epsilon((u^\epsilon,H^\epsilon),(\varphi,\phi)) + \beta^{-1} \int_{\Gamma^\epsilon} u^\epsilon \varphi \, dS = \int_{\Omega^\epsilon} \{1 - f^\epsilon\} \varphi \, dxdy - \int_{\Omega^\epsilon} g^\epsilon \phi \, dxdy, \quad \forall (\varphi,\phi) \in \mathcal{H}_\epsilon. \tag{3.1}
$$

Finally, we observe that, under appropriate conditions on the forcing terms f^ϵ and g^ϵ, the solutions are uniformly bounded.

PROPOSITION 3.1. Let us assume f^ϵ, $g^\epsilon \in L^2(\Omega^\epsilon)$ uniformly bounded in ϵ. Then, the family of solutions $(u^\epsilon, H^\epsilon) \in \mathcal{H}_\epsilon$ is also uniformly bounded in ϵ.

Proof. Let us take the solutions $(u^\epsilon, H^\epsilon) \in \mathcal{H}_\epsilon$ given by Remark 3. Then, by Remark 2, we have

$$\begin{aligned}\|(u^\epsilon, H^\epsilon)\|^2_{\mathcal{H}_\epsilon} &\le a_\epsilon((u^\epsilon, H^\epsilon),(u^\epsilon, H^\epsilon)) + \beta^{-1}\int_{\Gamma_\epsilon} {u^\epsilon}^2 dS \\ &\le \|1-f^\epsilon\|_{L^2(\Omega^\epsilon)}\|u^\epsilon\|_{L^2(\Omega^\epsilon)} + \|g^\epsilon\|_{L^2(\Omega^\epsilon)}\|H^\epsilon\|_{L^2(\Omega^\epsilon)} \qquad (3.2)\\ &\le C_P \max\{\|1-f^\epsilon\|_{L^2(\Omega^\epsilon)}, \|g^\epsilon\|_{L^2(\Omega^\epsilon)}\}\left(\|\nabla u^\epsilon\|_{L^2(\Omega^\epsilon)} + \|\nabla H^\epsilon\|_{L^2(\Omega^\epsilon)}\right).\end{aligned}$$

Consequently, the family of solutions are uniformly bounded in $\mathcal{H}_\epsilon$. □

REMARK 4. Notice that the proof of Proposition 3.1 also yields

$$\beta^{-1}\int_{\Gamma_\epsilon} {u^\epsilon}^2 dS \le C_P \max\{\|1-f^\epsilon\|_{L^2(\Omega^\epsilon)}, \|g^\epsilon\|_{L^2(\Omega^\epsilon)}\}\left(\|\nabla u^\epsilon\|_{L^2(\Omega^\epsilon)} + \|\nabla H^\epsilon\|_{L^2(\Omega^\epsilon)}\right).$$

Hence, $\|u^\epsilon\|_{L^2(\Gamma_\epsilon)}$ is uniformly bounded in ϵ as well.

3.2. *Extension operators.* In order to pass to limit in the solutions, we need to introduce an appropriate functional setting. It should be observed that the perturbed open sets Ω^ϵ vary with respect to ϵ, and, thus, we need a method to compare families of functions posed in different domains to obtain the effective equations when ϵ goes to zero.

Here we will compare the solutions in two different manners. If the parameter α introduced in (1.1) (the order of the boundary roughness) is less than or equal to one, we will extend all the solutions from Ω^ϵ into a fixed bounded open rectangle $U \subset \mathbb{R}^2$ satisfying $\Omega^\epsilon \subset U$ for all $\epsilon \in [0,1]$. We set $U = (-c,c)\times(-1,a)$ with $a \ge 1$ given by

$$a = \max\left\{1, \max_{(x,y)\in[-c,c]\times\mathbb{R}} 1 + \psi(x)G(y)\right\}. \qquad (3.3)$$

On the other hand, if $\alpha > 1$, we will extend the solutions a fixed domain $K \subset \Omega$ into Ω^ϵ. For the case $\alpha \in [0,1]$, we will work in according to [2,25,26]. If $\alpha > 1$, we need to change approach and follow [3,12] (see also [18]). This is necessary because we are allowing large roughness on the boundary which is performed when $\alpha > 1$. In this case, it does not exist an available extension operator from Ω^ϵ into $\mathbb{R}^2$ whose embedding constants of the Sobolev inequalities are uniformly bounded for $\epsilon \in [0,1]$.

We have the following result regarding the extension operators:

PROPOSITION 3.2. Let Ω^ϵ be the family of domains given by (1.1) and $U = (-c,c)\times(-1,a) \subset \mathbb{R}^2$ with a given by (3.3). Then, $\Omega^\epsilon \subset U$ for all $\epsilon \in [0,1]$, and,

(i) if $\alpha \in [0,1]$, there exists a family of extension operators $P_\epsilon : L^2(\Omega^\epsilon) \mapsto L^2(U)$ satisfying

$$\|P_\epsilon u^\epsilon\|_{L^2(U)} \le C_0\|u^\epsilon\|_{L^2(\Omega^\epsilon)} \quad \text{and} \quad \|P_\epsilon u^\epsilon\|_{H^1(U)} \le C_1\|u^\epsilon\|_{H^1(\Omega^\epsilon)}$$

for some positive constants C_0 and C_1 independent of ϵ.

(ii) In particular, for any $\alpha \ge 0$, there exist $P_\eta : L^2(K_\eta) \mapsto L^2(\Omega^\epsilon)$ such that

$$\|P_\eta u\|_{L^2(\Omega^\epsilon)} \le C_0\|u\|_{L^2(K_\eta)} \quad \text{and} \quad \|P_\eta u\|_{H^1(\Omega^\epsilon)} \le C_1\|u\|_{H^1(K_\eta)}$$

for positive constants C_0 and C_1 independent of ϵ where $K_\eta = (-c,c)\times(-1,1-\eta)$ for $\eta \ge 0$.

Proof. The proof of the item (i) can be found in [2, 4]. Item (ii) refers to the combination between the extension operator from a fixed Lipschitz domain K_η into the whole space (see for instance [27]) and the restriction operator from of the whole space to Ω^ϵ. □

Next, we state two technical results needed to pass to the limit on functions defined on the rough boundary Γ^ϵ. The first one is related to the case $\alpha \in [0,1]$, the second one for $\alpha > 1$.

LEMMA 3.1. Let $\varphi^\epsilon \in H^1(U)$ be a sequence with $\varphi^\epsilon \rightharpoonup \varphi$ weakly in $H^1(U)$. Then, if $\alpha \in [0,1]$,

$$\int_{\Gamma^\epsilon} \varphi^\epsilon \, dS \to \int_\Gamma \gamma \, \varphi \, dS \qquad \text{as } \epsilon \to 0$$

where $\gamma \in L^\infty(-c,c)$ and is given by

$$\gamma(x) = \begin{cases} \int_0^1 \sqrt{1+\psi(x)^2 G'(y)^2}\, dy & \text{if } \alpha = 1, \\ 1 & \text{as } \alpha \in [0,1). \end{cases} \tag{3.4}$$

Proof. First, let us observe that

$$\int_{\Gamma^\epsilon} \varphi^\epsilon \, dS = \int_{-c}^{c} \varphi^\epsilon(x, 1+\epsilon\psi(x)G(x/\epsilon^\alpha)) \sqrt{1+\left(\epsilon\psi'(x)\,G(x/\epsilon^\alpha)+\epsilon^{1-\alpha}\psi(x)\,G'(x/\epsilon^\alpha)\right)^2}\, dx.$$

Now, since φ^ϵ weakly converges to φ in $H^1(U)$, $\varphi^\epsilon(x, 1+\epsilon\psi(x)G(x/\epsilon^\alpha))$ strongly converges to $\varphi(x,1)$ in $L^2(-c,c)$. Also, we have, (see e.g. [1, Lemma 2.3]), that

$$\sqrt{1+\left(\epsilon\psi'(x)\,G(x/\epsilon^\alpha)+\epsilon^{1-\alpha}\psi(x)\,G'(x/\epsilon^\alpha)\right)^2} \rightharpoonup \gamma(x) \qquad \text{weakly}^* \text{ in } L^\infty(-c,c).$$

Thus, $\int_{\Gamma^\epsilon} \varphi^\epsilon \, dS \to \int_\Gamma \varphi \, dS$ proving the result. We also refer the reader to [2, Proposition 5.1]. □

LEMMA 3.2. Assume $\alpha > 1$ and $\psi > 0$ in $(-c,c)$ with $\psi(-c) = \psi(c) = 0$. Then, for each $t > 0$,

$$O_t = \left\{ x \in (-c,c) \, : \, \left| \sqrt{1+\left(\epsilon\psi'(x)\,G(x/\epsilon^\alpha)+\epsilon^{1-\alpha}\psi(x)\,G'(x/\epsilon^\alpha)\right)^2} \right| \le t \right\}$$

satisfies that its 1-dimensional measure goes to zero as $\epsilon \to 0$.

Proof. First, we observe that the result is equivalent to

$$\Phi(\epsilon) = \int_{-c}^{c} 1 + \left(\epsilon\psi'(x)\,G(x/\epsilon^\alpha)+\epsilon^{1-\alpha}\psi(x)\,G'(x/\epsilon^\alpha)\right)^2 dx \to \infty \qquad \text{as } \epsilon \to 0. \tag{3.5}$$

Thus, to prove the lemma, it is enough to confirm the assertion (3.5). Notice

$$\Phi(\epsilon) = \int_{-c}^{c} \Big\{ 1 + \epsilon\left(\psi'(x)G(x/\epsilon^\alpha)\right)^2 + \epsilon^{2(1-\alpha)}\left(2\epsilon^\alpha\psi'(x)\psi(x)G'(x/\epsilon)G(x/\epsilon) + \left(\psi(x)G'(x/\epsilon^\alpha)\right)^2\right)\Big\}\, dx.$$

Hence, as ψ and $G \in W^{1,\infty}(-c,c)$, to prove (3.5) we need to show that

$$\lim_{\epsilon\to 0+} \int_{-c}^{c} \left(\psi(x)G'(x/\epsilon^\alpha)\right)^2 dx > 0.$$

Indeed, from [5, Lemma 4.2], since G' is also 1-periodic, we have that

$$\lim_{\epsilon\to 0+}\int_{-c}^{c}\left(\psi(x)G'(x/\epsilon^{\alpha})\right)^2dx=\int_{-c}^{c}\psi(x)^2\int_0^1 G'(y)^2dydx>0$$

which finishes the proof. □

4. Main results. In this section we derive the limit system of our perturbed problem (2.1)–(2.5) describing the effective flow. As stated before, the asymptotic behavior of the flow strongly depends on the order of the boundary roughness set by $\alpha\geq 0$. We first assume $\alpha\in[0,1]$ which covers both the resonant case ($\alpha=1$) and the case of slightly corrugated boundary ($\alpha\in[0,1)$).

4.1. *The case* $\alpha\in[0,1]$.

THEOREM 4.1. Let $(u^\epsilon,H^\epsilon)\in\mathcal{H}_\epsilon$ be the solution of (2.1)–(2.5) for some $\alpha\in[0,1]$ and for all $\epsilon\in[0,1]$ with the forcing terms f^ϵ and $g^\epsilon\in L^2(U)$ satisfying

$$f^\epsilon\rightharpoonup f\quad\text{ and }\quad g^\epsilon\rightharpoonup g\quad\text{ weakly in } L^2(U)$$

for some f and $g\in L^2(U)$. Then, there exists $(u,H)\in\mathcal{H}_0$ such that

$$\|u^\epsilon-u\|_{H^1(\Omega^\epsilon\cap\Omega)}+\|H^\epsilon-H\|_{H^1(\Omega^\epsilon\cap\Omega)}\to 0\quad\text{ as } \epsilon\to 0.$$

Moreover, $(u,H)\in\mathcal{H}_0$ is the unique solution of the MHD system

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+M\frac{\partial H}{\partial y}+1=f,\qquad\text{in }\ \Omega,\tag{4.1}$$

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+M\frac{\partial u}{\partial y}=g,\qquad\text{in }\ \Omega\tag{4.2}$$

endowed with the following boundary conditions

$$\gamma\, u+\beta\frac{du}{d\mathbf{n}}=0,\quad H=0,\qquad\text{for }\ y=1,\tag{4.3}$$

$$u=0,\ \ H=0,\qquad\text{for }\ y=-1,\tag{4.4}$$

$$u=0,\ \ \frac{\partial H}{\partial x}=0,\qquad\text{for }\ x=-c,c\tag{4.5}$$

where the boundary coefficient $\gamma\in L^\infty(-c,c)$ is given by (3.4).

Proof. From Proposition 3.1, it follows that the solutions $(u^\epsilon,H^\epsilon)\in\mathcal{H}_\epsilon$ are uniformly bounded. Hence, their extensions $(P_\epsilon u^\epsilon,P_\epsilon H^\epsilon)$ are also uniformly bounded in $H^1(U)\times H^1(U)$ by Proposition 3.2. Thus, we can extract a convergent subsequence, still denoted by (u^ϵ,H^ϵ), such that

$$\begin{aligned}(u^\epsilon,H^\epsilon)\rightharpoonup(u,H)\quad&\text{weakly in } H^1(U)\times H^1(U)\quad\text{ and}\\ (u^\epsilon,H^\epsilon)\to(u,H)\quad&\text{strongly in } L^2(U)\times L^2(U).\end{aligned}\tag{4.6}$$

Also, from the convergence conditions on f^ϵ and g^ϵ, we have for any measurable set $A\subset U$ that

$$\|f^\epsilon-f\|_A\quad\text{ and }\quad\|g^\epsilon-g\|_A\to 0,\qquad\text{as } \epsilon\to 0.\tag{4.7}$$

Now, let us characterize (u, H). To accomplish that, we will pass to the limit in (3.1). Let $\eta > 0$, $K_\eta = (-c, c) \times (-1, 1-\eta) \subset \Omega$ and $\epsilon_\eta > 0$ such that $K_\eta \subset \Omega^\epsilon$ for all $\epsilon \in [0, \epsilon_\eta]$. Next, let us rewrite (3.1) as

$$\begin{aligned} &\int_{K_\eta} \{\nabla u^\epsilon \nabla\varphi + \nabla H^\epsilon \nabla\phi - M\left(\varphi\partial_y H^\epsilon + \phi\partial_y u^\epsilon\right)\} \, dydx + \beta^{-1} \int_{\Gamma^\epsilon} u^\epsilon \varphi \, dS \\ &\qquad + \int_{\Omega^\epsilon \setminus K_\eta} \{\nabla u^\epsilon \nabla\varphi + \nabla H^\epsilon \nabla\phi - M\left(\varphi\partial_y H^\epsilon + \phi\partial_y u^\epsilon\right)\} \, dydx \\ &= \int_{K_\eta} \{1 - f^\epsilon\} \, \varphi \, dxdy - \int_{K_\eta} g^\epsilon \, \phi \, dxdy + \int_{\Omega^\epsilon \setminus K_\eta} \{1 - f^\epsilon\} \, \varphi \, dxdy - \int_{\Omega^\epsilon \setminus K_\eta} g^\epsilon \, \phi \, dxdy. \end{aligned}$$

Notice that $|\Omega^\epsilon \setminus K_\eta| \to 0$ as $(\eta, \epsilon) \to (0, 0)$. Consequently, since f^ϵ, g^ϵ, ∇u^ϵ and ∇H^ϵ are uniformly bounded in $L^2(\Omega^\epsilon)$, the integrals on $\Omega^\epsilon \setminus K_\eta$ go to zero when $(\eta, \epsilon) \to (0, 0)$. Thus, from (4.6), (4.7) and Lemma 3.1, we obtain that

$$a_0((u, H), (\varphi, \phi)) + \beta^{-1} \int_\Gamma \gamma \, u\varphi \, dS = \int_\Omega \{1 - f\} \, \varphi \, dxdy - \int_\Omega g \, \phi \, dxdy \tag{4.8}$$

for all φ, $\phi \in H^1(U)$ satisfying the boundary conditions (4.4)–(4.5) with $\phi = 0$ on $y = 1$. Since (4.8) is well posed, $(u, H) \in \mathcal{H}_0$ is the unique solution of the limit system, and then, the sequence $(u^\epsilon, H^\epsilon) \in \mathcal{H}_\epsilon$ weakly converges. Finally, let us prove its strong convergence. For that, we use the semi-continuity of the norm. We have

$$\begin{aligned} \|(u, H)\|^2_{\mathcal{H}_0} &\leq \liminf_{\epsilon \in [0,1]} \|(u^\epsilon, H^\epsilon)\|^2_{\mathcal{H}_\epsilon} \leq \limsup_{\epsilon \in [0,1]} \|(u^\epsilon, H^\epsilon)\|^2_{\mathcal{H}_\epsilon} \\ &\leq \limsup_{\epsilon \in [0,1]} \left\{ \int_{\Omega^\epsilon} \{1 - f^\epsilon\} \, \varphi \, dxdy - \int_{\Omega^\epsilon} g^\epsilon \, \phi \, dxdy - \beta^{-1} \int_{\Gamma^\epsilon} u^\epsilon \varphi \, dS \right\} \\ &= \int_\Omega \{1 - f\} \, \varphi \, dxdy - \int_\Omega g \, \phi \, dxdy - \beta^{-1} \int_\Gamma \gamma \, u\varphi \, dS \\ &= \|(u, H)\|^2_{\mathcal{H}_0}. \end{aligned} \tag{4.9}$$

Hence, $\|(u^\epsilon, H^\epsilon)\|_{\mathcal{H}_\epsilon} \to \|(u, H)\|_{\mathcal{H}_0}$. Since $\mathcal{H}_\epsilon$ is a Hilbert space, the proof is completed. □

Remark 5. Theorem 4.1 provides the limit problems for the so-called weak and the critical regime of the roughness, given respectively by $\alpha \in [0, 1)$ and $\alpha = 1$. In the case of weak roughness, the same limit system is obtained as the starting one, now posed in the fixed ϵ-independent domain Ω. In particular, the slip boundary condition on the upper wall is kept, but with no effects captured from the rough boundary, as $\epsilon \to 0$. On the other hand, setting $\alpha = 1$, we are in the critical regime, where the reaction coefficient term $\gamma > 1$ appears in the slip boundary condition showing how the roughness on the boundary is affecting the system (see (4.3)). These findings differ from the results for a purely viscous fluid flow (without the action of the magnetic field). In the case of weak roughness, the slip boundary condition is kept in the limit only for certain ratio between the amplitude and the period of the roughness (see e.g. [10] for details). For the corrugated boundary with the same order of period and amplitude ($\alpha = 1$), it is known that the asymptotic behavior of a viscous fluid satisfying the slip boundary condition is the same as if we assume the no-slip boundary condition (see e.g. [9]).

Remark 6. If $\Omega \subset \Omega^\epsilon$ for all $\epsilon > 0$, which is called exterior perturbation, the convergence obtained by Theorem 4.1 is given in $\mathcal{H}_0$, that is, in the fixed perturbed domain Ω.

4.2. *The case* $\alpha > 1$. In this section we deal with large roughness on the upper boundary. Without loss of generality, we will assume from now on that the roughness occurs throughout Γ_ϵ, i.e.,

$$\psi > 0 \text{ in } (-c, c) \text{ with } \psi(-c) = \psi(c) = 0.$$

We show that the limit system satisfies homogeneous Dirichlet boundary conditions on $y = 1$ meaning that the fluid under the action of the magnetic field behaves as if the velocity vanishes on the whole boundary.

Theorem 4.2. Let $(u^\epsilon, H^\epsilon) \in \mathcal{H}_\epsilon$ be the solution of (2.1)–(2.5) for some $\alpha > 1$ and for all $\epsilon \in [0, 1]$ with the forcing terms f^ϵ and $g^\epsilon \in L^2(U)$ satisfying

$$f^\epsilon \rightharpoonup f \quad \text{ and } \quad g^\epsilon \rightharpoonup g \quad \text{ weakly in } L^2(U)$$

for some f and $g \in L^2(U)$.

Then, there exists $(u, H) \in \mathcal{H}_D$ with

$$\mathcal{H}_D = \left\{ (\varphi, \phi) \in H^1(\Omega) \times H^1(\Omega) \ : \ \varphi = 0 \text{ on } \partial\Omega \text{ and } \phi = 0 \text{ on } \Gamma_0 \cup \Gamma_l \right\}$$

such that

$$\|u^\epsilon - u\|_{H^1(\Omega^\epsilon \cap \Omega)} + \|H^\epsilon - H\|_{H^1(\Omega^\epsilon \cap \Omega)} \to 0 \quad \text{ as } \epsilon \to 0.$$

Also, (u, H) is the unique solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + M \frac{\partial H}{\partial y} + 1 = f, \qquad \text{in } \Omega\,, \tag{4.10}$$

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + M \frac{\partial u}{\partial y} = g, \qquad \text{in } \Omega \tag{4.11}$$

endowed with the boundary conditions

$$u = 0, \quad H = 0, \qquad \text{for } y = 1\,, \tag{4.12}$$

$$u = 0, \quad H = 0, \qquad \text{for } y = -1\,, \tag{4.13}$$

$$u = 0, \quad \frac{\partial H}{\partial x} = 0, \qquad \text{for } x = -c, c\,. \tag{4.14}$$

Proof. As in the proof of Theorem 4.1, given $\eta > 0$, there exists $\epsilon_\eta > 0$ such that $K_\eta = (-c, c) \times (-1, 1 - \eta) \subset \Omega^\epsilon$ for all $\epsilon \in [0, \epsilon_\eta]$. Next, we rewrite (3.1) as

$$\begin{aligned} &\int_{K_\eta} \{\nabla u^\epsilon \nabla \varphi + \nabla H^\epsilon \nabla \phi - M\,(\varphi \partial_y H^\epsilon + \phi \partial_y u^\epsilon)\}\, dydx + \beta^{-1} \int_{\Gamma^\epsilon} u^\epsilon \varphi\, dS \\ &\quad + \int_{\Omega^\epsilon \setminus K_\eta} \{\nabla u^\epsilon \nabla \varphi + \nabla H^\epsilon \nabla \phi - M\,(\varphi \partial_y H^\epsilon + \phi \partial_y u^\epsilon)\}\, dydx \\ &= \int_{K_\eta} \{1 - f^\epsilon\}\, \varphi\, dxdy - \int_{K_\eta} g^\epsilon\, \phi\, dxdy + \int_{\Omega^\epsilon \setminus K_\eta} \{1 - f^\epsilon\}\, \varphi\, dxdy - \int_{\Omega^\epsilon \setminus K_\eta} g^\epsilon\, \phi\, dxdy. \end{aligned} \tag{4.15}$$

As before, we have $|\Omega^\epsilon \setminus K_\eta| \to 0$ as $(\eta, \epsilon) \to (0, 0)$. Thus, since f^ϵ, g^ϵ, ∇u^ϵ and ∇H^ϵ are uniformly bounded in $L^2(\Omega^\epsilon)$, the integrals on $\Omega^\epsilon \setminus K_\eta$ go to zero when $(\eta, \epsilon) \to (0, 0)$.

Also, for each $\eta > 0$, the restriction of (u^ϵ, H^ϵ) to K_η is uniformly bounded, and then, we can extract a convergent subsequence which we still denote by $(u^\epsilon, H^\epsilon)|_{K_\eta}$. Notice that, for each $\eta > 0$, there exists $(u_\eta, H_\eta) \in H^1(K_\eta) \times H^1(K_\eta)$ such that, as $\epsilon \to 0$,

$$(u^\epsilon, H^\epsilon)|_{K_\eta} \rightharpoonup (u_\eta, H_\eta) \quad \text{weakly in } H^1(K_\eta) \times H^1(K_\eta).$$

Next, let us assume the test function φ satisfies $\varphi = 0$ if $y \leq 1 - \eta$. Hence, there exists $R(\eta) \in \mathbb{R}$, $R(\eta) \to 0$ as $\eta \to 0$, such that, for each $\eta > 0$, we can pass to the limit in (4.15) as $\epsilon \to 0$ obtaining

$$\begin{aligned} &\int_{K_\eta} \{\nabla u_\eta \nabla \varphi + \nabla H_\eta \nabla \phi - M(\varphi \partial_y H_\eta + \phi \partial_y u_\eta)\}\, dydx \\ &\qquad = \int_{K_\eta} \{1 - f\}\, \varphi\, dxdy - \int_{K_\eta} g\, \phi\, dxdy + R(\eta). \end{aligned} \tag{4.16}$$

From item (i) in Proposition 3.2, there exists an extension operator $P_\eta : H^1(K_\eta) \mapsto H^1(U)$ with $\|P_\eta \varphi\|_{H^1(U)} \leq C_1 \|\varphi\|_{H^1(K_\eta)}$ for some positive constant C_1. Hence, there exists $(u, H) \in H^1(\Omega) \times H^1(\Omega)$, such that, the restriction of $P_\eta u_\eta$ and $P_\eta H_\eta$ to Ω, weakly converges to (u, H), i.e. such that

$$(P_\eta u_\eta, P_\eta H_\eta)|_\Omega \rightharpoonup (u, H) \quad \text{weakly in } H^1(\Omega) \times H^1(\Omega). \tag{4.17}$$

Consequently, it follows from (4.16) that (u, H) satisfies

$$\begin{aligned} &\int_\Omega \{\nabla u \nabla \varphi + \nabla H \nabla \phi - M(\varphi \partial_y H + \phi \partial_y u)\}\, dydx \\ &\qquad = \int_\Omega \{1 - f\}\, \varphi\, dxdy - \int_\Omega g\, \phi\, dxdy \quad \forall (\varphi, \phi) \in \mathcal{H}_D. \end{aligned}$$

Now, in order to finish the proof, we need to show that $u = 0$ on $y = 1$ (see that the convergence of the norms can be performed as in (4.9)). Here, we proceed as in [3, Proposition 4.2]. First, let us notice that, for any $\varphi \in L^\infty(\Omega)$, we have

$$\begin{aligned} \left| \int_\Gamma u\varphi \right| \leq &\left| \int_\Gamma u\varphi - \int_{\Gamma_\eta} \varphi u \right| + \left| \int_{\Gamma_\eta} \varphi (u - u^\epsilon) \right| \\ &+ \left| \int_{\Gamma_\eta} \varphi (u^\epsilon - u^\epsilon \circ \Psi_\eta^\epsilon) \right| + \left| \int_{\Gamma_\eta} \varphi\, (u^\epsilon \circ \Psi_\eta^\epsilon) \right|, \end{aligned}$$

where Γ_η is the upper boundary of K_η, given by $\Gamma_\eta = \{(x, 1 - \eta) : x \in (-c, c)\}$, and Ψ_η^ϵ is the map

$$\Psi_\eta^\epsilon(x, y) = \begin{cases} (x, y + (1 + y)\epsilon\phi(x)G(x/\epsilon^\alpha)) & \text{for } y \in (-1, 0) \\ (x, y + (1 - y)\epsilon\phi(x)G(x/\epsilon^\alpha)) & \text{for } y \in [0, 1) \end{cases}.$$

Since u and ϕ are fixed functions on Ω and u^ϵ satisfies (4.17), given $\delta > 0$, there exist $\eta > 0$ such that

$$\left| \int_\Gamma u\varphi - \int_{\Gamma_\eta} \varphi u \right| + \left| \int_{\Gamma_\eta} \varphi (u - u^\epsilon) \right| \leq 2\delta.$$

Also, from [3, Lemma 3.2] and [3, Inequalities (4.9) and (4.10)], we get respectively

$$\left|\int_{\Gamma_\eta} \varphi(u^\epsilon - u^\epsilon \circ \Psi^\epsilon_\eta)\right| \le \|\varphi\|_{L^\infty(\Omega)} \|u^\epsilon - u^\epsilon \circ \Psi^\epsilon_\eta\|_{L^2(\Gamma_\eta)} \to 0 \quad \text{as } \epsilon \to 0$$

$$\text{and} \quad \left|\int_{\Gamma_\eta} \varphi\left(u^\epsilon \circ \Psi^\epsilon_\eta\right)\right| \to 0 \quad \text{as } (\epsilon, \eta) \to (0,0).$$

Thus, there exist $\eta > 0$ and $\epsilon_0 > 0$, such that, for each $\varphi \in L^\infty(\Omega)$, we have

$$\left|\int_\Gamma u\varphi\right| \le 4\delta \quad \forall \epsilon \in [0, \epsilon_0].$$

Hence, we can conclude that $u = 0$ on Γ finishing the proof. □

5. The effect of the slip length on the MHD system. Motivated by the results from [32], in this section we focus on the effects of the dimensionless slip length β on the MHD equations (2.1)–(2.2) posed in the fixed rectangle $\Omega = (-c, c) \times (-1, 1)$ without boundary distortion. We set $\beta = \epsilon^\delta$, for some $\delta \in \mathbb{R}$, and consider

$$\frac{\partial^2 u^\epsilon}{\partial x^2} + \frac{\partial^2 u^\epsilon}{\partial y^2} + M\frac{\partial H^\epsilon}{\partial y} + 1 = f^\epsilon, \qquad \text{in } \Omega\,, \tag{5.1}$$

$$\frac{\partial^2 H^\epsilon}{\partial x^2} + \frac{\partial^2 H^\epsilon}{\partial y^2} + M\frac{\partial u^\epsilon}{\partial y} = g^\epsilon, \qquad \text{in } \Omega \tag{5.2}$$

endowed with the following boundary conditions

$$u^\epsilon + \epsilon^\delta \frac{du^\epsilon}{d\mathbf{n}} = 0, \quad H = 0, \qquad \text{for } y = 1\,, \tag{5.3}$$

$$u^\epsilon = 0, \ H = 0, \qquad \text{for } y = -1\,, \tag{5.4}$$

$$u^\epsilon = 0, \ \frac{\partial H}{\partial x} = 0, \qquad \text{for } x = -c, c\,. \tag{5.5}$$

Notice that the existence and uniqueness of the solution are guaranteed by Remark 3 with $\psi \equiv 0$ in $(-c, c)$. Thereby, we have a family of solutions $(u^\epsilon, H^\epsilon) \in \mathcal{H}$ with $\mathcal{H} := \mathcal{H}_0$. Depending on the magnitude of the slip length, we prove the following result:

THEOREM 5.1. Let $(u^\epsilon, H^\epsilon) \in \mathcal{H}$ be the solution of (5.1)–(5.5) for $\epsilon \in [0, 1]$ with the forcing terms f^ϵ and $g^\epsilon \in L^2(\Omega)$ satisfying

$$f^\epsilon \rightharpoonup f \quad \text{and} \quad g^\epsilon \rightharpoonup g \quad \text{weakly in } L^2(\Omega)$$

for some f and $g \in L^2(\Omega)$. Then, there exists $(u, H) \in \mathcal{H}_0$ with

$$\|(u^\epsilon - u, H^\epsilon - H)\|_{\mathcal{H}} \to 0 \quad \text{as } \epsilon \to 0$$

where (u, H) is the unique solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + M\frac{\partial H}{\partial y} + 1 = f, \qquad \text{in } \Omega\,, \tag{5.6}$$

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + M\frac{\partial u}{\partial y} = g, \qquad \text{in } \Omega \tag{5.7}$$

endowed with the following boundary conditions

$$u = 0, \;\; H = 0, \qquad \text{for} \;\; y = -1\,, \tag{5.8}$$

$$u = 0, \;\; \frac{\partial H}{\partial x} = 0, \qquad \text{for} \;\; x = -c, c\,, \tag{5.9}$$

and, on $y = 1$, we have

$$u = 0, \quad H = 0, \qquad \text{if} \;\; \delta > 0\,, \tag{5.10}$$

$$u + \frac{du}{d\mathbf{n}} = 0, \quad H = 0, \qquad \text{if} \;\; \delta = 0\,, \tag{5.11}$$

$$\frac{du}{d\mathbf{n}} = 0, \quad H = 0, \qquad \text{if} \;\; \delta < 0\,. \tag{5.12}$$

Proof. Assuming $\psi \equiv 0$ in (1.1), we get from Proposition 3.1 that the family of solutions $(u^\epsilon, H^\epsilon) \in \mathcal{H}$ is uniformly bounded in ϵ. Also, we have

$$\begin{aligned}\|(u^\epsilon, H^\epsilon)\|^2_{\mathcal{H}_\epsilon} &\le a_\epsilon((u^\epsilon, H^\epsilon), (u^\epsilon, H^\epsilon)) + \epsilon^{-\delta} \int_\Gamma u^{\epsilon 2} dS \\ &\le C_P \max\{\|1 - f^\epsilon\|_{L^2(\Omega)}, \|g^\epsilon\|_{L^2(\Omega)}\} \left(\|\nabla u^\epsilon\|_{L^2(\Omega)} + \|\nabla H^\epsilon\|_{L^2(\Omega)}\right)\end{aligned}$$

implying as well

$$\epsilon^{-\delta} \int_\Gamma u^{\epsilon 2} dS \le C_P \max\{\|1 - f^\epsilon\|_{L^2(\Omega)}, \|g^\epsilon\|_{L^2(\Omega)}\} \left(\|\nabla u^\epsilon\|_{L^2(\Omega)} + \|\nabla H^\epsilon\|_{L^2(\Omega)}\right).$$

Thus, there exists $C > 0$ independent of ϵ such that

$$\epsilon^{-\delta} \int_\Gamma u^{\epsilon 2} dS \le C \quad \forall \epsilon \in [0, 1]. \tag{5.13}$$

Next, we us pass to the limit in

$$a_\epsilon((u^\epsilon, H^\epsilon), (\varphi, \phi)) + \epsilon^{-\delta} \int_\Gamma u^\epsilon \varphi \, dS = \int_{\Omega^\epsilon} \{1 - f^\epsilon\} \, \varphi \, dxdy - \int_{\Omega^\epsilon} g^\epsilon \, \phi \, dxdy, \quad \forall (\varphi, \phi) \in \mathcal{H}. \tag{5.14}$$

Since the solutions are uniformly bounded and $\partial\Omega$ is smooth enough, there exists a subsequence in $\mathcal{H}$, still denoted by (u^ϵ, H^ϵ), and $(u, H) \in \mathcal{H}$ such that

$$\begin{gathered}(u^\epsilon, H^\epsilon) \rightharpoonup (u, H) \quad \text{weakly in } \mathcal{H} \text{ and} \\ \|u^\epsilon\|_{L^2(\Gamma)} \text{ is uniformly bounded.}\end{gathered} \tag{5.15}$$

Now, let us pass to the limit in (5.14) as $\epsilon \to 0$. If $\delta > 0$, we take $\varphi = 0$ on $y = 1$ getting

$$a_0((u, H), (\varphi, \phi)) = \int_\Omega \{1 - f\} \, \varphi \, dxdy - \int_\Omega g \, \phi \, dxdy$$

whenever $(\varphi, \phi) \in \mathcal{H}$ with $\varphi = 0$ on $y = 1$. Hence, as $\delta > 0$, it follows from (5.13) that $\|u^\epsilon\|_{L^2(\Gamma)} \to 0$, and then, $u = 0$ on Γ and (u, H) satisfies Dirichlet boundary condition on $y = 1$. On the other hand, as $\delta < 0$, it follows from (5.15) that $\epsilon^{-\delta} \int_\Gamma u^{\epsilon 2} dS \to 0$. Hence, (5.14) becomes

$$a_0((u, H), (\varphi, \phi)) = \int_\Omega \{1 - f\} \, \varphi \, dxdy - \int_\Omega g \, \phi \, dxdy, \quad \forall (\varphi, \phi) \in \mathcal{H}$$

and the function u satisfies the homogeneous Neumann boundary conditions on $y = 1$. The proof of the strong convergence in $\mathcal{H}$ is analogous to (4.9) and is left to the interested reader. □

REMARK 7. The result from Theorem 5.1 can be interpreted in the following manner. Note that M is kept of order 1, whereas we allow the slip length to depend on the small parameter ϵ by putting $\beta = \mathcal{O}(\epsilon^\delta)$. Let us introduce the dimensionless parameter $S = \frac{\beta}{M}$ as the ratio between the slip length and the Hartmann number appearing in the equations (5.1)–(5.2). For $S \ll 1$ (i.e. $\delta > 0$), the effective flow does not differ from the classical MHD duct flow without the slip. If $S = \mathcal{O}(1)$, then both MHD and slip effects contribute to the effective flow, as we elaborated in detail throughout Section 4 by introducing the perturbed boundary as well. If $S \gg 1$ (i.e. $\delta < 0$) the MHD effects are still present, but the flow is not controlled by the slip phenomenon.

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