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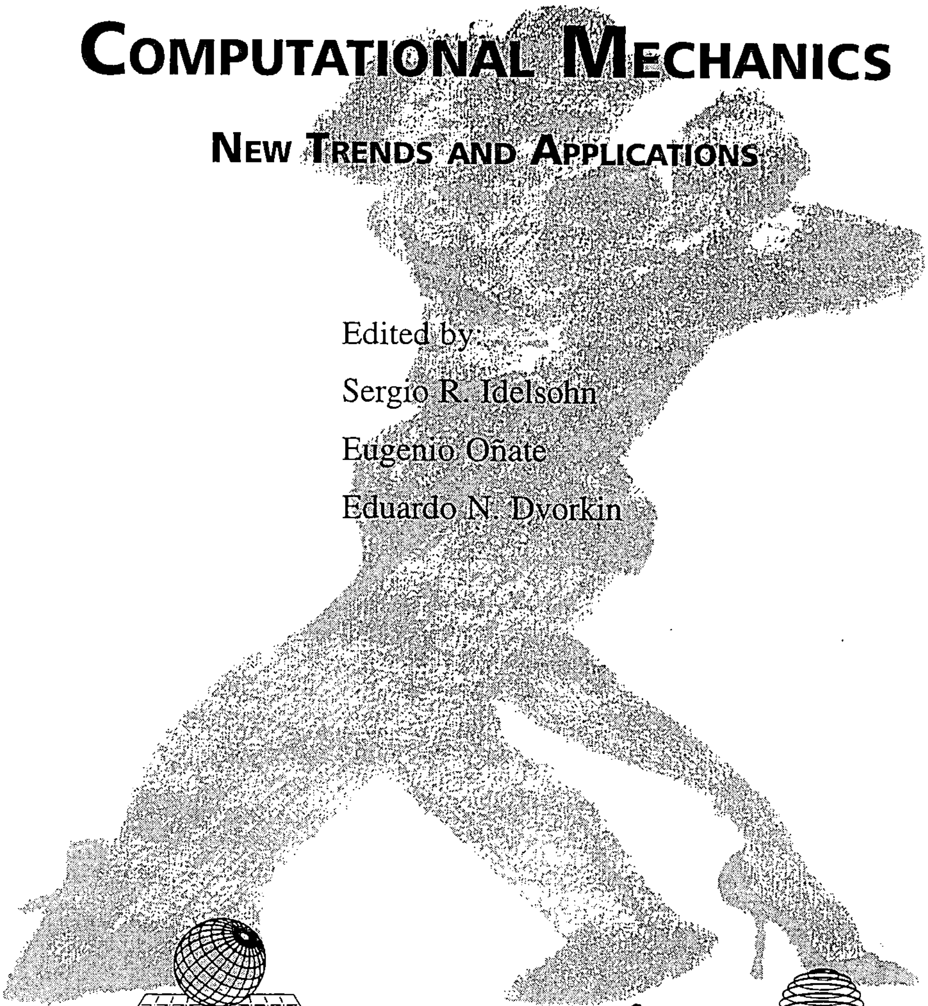

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COMPUTATIONAL MECHANICS
New Trends and Applications
S. Idelsohn, E. Oñate and E. Dvorkin (Eds.)
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BOUNDARY ELEMENT FORMULATION APPLIED TO LOCALISATION PROBLEMS

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Key Words, Boundary elements, Non-linear problems, Localization

Abstract. *The use of the boundary element method (BEM) to analyse non-linear problems is a interesting field of research with many possible applications.. For the particular case of elastoplastic boundary element analysis the formulation is known since the seventies. On this matter, many woks have been published so far emphasising possible practical applications of the technique and pointing out its accuracy. For any BEM non-linear formulation, the required integral representations for displacements and stresses are characterised by exhibiting domain integral terms, which are used to correct the stress field according to the adopted non-linear criterion.*

In this paper the elastoplastic boundary element formulation is extended to incorporate the analysis of bodies where the localisation phenomenon occurs. The plastic multiplier is assumed to be governed by the gradient theory, from which new integral representations are derived. After discretizing this new integral representation, for which boundary and cell meshes are required, the plastic multiplier field associated with the proper plastic region appears algebraically represented to be coupled together with the classical set of equations derived for the plastic problem. Numerical examples are then solved to illustrate the proposed formulation.

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1 INTRODUCTION

The Boundary Element Method (BEM) is nowadays a well established technique to deal with a large number of practical applications in engineering. In particular, the use of BEM to analyse non-linear problems has deserved special attention by BEM community. Fracture mechanics, just to mention one example, is a very suitable kind of problem to apply the BEM techniques, because only the crack lines have to be discretized, saving computer time and also increasing the accuracy of the results and the confidence in the global solution[1,2]. Another field of applications in engineering where the method has shown to be efficient is related to problems exhibiting infinite or large domains, in which only a small region concentrates the desired solution, soil-structure interactions, plate bending with concentrated loads, problems with stress concentration in general and others Some non-linear phenomena assumed smeared over the domain, for instance plasticity or visco-plasticity, can also be tackled by the technique[3]. Specially for those cases exhibiting stress or strain concentration, the boundary elements can be recommended. In general, the technique is able to represent well high gradients.

Strain localization may be seen among those cases. Thus, the BEM would be an efficient alternative for that kind of numerical analysis and therefore could be recommended. It is worth to stress that this problem exhibits small areas of interest inside the body, where the dissipation of energy occurs, as well as rather large displacement gradients, for which the method is expected to be efficient. Thus, for this case, using BEM requires the discretization of very small regions, consequently reducing the number of variables to be handled. In addition, the formulation could be further extended to include problems where plasticity is followed by the definition of crack surfaces.

Analysis of strain localization has been an important subject in the attempt to improve the numerical simulation of structure failures. The presence of strain softening in the constitutive laws brings great difficulties to classical (local) continuum theories[4,5,6,7]. The problem is no longer mathematically well posed after the onset of localization in strain-softening materials, because local continuum allows for an infinitely small band width in shear or in front of a crack tip[7,8]. At the numerical level, these difficulties translate in mesh dependence of solutions[9,10]. Different approaches have been proposed to overcome these difficulties. (see Armero & Garikipati[11]). One idea is to enrich the continuum with non conventional constitutive relations in such way that an internal or characteristic length scale is introduced. Non-local theories like that are the Cosserat continuum[12,13], the higher gradient theories[14], the integral theory or the gradient theory[15,16,17].

In this paper a gradient plasticity model is adopted together with the boundary element method. In finite element context, gradient plasticity has received a good amount of attention in the last years[17,18,19,20]. Apparently, the boundary element method has been very rarely adopted to analyse this kind of problem', but without result published so far[21].

The objective of the present work is to show the main steps of numerical formulations to deal with localization phenomena using the boundary element method. The only theory discussed here is the so called gradient plasticity adopted to define the plastic zone for strain softening materials. The yield function, written in terms of the hardening or softening parameter, as well as in terms of its gradient, will give origin to a second order partial differential equation, which can also be transformed into an integral equation.

2 CONSTITUTIVE EQUATIONS FOR GRADIENT PLASTICITY

The small strain gradient plasticity model taken for this work is a simple modification of the flow theory of plasticity[8,16]. This classical approach is given by the following relations:

- The Cauchy stress tensor increment

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) \tag{1}$$

where $\dot{\varepsilon}$ is the total strain rate, $\dot{\varepsilon}^p$ stands for the plastic strain and E is the elastic modulus tensor;

- The yield criterion

$$f(\sigma, R(p)) = 0 \tag{2}$$

where R is the size of the yield surface and p the cumulated plastic strain defined by:

$$\dot{p} = \sqrt{2\dot{\varepsilon}^p \dot{\varepsilon}^p} \tag{3}$$

- The flow rule

$$\dot{\varepsilon}^p = \lambda \frac{\partial F}{\partial \sigma} \tag{4}$$

with $F(\sigma, R)$ denoting the plastic potential and λ the plastic multiplier;

- The hardening rule

$$\dot{p} = \lambda \frac{\partial F}{\partial R} \tag{5}$$

The plastic multiplier in equations (4) and (5) satisfies the Kuhn-Tucker conditions:

$$\lambda \geq 0, \qquad f \leq 0, \qquad \lambda f = 0 \tag{6a,b,c}$$

being λ obtained when strictly positive and taking into account the consistency condition

$$\lambda \dot{f} = 0 \tag{7}$$

Using relations (1)-(5) one obtains:

$$\lambda = \frac{\frac{\partial f}{\partial \sigma} E \dot{\sigma}}{h + \frac{\partial f}{\partial \sigma} E \frac{\partial F}{\partial \sigma}} \tag{8}$$

with

$$h = -\frac{\partial f}{\partial R} \frac{\partial R}{\partial p} \frac{\partial F}{\partial R} \tag{9}$$

The small strain gradient plasticity is obtained just by modifying equation (2), to make R dependent on p, as well as on its successive gradients. For the sake of simplicity, we consider that R depends just on p and its Laplacian Δp, therefore the yield criterion becomes:

$$f(\sigma, p, \Delta p) = 0 \tag{10}$$

Thus, an explicit form for λ similar to equation (8) can not be derived. From the consistency condition, one is able to derive the following partial differential equation

$$\frac{\partial f}{\partial \sigma} E\dot{\varepsilon} - H\lambda + \omega\Delta\lambda = 0 \tag{11}$$

where $(\partial F / \partial R)$ is assumed constant and

$$H = h + \frac{\partial f}{\partial \sigma} E \frac{\partial F}{\partial \sigma} \qquad \omega = \frac{\partial f}{\partial R} \frac{\partial R}{\partial (\Delta p)} \frac{\partial F}{\partial R} \tag{12a,b}$$

From equation (11) one can realise that the dimension of ω is H times squared length, which gives: $\omega = \alpha \ell^2$, being ℓ a characteristic length and α a material parameter.

3 BOUNDARY ELEMENT FORMULATION

BEM formulations to deal with non-linear problems are nowadays well established. The simplest approach is based on integral representations written for particular collocation points defined along the boundary and inside the domain[22,23]. The formulation adopted here is the one that deals with smeared non-linearities, which are properly taken into account by the initial stress approach.

Let us first consider the elastic case, for which the BEM integral equations are derived by applying the Betti's principle (Green's second identity) taking into account two elastic states satisfying the Navier's equations, given here for completeness,

$$(-L_{ki} u_i) = -G u_{k,ii} - \frac{G}{1-2\nu} u_{i,ik} = b_k \tag{13}$$

where u_k represents displacements, G is the shear modulus, ν is the Poisson's ratio and L the Navier's operator.

For a domain Ω with boundary Γ, the following displacement and stress integral representations are easily derived[22,23].

$$c_{ik} u_k = -\int_\Gamma p^*_{ik} u_k d\Gamma + \int_\Gamma u^*_{ik} p_k d\Gamma + \int_\Omega u^*_{ik} b_k d\Omega \tag{14}$$

$$\beta \sigma_{ij} = -\int_\Gamma S_{ijk} u_k d\Gamma + \int_\Gamma D_{ijk} p_k d\Gamma + \int_\Omega D_{ijk} b_k d\Omega \tag{15}$$

where p_k and b_k are traction and body force components; the symbol "*" stands for the fundamental values; the free terms c_{ik} and β are dependent upon the boundary geometry; D_{ijk} and S_{ijk} are kernels derived from the corresponding ones in equation (14).

For non-linear problems, Betti's principle can not be directly applied. Moreover, in plasticity the state variables are functions of time. In this case, operator L is equivalent to the following equations satisfied by the state variable rates:

$$\frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}) = \dot{\varepsilon}_{ij}+\dot{\varepsilon}^p_{ij} \quad C_{ijkl}\dot{\varepsilon}_{kl} = \dot{\sigma}_{ij}+\dot{\sigma}^p_{ij} \quad -\dot{\sigma}_{ij,j} = \dot{b}_i \tag{16}$$

where $\dot{\varepsilon}^p$ and $\dot{\sigma}^p_{ij}$ are the plastic strain and stress rates, having the same meaning of initial strains and stresses in linear theory; and C_{ijkl} are the elastic constants.

In the initial stress formulation $\dot{\varepsilon}^p$ is zero, therefore equation (16) becomes:

$$-L_{kl}\dot{u}_l = \dot{b}_k - \dot{\sigma}^p_{kl,l} \tag{17}$$

Thus, the Somigliana's identity for plasticity with initial stress approach is:

$$c_{ik}\dot{u}_k = \int_\Gamma u^*_{ik}\dot{p}_k d\Gamma - \int_\Gamma p^*_{ik}\dot{u}_k d\Gamma + \int_\Omega u^*_{ik}\dot{b}_k d\Omega + \int_\Omega \varepsilon^*_{ijk}\dot{\sigma}^p_{jk} d\Omega \tag{18}$$

As for elasticity, the integral representation of stress rates can be obtained by differentiating (18) and applying Hooke's law, noting that in this case the relations between stresses and displacements are given by (16). Thus, one obtains,

$$\dot{\sigma}_{ij} = \int_\Gamma D_{ijk}\dot{p}_k d\Gamma - \int_\Gamma S_{ijk}\dot{u}_k d\Gamma + \int_\Omega D_{ijk}\dot{b} d\Omega + \int_\Omega E_{ijmk}\dot{\sigma}^p_{mk} d\Omega + g_{ij}\left(\dot{\sigma}^p_{mk}\right) \tag{19}$$

where the kernel E_{ijmk} comes from the differentiation of the plastic integral and $g_{ij}\left(\dot{\sigma}^p_{mk}\right)$ is the free-term that appeared due to the strong singularity of the original kernel.

A similar expression can be obtained for initial strain approach assuming $\dot{\sigma}^p_{ij}$ null in equation (16).

Equation (19) was derived only for internal points. For boundary nodes one must find the limit when q, internal collocation point, goes to Q, on the boundary. For this case, a similar integral representation is obtained. Alternatively, one can use a very often employed procedure, that consists of approaching the boundary stresses using boundary conditions to compute the normal and shear components, while the normal component in the boundary direction is achieved numerically by differentiating the displacements in this direction, after approaching them using shape functions and nodal values. This scheme is widely adopted and has proved to give good results, but the integral equation would be the exact representation for this value. In this work we preferred obtaining the exact integral representation of the boundary stresses; as we intend to use this formulation to solve another class of problems, the results have to be as accurate as possible.

In order to solve a gradient plasticity problem one has to take into consideration equation (11) to govern the plastic multiplier. In this case, the scalar value λ is not dependent only upon the local state of stress, as the classical procedure, equation (8), but is defined by a scalar partial differential equation which is coupled with the equilibrium equations.

Transforming equation (11) into an integral representation is not difficult. On must only follow the usual steps given in well known references[3]. For this work, we are going to use the well known collocation formulation. Even choosing this simple formulation we could have several representations by adopting different fundamental solutions. Using a particular fundamental solution, that takes into account the linear term in equation (11), one may get rid of some undesirable domain integrals, but the term due to the strain field, the first term in equation (11), certainly remains. The most simple fundamental solution choice to derive an integral representation is given by the solution of the Laplacian equation for the infinite 2D domain with an unit load applied at a single point which given by[3]:

$$\lambda^* = \frac{1}{2\pi}\ln(r) \qquad \frac{\partial\lambda^*}{\partial n} = -\frac{1}{2\pi} r_{,i} \tag{20a,b}$$

where λ^* and $\partial\lambda^*/\partial n$ are potential and flux fields due to the applied unit load; r is the distance between load and field points, respectively, and n stands for the outward normal vector.

From equation (11) one is able to derive an integral representation of potentials, λ, the plastic multiplier, using properly the fundamental values given in equations (20).

$$c\lambda = \int_\Gamma \lambda^* \frac{\partial\lambda}{\partial n} d\Gamma + \int_\Gamma \lambda^* \frac{\partial\lambda}{\partial n} d\Gamma - \frac{h}{\omega}\int_\Omega \lambda^*\lambda d\Omega + \frac{1}{\omega}\int_\Omega \lambda^* \frac{\partial f}{\partial\sigma} E\dot{\varepsilon} d\Omega \tag{21}$$

where c is the free term similar the one in equation to (18).

An alternative representation is could be derived if one finds the fundamental solution from the following equation:

$$-\Delta\bar{\lambda} + \frac{H}{\omega}\bar{\lambda} = \delta \tag{22}$$

where δ is the Dirac distribution.

In this case the integral representation of λ is given by:

$$\bar{c}\lambda = \int_\Gamma \bar{\lambda}\frac{\partial\lambda}{\partial n} d\Gamma - \int_\Gamma \frac{\partial\bar{\lambda}}{\partial n}\lambda d\Gamma + \frac{1}{\omega}\int_\Omega \bar{\lambda}\frac{\partial f}{\partial\sigma} E\dot{\varepsilon} d\Omega \tag{23}$$

As it has already been mentioned one domain integral remains, requiring therefore domain discretization and internal value approximation.

Equation1 (21) and (23) govern the plastic multiplier field of the body plastic zone, i.e. over the region where the plastic phenomenon takes place. Thus, Γ, in equations (21) and (23) represents the plastic zone boundary, which can move during loading process, leading to a moving boundary problem.

One important point to discuss is about the boundary conditions to be assumed to solve the this potential problem. There are clearly two types of boundaries to consider, the actual boundary of the body and the internal contours defined by the plastic zones. For the first case, the boundary conditions to be assumed at any node are very clear: the plastic zone can not move forward, therefore the outward normal flux must be zero, while the plastic multiplier boundary value will be unknown. Note that the plastic zone is physically limited by the actual boundary. In the second case, the boundary represents the plastic zone end, what enables us to assume that potential values are zero, i.e. plastic multipliers are zero. On the contrary, flux is the unknown boundary value. This value is related with the boundary velocity (the plastic zone is allowed to move forward).

On the body boundary and on the plastic zone interface one has the following boundary conditions, respectively:

$$\frac{\partial \lambda}{\partial n} = 0 \quad \lambda = 0 \tag{24a,b}$$

The proposed formulation do deal with the potential problem that represents the plastic multiplier problem requires a very skilful numerical approach. One may have only boundary moving in the outward direction. This is the case of problem where only monotonic crescent plastic zone problems are present. The scheme to identify the plastic zone must be more general. We may have boundaries moving in the inward direction; we may also find internal thin zones over which the plastic multiplier can go to zero. In order to overcome these difficulties, we are proposing another procedure that does not require finding the plastic zone boundary in the domain. Ones has to modify equation (22) to be possible to prescribe potentials at internal points. This is made by considering, in equation (11) and consequently in equation (21), the presence of a fictitious body force field represented by $\dot{b}$, that will play the role of displacement conjugate field. These fictitious body forces have no physical meaning. They are unknowns when the potentials are prescribed inside the domain, but their values are not used in the gradient plasticity solution model.

After introducing the body force term equation (11) becomes:

$$\dot{b} + \frac{\partial f}{\partial \sigma} E\dot{\varepsilon} - H\lambda + \omega\Delta\lambda = 0 \tag{25}$$

As a consequence the potential representation, equation (21), modifies to:

$$c\lambda = \int_\Gamma \lambda^* \frac{\partial \lambda}{\partial n} d\Gamma - \int_\Gamma \frac{\partial \lambda^*}{\partial n} \lambda d\Gamma - \frac{h}{\omega} \int_\Omega \lambda^* \lambda d\Omega + \frac{1}{\omega} \int_\Omega \lambda^* \frac{\partial f}{\partial \sigma} E\dot{\varepsilon} d\Omega + \frac{h}{\omega} \int_\Omega \lambda^* \dot{b} d\Omega \tag{26}$$

Note that equation (34) represents potential (plastic multiplier λ) at internal points as well, for which independent term c is one.

4 NUMERICAL IMPLEMENTATION

As it is well known, equations (18) and (19) of the precedent section can be transformed into algebraic representations by approximating $\dot{u}_k$ and $\dot{p}_k$ along the boundary duly divided into elements, as well as $\dot{b}$ and $\dot{\sigma}^p_{mk}$ over the domain now divided into cells. One can write as many algebraic representations as needed. Similarly, we can write an appropriate number of algebraic stress equations, the ones where the stress values are required to solve the problem. Thus, assuming boundary and domain divided into elements and cells respectively, as well as shape functions to approximate all variables, equations (18) and (19) become

$$\mathbf{H\dot{U}} = \mathbf{G\dot{P}} + \mathbf{T\dot{B}} + \mathbf{E}\underline{\dot{\sigma}^p} \tag{27}$$

$$\underline{\dot{\sigma}} = -\mathbf{A}^1\mathbf{X} + \mathbf{F}^1 + \mathbf{T}^1\mathbf{B} + \mathbf{E}^1\underline{\dot{\sigma}^p} \tag{28}$$

where $\mathbf{\dot{U}}$ and $\mathbf{\dot{P}}$ are vectors containing the nodal values for displacements and tractions, respectively; $\underline{\dot{\sigma}}$ and $\underline{\dot{\sigma}^p}$ are the stress and the initial stress vectors; $\mathbf{H}$, $\mathbf{H}^1$, $\mathbf{G}$, $\mathbf{G}^1$, $\mathbf{T}$, $\mathbf{T}^1$, $\mathbf{G}$ and $\mathbf{G}^1$ are the influence matrices computed by integrating elements and cells.

Applying the boundary conditions, equations (27) and (28) become

$$\mathbf{AX} = \mathbf{F} + \underline{\dot{\sigma}^p} \tag{29}$$

$$\underline{\dot{\sigma}} = -\mathbf{A}^1\mathbf{X} + \mathbf{F}^1 + \mathbf{T}^1\mathbf{B} + \mathbf{E}^1\underline{\dot{\sigma}^p} \tag{30}$$

where $\mathbf{A}$ and $\mathbf{A}^1$ contain the coefficients due to the unknown boundary values and $\mathbf{F}$ and $\mathbf{F}^1$ are independent vector due to the prescribed boundary conditions and body forces.

As it is well known equations (29) and (30) can be reduced to:

$$\mathbf{X} = \mathbf{M} + \mathbf{R}\underline{\dot{\sigma}^p} \quad \underline{\dot{\sigma}} = \mathbf{N} + \mathbf{S}\underline{\dot{\sigma}^p} \tag{31a,b}$$

where $\mathbf{M}$ and $\mathbf{N}$ contain the elastic solution due to the prescribed values and $\mathbf{R}$ and $\mathbf{S}$ contain the influences of the applied initial stresses.

This can be applied to solve any non-linear problem where the non-linearities can be assumed smeared over the domain.

In order to solve a plastic problem one has to consider the algebraic solution in terms of displacements and stresses given by equations (31), taking into account that the plastic stress must be appropriately computed by using the plastic multiplier governed by equation (8). Thus, taking into account that matrix equations (31) can be adopted for one increment and the plastic multiplier is given locally by equation (8), the presented formulation is appropriate for classical plasticity. In order to apply it to gradient plasticity, however, equation (21) or (26) must be used to describe the plastic multiplier field inside the plastic zone.

Expression (21) is the integral representation equivalent to equation (11), which governs the plastic multiplier in gradient plasticity. The discretized form of expression (21) can also be written assuming that boundary values, λ and $\partial\lambda/\partial n$, are approximated along elements. In addition, the domain integrals can be transformed as well into algebraic terms by dividing the domain into cells and adopting convenient shape functions to approximate the plastic multiplier λ and the independent term $\left[(\partial f/\partial\sigma)E\dot{\varepsilon}\right]$ .Similarly to the displacement algebraic representation one can find the algebraic representation of equation (21).

$$\mathbf{H}_p\underline{\Lambda}=\mathbf{G}_p\underline{\left(\frac{\partial\Lambda}{\partial \mathbf{n}}\right)}-\mathbf{h}\begin{bmatrix}\mathbf{S}_\Lambda & \mathbf{S}_\lambda\end{bmatrix}\begin{bmatrix}\underline{\Lambda}\\ \underline{\lambda}\end{bmatrix}+\begin{bmatrix}\mathbf{S}_\Lambda & \mathbf{S}_\lambda\end{bmatrix}\begin{bmatrix}\mathbf{B}_\Lambda\\ \mathbf{B}\lambda\end{bmatrix} \tag{32}$$

where $\underline{\Lambda}$ and $\underline{(\partial\Lambda/\partial\mathbf{n})}$ are the plastic multiplier nodal vector for boundary nodes and the corresponding flux vector; $\underline{\lambda}$ represents the plastic multiplier at internal points; $\mathbf{H}_p$ and $\mathbf{G}_p$ are the classical matrices of potential problems to consider the boundary values; and $\begin{bmatrix}\mathbf{S}_\Lambda & \mathbf{S}_\lambda\end{bmatrix}$ is the matrix to take into account the domain fields represented by their nodal vectors, either $\begin{bmatrix}\underline{\Lambda} & \underline{\lambda}\end{bmatrix}^t$ or $\begin{bmatrix}\mathbf{B}_\Lambda & \mathbf{B}_\lambda\end{bmatrix}^t$.

Similarly, expression (26) can be transformed into an algebraic representation. For this case the fictitious body force values have to be approximated over the domain cells. Thus, when domain unknown body forces are required to solve the problem equation (32) becomes:

$$\mathbf{H}_p\underline{\Lambda}=\mathbf{G}_p\underline{\left(\frac{\partial\Lambda}{\partial \mathbf{n}}\right)}-\mathbf{h}\begin{bmatrix}\mathbf{S}_\Lambda & \mathbf{S}_\lambda\end{bmatrix}\begin{bmatrix}\underline{\Lambda}\\ \underline{\lambda}\end{bmatrix}+\begin{bmatrix}\mathbf{S}_\Lambda & \mathbf{S}_\lambda\end{bmatrix}\begin{bmatrix}\mathbf{B}_\Lambda\\ \mathbf{B}\lambda\end{bmatrix}+\mathbf{S}_\lambda\overline{\mathbf{B}} \tag{33}$$

Note that the body force values $\dot{\vec{b}}$ are only taken at inside nodes. For boundary nodes, fluxes are the conjugate potential variable, therefore we do not require another the definition of a new value.

As new internal unknown values have been introduced, new representation must be written to have equalised the numbers of equations and unknowns. Thus, the plastic multiplier integral representation written for internal nodes are used to complete the final system of equations. Thus, ones can write

$$\underline{\lambda}=-\mathbf{H}_p^\mathrm{I}\underline{\Lambda}+\mathbf{G}_p^\mathrm{I}\underline{\left(\frac{\partial\Lambda}{\partial \mathbf{n}}\right)}-\mathbf{h}\begin{bmatrix}\mathbf{S}_\Lambda^\mathrm{I} & \mathbf{S}_\lambda^\mathrm{I}\end{bmatrix}\begin{bmatrix}\underline{\Lambda}\\ \underline{\lambda}\end{bmatrix}+\begin{bmatrix}\mathbf{S}_\Lambda^\mathrm{I} & \mathbf{S}_\lambda^\mathrm{I}\end{bmatrix}\begin{bmatrix}\mathbf{B}_\Lambda\\ \mathbf{B}_\lambda\end{bmatrix}+\mathbf{S}_\lambda^\mathrm{I}\overline{\mathbf{B}} \tag{34}$$

where the matrices $\mathbf{H}_p^\mathrm{I}$, $\mathbf{G}_p^\mathrm{I}$, and $\begin{bmatrix}\mathbf{S}_\Lambda^\mathrm{I} & \mathbf{S}_\lambda^\mathrm{I}\end{bmatrix}$ are the same ones given in equation (34) represented by different notation to indicate that they are refereed to internal collocation points.

Equations (33) and (34) can be assembled together to give:

$$\begin{bmatrix}\mathbf{H}_p & \underline{\mathbf{0}}\\ \mathbf{H}_p^\mathrm{I} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\underline{\Lambda}\\ \underline{\lambda}\end{bmatrix}=\begin{bmatrix}\mathbf{G}_p\\ \mathbf{G}_p^\mathrm{I}\end{bmatrix}\underline{\left(\frac{\partial\Lambda}{\partial \mathbf{n}}\right)}-\mathbf{h}\begin{bmatrix}\mathbf{S}_\Lambda & \mathbf{S}_\lambda\\ \mathbf{S}_\Lambda^\mathrm{I} & \mathbf{S}_\lambda^\mathrm{I}\end{bmatrix}\begin{bmatrix}\underline{\Lambda}\\ \underline{\lambda}\end{bmatrix}+\begin{bmatrix}\mathbf{S}_\Lambda & \mathbf{S}_\lambda\\ \mathbf{S}_\Lambda^\mathrm{I} & \mathbf{S}_\lambda^\mathrm{I}\end{bmatrix}\begin{bmatrix}\mathbf{B}_\Lambda\\ \mathbf{B}_\lambda\end{bmatrix}+\begin{bmatrix}\mathbf{S}_\Lambda\\ \mathbf{S}_\lambda\end{bmatrix}\overline{\mathbf{B}} \tag{35}$$

The vector $\overline{\mathbf{B}}$ with the fictitious body nodal values for internal points is evaluated by solving equation (35) for all internal points where the plastic multipliers are prescribed. The values obtained, as well as the fluxes along the boundaries, play no role for the plastic solution, although they necessarily appear when solving a mixed boundary condition problem.

Now, one can follow the steps required to find the plastic solution using the coupled system of equations (35). Due to the incremental nature of the plastic problem, equation (35) is written into its incremental form, $d\lambda$. For an increment of load, the problem is solved elastically and the elastic increment, **N**, is added to the actual stresses. For points that reaches the plastic stage the plastic, stress increment is computed from Hooke's law, equation (16). Note that the plastic strain increment is given by equation (4); $d\lambda$ comes from equation (8) in classical plasticity or from (11) in gradient plasticity. The actual stress increment is the difference between the elastic and the plastic increments. If the plastic increment values are sufficiently small another increment of load can be taken; otherwise, another iteration is carried out taking as new elastic increment the product of the matrix **S** , equation (31b) by the plastic stress increments.

Two different approaches have been implemented: one considering the coupled system of equations given equation (35), where λ equal to zero is specified over the elastic zone, the second scheme implemented the boundary conditions have been specified only along the body actual boundaries, and the plastic multiplier values have been taken only over the plastic zone; In this case double boundary conditions have been assumed along the elastic boundary. Both developed scheme have given identical results.

5 EXACT INTEGRAL REPRESENTATION FOR BOUNDARY STRESSES

Obtaining the exact representation for stresses along the boundary is not an easy task, due to the presence of strongly singular integrals (there are kernels of singularity order of: $1/r^2$ in 2D problems and $1/\ r^3$ in 3D problems). Several contributions on this subject can be found in the literature[24,25,26,27]. Here the approach given in reference 24 will be followed. In order to obtain an expression similar to equation (19) valid for boundary points, one must pay attention on the new singularities involved. As it has been show for equation (19), one has to start by differentiating (18) and then apply Hooke's law (16). Note that the differentiation is taken with reference to the load point now on the boundary. Thus, this differentiation will not give equation (19) again. A new free term will be achieved and the boundary integrals are now singular, requiring therefore a special treatment.

One can consider that the singularities are always at smooth points along the boundary. Even for elements situated at non smooth parts, one can avoid nodes at non smooth points, moving them from their position along the element. In 2D problems, the neighbourhood of a node is always a semi circle.

After considering the new position of the singular points, the free term g_{ij} for smooth points can be obtained (2D problems only):

$$g_{ij}\left(\dot{\sigma}^{p}_{mk}\right) = -\frac{1}{8}\left(2(1+\nu)\dot{\sigma}^{p}_{ij} + (1-3\nu)\dot{\sigma}^{p}_{ll}\delta_{ij}\right) \tag{46}$$

The domain integral of the plastic stress term has to be evaluated in the Cauchy's principal value sense. One can differentiate the other integrals (elastic ones) in the same way used to find the equation (19). Rigorously, one must carry out those differentiations before transforming the Betti's principle into the Somigliana's identity (18).

The integral of equation (18) requires special care due to the singularity order. The singularities arising when differentiating first and third integrals are weaker.

In order to differentiate the boundary integrals we can apply the approach proposed in reference 24. Essentially, it consists in removing a vanishing neighbourhood at the singular point. Then, the integral can therefore be performed on the boundary of the new domain without the part containing the singularity. After that the limit is carried out. Note that the integral must be performed as well along the boundary of the removed part that contains the singular point. They are treated separately, resulting into two free-terms c_{ijkl} and d_{ijk}. (One can also perform those integrals after expanding displacements and tractions into Taylor series centred at the singular point and then adding and subtracting the relevant terms to the corresponding functions in that integrals). c_{ijkl} can be obtained in the same way c_{ik} appeared in equation (18). d_{ijk} is the new part that deserves to be properly evaluated. For two dimensional problems it gives an unbounded term, when $\varepsilon \to 0$, which will be cancelled out with one of the terms resulting from the $p^*_{ij,k}$ integral over the original boundary. Thus, after performing properly all limits involved is this formulation, no unbounded term remains in the final stress representation. All the remaining integrals can be transformed into regular ones to be evaluated using standard gaussian quadrature rules, by simply expressing their kernels in polar co-ordinates centred at the singular points. In general, the main step before performing the limit is to expand the kernels, written in polar co-ordinates, using Laurent series (at the singular point) and expanding the polar co-ordinates of the singular point in Taylor series as well.

6 EXAMPLES

In order to illustrate the formulation we have only very preliminary results obtained by running a classical example already investigated by other authors who have implemented the localization phenomenon together with finite elements[18]. Thus, we have analysed the classical rectangle with displacements prescribed along to opposite sides as indicated in figure 1. In this figure, the solid geometric characteristics are specified together with the boundary internal discretizations. Several meshes have been tested, always using equal spaced boundary and domain sub-divisions. The discretization exhibited in Figure 1, the basic configuration used, has 128 internal cells and 24 boundary elements. Meshes with 256 and 512 cells generated from the basic one have been tested as well. The boundary values of displacements prescribe along the right end is equal to 0.036mm, enough to produce rather large plastic strains inside the solid. The material data are characterised by: Elastic modulus $E=2000kN/mm^2$, Poisson's ratio $\nu = 0.0$, yield stress, $\sigma_y = 2.0kN/mm^2$, softening slope h = -

0.05E and the localization parameter ω = 5000 N. A weaker region, shaded in figure 1, has been considered to start the localization process assuming, the yield stresses reduced by 10%.

The results displayed in Figure 2 have been computed by using the 256 cell mesh. They remain almost the same when the finer mesh has been adopted, with maximum changes around 1%. No significant differences appears in the end bar reactions when running the local and non-local plastic formulations. The non-local formulation shown a more smooth distributions of plastic strain, but not enough to modify the displacement × end reaction curve shown in figure 2.

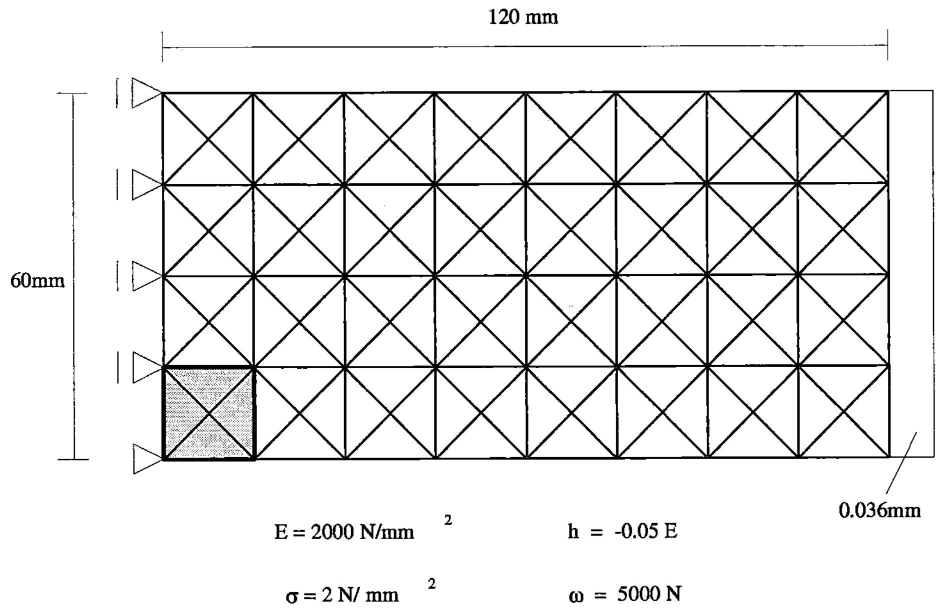

Figure 2. Plastic rectangle under analysis. Geometry and discretizations.

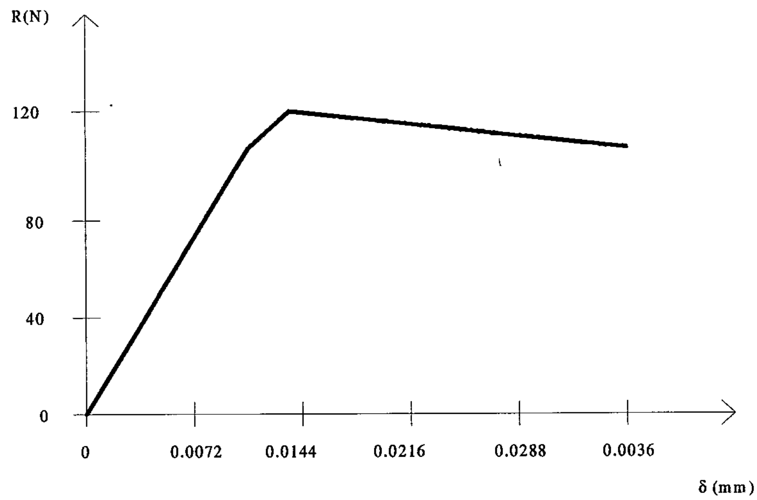

Figure 2. Displacements ×solid end reactions

7 CONCLUSION

A boundary element formulation for gradient plasticity has been presented. The proper integral representation to govern the plastic multiplier field has been derived from the corresponding differential equation. Boundary conditions of the plastic multiplier potential problem have been discussed, which gave origin to a simple scheme to solve the problem, not using moving boundaries but prescribing internal values of the potential field prescribing double boundary conditions along the elastic boundary. The two set of BEM algebraic relations for a single increment of load have been coupled to be solve the gradient plasticity problem. The exact stress integral representation has been derived to avoid inaccurate evaluation of the stress field. A simple example is presented to illustrate the preliminary results obtained as far.

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