

Nash meets Łukasiewicz: computing equilibria through logic

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Abstract

A Nash equilibrium is a strategy profile of a game in which none of the players involved has any gain by changing alone his/her own strategy. Given a two-player game, we show a codification of all of its Nash equilibria into Łukasiewicz infinitely-valued logic, that is, we derive a propositional theory in this logic whose models codify all the Nash equilibria. Based on such propositional theory, we derive a polynomial reduction from the problem of computing a Nash equilibrium to the problem of satisfiability of (sets of) formulas of Łukasiewicz infinitely-valued logic. These applications of logic to game theory lead to new methods for computing Nash equilibria.

Keywords: Nash equilibrium, Łukasiewicz infinitely-valued logic, satisfiability

1 Introduction

Game theory deals with the scenarios, called games, where two or more players have two or more choices of actions on how to behave. The utility or gain for each individual may be measured in accordance with the possible action profiles by all of them. The aim of game theory is to study the solutions of such games by, for instance, establishing what are the action profiles that have some kind of balance, called equilibrium.

A well-known notion of balance is Nash equilibrium. First, a pure Nash equilibrium is an action profile for which no player would gain by changing alone his/her choice of action. Unfortunately, such concept of equilibrium lacks universality, since not all games are guaranteed to have such an equilibrium. To work around this flaw, another concept of equilibrium is proposed. Instead of looking solely at the actions, a strategy for the behavior of a player is considered to be a probability distribution over his/her possible actions. In this way, the player is expected to randomly choose an action in each instance of the game that he/she faces according to such a probability assignment. The player's utilities or gains are now the probabilistic averages of the utilities of action profiles. Then, a (mixed) Nash equilibrium is a (probabilistic) strategy profile where no player would gain by changing alone his/her probabilities over actions. The famous result from John Nash establishes that any game has at least one Nash equilibrium [20].

From the perspective of complexity theory, the problem of computing an ε -Nash equilibrium, which is a strategy profile where a player would gain at most the value ε by changing his/her

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probabilities over actions, is **PPAD**-complete [5]. The class **PPAD** is the subclass of (the search version¹ of) **NP** that contains the search problems for which (i) all instances have a solution (differing, for instance, from the problems of finding a satisfiable assignment for logical formulas); and (ii) the proof of existence of solution uses the non-constructive *parity argument for directed graphs*: ‘If a directed graph has an unbalanced node (a vertex with different in-degree and out-degree), then it must have another’.

As far as (non ε -) Nash equilibria are concerned, their approximate computation for games with two players is **PPAD**-complete [2]. However, for games with three or more players, a result that places the computation of an approximate Nash equilibrium in **NP** would imply a breakthrough in complexity theory since it would also place in **NP** the *square root sum problem*: given positive integers $d_1, \dots, d_n$ and k , is it the case that $\sum_{i=1}^n \sqrt{d_i} \leq k$? Whether the square root sum problem is solvable in **NP** is an open problem since the 1970s [7].

Many algorithms have been proposed for computing Nash equilibria for two-player games. An important one is the Lemke–Howson Algorithm that proceeds by following a path of vertices standing for strategies in a pair of polytopes until it reaches a vertex that codifies a Nash equilibrium [14]. Such approach is possible due to the linearity of the constraints that determine Nash equilibria in two-player games.

In this work, we exploit such a linear nature of Nash equilibria in order to codify them into Łukasiewicz infinitely-valued logic (L_∞), a propositional logical system that is arguably one of the best studied many-valued logics [3]. Łukasiewicz infinitely-valued logic has several interesting properties such as a continuous truth-functional semantics, whose truth values lie in the unit interval $[0, 1]$, classical logic as a limit case and well developed proof-theoretical and algebraic presentations.

Given a two-player game G , we show how to build a set Φ_G of formulas of L_∞ such that a valuation v satisfies Φ_G if, and only if, v codifies a Nash equilibrium of G ; in other words, Φ_G is the L_∞ -propositional theory whose models encode the Nash equilibria of G . In Section 3, we codify probability distributions and a mechanism for guaranteeing that a strategy profile satisfies the conditions in a characterization of Nash equilibria. In order for such conditions to be fully expressed in the language of L_∞ , linear equations and inequalities with rational coefficients need to be codified in this logical system. The exact codification of such equations and inequalities is postponed to Section 4.

For codifying probability distributions in Section 3, we interpret probability values as truth values of propositional variables in the language of L_∞ . Such an approach is not novel, translations from probability statements to fresh propositional variables in the language of L_∞ were already used to show the completeness of probability logic systems, as $FP(L)$ and $FP(L, L)$, with respect to probability models [9, 10, 13]. More recently, these translations were also employed in the codification of de Finetti’s coherence criterion in L_∞ [11].

The problem of finding a satisfiable valuation for a set of formulas in L_∞ , if some exists (L_∞ -Satisfiability), is **NP**-complete [18]. Thus, as computing a Nash equilibrium for two-player games is a problem in **NP** (because **PPAD** $\subseteq$ **NP**), the Cook–Levin Theorem states that there is a polynomial reduction from this problem to L_∞ -Satisfiability [4]. Indeed, we show in Section 5 that the codification of Nash equilibria into L_∞ yields such a polynomial reduction.

Other proposals for encoding Nash equilibria into Łukasiewicz logics are found in the literature. Marchioni and Wooldridge [15] show how to encode pure Nash equilibria in finitely and infinitely-valued Łukasiewicz logics and some extensions of these systems. Moreover, Běhounek et al.

¹Deciding if instances of these problems have a solution is trivial, since it is assumed that they have. For a detailed treatment of the search version of **NP**, see [12].

[1] propose an encoding of (mixed) Nash equilibria in infinitely-valued product Łukasiewicz logic and its extensions. Such logical systems express real numbers product in addition to the traditional Łukasiewicz operators. While such expressiveness allows codifications for games with any number of players, it also leads to an increase in the complexity of Satisfiability. Note that solving Satisfiability for infinitely-valued product Łukasiewicz logic might involve finding roots for multivariate polynomials of any degree. The codification of Běhounek [1] is discussed in Section 6.

Then, this work shows an explicit reduction from the problem of computing Nash equilibria in two-player games to L_∞ -Satisfiability, even though L_∞ lacks a numbers product operator. Also, such is a reduction to a problem in the same complexity class as the reduced one (that is, **NP**).

Moreover, the results in this work advance the study of Nash equilibrium since new methods for computing it for two-player games might be derived resting on methods for deciding L_∞ -Satisfiability. And approximation techniques for L_∞ , as the one in [8], immediately lead to approximations of Nash equilibria for two-player games.

In addition to the aforementioned sections, we also introduce the necessary background about Łukasiewicz infinitely-valued logic and game theory in Section 2 and draw some conclusions in Section 7.

2 Preliminaries

2.1 Games and nash equilibrium

In this work, we call a *game*² a triple $G = \langle P, A, u \rangle$, where $P = \{1, \dots, n\}$ is a set of n players, $A = A_1 \times \dots \times A_n$ is a set of *action profiles*, in which each A_i is a finite set of all the possible *actions* for player i , and $u = \langle u_1, \dots, u_n \rangle$ is a sequence of *utility functions*, in which $u_i : A \rightarrow \mathbb{Q}$ is the utility function for player i (that assigns the player's gain to each possible action profile).

A (*mixed*) *strategy* for player i is a probability distribution σ_i over the set A_i of actions for player i and $\Sigma = \Sigma_1 \times \dots \times \Sigma_n$ is the set of *strategy profiles*, in which each Σ_i is the set of all possible strategies for player i . The set of actions with non-zero probability in a strategy σ_i is its *support*. We call *pure strategy* a strategy whose support is a singleton and we abuse the notation by identifying actions with pure strategies. We also identify strategy profiles that only have pure strategies with action profiles.

It is assumed that each player's choice of strategy is independent from all other players' choices, so the *expected utility function* U_i for player i is given by

$$U_i(\sigma) = \sum_{a \in A} u_i(a) \prod_{j \in P} \sigma_j(a_j),$$

where $\sigma \in \Sigma$ and $a = \langle a_1, \dots, a_n \rangle$, with $a_j \in A_j$.

A strategy profile $\sigma = \langle \sigma_1, \dots, \sigma_i, \dots, \sigma_n \rangle$ is a *Nash equilibrium* if, for every player i ,

$$U_i(\sigma) \geq U_i(\sigma_1, \dots, \sigma'_i, \dots, \sigma_n),$$

for every $\sigma'_i \in \Sigma_i$; each σ_i in σ is called a *best response* for player i in relation to the other players' strategies in σ because no player would gain by changing alone his/her strategy. Then, a strategy profile is a Nash equilibrium if, and only if, it is composed by best responses for all players. A game G always has at least one Nash equilibrium [20].

²These kind of games are usually referred to as *strategic-form games*. Since these are the only games we address in this work, we refer to them simply as *games*.

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TABLE 1 Utility functions for game G_e

	a_2^1	a_2^2
a_1^1	3, 2	0, 0
a_1^2	0, 0	2, 3

EXAMPLE 1.

Table 1 has the values of the utility functions u_1 and u_2 in the two-player game $G_e = \langle P, A, u \rangle$, where $P = \{1, 2\}$, $A = A_1 \times A_2 = \{a_1^1, a_1^2\} \times \{a_2^1, a_2^2\}$ and $u = \langle u_1, u_2 \rangle$. The lines in Table 1 stand for player 1 and the columns for player 2; for each action profile $\langle a_1^i, a_2^j \rangle$, the corresponding entry has the utility for player 1 followed by the utility for player 2. Game G_e has three Nash equilibria, two of which are pure strategy profiles: $\langle a_1^1, a_2^1 \rangle$ and $\langle a_1^2, a_2^2 \rangle$. The other Nash equilibrium is the strategy profile $\langle \sigma_1, \sigma_2 \rangle$, where

$$\sigma_1(a_1^1) = \frac{3}{5}, \quad \sigma_1(a_1^2) = \frac{2}{5}, \quad \sigma_2(a_2^1) = \frac{2}{5} \quad \text{and} \quad \sigma_2(a_2^2) = \frac{3}{5}. \quad (1)$$

Let $\sigma = \langle \sigma_1, \dots, \sigma_n \rangle \in \Sigma$ be a strategy profile and $U_i(\sigma | a_i)$ be player i 's expected utility for the strategy profile $\sigma' = \langle \sigma_1, \dots, a_i, \dots, \sigma_n \rangle$, where action $a_i \in A_i$ acts as a pure strategy (remember that we identify actions with pure strategies). It is easy to see that the following equation holds:

$$U_i(\sigma) = \sum_{a_i \in A_i} \sigma_i(a_i) U_i(\sigma | a_i). \quad (2)$$

Then, the following result highlights the combinatorial nature of the problem of computing Nash equilibria.

PROPOSITION 1. (Papadimitriou [21], Theorem 2.1)

A strategy is a best response if, and only if, all actions in its support are best responses.

PROOF. Suppose that some action in the support of a best response strategy is not itself a best response as a pure strategy. Then, in view of equation (2), the player would have a better expected utility by decreasing the probability of such an action and proportionally increasing the probability of another action in the support that is a best response. Conversely, if all actions in a strategy support are best responses, each one as a pure strategy has the same expected utility. This expected utility is the same for the strategy with such actions in its support. $\square$

It follows that, to compute a Nash equilibrium, one may search the possible supports for a strategy profile that satisfies the best response constraints. That is, if A is the set of actions available for a player, one may search for a subset of actions $S \subseteq A$ such that: the expected utilities for all actions in S are equal; and the expected utilities for the actions in S are, at least, the expected utilities for the actions in $A \setminus S$. Such a set S satisfying such conditions is the support of a best response.

Let k_i be the number of actions in the support of σ_i . For each player i , the equations

$$U_i(\sigma | a_i') = U_i(\sigma | a_i''), \quad (3)$$

for a_i' and a_i'' in σ_i 's support, state that player i 's k_i expected utilities $U_i(\sigma | a_i)$, for all actions a_i in the support of σ_i , are equal. And the inequalities:

$$U_i(\sigma | a_i') \leq U_i(\sigma | a_i''), \quad (4)$$

for a'_i outside σ_i 's support and a''_i in σ_i 's support, state that player i 's k_i expected utilities $U_i(\sigma | a_i)$, for all actions a_i in the support of σ_i , have value at least as the $|A_i| - k_i$ expected utilities $U_i(\sigma | a_i)$, for the actions a_i not in σ_i 's support.

Then, for σ to be a Nash equilibrium, the probability values in σ must satisfy

- A system of $\sum_{i \in P} k_i^2$ equations as (3), which we call *utility equations*; and
- A system of $\sum_{i \in P} (|A_i| - k_i) \cdot k_i$ inequalities as (4), which we call *utility inequalities*.

Therefore, a search procedure for finding a Nash equilibrium comes down to jointly solving such systems of utility equations and inequalities for each possible support configuration. When a solution for the systems is found, its values are the probabilities in a strategy profile that is a Nash equilibrium.

Note that, in case the game has only two players, such systems are linear, which guarantees that there are polynomially bounded equilibria comprehending rational distributions. Therefore, the problem of computing a Nash equilibrium is placed in **NP**.

As our results are about two-player games, from now on we use the notation a and b instead of 1 and 2 to name the players. Thus, we also use the cleaner notation a^i and b^j instead of a_1^i and a_2^j to refer to player's actions.

EXAMPLE 2.

For the game G_e in Example 1, a strategy $\sigma = \langle \sigma_a, \sigma_b \rangle$ for which the supports for players a and b are the entire sets of actions A_a and A_b must satisfy the following system of equations and inequalities (the notation for game G_e is already adapted to the newly introduced convention):

$$\begin{aligned} U_a(\sigma | a^1) = 3 \cdot \sigma_b(b^1) &= 2 \cdot \sigma_b(b^2) = U_a(\sigma | a^2) \\ U_b(\sigma | b^1) = 2 \cdot \sigma_a(a^1) &= 3 \cdot \sigma_a(a^2) = U_b(\sigma | b^2) \\ \sigma_a(a^1) + \sigma_a(a^2) &= 1 \\ \sigma_b(b^1) + \sigma_b(b^2) &= 1 \\ \sigma_a(a^1), \sigma_a(a^2), \sigma_b(b^1), \sigma_b(b^2) &\geq 0 \end{aligned}$$

The last three lines force probability distributions. The values given in the strategy in (1) satisfy the system. Note that we have omitted the trivial and redundant equations; also, as the supports in σ are maximal, there are no utility inequalities.

2.2 Łukasiewicz infinitely-valued logic

The basic language of Łukasiewicz infinitely-valued logic ($\mathbb{L}_\infty$) is built from a countable set of propositional symbols $\mathbb{P}$ and the disjunction ($\oplus$) and negation ($\neg$) operators. For the semantics, define a valuation $v : \mathbb{P} \rightarrow [0, 1]$, which maps propositional symbols to a value in the unit interval $[0, 1]$. Then v is extended to all formulas as follows:

$$\begin{aligned} v(\varphi \oplus \psi) &= \min(1, v(\varphi) + v(\psi)); \\ v(\neg \varphi) &= 1 - v(\varphi). \end{aligned}$$

The semantics of formulas of $\mathbb{L}_\infty$ represent all the $[0, 1]$ -valued continuous piecewise linear functions with integer coefficients over some unit cube $[0, 1]^n$ —the so-called McNaughton functions—and only those, as stated by McNaughton's Theorem [17, 19]. From the basic operators, one usually

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derives the following:

$$\text{Conjunction : } \varphi \odot \psi =_{\text{def}} \neg(\neg\varphi \oplus \neg\psi),$$

$$v(\varphi \odot \psi) = \max(0, v(\varphi) + v(\psi) - 1);$$

$$\text{Implication : } \varphi \rightarrow \psi =_{\text{def}} \neg\varphi \oplus \psi,$$

$$v(\varphi \rightarrow \psi) = \min(1, 1 - v(\varphi) + v(\psi));$$

$$\text{Maximum : } \varphi \vee \psi =_{\text{def}} \neg(\neg\varphi \oplus \psi) \oplus \psi,$$

$$v(\varphi \vee \psi) = \max(v(\varphi), v(\psi));$$

$$\text{Minimum : } \varphi \wedge \psi =_{\text{def}} \neg(\neg\varphi \vee \neg\psi),$$

$$v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi));$$

$$\text{Bi-implication : } \varphi \leftrightarrow \psi =_{\text{def}} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi),$$

$$v(\varphi \leftrightarrow \psi) = 1 - |v(\varphi) - v(\psi)|.$$

Note that $v(\varphi \rightarrow \psi) = 1$ iff $v(\varphi) \leq v(\psi)$ and $v(\varphi \leftrightarrow \psi) = 1$ iff $v(\varphi) = v(\psi)$. Let x be a propositional variable, then $v(x \odot \neg x) = 0$, for any $v \in \mathbf{Val}$; we define the constant $\mathbf{0}$ by $x \odot \neg x$. We also define $0\varphi =_{\text{def}} \mathbf{0}$, $1\varphi =_{\text{def}} \varphi$ and $n\varphi =_{\text{def}} \varphi \oplus \dots \oplus \varphi$, n times, for $n \in \mathbb{N} \setminus \{0, 1\}$; and $\bigoplus_{i \in \mathcal{O}} \varphi_i =_{\text{def}} \mathbf{0}$.

A formula φ is *satisfiable* if there exists a valuation v such that $v(\varphi) = 1$; otherwise it is *unsatisfiable*. A set of formulas Φ is satisfiable if there exists a valuation v such that $v(\varphi) = 1$, for all $\varphi \in \Phi$. We also call a set of formulas Φ a *propositional theory*; the valuations that satisfy Φ are the *models* of Φ .

The computational problem of exhibiting a valuation that satisfies a formula (or a set of formulas) or pointing out that it is unsatisfiable is the L_∞ -Satisfiability problem, which is an **NP**-complete problem [18].

3 Codifying Nash equilibria in L_∞

In this section, we begin to develop the construction of a set of formulas Φ_G from a two-player game $G = \langle P, A, u \rangle$ such that the valuations that satisfy Φ_G encode all the Nash equilibria of G .

Let $P = \{a, b\}$ be the set of two players, $A = A_a \times A_b$ be a set of action profiles, where $A_a = \{a^1, \dots, a^\alpha\}$ and $A_b = \{b^1, \dots, b^\beta\}$ are the sets of actions for players a and b , and $u = \langle u_a, u_b \rangle$ be the pair of utility functions, where $u_a : A \rightarrow \mathbb{Q}$ and $u_b : A \rightarrow \mathbb{Q}$ are utility functions for players a and b . Let $\sigma = \langle \sigma_a, \sigma_b \rangle$ be a generic strategy profile, where $\sigma_a : A_a \rightarrow [0, 1] \cap \mathbb{Q}$ and $\sigma_b : A_b \rightarrow [0, 1] \cap \mathbb{Q}$ are generic strategies for the players a and b .

To each probability value $\sigma_a(a^i)$ and $\sigma_b(b^j)$, we associate propositional variables p_i^a and p_j^b . Thus, the formulas in Φ_G have $p_1^a, \dots, p_\alpha^a, p_1^b, \dots, p_\beta^b$ among their propositional variables; the valuations that satisfy Φ_G are intended to represent strategy profiles in Nash equilibrium by assigning to such propositional variables truth values standing for their associated probability values.

In the following, we build the formulas of Φ_G according to player a . The formulas according to player b are analogous and also belong to Φ_G . Moreover, we state and prove results concerning these formulas for player a , which are completely analogous for player b .

3.1 Probability distributions

To assure that the propositional variables actually represent probability distributions, we add to Φ_G the formulas:

- (i) $p_1^a \oplus \dots \oplus p_\alpha^a$;
- (ii) $\neg(p_1^a \odot p_2^a), \neg[(p_1^a \oplus p_2^a) \odot p_3^a], \dots, \neg[(p_1^a \oplus \dots \oplus p_{\alpha-1}^a) \odot p_\alpha^a]$.

A valuation v satisfying item (i) guarantees that the represented probability values add up to at least 1. If a valuation v satisfies a formula $\neg[(p_1^a \oplus \dots \oplus p_i^a) \odot p_{i+1}^a]$ in item (ii), it guarantees that the values $v(p_1^a \oplus \dots \oplus p_i^a)$ and $v(p_{i+1}^a)$ add up to at most 1, for $i \in \{1, \dots, \alpha - 1\}$.

LEMMA 1.

Formulas in items (i) and (ii) are jointly satisfied by a valuation v if, and only if, values $v(p_1^a), \dots, v(p_\alpha^a)$ stand for a probability distribution.

PROOF. First, note that $v(p_1^a \oplus \dots \oplus p_\alpha^a) = 1$ iff $\min(1, v(p_1^a) + \dots + v(p_\alpha^a)) = 1$ iff $v(p_1^a) + \dots + v(p_\alpha^a) \geq 1$. Now, let φ and ψ be formulas; $v(\neg(\varphi \odot \psi)) = 1$ iff $v(\varphi \odot \psi) = 0$ iff $\max(0, v(\varphi) + v(\psi) - 1) = 0$ iff $v(\varphi) + v(\psi) \leq 1$. More generally, a valuation v satisfies formulas in item (ii) if, and only if, $v(p_1^a) + \dots + v(p_\alpha^a) \leq 1$. Therefore, as a valuation assigns only nonnegative values, v satisfies (i) and (ii) if, and only if, $v(p_1^a), \dots, v(p_\alpha^a)$ determine a probability distribution. $\square$

3.2 Utility equations

Let us abbreviate by $\ulcorner U_a(\sigma | a^i) = U_a(\sigma | a^k) \urcorner$ the formula that only is satisfied by a valuation v that encodes a strategy profile σ for which the player a 's expected utilities $U_a(\sigma | a^i)$ and $U_a(\sigma | a^k)$ are equal. Such type of formulas will be explicitly defined in Section 4. Then, for each player a 's action a^i , we build the following formula:

$$\left(\bigoplus_{k=1, \dots, \alpha} \ulcorner U_a(\sigma | a^i) = U_a(\sigma | a^k) \urcorner \odot p_k^a \right) \odot p_i^a,$$

which we abbreviate by $\xi_a(a^i)$.

LEMMA 2.

Let v be a valuation satisfying formulas (i) and (ii) for both players a and b , therefore representing a strategy profile σ . Then, formula $\xi_a(a^i)$ is evaluated by v with the exact same value as p_i^a , if one of the following two excluding cases occurs:

- a^i is not in the strategy σ_a 's support;
- a^i is in the strategy σ_a 's support, in which case the expected utility $U_a(\sigma | a^i)$ for pure strategy a^i is equal to the expected utilities $U_a(\sigma | a^k)$, for all a^k in σ_a 's support.

And, if none of the above cases occurs, v evaluates $\xi_a(a^i)$ strictly less than it evaluates p_i^a .

PROOF. First, let us fix the notation:

$$\begin{aligned} \psi_k &\doteq \ulcorner U_a(\sigma | a^i) = U_a(\sigma | a^k) \urcorner; \\ \varphi_k &\doteq \ulcorner U_a(\sigma | a^i) = U_a(\sigma | a^k) \urcorner \odot p_k^a; \\ \phi &\doteq \bigoplus_{k=1, \dots, \alpha} \ulcorner U_a(\sigma | a^i) = U_a(\sigma | a^k) \urcorner \odot p_k^a. \end{aligned}$$

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In case a^i is not in σ_a 's support, $v(p_i^a) = 0$ and

$$v(\xi_a(a^i)) = \max(0, v(\varphi) + v(p_i^a) - 1) = 0 = v(p_i^a).$$

In case a^i is an action in σ_a 's support and $U_a(\sigma|a^i) = U_a(\sigma|a^k)$, for all a^k in σ_a 's support, we have that $v(\psi_k) = 1$, when a^k is in σ_a 's support, and $v(p_k^a) = 0$, when a^k is not in σ_a 's support. Then,

$$v(\varphi_k) = \begin{cases} v(p_k^a), & a^k \text{ is in } \sigma_a \text{'s support;} \\ 0 = v(p_k^a), & a^k \text{ is not in } \sigma_a \text{'s support.} \end{cases}$$

Therefore, as $\sum_{k=1}^{\alpha} v(p_k^a) = 1$, we have $v(\varphi) = 1$ and

$$v(\xi_a(a^i)) = \max(0, v(\varphi) + v(p_i^a) - 1) = v(p_i^a).$$

Finally, if none of the above cases occurs, a^i is in σ_a 's support, but $U_a(\sigma|a^i) \neq U_a(\sigma|a^j)$, for some a^j also in σ_a 's support. Then, $v(\psi_j) < 1$ and

$$v(\varphi_j) = \max(0, v(\psi_j) + v(p_j^a) - 1) < v(p_j^a).$$

For $k \neq j$, we have

$$v(\varphi_k) = \max(0, v(\psi_k) + v(p_k^a) - 1) \leq v(p_k^a).$$

Therefore, $v(\varphi) < 1$ and

$$v(\xi_a(a^i)) = \max(0, v(\varphi) + v(p_i^a) - 1) < v(p_i^a).$$

□

In view of the above result, we add to Φ_G the following formula:

$$(iii) \quad \xi_a(a^1) \oplus \dots \oplus \xi_a(a^\alpha),$$

which is only satisfied by a valuation satisfying the formulas in items (i) and (ii) if all the actions a^i fall into one of the two cases highlighted in Lemma 2.

LEMMA 3.

Let v be a valuation satisfying formulas in items (i)-(ii) for both players a and b . Then, v satisfies the formula in item (iii) if, and only if, the probabilities encoded by v satisfy player a 's utility equations (3).

PROOF. Suppose v satisfies the formulas in items (i)-(iii). By Lemmas 1 and 2, in order to satisfy (iii), we must have $v(\xi_a(a^i)) = v(p_i^a)$, for $i = 1, \dots, \alpha$. However, such values may only be achieved if one of the two cases stated by Lemma 2 holds. Now, again by Lemma 2, we have that $U_a(\sigma|a^i) = U_a(\sigma|a^k)$, for all a^i and a^k in σ_a 's support. Conversely, suppose that the strategy profile codified by a valuation v is such that player a 's utility equations hold. Then, by Lemma 2, $v(\xi_a(a^i)) = v(p_i^a)$, for $i = 1, \dots, \alpha$, and formula (iii) is satisfied. □

3.3 Utility inequalities

Now, we abbreviate by $\lceil U_a(\sigma|a^i) \leq U_a(\sigma|a^k) \rceil$ the formula that is only satisfied by a valuation v that encodes a strategy profile σ for which player a 's expected utility $U_a(\sigma|a^i)$ is at most his/her expected utility $U_a(\sigma|a^k)$. We will also explicitly define these formulas in Section 4. Then, we build

the following formula, for each player a 's action a^i :

$$\bigoplus_{k=1, \dots, \alpha} \ulcorner U_a(\sigma | a^i) \leq U_a(\sigma | a^k) \urcorner \odot p_k^a,$$

which we abbreviate by $\chi_a(a^i)$.

LEMMA 4.

Let v be a valuation satisfying all the formulas in items (i)-(iii) for both players a and b . Then, formula $\chi_a(a^i)$ is satisfied by v if, and only if, player a 's expected utility $U_a(\sigma | a^i)$ for pure strategy a^i is, at most, the expected utilities $U_a(\sigma | a^k)$, for all his/her pure strategies a^k in σ_a 's support.

PROOF. First, let us fix the notation:

$$\psi_k \doteq \ulcorner U_a(\sigma | a^i) \leq U_a(\sigma | a^k) \urcorner;$$

$$\varphi_k \doteq \ulcorner U_a(\sigma | a^i) \leq U_a(\sigma | a^k) \urcorner \odot p_k^a.$$

Let $U_a(\sigma | a^i) \leq U_a(\sigma | a^k)$, for all a^k in σ_a 's support. Then, for such a^k , $v(\psi_k) = 1$ and

$$v(\varphi_k) = \max(0, v(\psi_k) + v(p_k^a) - 1) = v(p_k^a).$$

Now, for action a^k not in σ_a 's support, $v(p_k^a) = 0$ and

$$v(\varphi_k) = \max(0, v(\psi_k) + v(p_k^a) - 1) = 0 = v(p_k^a).$$

Therefore, as $\sum_{k=1}^{\alpha} v(p_k^a) = 1$, we have $v(\chi_a(a^i)) = 1$. Conversely, assume that there is an action a^k in σ_a 's support such that $U(\sigma | a^i) > U(\sigma | a^k)$. Then, $v(\psi_k) < 1$ and

$$v(\varphi_k) = \max(0, v(\psi_k) + v(p_k^a) - 1) < v(p_k^a).$$

Therefore, $v(\chi_a(a^i)) < 1$. □

We also add to Φ_G the following formulas:

$$(iv) \quad \chi_a(a^1), \dots, \chi_a(a^\alpha).$$

COROLLARY 1.

Let v be a valuation satisfying formulas in items (i)-(iii) for both players a and b . Then, v satisfies the formulas in item (iv) if, and only if, player a 's utility inequalities (4) hold.

3.4 Codification

By the development of the set Φ_G and the discussion in Section 2, if a valuation v encoding a strategy $\sigma = \langle \sigma_a, \sigma_b \rangle$ satisfies Φ_G , σ_a is a best response. As we analogously add to Φ_G all the formulas (i)-(iv) concerning player b , σ_b is also a best response. In such case, σ is a Nash equilibrium and we state the following result.

THEOREM 1.

Given a two-player game G and a strategy profile σ encoded by a valuation v , σ is a Nash equilibrium if, and only if, v satisfies Φ_G .

COROLLARY 2.

Given a two-player game G , Φ_G is a propositional theory whose models encode all of its Nash equilibria.

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EXAMPLE 3.

Formulas in items (i)–(iv) for player a_1 in the game G_e in Example 1 are

1. $p_1^a \oplus p_2^a$;
2. $\neg(p_1^a \odot p_2^a)$;
3. $\{ \begin{bmatrix} (\ulcorner U_a(\sigma|a^1) = U_a(\sigma|a^1)^\neg \odot p_1^a \urcorner \oplus \\ (\ulcorner U_a(\sigma|a^1) = U_a(\sigma|a^2)^\neg \odot p_2^a \urcorner) \end{bmatrix} \odot p_1^a \} \oplus$
 $\{ \begin{bmatrix} (\ulcorner U_a(\sigma|a^2) = U_a(\sigma|a^1)^\neg \odot p_1^a \urcorner \oplus \\ (\ulcorner U_a(\sigma|a^2) = U_a(\sigma|a^2)^\neg \odot p_2^a \urcorner) \end{bmatrix} \odot p_2^a \}$;
4. $(\ulcorner U_a(\sigma|a^1) \leq U_a(\sigma|a^1)^\neg \odot p_1^a \urcorner \oplus$
 $(\ulcorner U_a(\sigma|a^1) \leq U_a(\sigma|a^2)^\neg \odot p_2^a \urcorner),$
 $(\ulcorner U_a(\sigma|a^2) \leq U_a(\sigma|a^1)^\neg \odot p_1^a \urcorner \oplus$
 $(\ulcorner U_a(\sigma|a^2) \leq U_a(\sigma|a^2)^\neg \odot p_2^a \urcorner).$

4 Equations and inequalities in $\mathbb{L}_\infty$

To completely determine the set of formulas Φ_G developed in the former section, we still need to explicitly write formulas like $\ulcorner U_a(\sigma|a^i) = U_a(\sigma|a^k)^\neg \urcorner$ and $\ulcorner U_a(\sigma|a^i) \leq U_a(\sigma|a^k)^\neg \urcorner$; in this section, such constructions are done based on the work about function representation by Preto and Finger [22, 23]. We continue to consider only player a since the formulas for player b are analogous. Player a 's expected utilities $U_a(\sigma|a^i)$, for actions a^i , are given by

$$U_a(\sigma|a^i) = \sum_{j=1}^{\beta} u_a(a^i, b^j) \sigma_b(b^j).$$

Thus, our goal is to represent linear equations and inequalities in $\mathbb{L}_\infty$ with variables standing for $\sigma_b(b^j)$ and rational coefficients $u_a(a^i, b^j)$. By representing an equation (or an inequality) in $\mathbb{L}_\infty$, we mean building a formula or a set of formulas that is satisfiable by a valuation v if, and only if, v encodes a solution to the equation (or inequality). Let us treat the general case of an equation

$$\gamma_1 x_1 + \dots + \gamma_n x_n = 0 \tag{5}$$

with variables x_i and non simultaneously zero rational fractions γ_i , for which we are interested in solutions in $[0, 1]^n$.

Before building the representation, we put this equation in an equivalent *normal form* defined in two steps. First we turn (5) into

$$\sum_{i \in I} \frac{\tilde{\gamma}_i}{m} x_i = \sum_{j \in J} \frac{\tilde{\gamma}_j}{m} x_j,$$

where $i \in I$, if $\gamma_i \geq 0$, and $j \in J$, if $\gamma_j < 0$, with $I \cup J = \{1, \dots, n\}$; m is the least common multiple of all denominators in fractions γ_k , for $k = 1, \dots, n$; $\frac{\tilde{\gamma}_i}{m}$ are equivalent fractions to γ_i , for $i \in I$; and $\frac{\tilde{\gamma}_j}{m}$ are equivalent fractions to $-\gamma_j$, for $j \in J$. Note that $\tilde{\gamma}_k$ and m are positive integers, for $k = 1, \dots, n$.

The second step consists in turning (5) into

$$\sum_{i \in I} \frac{\tilde{\gamma}_i}{M} x_i = \sum_{j \in J} \frac{\tilde{\gamma}_j}{M} x_j,$$

where $M = \tilde{\gamma}_1 + \dots + \tilde{\gamma}_n \neq 0$. In the final normal form, both sides of the equation take values in $[0, 1]$ for any vector $\langle x_1, \dots, x_n \rangle \in [0, 1]^n$.

We associate to equation variables x_k propositional variables with same name x_k , for $k = 1, \dots, n$, and we build the following set of formulas that represents (5), using the auxiliary propositional variables $c_{\frac{1}{M}}$ and $\tilde{x}_k$, for $k = 1, \dots, n$:

$$\bigoplus_{i \in I} \tilde{\gamma}_i \tilde{x}_i \leftrightarrow \bigoplus_{j \in J} \tilde{\gamma}_j \tilde{x}_j; \quad (6)$$

$$c_{\frac{1}{M}} \leftrightarrow \neg(M - 1)c_{\frac{1}{M}}; \quad (7)$$

$$\tilde{x}_k \rightarrow c_{\frac{1}{M}}, \text{ for } k = 1, \dots, n; \quad (8)$$

$$x_k \leftrightarrow M\tilde{x}_k, \text{ for } k = 1, \dots, n. \quad (9)$$

LEMMA 5.

Let v be a valuation satisfying formulas (7)-(9). Then, $v(\tilde{x}_k) = \frac{v(x_k)}{M}$.

PROOF. First, suppose that v satisfies formula (7). Then, if $v(c_{\frac{1}{M}}) > \frac{1}{M}$, we have that $v(\neg(M - 1)c_{\frac{1}{M}}) < \frac{1}{M}$, which contradicts the satisfiability of bi-implication. An analogous contradiction is achieved by supposing $v(c_{\frac{1}{M}}) < \frac{1}{M}$. Then, as $v(c_{\frac{1}{M}}) = \frac{1}{M}$ satisfies (7), this is the only possibility for such a valuation v . We already know that if v satisfies (8), then $v(\tilde{x}_k) \leq v(c_{\frac{1}{M}})$. Thus, if v simultaneously satisfies (7)-(9), we have that $v(\tilde{x}_k) \leq \frac{1}{M}$ and $v(x_k) = \min(1, M \cdot v(\tilde{x}_k)) = M \cdot v(\tilde{x}_k)$. Therefore, $v(\tilde{x}_k) = \frac{v(x_k)}{M}$. $\square$

THEOREM 2.

Let v be a valuation satisfying formulas (7)-(9). Then, v satisfies formula (6) if, and only if, $\langle v(x_1), \dots, v(x_n) \rangle$ satisfies equation (5).

PROOF. A valuation v satisfies (6) if, and only if,

$$\min \left(1, \sum_{i \in I} \tilde{\gamma}_i v(\tilde{x}_i) \right) = \min \left(1, \sum_{j \in J} \tilde{\gamma}_j v(\tilde{x}_j) \right).$$

But, if v satisfies (7)-(9), such equation is equivalent to

$$\min \left(1, \sum_{i \in I} \tilde{\gamma}_i \frac{v(x_i)}{M} \right) = \min \left(1, \sum_{j \in J} \tilde{\gamma}_j \frac{v(x_j)}{M} \right).$$

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Now, by the definition of M , $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$, for $i \in I$ and $j \in J$, we have that $\sum_{i \in I} \tilde{\gamma}_i \frac{v(x_i)}{M} \leq 1$ and $\sum_{j \in J} \tilde{\gamma}_j \frac{v(x_j)}{M} \leq 1$. Therefore, the last equation is equivalent to

$$\sum_{i \in I} \tilde{\gamma}_i \frac{v(x_i)}{M} = \sum_{j \in J} \tilde{\gamma}_j \frac{v(x_j)}{M}.$$

Then, we have that a valuation v satisfies (6)-(9) if, and only if, $\langle v(x_1), \dots, v(x_n) \rangle$ is a solution to the normal form of equation (5), which is equivalent to (5). $\square$

In view of the above theorem, we are able to explicitly write formulas $\lceil U_a(\sigma|a^i) = U_a(\sigma|a^k) \rceil$ using the above representation technique for equation (5) remembering that we have already identified the equation variables $\sigma_b(b^j)$ with the propositional variables p_j^b . For that, we may use the minimum ($\wedge$) of the formulas (6)-(9) as $\lceil U_a(\sigma|a^i) = U_a(\sigma|a^k) \rceil$ or even only use the formula (6) and add the formulas (7)-(9) to the set Φ_G .

To explicitly write formulas $\lceil U_a(\sigma|a^i) \leq U_a(\sigma|a^k) \rceil$, we use *mutatis mutandis* this very same technique considering

$$\gamma_1 x_1 + \dots + \gamma_n x_n \leq 0$$

instead of equation (5) and using

$$\bigoplus_{i \in I} \tilde{\gamma}_i \tilde{x}_i \rightarrow \bigoplus_{j \in J} \tilde{\gamma}_j \tilde{x}_j$$

instead of formula (6).

The drawback in the representations just presented is that formulas (6), (7) and (9) have exponential size in the binary representation of $\tilde{\gamma}_k$, $M - 1$ and M due to the several times ($\tilde{\gamma}_k$, $M - 1$ and M) we need to add Łukasiewicz disjunctions of the same propositional variable.

This situation may be circumvented by taking advantage of binary representation in writing these disjunctions. Thus, instead of formula Nx , for $N \in \mathbb{N} \setminus \{0, 1\}$ and x a propositional variable, we write

$$\bigoplus_{k \in K} x_{2^k},$$

where x_{2^k} , for $k = 0, \dots, \lfloor \log N \rfloor$, are new propositional variables and, for $n_{2^k} \in \{0, 1\}$ coming from the binary expansion

$$N = \sum_{k=0}^{\lfloor \log N \rfloor} 2^k n_{2^k},$$

we define the set

$$K = \left\{ k \in \{0, \dots, \lfloor \log N \rfloor\} \mid n_{2^k} = 1 \right\}.$$

We also need to define the new propositional variables by adding the following $\lfloor \log N \rfloor + 1$ formulas to the original collection (6)-(9):

$$x_{2^0} \leftrightarrow x; \quad (10)$$

$$x_{2^k} \leftrightarrow x_{2^{k-1}} \oplus x_{2^{k-1}}, \quad \text{for } k = 1, \dots, \lfloor \log N \rfloor. \quad (11)$$

In view of all the discussion in this section, we are able to state the following general result.

THEOREM 3.

Linear equations and inequalities with rational coefficients may be represented in $\mathbb{L}_\infty$ by formulas with polynomial size in the size of the binary representation of such equations and inequalities coefficients.

EXAMPLE 4.

We build the formula $\ulcorner U_a(\sigma|a^1) = U_a(\sigma|a^2) \urcorner$ in Example 2 related to game G_e in Example 1 in the following. Let $\sigma_b(b^1) = p_1$ and $\sigma_b(b^2) = p_2$, so we need to represent in $\mathbb{L}_\infty$ the equation

$$3p_1 = 2p_2,$$

whose normal form is

$$\frac{3}{5}p_1 = \frac{2}{5}p_2.$$

Note that the propositional variables p_1^b and p_2^b stand for the equational variables p_1 and p_2 , respectively, in the representation. Thus, $\ulcorner U_a(\sigma|a^1) = U_a(\sigma|a^2) \urcorner$ is the minimum ($\wedge$) of the following formulas:

- $\tilde{p}_{1,2^0}^b \oplus \tilde{p}_{1,2^1}^b \leftrightarrow \tilde{p}_{2,2^1}^b$;
- $\tilde{p}_{1,2^0}^b \leftrightarrow \tilde{p}_1^b, \tilde{p}_{1,2^1}^b \leftrightarrow \tilde{p}_{1,2^0}^b \oplus \tilde{p}_{1,2^0}^b$;
- $\tilde{p}_{2,2^0}^b \leftrightarrow \tilde{p}_2^b, \tilde{p}_{2,2^1}^b \leftrightarrow \tilde{p}_{2,2^0}^b \oplus \tilde{p}_{2,2^0}^b$;
- $c_{\frac{1}{5}} \leftrightarrow \neg c_{\frac{1}{5},2^2}$;
- $c_{\frac{1}{5},2^0} \leftrightarrow c_{\frac{1}{5}}, c_{\frac{1}{5},2^1} \leftrightarrow c_{\frac{1}{5},2^0} \oplus c_{\frac{1}{5},2^0}, c_{\frac{1}{5},2^2} \leftrightarrow c_{\frac{1}{5},2^1} \oplus c_{\frac{1}{5},2^1}$;
- $\tilde{p}_1^b \rightarrow c_{\frac{1}{5}}, \tilde{p}_2^b \rightarrow c_{\frac{1}{5}}$;
- $p_1^b \leftrightarrow \tilde{p}_{1,2^0}^b \oplus \tilde{p}_{1,2^2}^b, p_2^b \leftrightarrow \tilde{p}_{2,2^0}^b \oplus \tilde{p}_{2,2^2}^b$;
- $\tilde{p}_{1,2^2}^b \leftrightarrow \tilde{p}_{1,2^1}^b \oplus \tilde{p}_{1,2^1}^b$;
- $\tilde{p}_{2,2^2}^b \leftrightarrow \tilde{p}_{2,2^1}^b \oplus \tilde{p}_{2,2^1}^b$.

5 From Nash equilibrium to $\mathbb{L}_\infty$ -Satisfiability

Two-player games are also known as *bimatrix games* because they may be represented by two matrices whose entries are the player's utilities standing for a table representation like Table 1 in Example 1. In such example, the game G_e has the following matrices for players a and b , respectively:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus, a two-player game G may be encoded for algorithmic purposes by two matrices with entries given in binary representation; we assume that the values of entries are given in the format of fractions. We write $|G|$ for the size of such representation.

In this way, for players a and b in the bimatrix game G assumed in Sections 3 and 4, the formulas in (i) may be, respectively, built in times $O(\alpha)$ and $O(\beta)$ and the formulas in (ii) may be, respectively, built in times $O(\alpha^2)$ and $O(\beta^2)$.

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Let us assume that formulas like $\lceil U_a(\sigma|a^i) = U_a(\sigma|a^k) \rceil$ and $\lceil U_a(\sigma|a^i) \leq U_a(\sigma|a^k) \rceil$ may be built in time $p(|G|)$. Then, both the formula in (iii) and the formulas in (iv) may be built in times $O(\alpha^2 \cdot p(|G|))$ and $O(\beta^2 \cdot p(|G|))$, for players a and b , respectively.

By the discussion above, building the set Φ_G from a game G may be done in polynomial time as long as formulas like $\lceil U_a(\sigma|a^i) = U_a(\sigma|a^k) \rceil$ and $\lceil U_a(\sigma|a^i) \leq U_a(\sigma|a^k) \rceil$ may also be computed in polynomial time. The equations and inequalities necessary for building representations in $\mathbb{L}_\infty$ may surely be computed from a bimatrix game G in polynomial time. Given a linear equation (or inequality), its transformation to the normal form may as well be computed in polynomial time. From an equation (or inequality) in normal form, the version of formulas (6)-(9) where multiple disjunctions are written using the technique given by (10)-(11) may also be built in polynomial time.

Therefore, on the one hand, we have associated to each bimatrix game G a polynomial time computable set Φ_G . On the other hand, from any valuation v that satisfies Φ_G , it is easy to compute in polynomial time a strategy profile σ_v that is a Nash equilibrium in G . We have established the following result.

THEOREM 4.

The problem of computing a Nash equilibrium in a two-player (or bimatrix) game is polynomial time reducible to $\mathbb{L}_\infty$ -Satisfiability.

6 Related work: Nash equilibria in product Łukasiewicz logics

Běhounek et al. [1] express Nash equilibria in systems that extend $\mathbb{L}_\infty$ with a product operator. Let us present such approach focusing in the case of product Łukasiewicz infinitely-valued logic (PL_∞) [6], however other logics that extend $\mathbb{L}_\infty$ with a product operator may be employed as well. The language of PL_∞ extends the language of $\mathbb{L}_\infty$ with the product operator ($\otimes$) and PL_∞ -semantics generalizes $\mathbb{L}_\infty$ -semantics adding the following rule for $\otimes$:

$$v(\varphi \otimes \psi) = v(\varphi) \cdot v(\psi),$$

where $v : \mathbb{P} \rightarrow [0, 1]$ is a PL_∞ -valuation.

Let $G = \langle P, A, u \rangle$ be a game that is called a *finite expressible logical PL_∞ -game*, which means that, in addition to P being a set of n players, $A = A_1 \times \dots \times A_n$ is a set of action profiles in which the set of actions for player i

$$A_i = \left\{ v_i^j : V_i \rightarrow [0, 1] \mid j = 1, \dots, |A_i| \right\}$$

is a finite set of assignments to the propositional variables in a finite set $V_i = \{x_i^1, \dots, x_i^{|V_i|}\}$. We must have $V_i \cap V_j = \emptyset$, if $i \neq j$; then, we define $V \doteq \bigcup_{i=1}^n A_i$. Also, each utility function $u_i : A \rightarrow \mathbb{Q}$ is given by a PL_∞ -formula φ_i over variables from V in a way that

$$u_i(v_1, \dots, v_n) = v(\varphi_i),$$

where $v_j \in A_j$, for $j \in P$, and v is a PL_∞ -valuation such that $v(p) = v_j(p)$, if $p \in V_j$. Such a game is said to be *finite* because sets of actions are finite and *expressible* because constant values may be represented in the language of PL_∞ ; given $c \in [0, 1] \cap \mathbb{Q}$, let $\bar{c}$ be the *truth constant* that is evaluated as c .

Logical games as the ones defined above are said to realize a compact representation of games. That is because, in many cases, the necessary space for representing formulas φ_i is way smaller than explicit table representations for the utility functions u_i .

The codification of equilibria in PL_∞ is based on the following result.

PROPOSITION 2. ([16], Corollary 5.8)

Let $G = \langle P, A, u \rangle$ be a game. A strategy profile σ is a Nash equilibrium if, and only if,

$$U_i(\sigma|a_i) \leq U_i(\sigma),$$

for every player $i \in P$ and every pure strategy $a_i \in A_i$.

The codification proceeds as follows: let $a = \langle v_1^{\alpha_1}, \dots, v_n^{\alpha_n} \rangle$ be an action profile, where $\alpha_i \in \{1, \dots, |A_i|\}$; we denote by $\varphi_i(a)$ the PL_∞ -formula obtained from φ_i by replacing each occurrence of a propositional variable $x_j \in V_j$ by the truth constant representing value $v_j^{\alpha_j}(x_j)$. Let $\sigma = \langle \sigma_1, \dots, \sigma_n \rangle$ be a strategy profile; to each probability value $\sigma_i(v_i^\alpha)$, for all $v_i^\alpha \in A_i$, we associate the propositional variable $p_{\alpha_i}^i$. Then, formula

$$\bigoplus_{a \in A} \left(\varphi_i(a) \otimes \bigotimes_{j \in P} p_{\alpha_j}^j \right),$$

which we abbreviate by ζ_i , evaluates to player i 's utility $U_i(\sigma)$ by PL_∞ -valuation v such that $v(p_{\alpha_j}^j) = \sigma_j(v_j^{\alpha_j})$.

Now, for an action (pure strategy) $v_i \in A_i$, let $A' \subseteq A$ be a set of strategy profiles where player i only uses pure strategy v_i . Then, formula

$$\bigoplus_{a \in A'} \left(\varphi_i(a) \otimes \bigotimes_{j \in P \setminus \{i\}} p_{\alpha_j}^j \right),$$

which we abbreviate by $\eta_i(v_i)$, evaluates to player i 's utility $U_i(\sigma|v_i)$ by PL_∞ -valuation v such that $v(p_{\alpha_j}^j) = \sigma_j(v_j^{\alpha_j})$, for $j \in P \setminus \{i\}$.

Finally, by Proposition 2, game G has a Nash equilibrium encoded by PL_∞ -valuation v if, and only if, v satisfies formula

$$\bigwedge_{i \in P} \left(\mathbf{prob}_i(p_1^i, \dots, p_{|A_i|}^i) \wedge \bigwedge_{v_i \in A_i} (\eta_i(v_i) \rightarrow \zeta_i) \right), \quad (12)$$

where $\mathbf{prob}_i(p_1^i, \dots, p_{|A_i|}^i)$ is satisfied by a PL_∞ -valuation v iff the values v assign to $p_1^i, \dots, p_{|A_i|}^i$ represent a probability distribution [1, Theorem 4.7].

Note that expression $\mathbf{prob}_i$ might be based, for instance, in the techniques in Section 3.1. Also, despite finite expressible logical PL_∞ -games being a compact representation for games, formula (12) has occurrences of all formulas $\varphi_i(a)$, for all $i \in P$ and $a \in A$.

For two-player games, formula $\eta_1(v_1)$ comes down to

$$\bigoplus_{a \in A'} \left(\varphi_1(a) \otimes p_{\alpha_2}^2 \right).$$

As $\varphi_1(a)$ has constant value under any valuation v , let such value be $\frac{c}{d} \in \mathbb{Q}$. Then, $\varphi_1(a) \otimes p_{\alpha_2}^2$ may be replaced by

$$c\tilde{p}_{\alpha_2}^2 \wedge (c\frac{1}{d} \leftrightarrow \neg(d-1)c\frac{1}{d}) \wedge (\tilde{p}_{\alpha_2}^2 \rightarrow c\frac{1}{d}) \wedge (p_{\alpha_2}^2 \leftrightarrow d\tilde{p}_{\alpha_2}^2),$$

in an approach based on Lemma 5. In this way, formulas $\eta_i(v_i)$ may be replaced by formulas without the product operator. This is the case because such formulas express a multiplication of a constant value by a variable value in the setting of two-player games.

However, even for two-player games, the same technique is not available for formulas ζ_i , which express the multiplication of, at least, two variable values.

7 Conclusions

We have presented a recipe to build a L_∞ -propositional theory Φ_G whose models encode all the Nash equilibria of a given two-player game G . Such theory Φ_G has polynomial size in the representation of G and may be built in polynomial time from G .

From an algebraic point of view, we have provided equational constraints for Nash equilibria over the MV-algebra $\langle [0, 1], \oplus, \neg, 0 \rangle$, that is the algebraic structure where the unit interval $[0, 1]$ is the domain, $\oplus$ and $\neg$ are the truncated sum and the complement operations over the unit interval and 0 is the identity element of the truncated sum.

Computationally, we have provided a specific polynomial reduction from the problem of computing a Nash equilibrium to L_∞ -Satisfiability, which was guaranteed to exist by the Cook–Levin Theorem. Thus, new algorithms for computing Nash equilibria through methods for computing satisfiable valuations to formulas may be derived. Also, the study of approximation of Nash equilibria might take advantage of approximation techniques for L_∞ [8].

The comparison in practice of the performances of implementations of such new algorithms (coupling the reduction discussed in this work and methods for L_∞ -Satisfiability) with implementations of the already known ones (e.g. the Lemke–Howson Algorithm) is a possible path to pursue in the future. Also, the discussed reduction may be used for generating benchmarks intended for testing implementations of methods for L_∞ -Satisfiability.

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