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LOOP RINGS OF I. P. LOOPS

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ABSTRACT

Given an associative and commutative ring R with unity and two loops L and M with inverse property, we investigate conditions under which the loop ring RL is isomorphic to RM. In particular, we prove that $GF(2)$, the field with two elements, determines the five Moufang loops of order 24. Also we give a description of the decomposition as a direct sum of simple algebras of the loop algebra of the smallest Moufang loop over a field with characteristic different from 2 and 3.

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1 Introduction

A loop is a set L together with a binary operation $(g, h) \longrightarrow g.h$ for which there is a two-sided identity and with the property that the left and right multiplication maps determined by any element of L are one–one and onto.

Over an associative and commutative ring R with identity, one can construct the *loop ring* RL in the same way we construct a group ring.

A loop L is said to be a *loop with inverse property*, or a *I.P. loop* if for all $x, y \in L$ the following identities hold:

$$x^{-1}.(xy) = y$$
$$(xy).y^{-1} = x.$$

The *isomorphism problem for loop rings* is a version for loops of a classical question set by R.M. Thrall in 1947 for group rings, which here we may roughly state as follows: given a ring R and two loops L and M when will the ring isomorphism $RL \cong RM$ imply the loop isomorphism $L \cong M$? In an affirmative answer we are used to saying that the ring R *determines* the loop L.

Some answers are given in recent years for certain classes of loops (see, [11], [12], [1], [2], [3], [4]). In this paper we prove some results for loop rings of I.P. loops.

2 Background and Notation

The *(loop) commutator* of two elements x and y of a loop L is the element in L, denoted by (x, y), such that $xy = (yx).(x, y)$. The subloop generated by all commutators of a loop L is called its *commutator subloop* and we will denote it by L'.

The *(loop) associator* of three elements x, y and z of a loop L is the element in L, denoted by (x, y, z), such that $(xy)z = (x(yz)).(x, y, z)$. The subloop generated by all associators of a loop L is called its *associator subloop* and we will denote it by $A(L)$.

We observe that if L is a I.P. loop then for all x, y and z in L, we have that

$$(x, y) = (yx)^{-1}.(xy)$$
$$(x, y, z) = ((xy)z)^{-1}.(x(yz)).$$

A subloop N of a loop L is said to be a *normal subloop* of L if for all x and y in L we have that $x(yN) = (xy)N$

$$(Nx)y = N(xy)$$
$$xN = Nx \quad .$$

If N is a normal subloop of a loop L we can define the *quocient loop* $\frac{L}{N}$.

If R is an associative and commutative ring with identity and N is a normal subloop of a loop L, the natural epimorphism $L \to \frac{L}{N}$ extends to a ring epimorphism $RL \to R\left[\frac{L}{N}\right]$ whose kernel, denoted $\Delta_R(L:N)$, is the ideal of RL generated by elements of the form $1-n$, $n \in N$. In the special case $L = N$, the homomorphism above described maps $\sum \alpha_g g \in RL$ to $\sum \alpha_g \in R$. This map called the *augmentation map* , has a kernel written $\Delta_R(L)$ rather than $\Delta_R(L:L)$, which is known as the *augmentation ideal* of RL.

From these definitions it follows that for a normal subloop N of L,

$$\Delta_R(L:N) = RL.\Delta_R(N) \quad \text{and} \quad R\left[\frac{L}{N}\right] \cong \frac{RL}{\Delta_R(L:N)} \quad .$$

In [6], R.H. Bruck showed that for any loop L, the subloop $B(L)$ generated by all associators and all commutators of L is a normal subloop. We will call $B(L)$, the *associator-commutator subloop* of L .

Then for I.P. loops we can state the following proposition which generalizes a result by D. Coleman [9] for group rings.

Proposition 2.1 *Let R be an associative and commutative ring with identity. Let L and M be I.P. loops with associator-commutator subloops $B(L)$ and $B(M)$ respectively. Then $RL \cong RM$ implies that* $R\left[\frac{L}{B(L)}\right] \cong R\left[\frac{M}{B(M)}\right]$.

Proof: Consider the natural epimorphism $L \to \frac{L}{B(L)}$ and its ring extension $RL \to R\left[\frac{L}{B(L)}\right]$ whose kernel is $\Delta_R(L:B(L)) = RL.\Delta_R(B(L))$.

For all α, β, γ in RL, let $[RL, RL]$ denote the left ideal of RL generated by the elements of the form $\alpha\beta - \beta\alpha$, and $[RL, RL, RL]$ denote the left ideal of RL generated by the elements of the form $\alpha(\beta\gamma) - (\alpha\beta)\gamma$.

We claim that $\Delta_R(L:B(L)) = [RL, RL] + [RL, RL, RL]$.

In fact, $[RL, RL]$ is generated over RL by elements of the form $xy - yx$ with x, y in L and, since $xy - yx = xy - xy.(y,x) = xy.(1-(y,x))$, it follows that $[RL, RL] \subset \Delta_R(L:B(L))$. In the same way, $[RL, RL, RL]$ is generated over RL by elements of the form $x(yz) - (xy)z$ with x, y and z in L and, since

$x(yz) - (xy)z = x(yz) - (x(yz)).(x,y,z) = (x(yz)).(1-(x,y,z))$, it follows that $[RL, RL, RL] \subset \Delta_R(L : B(L))$. Hence $[RL, RL] + [RL, RL, RL] \subset \Delta_R(L : B(L))$.

On the other hand, the equality $1 - gh = (1-g) + g.(1-h)$ for all g, h in L shows that $\Delta_R(L : B(L))$ is generated over RL by elements of the form $1-(x,y)$ or $1-(x,y,z)$ with x, y and z in L.

Since $1-(x,y) = 1-(yx)^{-1}.(xy) = (yx)^{-1}.(yx - xy)$ and, since $1-(x,y,z) = 1-((xy)z)^{-1}.(x(yz)) = ((xy)z)^{-1}.[x(yz)-(xy)z]$, it's easy to see that $\Delta_R(L : B(L)) \subset [RL, RL] + [RL, RL, RL]$.

Now given an isomorphism $\varphi : RL \to RM$, we have that
$\varphi(\Delta_R(L : B(L))) = \varphi([RL, RL] + [RL, RL, RL]) =$
$[RM, RM] + [RL, RL, RL] = \Delta_R(M : B(M))$. Consequently, φ induces an isomorphism between the corresponding factor rings and finally we have that

$$R\left[\frac{L}{B(L)}\right] \cong \frac{RL}{\Delta_R(L : B(L))} \cong \frac{RM}{\Delta_R(M : B(M))} \cong R\left[\frac{M}{B(M)}\right] .$$

□

Observing that $\dfrac{L}{B(L)}$ is an abelian group we can state the following

Corollary 2.2 *Let* $\mathbf{Q}$ *be the rational field. Let* L *and* M *be I.P. loops with associator-commutator subloops* $B(L)$ *and* $B(M)$ *respectively. Then* $\mathbf{Q}L \cong \mathbf{Q}M$ *implies that* $\dfrac{L}{B(L)} \cong \dfrac{M}{B(M)}$.

Proof: From proposition 2.1 we have $\mathbf{Q}\left[\dfrac{L}{B(L)}\right] \cong \mathbf{Q}\left[\dfrac{M}{B(M)}\right]$. The result follows by a classical result by S.Perlis and G.L.Walker [13]. □

Corollary 2.3 *Let* F *be a prime field with characteristic* p. *Let* L *and* M *be I.P. loops with associator-commutator subloops* $B(L)$ *and* $B(M)$, *respectively, such that* $\dfrac{L}{B(L)}$ *is a* p*-group. Then* $FL \cong FM$ *implies that* $\dfrac{L}{B(L)} \cong \dfrac{M}{B(M)}$.

Proof: From proposition 2.1 we have $F\left[\dfrac{L}{B(L)}\right] \cong F\left[\dfrac{M}{B(M)}\right]$. The result follows by a classical result by W.E.Deskins [10, Theorem 8]. □

A loop L is said to be a *Moufang loop* if for all x, y, z in L one of the following equivalent *Moufang identities* holds:

$$(xy)(zx) = (x(yz))x$$
$$((zx)y)x = z(x(yx))$$
$$((xy)x)z = x(y(xz))$$

According to [14, Theorem IV.1.4], Moufang loops are P.I. loops, and Lagrange's theorem holds for them [7].

In [8], O. Chein and E.G. Goodaire defined $RA2$ loops as being loops whose loop ring over any field of characteristic 2 is an alternative nonassociative ring. There they proved that $RA2$ loops are Moufang loops. They also proved (Corollary 2.12, p.667) that for any $RA2$ loop L, $A(L) \subset L'$. Then, in this case $B(L) = L'$.

As a particular case of corollary 2.2, we can state the following

Corollary 2.4 *Let* $\mathbf{Q}$ *be the rational field and* L *and* M *be* $RA2$ *loops such that* $\mathbf{Q}L \cong \mathbf{Q}M$. *Then* $\frac{L}{L'} \cong \frac{M}{M'}$.

And as a particular case of corollary 2.3, we can state the following

Corollary 2.5 *Let* F *be a prime field with characteristic* p. *Let* L *and* M *be* $RA2$ *loops such that* $\frac{L}{L'}$ *is a* p*-group. Then* $FL \cong FM$ *implies* $\frac{L}{L'} \cong \frac{M}{M'}$.

In [7], O. Chein classified all nonassociative Moufang loops whose order is smaller than 32. There are 5 nonassociative Moufang loops of order 24, which we have displayed in table 1, using the notation for loops and groups as in [7].

L	$B(L)$	$\frac{L}{B(L)}$	$RA2$ *loop?*
$L_1 = M_{24}(D_6, 2)$	C_3	$C_2 \times C_2 \times C_2$	*yes*
$L_2 = M_{24}(G_{12}, C_2 \times C_4)$	C_3	$C_2 \times C_4$	*yes*
$L_3 = M_{24}(A_4, 2)$	A_4	C_2	*no*
$L_4 = M_{24}(G_{12}, 2)$	C_6	$C_2 \times C_2$	*no*
$L_5 = M_{24}(G_{12}, Q)$	C_6	$C_2 \times C_2$	*no*

Table 1

In the presence of corollary 2.3, table 1 shows us that to study the isomorphism problem for those 5 loops over the field $F = GF(2)$, the field with 2 elements, we just need to study it for the two last loops of the table. Using a computer program, we count the non-null elements in the respective loop algebra whose square is null. In FL_4 we have found 67,583 such elements and in FL_5 we have found 22,527 such elements. Then we can establish the following

Corollary 2.6 *The field $GF(2)$ determines all nonassociative Moufang loops of order 24.*

3 The Semisimple Case

When the characteristic of a field F does not divide the order of a loop L, R.H.Bruck in [5, Th.7A, p.160] shows that the loop algebra FL is semisimple. In this section we will consider this case.

We begin with a result which holds for loop rings in general and it is similar to one by D.Coleman for group rings in [9].

Proposition 3.1 *Let N be a finite normal subloop of a loop L and R be an associative and commutative ring with identity such that $|N|$ is invertible in R. Then*

$$RL \cong R\left[\frac{L}{N}\right] \bigoplus \Delta_R(L:N)$$

Proof: Since $|N|$ is invertible in R, we can define the loop ring element $\widehat{N} = \frac{1}{|N|} \cdot \sum_{n \in N} n$. It is easy to prove that $\widehat{N}$ is a central idempotent element in RL, and we have that $RL = RL.\widehat{N} \bigoplus RL.(1-\widehat{N})$.

Firstly we observe that $RL.(1-\widehat{N}) \subset \Delta_R(L:N)$. Since $\Delta_R(L:N) = RL.\Delta_R(N)$ and $\Delta_R(N)$ is generated by elements of the form $1-n$, $n \in N$, the equalities $1-n = (1-n).(1-\widehat{N})$ for all $n \in N$ imply that the reverse inclusion holds. So we have that $RL.(1-\widehat{N}) = \Delta_R(L:N)$.

Now we observe that

$$RL.\widehat{N} \cong \frac{RL}{RL.(1-\widehat{N})} = \frac{RL}{\Delta_R(L:N)} \cong R\left[\frac{L}{N}\right].$$

□

Proposition 3.2 *Let L be an I.P. loop and $B(L)$ be its associator-commutator subloop. Let R be an associative and commutative ring with identity ring such that $|B(L)|$ is invertible in R. Suppose I and J are two-sided ideals of RL with $RL = I \bigoplus J$. Then I is associative and commutative if and only if $J \supset \Delta_R(L:B(L))$.*

Proof: If $I \cong \dfrac{RL}{J}$ is associative and commutative, then we have that
i) For all x and y in L, $xy - yx = xy - xy.(y,x) = xy.(1-(y,x)) \in J$, which implies that $(xy)^{-1}.[xy.(1-(y,x))] = 1-(y,x) \in J$.

ii) For all x, y and z in L, $x(yz) - (xy)z = x(yz) - (x(yz)).(x,y,z) = (x(yz)).(1-(x,y,z)) \in J$, which implies that $(x(yz))^{-1}.[(x(yz)).(1-(x,y,z))] = 1-(x,y,z) \in J$.

From i) and ii) we conclude that $\Delta_R(L : B(L)) \subset J$.

On the other hand, if $J \supset \Delta_R(L : B(L))$, from
$R\left[\dfrac{L}{B(L)}\right] \cong \dfrac{RL}{\Delta_R(L : B(L))} = \dfrac{I \bigoplus J}{\Delta_R(L : B(L))} \cong I \bigoplus \dfrac{J}{\Delta_R(L : B(L))}$, we can conclude that I is associative and commutative. □

As a direct consequence of the proposition 3.2, we have the following results:

Theorem 3.3 *Let L and M be I.P. loops with associator-commutator subloops $B(L)$ and $B(M)$ respectively. Let R be an associative and commutative ring with identity such that $|B(L)|$ and $|B(M)|$ are invertible in R. Then $RL \cong RM$ if and only if $R\left[\dfrac{L}{B(L)}\right] \cong R\left[\dfrac{M}{B(M)}\right]$ and $\Delta_R(L : B(L)) \cong \Delta_R(M : B(M))$.*

Corollary 3.4 *Let L and M be RA2 loops with commutator subloops L' and M' respectively. Let F be a field such that $char(F) \nmid |L|$. Then $FL \cong FM$ if and only if $F\left[\dfrac{L}{L'}\right] \cong F\left[\dfrac{M}{M'}\right]$ and $\Delta_F(L : L') \cong \Delta_F(M : M')$.*

Corollary 3.5 *Let L and M be RA2 loops with commutator subloops L' and M' respectively. Let $\mathbf{Q}$ be the rational field. Then $\mathbf{Q}L \cong \mathbf{Q}M$ if and only if $\dfrac{L}{L'} \cong \dfrac{M}{M'}$ and $\Delta_{\mathbf{Q}}(L : L') \cong \Delta_{\mathbf{Q}}(M : M')$.*

4 The Structure of Semisimple Loop Algebras of RA2 Loops

For *RA2* loops, proposition 3.2 becomes

Proposition 4.1 *Let L be an RA2 loop and L' be its commutator subloop. Let R be an associative and commutative ring with identity ring such that $|L'|$ is invertible in R. Suppose I and J are two-sided ideals of RL with*

$RL = I \bigoplus J$. *Then I is associative and commutative if and only if* $J \supset \Delta_R(L : L')$.

Then we can establish the following

Theorem 4.2 *Let L be an $RA2$ loop with commutator subloop L'. Let F be a field such that the characteristic of F does not divide the order of L. Then the semisimple loop algebra FL decomposes as a direct sum of simple components which are extensions of F by a primitive root of unity or simple algebras that are not commutative nor associative.*

Proof: From proposition 3.1 we have that $FL \cong F\left[\frac{L}{L'}\right] \bigoplus \Delta_F(L : L')$.

Since $\frac{L}{L'}$ is an abelian group, then $F\left[\frac{L}{L'}\right] \cong \bigoplus_{i=1}^{n} F(\xi_i)$, where ξ_i is a i^{th}-primitive root of unity, $i = 1, ..., n$, by a classical result by S. Perlis and G.L. Walker [13] for commutative group algebras.

Since FL is semisimple, then $\Delta_F(L : L')$ is also semisimple, and we can write $\Delta_F(L : L') \cong \bigoplus_{i=1}^{t} A_i$, where A_i is a simple algebra for $i = 1, ..., t$.

From proposition 4.1, we have that A_i is neither commutative nor associative. □

O. Chein has proved in [7, Th. 1, p.35] the following result about Moufang loops, which we will need in the sequence.

Theorem 4.3 *If L is a nonassociative Moufang loop for which every minimal set of generators contains an element of order 2, then there exists a nonabelian group G, and an element u of order 2 in L, such that each element of L may be uniquely expressed in the form $g.u^{\alpha}$, where $g \in G$, $\alpha = 0, 1$, and the product of two elements of L is given by*

$$g_1.(g_2.u) = (g_2 g_1).u$$
$$(g_1.u).g_2 = (g_1 {g_2}^{-1}).u$$
$$(g_1.u).(g_2.u) = {g_2}^{-1} g_1$$

Conversely, given any nonabelian group G, the loop L constructed as indicated above is a nonassociative Moufang loop.

Using the symmetric group S_3 of order 6 we can construct, as in theorem 4.3, the loop $L = M_{12}(S_3, 2)$, the smallest Moufang loop according to [7], which is also the smallest $RA2$ loop.

Labeling $G = S_3 = \{1, g_2, g_3, g_4, g_5, g_6\}$ and calling $u = g_7$, $g_2.u = g_8$, $g_3.u = g_9$, $g_4.u = g_{10}$, $g_5.u = g_{11}$ and $g_6.u = g_{12}$, we have the following loop table for L:

1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
g_2	g_3	1	g_5	g_6	g_4	g_8	g_9	g_7	g_{12}	g_{10}	g_{11}
g_3	1	g_2	g_6	g_4	g_5	g_9	g_7	g_8	g_{11}	g_{12}	g_{10}
g_4	g_6	g_5	1	g_3	g_2	g_{10}	g_{11}	g_{12}	g_7	g_8	g_9
g_5	g_4	g_6	g_2	1	g_3	g_{11}	g_{12}	g_{10}	g_9	g_7	g_8
g_6	g_5	g_4	g_3	g_2	1	g_{12}	g_{10}	g_{11}	g_8	g_9	g_7
g_7	g_9	g_8	g_{10}	g_{11}	g_{12}	1	g_3	g_2	g_4	g_5	g_6
g_8	g_7	g_9	g_{11}	g_{12}	g_{10}	g_2	1	g_3	g_6	g_4	g_5
g_9	g_8	g_7	g_{12}	g_{10}	g_{11}	g_3	g_2	1	g_5	g_6	g_4
g_{10}	g_{11}	g_{12}	g_7	g_9	g_8	g_4	g_6	g_5	1	g_2	g_3
g_{11}	g_{12}	g_{10}	g_8	g_7	g_9	g_5	g_4	g_6	g_3	1	g_2
g_{12}	g_{10}	g_{11}	g_9	g_8	g_7	g_6	g_5	g_4	g_2	g_3	1

From this table it is easy to see that for $L = M_{12}(S_3, 2)$ it holds that $L' = \{1, g_2, g_3\} \cong C_3$, the cyclic group of order 3 and $\dfrac{L}{L'} \cong C_2 \times C_2$, the Klein group.

Let F be a field and suppose that the characteristic of F is different from 2 and 3. Then the loop algebra FL is semisimple. We will show the decomposition of FL as a direct sum of simple algebras.

Applying proposition 3.1 for $N = L'$ we have

$$FL \cong F\left[\frac{L}{L'}\right] \bigoplus \Delta_F(L : L') \cong F[C_2 \times C_2] \bigoplus \Delta_F(L : L') \cong$$
$$\cong F \bigoplus F \bigoplus F \bigoplus F \bigoplus \Delta_F(L : L') \ .$$

We are going to exhibit a F-basis for the algebra $\Delta_F(L : L')$.

Define

$$f_o = 1 - \widehat{L'} = 1 - \frac{1}{3}.(1 + g_2 + g_3) = \frac{1}{3}.(2 - g_2 - g_3)$$
$$f_1 = (g_2 - g_3).f_o = g_2 - g_3$$
$$f_2 = g_4.f_o = \frac{1}{3}.(2g_4 - g_5 - g_6)$$
$$f_3 = (g_2 - g_3).g_4.f_o = g_5 - g_6$$
$$f_4 = g_7.f_o = \frac{1}{3}.(2g_7 - g_8 - g_9)$$
$$f_5 = ((g_2 - g_3).g_7).f_o = g_8 - g_9$$

$f_6 = g_{10}.f_o = \frac{1}{3}.(2g_{10} - g_{11} - g_{12})$
$f_7 = ((g_2 - g_3).g_{10}).f_o = g_{12} - g_{11}$

It's easy to see that $\{f_o, f_1, ..., f_7\}$ is linearly independent and generates $\Delta_F(L : L')$ over F. Its multiplication table is

	f_o	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_o	f_o	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	f_1	$-3f_o$	f_3	$-3f_2$	f_5	$-3f_4$	f_7	$-3f_6$
f_2	f_2	$-f_3$	f_o	$-f_1$	f_6	$-f_7$	f_4	$-f_5$
f_3	f_3	$3f_2$	f_1	$3f_o$	$-f_7$	$-3f_6$	$-f_5$	$-3f_4$
f_4	f_4	$-f_5$	f_6	$-f_7$	f_o	$-f_1$	f_2	$-f_3$
f_5	f_5	$3.f_4$	$-f_7$	$-3f_6$	f_1	$3f_o$	$-f_3$	$-3f_2$
f_6	f_6	$-f_7$	f_4	$-f_5$	f_2	$-f_3$	f_o	$-f_1$
f_7	f_7	$3.f_6$	$-f_5$	$-3f_4$	$-f_3$	$-3f_2$	f_1	$3f_o$

The inequality $f_2.f_1 = -f_3 \neq f_1.f_2 = f_3$ shows that $\Delta_F(L : L')$ is not commutative and the inequality $(f_2 + f_4).[(f_2 + f_4).f_5)] = 2f_3 + 2f_5 \neq [(f_2 + f_4).(f_2 + f_4)].f_5 = 2f_5 - 2f_3$ shows that $\Delta_F(L : L')$ is not associative. A direct calculation shows that $\Delta_F(L : L')$ is simple. □

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