

Layered Systems at the Mean Field Critical Temperature

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Abstract We consider the Ising model on $\mathbb{Z} \times \mathbb{Z}$ where on each horizontal line $\{(x, i), x \in \mathbb{Z}\}$ the interaction is given by a ferromagnetic Kac potential with coupling strength $J_\gamma(x, y) \sim \gamma J(\gamma(x - y))$ at the mean field critical temperature. We then add a nearest neighbor ferromagnetic vertical interaction of strength ϵ and prove that for every $\epsilon > 0$ the systems exhibits phase transition provided $\gamma > 0$ is small enough.

Keywords Kac potentials · Lebowitz–Penrose free energy functional · Phase transition

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1 Introduction

We consider an Ising model on the lattice $\mathbb{Z} \times \mathbb{Z}$, whose points we denote by (x, i) . The spins $\sigma(x, i)$ take values in $\{-1, +1\}$ and on each horizontal line, also called *layer*, $\{(x, i), x \in \mathbb{Z}\}$, the interaction is given by a ferromagnetic Kac potential, that is, the interaction between the spins at (x, i) and (y, i) is given by

$$-\frac{1}{2}J_\gamma(x, y)\sigma(x, i)\sigma(y, i), \quad \sum_{y \neq x} J_\gamma(x, y) = 1, \quad (1.1)$$

where $J_\gamma(x, y) = c_\gamma \gamma J(\gamma(x - y))$; $J(r)$, $r \in \mathbb{R}$, is a symmetric probability density with continuous derivative and support in $[-1, 1]$, $\gamma > 0$ is a scale parameter, c_γ is the normalization constant (c_γ tends to 1 as $\gamma \rightarrow 0$). We also suppose that $J(0) > 0$. $H_{\gamma,0}$ denotes the Hamiltonian with only the interactions (1.1) on each layer, so that different layers do not interact with each other, the system is essentially one dimensional and does not have phase transitions.

We fix the inverse temperature at the mean field critical value $\beta = 1$ so that also in the Lebowitz–Penrose limit no phase transition is present. Purpose of this paper is to study what happens if we put a *small nearest neighbor vertical interaction*

$$- \epsilon \sigma(x, i)\sigma(x, i + 1). \quad (1.2)$$

The main result in this paper is the following.

Theorem 1 *Given any $\epsilon > 0$, for any $\gamma > 0$ small enough $\mu_\gamma^+ \neq \mu_\gamma^-$, $\mu_\gamma^\pm$ the plus-minus DLR measures defined as the thermodynamic limits of the Gibbs measures with plus, respectively minus, boundary conditions.*

It is worth mentioning that a version of Theorem 1 holds for $\beta > 1$ with $\epsilon = \gamma^A$ for any A (see [5] where indeed the above result has been conjectured).

In many cases it has been proved that if in the Lebowitz–Penrose limit there is a phase transition then in dimension $d \geq 2$ there is also a phase transition at small $\gamma > 0$ (i.e. without taking the limit $\gamma \rightarrow 0$). We cannot follow this route here because we do not know the phase diagram for our model in the limit $\gamma \rightarrow 0$: a “Lebowitz–Penrose theorem” for our system is an interesting open problem that our analysis does not solve. If the support of the Kac interaction would contain two dimensional balls (i.e. layers at distance of order γ^{-1} interact with each other) then the Lebowitz–Penrose analysis would apply and therefore the free energy in the limit $\gamma \rightarrow 0$ would be the convex envelope (i.e. the Legendre transform of the Legendre transform)

$$\left(f_\epsilon(m) - \frac{m^2}{2}\right)^{**}, \quad (1.3)$$

where $f_\epsilon(m)$ is the free energy of the one dimensional Ising model with nearest neighbor interaction of strength ϵ . (1.3) yields a phase transition if $\epsilon > 0$. Does (1.3) remain valid also when the Kac interaction is only horizontal? We do not know the answer but our analysis shows that indeed our system has a phase transition as indicated by (1.3).

The proof of Theorem 1 requires a non trivial extension of previous results on Kac potentials and it is given in complete details in this paper. It is obtained by proving Peierls bounds for suitably defined contours. The bounds are established via a Lebowitz–Penrose coarse graining procedure which however is not straightforward for the reasons explained before (due to the local nature of the vertical interaction and the strictly horizontal structure of the

Kac interaction). The trick is to use ferromagnetic inequalities to compare the magnetization under μ_γ^+ with that under the corresponding Gibbs measure for which the vertical interaction is removed in a chessboard fashion. To this new system (which is more decoupled but not so much as to lose the phase transition) we can apply the Lebowitz–Penrose coarse graining strategy. In this way we reduce the proof of the Peierls bound to the analysis of variational problems for a suitable free energy functional.

The model we are considering is related to a $d = 1$ quantum spin model with transverse field, whose hamiltonian is:

$$H(\sigma) = - \sum_{x \neq y} J_\gamma(x, y) \hat{\sigma}^3(x) \hat{\sigma}^3(y) - \alpha \sum_x \hat{\sigma}^1(x)$$

in its stochastic representation via Feynman–Kac, [1], [3] and [7]. We are indebted to D. Ioffe for pointing out the connection and for useful comments. We also thank A. Levit for conversations about the related quantum Curie–Weiss model.

At this point we state two conjectures.

The first question is: can we choose $\epsilon = \epsilon(\gamma)$ so that $\epsilon(\gamma) \rightarrow 0$ as $\gamma \rightarrow 0$ and still have for all γ small enough a phase transition? Is there a critical choice for $\epsilon(\gamma)$? The conjecture is that setting $\epsilon(\gamma) = \kappa \gamma^{2/3}$, $\kappa > 0$, we have a phase transition if κ is large enough and no phase transition for κ small.

This is related to the next conjecture. Consider the system where on each layer we have a process $m(r, i) \in \mathbb{R}$, $r \in \mathbb{R}$, $i \in \mathbb{Z}$. The formal Gibbs measure that we want to study is:

$$e^{-\sum_i \{\int dr \kappa m(r, i) m(r, i+1)\}} \prod_i P(dm(r, i)), \quad (1.4)$$

where P is the Euclidean ϕ_1^4 probability measure, namely the stationary solution of the real valued stochastic PDE

$$du(r, t) = \left(\frac{1}{2} u''(r, t) - u^3(r, t) \right) dt + dw, \quad (1.5)$$

dw white noise in space-time. The conjecture is that there is a phase transition for κ large and no phase transition for κ small.

The measure in (1.4) is the formal scaling limit of the Gibbs distribution of the empirical magnetization when we scale space as $x \rightarrow r = \gamma^{1+1/3} x$ and renormalize the averages by a factor $\gamma^{-1/3}$ as proved in [2] and [6]; see also [4], where (1.5) without the second derivative term is derived by studying the critical fluctuations in the mean field version of the model.

More precisely, in both papers the question is about the analysis of the long space-time fluctuations of the $d = 1$ Ising model with Glauber dynamics and Kac potential at $\beta = 1$ (like ours here). Namely the analysis of the fluctuations field

$$\gamma \sum_x \phi(\gamma^{1+1/3} x) \sigma(x, \gamma^{-2/3} t),$$

with ϕ a test function. This is the right normalization because one can prove that at such long times the typical values of the empirical magnetization (in the scale γ^{-1}) in the limit $\gamma \rightarrow 0$ have order $\gamma^{1/3}$ and not the normal values $\gamma^{1/2}$ of the finite time fluctuations. It is then proved that the above fluctuations field converges to $\int \phi(r) u(r, t) dr$, where u solves (1.5).

Outline of the paper: As already mentioned, our proof involves the study of the Gibbs measures for another Hamiltonian, denoted by $H_{\gamma, \epsilon}$ and defined in (2.3), where the vertical interactions are removed in a convenient chessboard fashion. This yields a two dimensional

system with pairs of long segments of parallel layers interacting vertically within the pair (but not with the outside), plus the horizontal Kac interaction. For this system we can exploit the spontaneous magnetization that emerges when two parallel one dimensional Kac models at mean field critical temperature interact vertically as in our case, as studied in Sect. 5. This spontaneous magnetization plays a natural role in the definition of contours (as in the analysis of the one dimensional Kac interactions below the mean field critical temperature). The main point is that for this chessboard Hamiltonian, and after a proper coarse graining procedure, we are able to implement the Lebowitz–Penrose procedure: the corresponding free energy functional is defined in Sect. 3 and the problem of getting the corresponding Peierls bounds for the weight of contours is transformed in variational problems for the free energy functional. This is the content of Theorem 3, whose proof involves, as a preliminary step, the study of the free energy function of two layers and its minimizers (determining the spontaneous magnetization). There are delicate choices of scales so as to allow the implementation of this procedure, as explained in Sect. 2. In Sect. 4 we get an upper bound for the restricted partition function that appears in the numerator of the weight of a contour. Section 5 is dedicated to the study of the minimizers of the free energy functional of Sect. 3, which then will play a crucial role in the lower bound for the denominator in the weight of a contour, completed in Sect. 6. In Sect. 7 we combine the two estimates to conclude the proof of the main theorem. The analysis of the mean field free energy function for two layers and the crucial estimates used in Sect. 5 are carried out in the Appendices.

2 Contours

For $i \in \mathbb{Z}$, we call i -th layer the set $\mathbb{Z} \times \{i\}$. As mentioned in the introduction we shall extensively use coarse graining, for which we start by partitioning each layer into intervals of length $\ell \in \{2^n, n \in \mathbb{Z}\}$. Let $\mathcal{D}^{\ell,i} = \{C_{k\ell}^{\ell,i}, k \in \mathbb{Z}\}$ denote the partition of the i -th layer:

$$C_x^{\ell,i} = C_x^\ell \times \{i\} := ([k\ell, (k+1)\ell) \cap \mathbb{Z}) \times \{i\}, \text{ where } k = \lfloor x/\ell \rfloor \quad (2.1)$$

and, as usual, $\lfloor s \rfloor = \max\{x \in \mathbb{Z} : x \leq s\}$. To simplify notation we restrict γ to belong to $\{2^{-n}, n \in \mathbb{N}\}$.

For the coarse grained description we shall use three length scales and an accuracy parameter $\zeta > 0$ which all depend on γ :

$$\gamma^{-1/2}, \quad \ell_{\pm} = \gamma^{-(1 \pm \alpha)}, \quad \zeta = \gamma^a, \quad 1 \gg \alpha \gg a > 0. \quad (2.2)$$

The smallest scale, $\gamma^{-1/2}$, will be used to implement the Lebowitz–Penrose procedure to define free energy functionals. Together with ζ , the scales ℓ_- and ℓ_+ will be used to define, at the spin level, the *plus* and *minus* regions and then the contours.

For notational simplicity we suppose that also $\gamma^{-\alpha}$ and all the above lengths belong to $\{2^n, n \in \mathbb{N}_+\}$: this is a restriction on α that could be removed by changing “slightly” α with γ .

We shall prove Theorem 1 for the “chessboard” Hamiltonian:

$$H_{\gamma,\epsilon} = -\frac{1}{2} \sum_{x \neq y,i} J_\gamma(x,y) \sigma(x,i) \sigma(y,i) - \epsilon \sum_{x,i} \chi_{i,x} \sigma(x,i) \sigma(x,i+1), \quad (2.3)$$

where

$$\chi_{x,i} = \begin{cases} 1 & \text{if } \lfloor x/\ell_+ \rfloor + i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

Definition 1 When $\chi_{x,i} = 1$, according to (2.4), we say that (x, i) and $(x, i + 1)$ interact vertically and denote by $v_{x,i}$ the site (x, j) which interacts vertically with (x, i) .

By the GKS correlation inequalities (see e.g. [9, Theorem 1.21 in Chapter IV]), the plus state for $H_{\gamma,\epsilon}$ is less magnetized than the one for the full Hamiltonian (with $\chi_{i,x}$ replaced by 1 everywhere). Hence Theorem 1 will follow once we prove that the magnetization in the plus state of the Hamiltonian given by (2.3) is strictly positive.

For the chessboard Hamiltonian, we shall see via a Lebowitz–Penrose analysis that in the limit as $\gamma \rightarrow 0$ there is a spontaneous magnetization equal to some $m_\epsilon > 0$ in the plus state and $-m_\epsilon$ in the minus state. This will follow from the analysis in Sects. 3–5. This value m_ϵ is used to define contours, as we now explain (taking $m_\epsilon > 0$ for granted).

Define first the empirical magnetization on a scale $\ell \in \{2^n, n \in \mathbb{N}\}$ in the layer i as

$$\sigma^{(\ell)}(x, i) := \frac{1}{\ell} \sum_{y \in C_x^\ell} \sigma(y, i). \quad (2.5)$$

We also consider the partition of $\mathbb{Z}^2$ into rectangles $\{Q_\gamma(k, j) : k, j \in \mathbb{Z}\}$, where

$$Q_\gamma(k, j) = ([k\ell_+, (k+1)\ell_+) \times [j\gamma^{-\alpha}, (j+1)\gamma^{-\alpha}]) \cap \mathbb{Z}^2 \text{ if } k \text{ is even}$$

and

$$Q_\gamma(k, j) = ([k\ell_+, (k+1)\ell_+) \times (j\gamma^{-\alpha}, (j+1)\gamma^{-\alpha}]) \cap \mathbb{Z}^2 \text{ if } k \text{ is odd.}$$

For convenience we sometimes write $Q_{x,i} = Q_\gamma(k, j)$ if $(x, i) \in Q_\gamma(k, j)$. The important feature of this definition (frequently exploited in the sequel) is that the spins in each rectangle $Q_{x,i}$ do not interact vertically with the spins of the complement, namely recalling the definition of $v_{x,i}$ and that $\gamma^{-\alpha}$ is even, we see that $v_{x,i} \in Q_{x,i}$ for all (x, i) . Notice also that the rectangles $Q_\gamma(k, j)$ become squares if lengths are measured in interaction length units: in fact in such units the horizontal side of a rectangle has length $\ell_+/\gamma^{-1} = \gamma^{-\alpha}$ and the vertical side has also length $\gamma^{-\alpha}$ as the vertical interaction length is equal to 1. The other important feature behind the definition of rectangles is that their size in interaction length units diverges as $\gamma \rightarrow 0$: this will be exploited to prove decay of correlations from the boundaries.

The random variables $\eta(x, i)$, $\theta(x, i)$ and $\Theta(x, i)$ are then defined as follows:

- $\eta(x, i) = \pm 1$ if $|\sigma^{(\ell_-)}(x, i) \mp m_\epsilon| \leq \zeta$; $\eta(x, i) = 0$ otherwise.
- $\theta(x, i) = 1, [-1]$, if $\eta(y, j) = 1, [-1]$, for all $(y, j) \in Q_{x,i}$; $\theta(x, i) = 0$ otherwise.
- $\Theta(x, i) = 1, [-1]$, if $\eta(y, j) = 1, [-1]$, for all $(y, j) \in \cup_{u,v \in \{-1,0,1\}} Q_\gamma(k+u, j+v)$, with (k, j) determined by $Q_{x,i} = Q_\gamma(k, j)$.

Namely, for the Θ variables we consider a “block” 3×3 of Q -rectangles.

The plus phase is the union of all the rectangles $Q_{x,i}$ such that $\Theta(x, i) = 1$, the minus phase is where $\Theta(x, i) = -1$, in the complement the phase is *undetermined*.

Two rectangles $Q_\gamma(k, j)$ and $Q_\gamma(k', j')$ are said to be connected if (k, j) and (k', j') are $*$ -connected, i.e. $|k - k'| \vee |j - j'| \leq 1$. By choosing suitable boundary conditions, we shall restrict in the sequel to spin configurations such that $\Theta = 1$ outside of a compact (the case when $\Theta = -1$ can be recovered via spin flip). Given such a σ , we call *contours* the pairs $\Gamma = (\text{sp}(\Gamma), \eta_\Gamma)$, where $\text{sp}(\Gamma)$ is a maximal connected component of the undetermined region, called *the spatial support of Γ* , and η_Γ is the restriction of η to $\text{sp}(\Gamma)$, called *the specification of Γ* .

Geometry of Contours Denote by $\text{ext}(\Gamma)$ the maximal unbounded connected component of the complement of $\text{sp}(\Gamma)$ and $\partial_{\text{out}}(\Gamma)$ the union of the rectangles in $\text{ext}(\Gamma)$ which are

connected to $\text{sp}(\Gamma)$. $\partial_{\text{in}}(\Gamma)$ is instead the union of the rectangles in $\text{sp}(\Gamma)$ which are connected to $\text{ext}(\Gamma)$. Θ is constant and different from 0 on $\partial_{\text{out}}(\Gamma)$ and we call *plus* a contour Γ when $\Theta = 1$ on $\partial_{\text{out}}(\Gamma)$ and *minus* otherwise. Observe that in a plus contour $\eta = 1$ on $\partial_{\text{in}}(\Gamma)$.

Analogously we call $\text{int}_k(\Gamma)$, $k = 1, \dots, k_\Gamma$ the bounded maximal connected components (if any) of the complement of $\text{sp}(\Gamma)$, $\partial_{\text{in},k}(\Gamma)$ the union of all rectangles in $\text{sp}(\Gamma)$ which are connected to $\text{int}_k(\Gamma)$. $\partial_{\text{out},k}(\Gamma)$ is the union of all the rectangles in $\text{int}_k(\Gamma)$ which are connected to $\text{sp}(\Gamma)$. Then Θ is constant and different from 0 on each $\partial_{\text{out},k}(\Gamma)$ and we write $\partial_{\text{out},k}^\pm(\Gamma)$, $\text{int}_k^\pm(\Gamma)$ and $\partial_{\text{in},k}^\pm(\Gamma)$ if $\Theta = \pm 1$ on the former, observing that $\eta = \pm 1$ on $\partial_{\text{in},k}^\pm(\Gamma)$, respectively. We also call

$$c(\Gamma) = \text{sp}(\Gamma) \cup \bigcup_k \text{int}_k(\Gamma). \quad (2.6)$$

Diluted Partition Functions Let Λ be a bounded region which is an union of Q -rectangles. The plus diluted partition function in Λ with boundary conditions $\bar{\sigma}$ is

$$Z_{\Lambda, \bar{\sigma}}^+ = \sum_{\sigma_\Lambda} \mathbf{1}_{\{\Theta=1 \text{ on } \partial_{\text{in}}(\Lambda)\}} e^{-H_{\gamma, \epsilon}(\sigma_\Lambda | \bar{\sigma})} =: Z_{\Lambda, \bar{\sigma}}(\Theta = 1 \text{ on } \partial_{\text{in}}(\Lambda)), \quad (2.7)$$

where $\bar{\sigma}$ is a configuration on the complement of Λ ; Θ is computed on the configuration $(\sigma_\Lambda, \bar{\sigma})$ and $\partial_{\text{in}}(\Lambda)$ is the union of all Q -rectangles in Λ connected to Λ^c . Minus diluted partition functions are defined analogously. As a rule we denote by $Z_{\Lambda, \bar{\sigma}}(\mathcal{A})$ the partition function with the constraint $\mathcal{A}$, $\mathcal{A}$ a set of configurations. Notice that there is no vertical interaction between the spins in Λ and those in its complement because Λ is union of rectangles.

The plus diluted Gibbs measure (with boundary conditions $\bar{\sigma}$ such that $\eta = 1$ on $\partial_{\text{out}}(\Lambda)$) is defined in the usual way, namely, given a configuration of spins σ_Λ on Λ , the weight assigned to σ_Λ by the plus Gibbs measure is given by

$$\mu_{\Lambda, \bar{\sigma}}^+(\sigma_\Lambda) = \frac{e^{-H_{\gamma, \epsilon}(\sigma_\Lambda | \bar{\sigma})}}{Z_{\Lambda, \bar{\sigma}}^+} \mathbf{1}_{\{\Theta=1 \text{ on } \partial_{\text{in}}(\Lambda)\}}. \quad (2.8)$$

The minus diluted Gibbs measure is defined analogously.

We shall prove the Peierls estimates for the plus and minus diluted Gibbs measures, which, as a consequence, have distinct thermodynamic limits; Theorem 1 will then follow.

Weight of a Contour We are now ready to define the fundamental notion of *weight of a contour*. Let Γ be a plus contour (the definition for minus contours is obtained by spin flip) and $\bar{\sigma}$ a configuration on the complement of $c(\Gamma)$ such that $\eta = 1$ on $\partial_{\text{out}}(\Gamma)$ (in agreement with the definition of a plus contour). Then the weight of Γ with boundary conditions $\bar{\sigma}$ is

$$W_\Gamma(\bar{\sigma}) := \frac{Z_{c(\Gamma); \bar{\sigma}}(\eta = \eta_\Gamma \text{ on } \text{sp}(\Gamma); \Theta = \pm 1 \text{ on each } \partial_{\text{out},k}^\pm(\Gamma))}{Z_{c(\Gamma); \bar{\sigma}}(\Theta = 1 \text{ on } \text{sp}(\Gamma) \text{ and on each } \partial_{\text{out},k}^\pm(\Gamma))}, \quad (2.9)$$

where $Z_{\Lambda, \bar{\sigma}}(\mathcal{A})$ is the partition function in Λ with Hamiltonian $H_{\gamma, \epsilon}$, with boundary conditions $\bar{\sigma}$ and constraint $\mathcal{A}$. In the next sections we shall prove the following theorem

Theorem 2 (The Peierls bounds) *There are $c > 0$, $\epsilon_0 > 0$ and $\gamma : (0, \infty) \rightarrow (0, \infty)$ so that for any $0 < \epsilon \leq \epsilon_0$, $0 < \gamma \leq \gamma_\epsilon$ and any contour Γ with boundary spins $\bar{\sigma}$*

$$W_\Gamma(\bar{\sigma}) \leq e^{-c|\text{sp}(\Gamma)|\gamma^{2\alpha+4\alpha}}. \quad (2.10)$$

In Sect. 7 we shall see how to prove Theorem 1 using the Peierls bounds (2.10).

3 Reduction to a Variational Problem

The goal of this section is to introduce the Lebowitz–Penrose free energy functional and to set the variational problem that emerges in the estimates of the partition functions in (2.9). We start by the next proposition which deals with the very simple situation of two layers of ± 1 spins whose unique interaction is the nearest neighbor vertical one. It is just a chain of independent pairs of spins. Therefore the multi-canonical partition function, where we fix the magnetization on each layer, is studied by very simple tools. This first result, proved in Appendix 1 for sake of completeness, describes its convergence (in the thermodynamic limit) to the infinite volume free energy $\hat{\phi}_\epsilon(m_1, m_2)$ and finite volume corrections. We then state and prove a Lebowitz–Penrose theorem for the spin model associated to the chessboard Hamiltonian $H_{\gamma, \epsilon}$.

Proposition 1 *Let n be a positive integer, $X_n = \{-1, 1\}^n$. For $i = 1, 2$, let $m_i \in \{-1 + \frac{2j}{n} : j = 1, \dots, n-1\}$ and set*

$$Z_{\epsilon, n}(m_1, m_2) = \sum_{(\sigma_1, \sigma_2) \in X_n \times X_n} \mathbf{1}_{\{\sum_{i=1}^n \sigma_i(x) = nm_i \mid i=1, 2\}} e^{\epsilon \sum_{x=1}^n \sigma_1(x) \sigma_2(x)}. \quad (3.1)$$

There is a continuous and convex function $\hat{\phi}_\epsilon$ defined on $[-1, 1] \times [-1, 1]$, with bounded derivatives on each $[-r, r] \times [-r, r]$ for $|r| < 1$, and a constant $c > 0$ so that

$$-\hat{\phi}_\epsilon(m_1, m_2) - c \frac{\log n}{n} \leq \frac{1}{n} \log Z_{\epsilon, n}(m_1, m_2) \leq -\hat{\phi}_\epsilon(m_1, m_2). \quad (3.2)$$

We shall next use the above proposition to study the partition functions which enter in the definition of contours. We thus consider a region Λ which in the applications will be the spatial support of a contour. Here it only matters that Λ is a connected set union of Q -rectangles. We want to bound from above and below the partition function

$$Z_{\Lambda, \bar{\sigma}}(\mathcal{A}) := \sum_{\sigma_\Lambda \in \mathcal{A}} e^{-H_{\gamma, \epsilon}(\sigma_\Lambda \mid \bar{\sigma})}, \quad (3.3)$$

where $\bar{\sigma}$ is a spin configuration in the complement of Λ and “the constraint” $\mathcal{A}$ is a set of configurations in Λ defined in terms of the values of η_Λ . We shall coarse-grain on the scale $\gamma^{-1/2}$. We thus call $M_{\gamma^{-1/2}}$ the possible values of the magnetization densities $\sigma^{(\gamma^{-1/2})}$, ($\sigma^{(\ell)}$ has been defined in (2.5)), namely

$$M_{\gamma^{-1/2}} = \{-1, -1 + 2\gamma^{1/2}, \dots, 1 - 2\gamma^{1/2}, 1\}$$

and we set

$$\mathcal{M}_\Lambda := \{m(\cdot) \in (M_{\gamma^{-1/2}})^\Lambda : m(\cdot) \text{ is constant on each } C^{\gamma^{-1/2}, i} \subset \Lambda\}. \quad (3.4)$$

The Lebowitz–Penrose free energy functional (on Λ with boundary conditions $\bar{m}$) is the following functional on $[-1, 1]^\Lambda$ (whose elements are denoted in short by m)

$$\begin{aligned} F_{\Lambda, \gamma}(m \mid \bar{m}) &= \frac{1}{2} \sum_{(x, i) \in \Lambda} \hat{\phi}_\epsilon(m(x, i), m(v_{x, i})) \\ &\quad - \frac{1}{2} \sum_{(x, i) \neq (y, i) \in \Lambda} J_\gamma(x, y) m(x, i) m(y, i) \\ &\quad - \sum_{(x, i) \in \Lambda, (y, i) \notin \Lambda} J_\gamma(x, y) m(x, i) \bar{m}(y, i), \end{aligned} \quad (3.5)$$

where $\bar{m} \in [-1, 1]^{\Lambda^c}$, $\hat{\phi}_\epsilon$ is the free energy function in (3.2) and $v_{x,i}$ is given in Definition 1. (Recall that $v_{x,i} \in \Lambda$ for each $(x, i) \in \Lambda$ since there are no vertical interactions between a Q -rectangle and the outside.)

Notational Remark The same formula is used when $\bar{m}$ is defined in a set Δ contained in the complement of Λ ; in such a case the sum over (y, i) in the last term is extended only to Δ .

By an abuse of notation we write, analogously to (2.5),

$$m^{(\ell)}(x, i) := \frac{1}{\ell} \sum_{y: (y, i) \in C_x^{\ell, i}} m(y, i) \quad (3.6)$$

and define $\eta(x, i; m) = \pm 1$ if $|m^{(\ell-)}(x, i) \mp m_\epsilon| \leq \zeta$ and $= 0$ otherwise. We still denote by $\mathcal{A}$ a constraint that depends on $\eta(\cdot; m)$ as for instance $\eta(\cdot; m) = \eta^*(\cdot)$ on Λ .

Theorem 3 *There is a constant c so that*

$$\log Z_\Lambda(\bar{\sigma}; \mathcal{A}) \leq - \inf_{m \in \mathcal{M}_\Lambda \cap \mathcal{A}} F_{\Lambda, \gamma}(m | \bar{m}) + c|\Lambda| \gamma^{1/2} \log \gamma^{-1}, \quad (3.7)$$

where, recalling (2.5), $\bar{m}(x, i) = \bar{\sigma}^{\gamma^{-1/2}}(x, i)$, $(x, i) \notin \Lambda$. Moreover, for any $m \in \mathcal{M}_\Lambda \cap \mathcal{A}$

$$\log Z_\Lambda(\bar{\sigma}; \mathcal{A}) \geq -F_{\Lambda, \gamma}(m | \bar{m}) - c|\Lambda| \gamma^{1/2} \log \gamma^{-1}. \quad (3.8)$$

Proof Here is essential the restriction to regions with no vertical interaction with the complement. We have, writing σ for a spin configuration in Λ ,

$$Z_\Lambda(\bar{\sigma}; \mathcal{A}) = \sum_{m \in \mathcal{A}} \sum_{\sigma: \sigma(\gamma^{-1/2}(\cdot)) = m(\cdot) \text{ on } \Lambda} e^{-H_{\gamma, \epsilon; \Lambda}(\sigma | \bar{\sigma})},$$

where $H_{\gamma, \epsilon; \Lambda}(\sigma | \bar{\sigma})$ is the Hamiltonian (2.3) in the region Λ interacting with $\bar{\sigma}$ outside Λ .

By the smoothness of J_γ we get

$$\left| \frac{1}{2} \sum_{(x, i) \in \Lambda} \sum_{y \neq x} (J_\gamma(x, y) - \hat{J}_\gamma(x, y)) \sigma(x, i) \sigma(y, i) \right| \leq c|\Lambda| \gamma^{1/2},$$

where $\sigma(y, i) = \bar{\sigma}(y, i)$ if $(y, i) \notin \Lambda$ and

$$\hat{J}_\gamma(x, y) := \frac{1}{\gamma^{-1}} \sum_{x' \in C_x^{\gamma^{-1/2}}} \sum_{y' \in C_y^{\gamma^{-1/2}}} J_\gamma(x', y').$$

Thus, recalling that there is no vertical interaction between Λ and its complement,

$$\left| H_{\gamma, \epsilon; \Lambda}(\sigma | \bar{\sigma}) - (H_{\gamma, 0; \Lambda}(m | \bar{m}) - \frac{1}{2} \epsilon \sum_{(x, i) \in \Lambda} \sigma(x, i) \sigma(v_{x, i})) \right| \leq c|\Lambda| \gamma^{1/2},$$

where $H_{\gamma, 0; \Lambda}(m | \bar{m})$ is the Hamiltonian $H_{\gamma, \epsilon; \Lambda}(\sigma | \bar{\sigma})$ with $\epsilon = 0$ and the spins replaced by $m(x, i)$ and $\bar{m}(x, i)$. We then get, using (3.2),

$$Z_\Lambda(\bar{\sigma}; \mathcal{A}) \leq |\mathcal{M}_\Lambda| \sup_{m \in \mathcal{A}} e^{-H_{\gamma, 0; \Lambda}(m | \bar{m}) - \frac{1}{2} \sum_{(x, i) \in \Lambda} \hat{\phi}_\epsilon(m(x, i), m(v_{x, i}))} e^{c|\Lambda| \gamma^{1/2}},$$

which proves (3.7) because $|\mathcal{M}_\Lambda| \leq (c\gamma^{-1/2})^{|\Lambda| \gamma^{1/2}}$ (for a suitable constant c). (3.8) is proved similarly. $\square$

The variations of J_γ on the scale $\gamma^{-1/2}$ give a contribution of the order $|\Lambda|\gamma^{1/2}$ to the errors in (3.7) and (3.8); in (3.7) there is also a contribution of order $|\Lambda|\gamma^{1/2} \log \gamma^{-1}$ coming from the cardinality of $\mathcal{M}_\Lambda$. In (3.8) we need to take into account the lower bound in Proposition 1. Of course in the upper bound of the partition function we can drop the condition that m takes values in $M_{\gamma^{-1/2}}$ and that it is constant in the intervals $C\gamma^{-1/2, i} \subset \Lambda$:

Corollary 1 *In the same context of Theorem 3*

$$\log Z_\Lambda(\bar{\sigma}; \mathcal{A}) \leq - \inf_{m \in [-1, 1]^\Lambda \cap \mathcal{A}} F_{\Lambda, \gamma}(m|\bar{m}) + c|\Lambda|\gamma^{1/2} \log \gamma^{-1}. \quad (3.9)$$

4 The Upper Bound

Let us now be more specific and see how (3.9) is used to get an upper bound for the numerator of (2.9). The key point will be to prove that the excess free energy due to the constraint $\eta = \eta_\Gamma$ is much larger than the errors in (3.7–3.8).

In the sequel we specify $\Lambda = \text{sp}(\Gamma)$, and refer to the paragraph “Geometry of contours” in Sect. 2. Because the notation gets clumsy in some formulae, we shorten it a bit as follows:

$$\Delta_{\text{in}} = \partial_{\text{in}}(\Gamma), \quad \Delta_k^\pm = \partial_{\text{in}, k}^\pm(\Gamma), \quad I_k^\pm = \text{int}_k^\pm(\Gamma),$$

recalling that the suffix $\pm$ here refers to the (constant) value of Θ on the corresponding $\partial_{\text{out}, k}(\Gamma)$, and that $\eta = \pm 1$ on $\Delta_k^\pm$. Set then

$$\Delta_0 = \text{sp}(\Gamma) \setminus (\Delta_{\text{in}} \cup \{\cup_k \Delta_k^+\} \cup \{\cup_k \Delta_k^-\})$$

so that one has the following partition of $c(\Gamma)$:

$$c(\Gamma) = \Delta_0 \cup \Delta_{\text{in}} \cup \{\cup_k \Delta_k^+\} \cup \{\cup_k \Delta_k^-\} \cup \{\cup_k I_k^+\} \cup \{\cup_k I_k^-\}. \quad (4.1)$$

Thus the function $\bar{m}$ in (3.9) is specified by the spins outside $c(\Gamma)$ and by those in the sets $I_k^\pm$. When necessary we write $\bar{m}_{\sigma_{\text{ext}}}$ and $\bar{m}_{\sigma_{I_k^\pm}}$ for its restriction to the complement of $c(\Gamma)$ and to $I_k^\pm$, respectively. Finally the constraint in Theorem 3 is $\mathcal{A} = \{\eta = \eta_\Gamma \text{ on } \Lambda\}$.

Following [10] (see Chapters 6 and 9), an important ingredient in the proof of Peierls bounds consists in showing that the minimizers of the free energy functional have good regularity properties even when constrained to have given magnetization values in *small boxes* (the multi-canonical constraints). The next proposition, proved in Appendix 3, shows that the infimum in (3.9) can be restricted to *smooth functions*.

Proposition 2 *There is a positive constant c so that, with the same notation as above and recalling that $\Lambda = \text{sp}(\Gamma)$,*

$$\inf_{m \in [-1, 1]^\Lambda \cap \mathcal{A}} F_{\Lambda, \gamma}(m|\bar{m}) = \inf_{m \in [-1, 1]^\Lambda \cap \mathcal{A} \cap S_{\Delta_0}} F_{\Lambda, \gamma}(m|\bar{m}), \quad (4.2)$$

where

$$S_{\Delta_0} := \{m : \sup_{(x, i) \in \Delta_0} |m(x, i) - m^{(\ell_-)}(x, i)| \leq c\gamma^\alpha\}. \quad (4.3)$$

Remark The smoothness request could be extended to the whole Λ without changing the infimum but we only need it in Δ_0 .

Let us write

$$-m(x, i)m(y, i) = \frac{1}{2} \left(-m(x, i)^2 - m(y, i)^2 + [m(x, i) - m(y, i)]^2 \right) \quad (4.4)$$

in some of the terms.

With the above notation, and recalling (3.5), we get

$$\begin{aligned} F_{\text{sp}(\Gamma),\gamma}(m|\bar{m}) &= F_{\Delta_0,\gamma}^*(m_{\Delta_0}) + F'_{\Delta_{\text{in}},\gamma}(m_{\Delta_{\text{in}}}|\bar{m}_{\sigma_{\text{ext}}}) \\ &\quad + \sum_k F'_{\Delta_k^+,\gamma}\left(m_{\Delta_k^+}|\bar{m}_{\sigma_{I_k^+}}\right) + \sum_k F'_{\Delta_k^-,\gamma}\left(m_{\Delta_k^-}|\bar{m}_{\sigma_{I_k^-}}\right) \\ &\quad + \frac{1}{2} \sum_{(x,i) \in \Delta_0} \sum_{(y,i) \notin \Delta_0} J_\gamma(x,y)[m(x,i) - m(y,i)]^2, \end{aligned} \quad (4.5)$$

where, writing m for m_{Δ_0} ,

$$\begin{aligned} F_{\Delta_0,\gamma}^*(m) &= \sum_{(x,i) \in \Delta_0} \left\{ -\frac{1}{2}m(x,i)^2 + \frac{1}{2}\hat{\phi}_\epsilon(m(x,i), m(v_{x,i})) \right\} \\ &\quad + \frac{1}{4} \sum_{(x,i) \neq (y,i) \in \Delta_0} J_\gamma(x,y)(m(x,i) - m(y,i))^2, \end{aligned} \quad (4.6)$$

while writing m for $m_{\Delta_{\text{in}}}$,

$$F'_{\Delta_{\text{in}},\gamma}(m|\bar{m}_{\sigma_{\text{ext}}}) = F_{\Delta_{\text{in}},\gamma}(m|\bar{m}_{\sigma_{\text{ext}}}) - \sum_{(x,i) \in \Delta_{\text{in}}} a_{x,i} \frac{m(x,i)^2}{2}, \quad (4.7)$$

where

$$a_{x,i} := \sum_{y:(y,i) \in \Delta_0} J_\gamma(x,y). \quad (4.8)$$

$F'_{\Delta_k^\pm,\gamma}(m_{\Delta_k^\pm}|\bar{m}_{\sigma_{I_k^\pm}})$ is defined analogously.

Remark Recalling the remark after (3.5), notice that the interaction between Δ_{in} and Δ_0 present in $F_{\text{sp}(\Gamma),\gamma}(m|\bar{m})$ is absent in $F_{\Delta_{\text{in}},\gamma}(m_{\Delta_{\text{in}}}|\bar{m}_{\sigma_{\text{ext}}})$. It is instead contained (and split via (4.4)) in the following terms:

- (i) the last term on the right hand side of (4.5),
- (ii) the second term in the right hand side of (4.7) with $m(x,i)^2$, $(x,i) \in \Delta_{\text{in}}$. Something analogous takes place to the interaction between $\Delta_k^\pm$ and Δ_0 , involving the last term on the right hand side of (4.4) and the corresponding terms in $F'_{\Delta_k^\pm,\gamma}(m_{\Delta_k^\pm}|\bar{m}_{\sigma_{I_k^\pm}})$.

By Proposition 2 and (4.5), we get, dropping the last term in (4.5),

Corollary 2

$$\begin{aligned} \inf_{m \in [-1,1]^{\Lambda \cap \mathcal{A}}} F_{\text{sp}(\Gamma),\gamma}(m|\bar{m}) &\geq \Phi_{\Delta_0} + \Phi_{\Delta_{\text{in}}}(\bar{m}_{\sigma_{\text{ext}}}) + \sum_k \Phi_{\Delta_k^+}(\bar{m}_{\sigma_{I_k^+}}) \\ &\quad + \sum_k \Phi_{\Delta_k^-}(\bar{m}_{\sigma_{I_k^-}}), \end{aligned} \quad (4.9)$$

where

$$\begin{aligned}
\Phi_{\Delta_0} &= \inf \left\{ F_{\Delta_0, \gamma}^*(m) \mid m \in [-1, 1]^{\Delta_0}, |m - m^{(\ell_-)}| \leq c\gamma^\alpha, \eta(\cdot; m) = \eta_\Gamma(\cdot) \right\}, \\
\Phi_{\Delta_{\text{in}}}^+(\bar{m}_{\sigma_{\text{ext}}}) &= \inf \left\{ F'_{\Delta_{\text{in}}, \gamma}(m | \bar{m}_{\sigma_{\text{ext}}}) \mid m \in [-1, 1]^{\Delta_{\text{in}}}, \eta(\cdot, m) = 1 \right\}, \\
\Phi_{\Delta_k^+}^+(\bar{m}_{\sigma_{I_k^+}}) &= \inf \left\{ F'_{\Delta_k^+, \gamma}(m | \bar{m}_{\sigma_{I_k^+}}) \mid m \in [-1, 1]^{\Delta_k^+}, \eta(\cdot, m) = 1 \right\}, \\
\Phi_{\Delta_k^-}^-(\bar{m}_{\sigma_{I_k^-}}) &= \inf \left\{ F'_{\Delta_k^-, \gamma}(m | \bar{m}_{\sigma_{I_k^-}}) \mid m \in [-1, 1]^{\Delta_k^-}, \eta(\cdot, m) = -1 \right\}.
\end{aligned} \quad (4.10)$$

Corollary 2 is useful for us because it allows to split the original variational problem on the left hand side of (4.9) into separated, localized variational problems, as on the right hand side of (4.9).

Recalling (3.9), we have the following upper bound for the partition function in the numerator of (2.9):

$$e^{-\Phi_{\Delta_0} + c|\Lambda|\gamma^{1/2} \log \gamma^{-1}} e^{-\Phi_{\Delta_{\text{in}}}(\bar{m}_{\sigma_{\text{ext}}})} \left\{ \prod Z^+(I_k^+) \right\} \left\{ \prod Z^-(I_k^-) \right\}, \quad (4.11)$$

where

$$Z^+(I_k^+) = \sum_{\sigma_{I_k^+}^+} e^{-H(\sigma_{I_k^+}^+) - \Phi_{\Delta_k^+}^+(\bar{m}_{\sigma_{I_k^+}^+})}. \quad (4.12)$$

The superscript $+$ in the sum means that the sum is restricted to spin configurations in I_k^+ such that a configuration made by $\sigma_{I_k^+}$ in I_k^+ and by any configuration with $\eta = 1$ in Δ_k^+ has $\Theta = 1$ on $\partial_{\text{out}, k}^+(\Gamma)$, see (2.9). $Z^-(I_k^-)$ is defined analogously.

Following the Peierls strategy we use at this point the spin flip symmetry to rewrite (4.11) in a more convenient way. In fact we have:

$$\Phi_{\Delta_k^-}^-(\bar{m}_{\sigma_{I_k^-}}) = \Phi_{\Delta_k^+}^+(\bar{m}_{-\sigma_{I_k^-}}) \quad (4.13)$$

and therefore $Z^-(I_k^-) = Z^+(I_k^-)$. The numerator in (2.9) is thus bounded by

$$\begin{aligned}
&Z_{c(\Gamma); \bar{\sigma}}(\eta = \eta_\Gamma \text{ on } \text{sp}(\Gamma); \Theta = \pm 1 \text{ on each } \partial_{\text{out}, k}^\pm(\Gamma)) \\
&\leq e^{-\Phi_{\Delta_0} + c|\Lambda|\gamma^{1/2} \log \gamma^{-1}} \\
&\quad \times e^{-\Phi_{\Delta_{\text{in}}}(\bar{m}_{\sigma_{\text{ext}}})} \left\{ \prod Z^+(I_k^+) \right\} \left\{ \prod Z^+(I_k^-) \right\}.
\end{aligned} \quad (4.14)$$

The key point is now to prove a lower bound on Φ_{Δ_0} so good as to kill the error terms in the first exponent and to give what is required by the Peierls bounds. The other factors in (4.14) will simplify with those coming from the lower bound modulo a small error. Preliminary to that is the analysis of the two layer free energy $\hat{f}_\epsilon(m_1, m_2)$. In Appendix 2 it is proved that:

Proposition 3 *For any $\epsilon > 0$ small enough*

$$\hat{f}_\epsilon(m_1, m_2) := -\frac{1}{2}(m_1^2 + m_2^2) + \hat{\phi}_\epsilon(m_1, m_2) \quad (4.15)$$

has two minimizers, $\pm m^{(\epsilon)} := \pm(m_\epsilon, m_\epsilon)$, and there is a constant c so that

$$|m_\epsilon - \sqrt{3\epsilon}| \leq c\epsilon^{3/2}. \quad (4.16)$$

Moreover, calling $\hat{f}_{\epsilon, \text{eq}}$ the minimum of $\hat{f}_\epsilon(m)$, for any $\zeta > 0$ small enough:

$$|\hat{f}_\epsilon(m) - \hat{f}_{\epsilon, \text{eq}}| \geq c\zeta^2, \quad \text{for all } m \notin U_\zeta, \quad (4.17)$$

where

$$U_\zeta := \left\{ (m_1, m_2) : |m_i - m_\epsilon| < \frac{\zeta}{2}, i = 1, 2 \right\} \cup \left\{ (m_1, m_2) : |m_i + m_\epsilon| < \frac{\zeta}{2}, i = 1, 2 \right\}.$$

The following lower bound for Φ_{Δ_0} follows from Proposition 3 and proves that the excess free energy $\Phi_{\Delta_0} - \hat{f}_{\epsilon, \text{eq}} \frac{|\Delta_0|}{2}$ grows at least like $c|\Delta_0|\gamma^{4\alpha+2a}$ (recall that $\hat{f}_{\epsilon, \text{eq}}$ is defined after (4.16) and that $F_{\Delta_0, \gamma}^*(m) = \hat{f}_{\epsilon, \text{eq}} \frac{|\Delta_0|}{2}$ when m is identically equal to m_ϵ or to $-m_\epsilon$). As desired such excess free energy $c|\Delta_0|\gamma^{4\alpha+2a}$ is much larger (for small γ) than the error term in the first exponent in (4.14) which is given by $c|\text{sp}(\Gamma)|\gamma^{1/2} \log \gamma^{-1}$.

Theorem 4 *There is $c > 0$ so that*

$$\Phi_{\Delta_0} \geq \hat{f}_{\epsilon, \text{eq}} \frac{|\Delta_0|}{2} + c \frac{|\Delta_0|}{\gamma^{-(1+\alpha)} \gamma^{-\alpha}} \gamma^{-(1-\alpha)} \min\{\gamma^\alpha; \gamma^{2a}\}. \quad (4.18)$$

Proof We rewrite (4.6) as

$$\begin{aligned} F_{\Delta_0, \gamma}^*(m) &= \frac{1}{2} \sum_{(x, i) \in \Delta_0} \hat{f}_\epsilon(m(x, i), m(v_{x, i})) \\ &\quad + \frac{1}{4} \sum_{(x, i) \neq (y, i) \in \Delta_0} J_\gamma(x, y) (m(x, i) - m(y, i))^2 \end{aligned} \quad (4.19)$$

and start by bounding from below the first term. We distinguish two cases:

- (i) when $\eta(x, i) = \eta(v_{x, i}) \neq 0$ we bound $\hat{f}_\epsilon(m(x, i), m(v_{x, i})) \geq \hat{f}_{\epsilon, \text{eq}}$;
- (ii) in the other cases for all $(y, i) \in C_x^{\ell-, i}$, $(m(y, i), m(v_{x, i})) \notin U_\zeta$ for γ small enough (as, by the smoothness condition, $|m - m^{(\ell-)}| \leq c\gamma^\alpha$). We then bound $\hat{f}_\epsilon(m(x, i), m(v_{x, i})) \geq \hat{f}_{\epsilon, \text{eq}} + c\zeta^2$.

Thus the first term on the right hand side of (4.19) is bounded from below by

$$\hat{f}_{\epsilon, \text{eq}} \frac{|\Delta_0|}{2} + N_{(ii)} c \gamma^{-(1-\alpha)} \zeta^2$$

where, writing $v_{x, i} = (x, i')$, $N_{(ii)}$ is the number of distinct pairs of intervals $C_x^{\ell-, i}$, $C_x^{\ell-, i'}$ where case (ii) occurs.

The second term on the right hand side of (4.19) is bounded from below by retaining only the terms where (x, i) and (y, i) are in two consecutive $C^{\ell-, i}$ intervals and $\eta(x, i; m) = -\eta(y, i; m) \neq 0$. Suppose for instance $\eta(x, i; m) = 1$, then $|m(x, i; m) - m_\epsilon| \leq 2\zeta$ (for γ small enough and using smoothness as before). Analogously $|m(y, i) + m_\epsilon| \leq 2\zeta$ and recalling the assumption that $J(\cdot)$ is strictly positive at the origin, we see that the contribution to (4.19) coming from any such pair of intervals is, for γ small enough, at least

$$c\gamma \frac{J(0)}{2} (\gamma^{-(1-\alpha)})^2 = \tilde{c} \gamma^{-(1-\alpha)} \gamma^\alpha.$$

To conclude we observe that by definition of contours for any Q -rectangle Q' in Δ_0 there is a rectangle Q in Δ_0 connected to Q' (or just Q' itself) with the following property. Either case (ii) occurs in Q or there are two consecutive $C^{\ell-, i}$ intervals one at least inside Q with opposite values of η , or both the above events occur. (4.18) is then obtained because the number of Q rectangles in Δ_0 is $\frac{|\Delta_0|}{\gamma^{-(1+\alpha)} \gamma^{-\alpha}}$. $\square$

5 Characterization of Minimizers

The lower bound of the denominator in (2.9) will be obtained by computing the free energy functional on a suitable test function m on $\text{sp}(\Gamma)$.

On Δ_0 , m will be a constant approximately equal to m_ϵ while it will be (approximately) equal to the minimizers of $\Phi_{\Delta_{\text{in}}}(\bar{m}_{\sigma_{\text{ext}}})$, $\Phi_{\Delta_k^+}(\bar{m}_{\sigma_{I_k^+}})$, and $\Phi_{\Delta_k^-}(\bar{m}_{-\sigma_{I_k^-}})$, in the respective sets Δ_{in} , Δ_k^+ , and Δ_k^- .

The main difficulty will be to estimate the last term in (4.5) which in the upper bound for the partition function could be neglected being non negative. We shall prove below that the term $[m(x, i) - m(y, i)]^2$ in (4.5) with $(x, i) \in \Delta_0$ and $(y, i) \notin \Delta_0$ is bounded from above by $e^{-c\gamma^{-\alpha}}$, with the above choices of $m(x, i)$ and $m(y, i)$. Thus the last term in (4.5) will then be negligible also in the lower bound.

The analysis of the minimizers is essentially the same for all of them and, for the sake of definiteness, we will just look at the minimizer of $\Phi_{\Delta_{\text{in}}}^+(\bar{m}_{\sigma_{\text{ext}}})$, referring to (4.10) and (4.7) for the definition.

Recalling (4.7), (3.5) and (4.8) we have

$$F'_{\Delta_{\text{in}}, \gamma}(m|\bar{m}) = \sum_{(x, i) \in \Delta_{\text{in}}} \frac{1}{2} \left(\hat{\phi}_\epsilon(m(x, i), m(v_{x, i})) - a_{x, i} m(x, i)^2 \right) - \sum_{(x, i) \in \Delta_{\text{in}}} m(x, i) \left(\frac{1}{2} \sum_{(y, i) \in \Delta_{\text{in}}} J_\gamma(x, y) m(y, i) + \sum_{(y, i) \notin \text{sp}(\Gamma)} J_\gamma(x, y) \bar{m}(y, i) \right). \quad (5.1)$$

Being a continuous function of $\{m(x, i), (x, i) \in \Delta_{\text{in}}\}$ the function $m \mapsto F'_{\Delta_{\text{in}}, \gamma}(m|\bar{m})$ attains a minimum when m varies in the compact set

$$K := \bigcap_{(x, i) \in \Delta_{\text{in}}} \left\{ |m^{\ell-}(x, i) - m_\epsilon| \leq \zeta \right\}. \quad (5.2)$$

We are going to prove that the minimizer is unique and will establish properties of the minimizer typical of the correlations in the Gibbsian high temperatures regime.

We fix arbitrarily a pair (x, i) and $(x, i') = v_{x, i}$ of vertically interacting sites in Δ_{in} , and regard $F'_{\Delta_{\text{in}}, \gamma}(m|\bar{m})$ in (5.1) as a function of $m(x, i)$ and $m(x, i')$ alone with all the other $m(y, j)$ considered as fixed parameters that we denote by $u(y, j)$. Let

$$\mathcal{N}_{x, i, i'} = \left\{ u(y, j), y \neq x, j = i, i' : (y, j) \in \Delta_0^c, u(y, j) \in (-1, 1) \text{ and } |u^{\ell-}(y, j) - m_\epsilon| \leq \zeta, \text{ for all } (y, j) \notin C_x^{\ell-, j} \right\}, \quad (5.3)$$

namely the set where the function $\eta(\cdot; u)$ is identically 1 except maybe on the intervals containing (x, i) or (x, i') where we do not impose conditions on the $u(\cdot)$.

For any $u \in \mathcal{N}_{x, i, i'}$ we introduce the function

$$g_\epsilon(m_i, m_{i'}) := \hat{\phi}_\epsilon(m_i, m_{i'}) - \frac{1}{2} (a_i m_i^2 + a_{i'} m_{i'}^2) - \lambda_i^u m_i - \lambda_{i'}^u m_{i'}, \quad (5.4)$$

where $(m_i, m_{i'}) \in (-1, 1) \times (-1, 1)$, a_j is a shorthand for $a_{x, j}$, $j = i, i'$, and

$$\lambda_j^u = \sum_{y \neq x: (y, j) \notin \Delta_0} J_\gamma(x, y) u(y, j), \quad j = i, i'. \quad (5.5)$$

In Appendix 4 we shall prove:

Proposition 4 *There are $\epsilon_0 > 0$ and $\gamma : (0, \infty) \rightarrow (0, \infty)$ such that for any $0 < \epsilon \leq \epsilon_0$ there are $r < 1$ and coefficients $C_{x,i,i'}(j, j')$, $j, j' \in \{i, i'\}$, so that the following holds for all $\gamma \leq \gamma_\epsilon$.*

- For any $u \in \mathcal{N}_{x,i,i'}$ there is a unique minimizer $m^{(u)} = (m_i^{(u)}, m_{i'}^{(u)})$ of g_ϵ .
- $\sum_{j'=i,i'} C_{x,i,i'}(j, j') \leq r$ for $j = i, i'$.
- $\left| m_j^{(u)} - m_\epsilon \right| \leq \sum_{j'=i,i'} C_{x,i,i'}(j, j') \frac{|\lambda_{j'}^u - \lambda_{j'}^{\text{eq}}|}{1 - a_{x,j'}}$, $j = i, i'$, where λ_j^{eq} is the value of λ_j when u is identically equal to m_ϵ .
- $\frac{|\lambda_j^u - \lambda_j^{\text{eq}}|}{1 - a_{x,j}} \leq \zeta + c\gamma^\alpha$ for $j = i, i'$.
- For any $u, v \in \mathcal{N}_{x,i,i'}$ and $j = i, i'$, $\left| m_j^{(u)} - m_j^{(v)} \right| \leq \sum_{j'=i,i'} C_{x,i,i'}(j, j') \frac{|\lambda_{x,j'}^u - \lambda_{x,j'}^v|}{1 - a_{x,j'}}$.

As a consequence of the above proposition we have:

Theorem 5 *In the same context of Proposition 4 and for γ small enough the following holds. There is a unique minimizer m^* of $F'_{\Delta_{\text{in}},\gamma}(\cdot|\bar{m})$ in K , see (5.2) and for any $(x, i) \in \Delta_{\text{in}}$*

$$\left| m^*(x, i) - m_\epsilon \right| < \zeta, \quad (5.6)$$

$$\left| m^*(x, i) - m_\epsilon \right| < 2r^n, \quad (5.7)$$

where n is the minimal number of steps required to go from (x, i) to the complement of $\text{sp}(\Gamma)$ when horizontal steps have length $\leq \gamma^{-1}$ while the vertical steps have length 1.

Proof We shall preliminary prove that for γ small enough the minimizer in Proposition 4 satisfies $|m_j^{(u)} - m_\epsilon| < \zeta$. Indeed:

$$\left| m_j^{(u)} - m_\epsilon \right| \leq \sum_{j'=i,i'} C_{x,i,i'}(j, j') \frac{|\lambda_{x,j'}^u - \lambda_{j'}^{\text{eq}}|}{1 - a_{j'}} \leq r\zeta + c\gamma^\alpha < \zeta, \quad (5.8)$$

having used the bounds on $C_{x,i,i'}(j, j')$ and $|\lambda_{j'}^u - \lambda_{j'}^{\text{eq}}|$ stated in Proposition 4. The last inequality $r\zeta + c\gamma^\alpha < \zeta$ holds for γ small enough, because $r < 1$ and by the choice of ζ and α .

Since $F'_{\Delta_{\text{in}},\gamma}(m|\bar{m})$ is a continuous function of the coordinates $m(x, i)$, $(x, i) \in \Delta_{\text{in}}$, it has a minimum in the compact set K . Let m be a minimizer, and $C_x^{\ell-i}$ a segment in Δ_{in} whose points are denoted (x_1, i) , ..., (x_N, i) . Let $m_{x_1;i}$ be the function obtained from m after replacing the elements $m(x_1, i)$ and $m(v_{x_1,i})$ by the minimizer of g_ϵ relative to the points (x_1, i) and $v_{x_1,i}$ and with $u = m$ on the complement of $\{(x_1, i), v_{x_1,i}\}$. We then define iteratively the sequence $m_{x_1,\dots,x_k;i}$, $k \leq N$, by applying the above procedure to $m_{x_1,\dots,x_{k-1};i}$. We claim that $m_{x_1,\dots,x_N;i} = m$. In fact $F'_{\Delta_{\text{in}},\gamma}(m_{x_1,\dots,x_k;i}|\bar{m})$ is non increasing in k (because we are relaxing the condition $\eta = 1$ in $C_{x,i}^{\ell-i}$ and because we are putting at each step the minimizer of the corresponding g_ϵ) and therefore

$$F'_{\Delta_{\text{in}},\gamma}(m_{x_1,\dots,x_N;i}|\bar{m}) \leq F'_{\Delta_{\text{in}},\gamma}(m|\bar{m}).$$

By (5.8) $|m_{x_1, \dots, x_N; i}(y, j) - m_\epsilon| < \zeta$ for all (y, j) in $C_{x, i}^{\ell-, i} \cup C_{v_{x, i}}^{\ell-, i}$. Therefore $m_{x_1, \dots, x_N; i} \in K$ and since m is a minimizer

$$F'_{\Delta_{\text{in}}, \gamma}(m|\bar{m}) \leq F'_{\Delta_{\text{in}}, \gamma}(m_{x_1, \dots, x_N; i}|\bar{m}).$$

This means that at each step

$$F'_{\Delta_{\text{in}}, \gamma}(m_{x_1, \dots, x_{k-1}; i}|\bar{m}) = F'_{\Delta_{\text{in}}, \gamma}(m_{x_1, \dots, x_k; i}|\bar{m})$$

and by the uniqueness of the minimizer of g_ϵ we conclude the proof of the claim. Observe that we have also proved a sort of DLR property, namely that if m is a minimizer then its values at (x, i) and (x, i') minimize the corresponding g_ϵ .

By the arbitrariness of the choice of $(x, i) \in \Delta_{\text{in}}$ in the above argument we deduce that for all $(x, i) \in \Delta_{\text{in}}$ $|m(x, i) - m_\epsilon| < \zeta$, and since $(m(x, i), m(v_{x, i}))$ is the minimizer of the corresponding g_ϵ then by the second property in Proposition 4 and by (5.5)

$$|m(x, i) - m_\epsilon| \leq \sum_{j'=i, i'} C_{x, i, i'}(i, j') \sum_{y \neq x: (y, j) \notin \Delta_0} \frac{J_\gamma(x, y)}{1 - a_{x, j'}} |m(y, j') - m_\epsilon|, \quad (5.9)$$

where $m = \bar{m}$ outside $\text{sp}(\Gamma)$. The inequality (5.9) can be iterated n times with n as in the text of the theorem and we get

$$\begin{aligned} |m(x, i) - m_\epsilon| &\leq \sum_{j_1=i, i'} C_{x, i, i'}(i, j_1) \sum_{y_1 \neq x: (y, j) \notin \Delta_0} \frac{J_\gamma(x, y_1)}{1 - a_{x, j_1}} \\ &\times \sum_{j_2=j_1, j'_1} C_{y_1, j_1, j'_1}(j_1, j_2) \sum_{y_2 \neq y_1: (y_2, j_2) \notin \Delta_0} \frac{J_\gamma(x, y_2)}{1 - a_{y_1, j_2}} \cdots |m(y_n, j_n) - m_\epsilon|. \end{aligned}$$

We bound the last factor by 2, the sum over the y_k is normalized to 1 hence $|m(x, i) - m_\epsilon| \leq 2r^n$.

It remains to prove the uniqueness of the minimizer of $F'_{\Delta_{\text{in}}, \gamma}(\cdot|\bar{m})$ in K . Suppose there are two minimizers m and m' , then by the last statement of Proposition 4

$$|m(x, i) - m'(x, i)| \leq \sum_{j'=i, i'} C_{x, i, i'}(i, j') \sum_{y \neq x: (y, j) \notin \Delta_0} \frac{J_\gamma(x, y)}{1 - a_{x, j'}} |m(y, j') - m'(y, j')|. \quad (5.10)$$

The inequality can be iterated n times. But now n is arbitrary because $m(y, j) = m'(y, j)$ outside $\text{sp}(\Gamma)$ and therefore $m(x, i) = m'(x, i)$. $\square$

6 The Lower Bound

In this section we will prove a lower bound for the denominator in (2.9).

We call *trial function* a function $m \in \mathcal{M}_{\text{sp}(\Gamma)}$, see (3.4), namely with values in $M_{\gamma-1/2}$ and constant on the intervals $C^{\gamma-1/2, i}$ contained in $\text{sp}(\Gamma)$. Denote by σ the collection of spins in $c(\Gamma)^c$ and in the sets $I_k^\pm$. For any such σ we choose a trial function m_σ and using (3.8) we get that the denominator in (2.9) is bounded from below by

$$e^{-c|\text{sp}(\Gamma)|\gamma^{1/2-a}} \sum_{\sigma_{I_k^+}, \sigma_{I_k^-}}^+ e^{-F_{\text{sp}(\Gamma), \gamma}(m_\sigma|\bar{m}_\sigma) - \sum_k H(\sigma_{I_k^+}) - \sum_k H(\sigma_{I_k^-})}, \quad (6.1)$$

having used the notation of (4.12). The whole point is now to reduce (6.1) to (4.14) and this is done with a good choice of the trial function. We fix σ and start with the function m' which in Δ_0 is identically equal to m_ϵ , in Δ_{in} it is the minimizer of $\Phi_{\Delta_{\text{in}}}^+(\bar{m}_{\sigma_{\text{ext}}})$ and in $I_k^\pm$ it coincides with the minimizer of $\Phi_{\Delta_k^\pm}^+(\bar{m}_\sigma)$. Since the values of m' are not necessarily in $M_{\gamma^{-1/2}}$, m' may not be a trial function. We then define m'' which at each (x, i) is equal to a value in $M_{\gamma^{-1/2}}$ which minimizes the distance of $m'(x, i)$ from $M_{\gamma^{-1/2}}$. m'' may not be constant on the intervals $C^{\gamma^{-1/2}, i}$ so that we define m_σ as

$$m_\sigma(x, i) = \gamma^{1/2} \sum_{y \in C_x^{\gamma^{-1/2}}} m''(y, i). \quad (6.2)$$

m_σ is a trial function and we can use it in (6.1). We claim that

$$F_{\text{sp}(\Gamma), \gamma}(m_\sigma | \bar{m}_\sigma) \leq F_{\text{sp}(\Gamma), \gamma}(m'' | \bar{m}_\sigma) + c\gamma^{1/2} |\text{sp}(\Gamma)|. \quad (6.3)$$

Proof Recalling the definition (3.5) of the free energy functional, we observe that by convexity

$$\sum_{(x, i) \in \text{sp}(\Gamma)} \hat{\phi}_\epsilon(m''(x, i), m''(v_{x, i})) \geq \sum_{(x, i) \in \text{sp}(\Gamma)} \hat{\phi}_\epsilon(m_\sigma(x, i), m_\sigma(v_{x, i})).$$

For the terms in (3.5) that contain J_γ , replacing m_σ by m'' gives an error that is bounded from above by $c\gamma^{1/2} |\text{sp}(\Gamma)|$, yielding (6.3). Similarly

$$F_{\text{sp}(\Gamma), \gamma}(m'' | \bar{m}_\sigma) \leq F_{\text{sp}(\Gamma), \gamma}(m' | \bar{m}_\sigma) + c\gamma^{1/2} |\text{sp}(\Gamma)|. \quad (6.4)$$

So far we have proved that the denominator in (2.9) is bounded from below by

$$e^{-c|\text{sp}(\Gamma)|\gamma^{1/2-a}} \sum_{\sigma_{I_k^+}, \sigma_{I_k^-}}^+ e^{-F_{\text{sp}(\Gamma), \gamma}(m' | \bar{m}_\sigma) - \sum_k H(\sigma_{I_k^+}) - \sum_k H(\sigma_{I_k^-})} \quad (6.5)$$

(with c a suitable constant which takes care of all the above errors). We next use (4.5) to get

$$\begin{aligned} F_{\text{sp}(\Gamma), \gamma}(m' | \bar{m}_\sigma) &\leq \hat{f}_{\epsilon, \text{eq}} \frac{|\Delta_0|}{2} + \Phi_{\Delta_{\text{in}}}^+(\bar{m}_\sigma) \\ &\quad + \sum_k \Phi_{\Delta_k^+}^+(\bar{m}_\sigma) + \sum_k \Phi_{\Delta_k^-}^+(\bar{m}_\sigma) + c|\text{sp}(\Gamma)|e^{-\gamma^{-\alpha}}, \end{aligned} \quad (6.6)$$

where we used that (i) $m' = m_\epsilon$ in Δ_0 ; (ii) m' is the minimizer of $\Phi_{\Delta_k^\pm}^+(\bar{m}_\sigma)$ and of $\Phi_{\Delta_{\text{in}}}^+(\bar{m}_\sigma)$ in the respective sets; (iii) the last term in (4.5) is bounded using (5.7).

In conclusion

$$\begin{aligned} Z_{c(\Gamma), \bar{\sigma}}(\eta = 1 \text{ on } \text{sp}(\Gamma); \Theta = \pm 1 \text{ on each } \partial_k^\pm(\Gamma)) \\ \geq e^{-\hat{f}_{\epsilon, \text{eq}} \frac{|\Delta_0|}{2} - c|\text{sp}(\Gamma)|\gamma^{1/2}} \\ \times e^{-\Phi_{\Delta_{\text{in}}}(\bar{m}_{\sigma_{\text{ext}}})} \left\{ \prod Z^+(I_k^+) \right\} \left\{ \prod Z^+(I_k^-) \right\}. \end{aligned} \quad (6.7)$$

Proof of Theorem 2. A comparison with the upper bound (4.14) and use of (4.18) then completes the proof of the Peierls bounds. $\square$

7 Proof of Theorem 1

The proof of Theorem 1 is based on the validity of the Peierls bounds (2.10) and it follows closely the well known proof for the nearest neighbor Ising model at low temperatures.

Let $\{\Lambda_n\}$ be an increasing sequence of bounded Q -measurable regions which invades the whole space and $\mu_{\Lambda_n, \bar{\sigma}}^+$ plus diluted Gibbs measures with boundary conditions $\bar{\sigma}$ such that $\eta = 1$ on $\partial \text{out}(\Lambda_n)$ ($\bar{\sigma}$ may depend on n). We want to prove that for γ small enough and all boundary conditions $\bar{\sigma}$ as in the paragraph of (2.9)

$$\lim_{n \rightarrow \infty} \mu_{\Lambda_n, \bar{\sigma}}^+ [\Theta(0) < 1] < \frac{1}{2}. \quad (7.1)$$

By the definition of plus diluted Gibbs measures the event in (7.1) can only occur if there is a contour Γ such that the origin belongs to $c(\Gamma)$. Call $N(\Gamma)$ the number of Q -rectangles contained in $\text{sp}(\Gamma)$. Then there is a horizontal translate of $\text{sp}(\Gamma)$ by $k\ell_+$, $k \leq N(\Gamma)$, so that the translate of $\text{sp}(\Gamma)$ contains the origin.

This means that

$$\mu_{\Lambda_n, \bar{\sigma}}^+ [\Theta(0) < 1] \leq \mu_{\Lambda_n, \bar{\sigma}}^+ \left[\bigcup_{\Gamma: \text{sp}(\Gamma) \ni 0} \bigcup_{k \leq N(\Gamma)} \{\Gamma_k \text{ is a contour}\} \right], \quad (7.2)$$

where Γ_k is obtained from Γ by translating it by $k\ell_+$. By subadditivity, the right hand side of (7.2) is bounded above by

$$\sum_{\Gamma: \text{sp}(\Gamma) \ni 0} \sum_{k \leq N(\Gamma)} \mu_{\Lambda_n, \bar{\sigma}}^+ [\Gamma_k \text{ is a contour}]. \quad (7.3)$$

Now the probability inside the double sum in (7.3) equals

$$\frac{Z_{c(\Gamma_k); \bar{\sigma}}(\eta = \eta_{\Gamma_k} \text{ on } \text{sp}(\Gamma_k); \Theta = \pm 1 \text{ on each } \partial_{\text{out}, i}^{\pm}(\Gamma_k))}{Z_{\Lambda_n, \bar{\sigma}}^+} \leq W_{\Gamma_k}(\bar{\sigma}), \quad (7.4)$$

the inequality justified by the fact that the denominator in (2.9) is bounded above by $Z_{\Lambda_n, \bar{\sigma}}^+$ (since the sum defining the latter quantity contains the terms in the sum defining the former one).

Therefore by (2.10), and using the fact that $|\text{sp}(\Gamma_k)| = |\text{sp}(\Gamma)|$, we find that

$$\mu_{\Lambda_n, \bar{\sigma}}^+ [\Theta(0) < 1] \leq \sum_{\Gamma: \text{sp}(\Gamma) \ni 0} N(\Gamma) e^{-c|\text{sp}(\Gamma)|\gamma^{2a+4\alpha}}. \quad (7.5)$$

On the other hand $|\text{sp}(\Gamma)| = N(\Gamma)\gamma^{-(1+\alpha)}\gamma^{-\alpha}$ and once $\text{sp}(\Gamma)$ is fixed, η_{Γ} can be chosen in at most $3^{|\text{sp}(\Gamma)|/\ell_-}$ ways, which would contribute with a factor $3^{N(\Gamma)\gamma^{-3\alpha}}$ at most. As α is very small ($5\alpha + 2a < 1$) we see that for γ small the sum on the right hand side of (7.5) can be bounded from above by the following sum below over all connected regions D made of unit cubes with vertices in $\mathbb{Z}^2$.

$$\mu_{\Lambda_n, \bar{\sigma}}^+ [\Theta(0) < 1] \leq \sum_{D \ni 0} |D| e^{-\frac{c}{2}|D|\gamma^{-1+2a+2\alpha}} \quad (7.6)$$

Since a and α are much smaller than 1, then the sum vanishes in the limit when $\gamma \rightarrow 0$, see for instance Lemma 3.1.2.4 in [10], so that (7.1) is proved.

By (7.1) and the spin flip symmetry it follows that there are at least two DLR measures, hence by ferromagnetic inequalities the plus and minus DLR measures $\mu_{\gamma}^{\pm}$ of Theorem 1 are

distinct and Theorem 1 is proved. There are many more consequences of the Peierls bounds, see for instance Chapter 12 in [10], but we shall not discuss such extensions here.

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Appendix 1: Proof of Proposition 1

We prove Proposition 1 via equivalence of ensembles. The grand-canonical partition function π_ϵ of the two layers is trivially equal to the logarithm of

$$\sum_{(\sigma_1, \sigma_2) \in \{-1, +1\}^2} e^{\{h_1 \sigma_1 + h_2 \sigma_2\} + \epsilon \sigma_1 \sigma_2} = 2\{e^\epsilon \cosh(h_+) + e^{-\epsilon} \cosh(h_-)\},$$

where $h_+ = h_1 + h_2$ and $h_- = h_2 - h_1$. Thus the pressure is given by

$$\pi_\epsilon(h_+, h_-) = \log(2Z), \quad Z = \{e^\epsilon \cosh(h_+) + e^{-\epsilon} \cosh(h_-)\}. \quad (8.1)$$

We can easily check that the function $(h_+, h_-) \mapsto \pi_\epsilon(h_+, h_-)$ is strictly convex, namely its Hessian, denoted here by $D^2\pi_\epsilon$, is a positive definite operator. Indeed, by computation we have:

$$\frac{\partial}{\partial h_+} \pi_\epsilon(h_+, h_-) = e^\epsilon \frac{\sinh(h_+)}{Z}, \quad \frac{\partial}{\partial h_-} \pi_\epsilon(h_+, h_-) = e^{-\epsilon} \frac{\sinh(h_-)}{Z}, \quad (8.2)$$

$$\begin{aligned} \frac{\partial^2}{\partial h_+^2} \pi_\epsilon(h_+, h_-) &= \frac{e^\epsilon}{Z} \left(\cosh(h_+) - e^\epsilon \frac{\sinh^2(h_+)}{Z} \right) > 0, \\ \frac{\partial^2}{\partial h_-^2} \pi_\epsilon(h_+, h_-) &= \frac{e^{-\epsilon}}{Z} \left(\cosh(h_-) - e^{-\epsilon} \frac{\sinh^2(h_-)}{Z} \right) > 0, \\ \frac{\partial^2}{\partial h_+ \partial h_-} \pi_\epsilon(h_+, h_-) &= -\frac{\sinh(h_+) \sinh(h_-)}{Z^2}. \end{aligned}$$

It then follows that the diagonal elements of $D^2\pi_\epsilon(h_+, h_-)$ and its determinant, given by

$$\begin{aligned} |D^2\pi_\epsilon(h_+, h_-)| &= Z^{-4} \left(1 + 2 \cosh(2\epsilon) \cosh(h_+) \cosh(h_-) + \cosh^2(h_+) \cosh^2(h_-) \right. \\ &\quad \left. - \sinh^2(h_+) \sinh^2(h_-) \right), \end{aligned} \quad (8.3)$$

are all positive, and therefore the 2×2 Hessian matrix is positive definite. We now consider the Legendre transform of π_ϵ :

$$\phi_\epsilon(m) := \sup_h \left(\frac{1}{2} \langle h, m \rangle - \pi_\epsilon(h) \right), \quad (8.4)$$

where $m = (m_+, m_-)$ and

$$\langle h, m \rangle := h_+ m_+ + h_- m_-, \quad (8.5)$$

and we have:

Lemma 1 For any $m = (m_+, m_-)$ such that $|m_i| < 1$, $i = 1, 2$, where

$$m_1 = (m_+ - m_-)/2, \quad m_2 = (m_+ + m_-)/2 \quad (8.6)$$

the sup in (8.4) is a maximum, achieved at a unique $h = (h_+, h_-)$, which is the unique solution of

$$m_+ = 2 \frac{\partial}{\partial h_+} \pi_\epsilon(h_+, h_-), \quad m_- = 2 \frac{\partial}{\partial h_-} \pi_\epsilon(h_+, h_-). \quad (8.7)$$

In other words, for this h

$$\phi_\epsilon(m) = \frac{1}{2} \langle h, m \rangle - \pi_\epsilon(h) = \sup_{h'} \left(\frac{1}{2} \langle h', m \rangle - \pi_\epsilon(h') \right). \quad (8.8)$$

Proof If $|m_i| < 1$, $i = 1, 2$, the function

$$\Gamma(h) := \frac{1}{2} \langle h, m \rangle - \pi_\epsilon(h)$$

goes to $-\infty$ when $|h| \rightarrow \infty$. Together with the continuity of $\Gamma(h)$ this implies that the supremum is a maximum, achieved at the critical point, hence (8.7). Uniqueness follows from the strict convexity of π_ϵ . $\square$

The following lemma is an immediate consequence of the strict convexity of π_ϵ and the properties of the Legendre transform.

Lemma 2 ϕ_ϵ is strictly convex. Writing $D\phi_\epsilon$ for its gradient, $m = (m_+, m_-)$ solves the equation $D\phi_\epsilon(m) = \frac{\theta}{2}$, $\theta = (\theta_+, \theta_-)$, if and only if

$$m = 2D\pi_\epsilon(\theta). \quad (8.9)$$

More explicitly:

$$m_+ = 2e^\epsilon \frac{\sinh(\theta_+)}{e^\epsilon \cosh(\theta_+) + e^{-\epsilon} \cosh(\theta_-)} \quad (8.10)$$

$$m_- = 2e^{-\epsilon} \frac{\sinh(\theta_-)}{e^\epsilon \cosh(\theta_+) + e^{-\epsilon} \cosh(\theta_-)}.$$

Changing back to coordinates (h_1, h_2) and (m_1, m_2) , let $\hat{\pi}_\epsilon(h_1, h_2)$ and $\hat{\phi}_\epsilon(m_1, m_2)$ denote the functions $\pi_\epsilon(h_+, h_-)$ and $\phi_\epsilon(m_+, m_-)$ when $h_\pm$ and $m_\pm$ are expressed in terms of (h_1, h_2) and respectively (m_1, m_2) . Thus $\hat{\pi}_\epsilon$ and $\hat{\phi}_\epsilon$ are the Legendre transform of each other:

$$\hat{\phi}_\epsilon(m_1, m_2) = \sup_{(h'_1, h'_2)} \left((h'_1 m_1 + h'_2 m_2) - \hat{\pi}_\epsilon(h'_1, h'_2) \right) = (h_1 m_1 + h_2 m_2) - \hat{\pi}_\epsilon(h_1, h_2) \quad (8.11)$$

where in the last equality (h_1, h_2) are functions of (m_1, m_2) via (8.7).

The following lemma is an immediate consequence of the above.

Lemma 3 The map $(m_1, m_2) \mapsto \hat{\phi}_\epsilon(m_1, m_2)$ is strictly convex on $(-1, 1)^2$ and $\hat{m} = (m_1, m_2)$ solves the equation $D\hat{\phi}_\epsilon(\hat{m}) = \hat{\theta}$, $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$, if and only if

$$\hat{m} = D\hat{\pi}_\epsilon(\hat{\theta}), \quad (8.12)$$

where $D\hat{\pi}_\epsilon$ and $D\hat{\phi}_\epsilon$ denote the gradient of $\hat{\pi}_\epsilon$ and $\hat{\phi}_\epsilon$. Moreover, calling

$$m = (m_+, m_-), \quad m_+ = \frac{\hat{m}_1 + \hat{m}_2}{2}, \quad m_- = \frac{\hat{m}_2 - \hat{m}_1}{2},$$

$$\theta = (\theta_+, \theta_-), \quad \theta_+ = \frac{\hat{\theta}_1 + \hat{\theta}_2}{2}, \quad \theta_- = \frac{\hat{\theta}_2 - \hat{\theta}_1}{2},$$

we have that $D\hat{\phi}_\epsilon(\hat{m}) = \hat{\theta}$ if and only if $D\phi_\epsilon(m) = \frac{\theta}{2}$.

Proof of Proposition 1 It remains to prove that $\hat{\phi}_\epsilon$ is the canonical free energy of the two layers system.

With (h_1, h_2) as in (8.11) and Z as in (8.1), we get

$$Z_{\epsilon,n}(m_1, m_2) = e^{-n(h_1 m_1 + h_2 m_2)} (2Z)^n P_{h_1, h_2, \epsilon} \left[\sum_{x=1}^n \sigma_i(x) = m_i n, \quad i = 1, 2 \right], \quad (8.13)$$

where $P_{h_1, h_2, \epsilon}$ is the Gibbs measure with Hamiltonian

$$-\sum_{x=1}^n \{h_1 \sigma_1(x) + h_2 \sigma_2(x)\} - \epsilon \sum_{x=1}^n \sigma_1(x) \sigma_2(x)$$

hence the upper bound in (3.2). The lower bound follows by an application of Lemma 4 below, which gives an elementary local limit theorem lower bound for the product measure $P_{h_1, h_2, \epsilon}$ of the form $\text{const } n^{-3/2}$, with a uniform constant over n and m . $\square$

Lemma 4 *With notation as above (see 8.13), there is a constant $c > 0$ independent of $m = (m_1, m_2) \in \{-1 + \frac{2}{n}, \dots, 1 - \frac{2}{n}\}^2$ and $n \geq 1$ such that*

$$P_{h_1, h_2, \epsilon} \left[\sum_{x=1}^n \sigma_i(x) = m_i n, \quad i = 1, 2 \right] \geq cn^{-3/2}. \quad (8.14)$$

Remark The sharp bound is cn^{-1} , but (8.14) is enough for our purposes, and requires a shorter argument, given next.

Proof of Lemma 4. We first note that $\sum_{x=1}^n (\sigma_1(x), \sigma_2(x))$ is a two dimensional random walk with jumps to the nearest diagonals. We rotate by $-\pi/4$ radians and rescale space by $1/\sqrt{2}$ in order to get a simple 2-d random walk, denoted by $X = (X(n) = (X_1(n), X_2(n)))_{n \geq 0}$, with mean jump $m_* = (m_+, m_-)/2$. Notice two things:

1. $m_* \in \{(i, j)/n : i, j = -n + 2, -n + 3, \dots, n - 3, n - 2; i, j \text{ have the same parity}\}$;
2. $X_1(n)$ and $X_2(n)$ have the same parity for every $n \geq 0$.

In these terms we want to prove a lower bound for

$$P_{h_1, h_2, \epsilon}(X(n) = m_* n). \quad (8.15)$$

Let p_1, q_1, p_2, q_2 denote the jump probabilities of X to the right, left, up and down, respectively. Notice that $m_+ = p_1 - q_1$ and $m_- = p_2 - q_2$.

Let $H(n)$ denote the number of horizontal steps given by X in the first n steps. Then $H(n)$ has a binomial distribution with success probability $h := p_1 + q_1$. Let us assume that $h \leq 1/2$ for the remainder of the argument. A similar reasoning holds in the other case. Now given $H(n) = k$, we have that $(X_1(n), X_2(n)) = (Y_1(k), Y_2(n - k))$, where Y_1, Y_2 are independent

simple random walks in 1d with respective jump probabilities to the right $r = p_1/(p_1 + q_1)$ and $s = p_2/(p_2 + q_2)$. Notice that $p_1 + q_1 > 0$, $p_2 + q_2 > 0$.

We thus have that for any $k = 0, 1, \dots, n$

$$\begin{aligned} P_{h_1, h_2, \epsilon}(X(n) = m_+ n) &\geq P_{h_1, h_2, \epsilon}(X(n) = m_* n, H(n) = k) \\ &= P_{h_1, h_2, \epsilon}(Y_1(k) = m_+ n) P_{h_1, h_2, \epsilon}(Y_2(n - k) = m_- n) P_{h_1, h_2, \epsilon}(H(n) = k). \end{aligned} \quad (8.16)$$

Let now $Y'_1(k) = (Y_1(k) + k)/2$. Then $Y'_1(k)$ is binomial with parameters k and r , and the first probability in (8.16) equals

$$P_{h_1, h_2, \epsilon}(Y'_1(k) = (m_+ n + k)/2). \quad (8.17)$$

Similarly, the second probability in (8.16) equals

$$P_{h_1, h_2, \epsilon}(Y'_2(n - k) = (m_- n + n - k)/2), \quad (8.18)$$

where $Y'_2(n - k)$ is binomial with parameters $n - k$ and s .

We will now choose $k = k_n$ either $\lfloor hn \rfloor$ or $\lfloor hn \rfloor + 1$ so that k_n has the same parity as $m_+ n$. Notice that in this case $k_n \geq 1$ and $(m_+ n + k_n)/2$ is an integer.

A straightforward recourse to Stirling shows that the last probability in (8.16) is bounded from below by a constant times $1/\sqrt{n}$. We next argue that the same holds for the probabilities in (8.17) and (8.18), and we will be done. We consider the first such probability; the second one can be similarly treated.

We have that $k_n = (p_1 + q_1)n + \theta_n$, where $\theta_n \in (-1, 1]$, and $(m_+ n + k_n)/2 = p_1 n + \frac{\theta_n}{2}$. Notice that $Y'_1(k_n)$ has mean $k_n r = p_1 n + r\theta_n$. It again follows readily from Stirling that the probability in (8.17) is bounded from below by a constant times $1/\sqrt{n}$ (notice that if $p_1 = 0$, then θ_n also vanishes). $\square$

Appendix 2: Properties of the Mean Field Free Energy

To prove Proposition 3 we study the *free energy* given by (4.15) (two layers with a small vertical n.n. interaction).

We shall exploit the smallness of ϵ , observing that for $\epsilon = 0$ we have the well known explicit expression:

$$\hat{f}_0(m_1, m_2) := -\frac{m_1^2}{2} - I(m_1) - \frac{m_2^2}{2} - I(m_2), \quad (9.1)$$

where the entropy $I(m)$ is given by

$$I(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2}, \quad m \in [-1, 1]. \quad (9.2)$$

The function $-\frac{1}{2}m^2 - I(m)$ is a symmetric convex function of m with a quartic minimum at 0 so that

$$\hat{f}_0(m_1, m_2) \geq \hat{f}_0(0, 0) + c(m_1^4 + m_2^4). \quad (9.3)$$

Lemma 5

$$\hat{f}_0(m_1, m_2) - \epsilon \leq \hat{f}_\epsilon(m_1, m_2) \leq \hat{f}_0(m_1, m_2) + \epsilon. \quad (9.4)$$

Proof It follows at once from (8.11) and (8.1) since $\hat{\pi}_0(\cdot) - \epsilon \leq \hat{\pi}_\epsilon(\cdot) \leq \hat{\pi}_0(\cdot) + \epsilon$. $\square$

Corollary 3 *There is $c' > 0$ so that*

$$\inf_{m_1, m_2} \hat{f}_\epsilon(m_1, m_2) = \inf_{(m_1, m_2) \in \mathcal{G}_{c'}} \hat{f}_\epsilon(m_1, m_2) \quad (9.5)$$

where

$$\mathcal{G}_c = \{(m_1, m_2) \in [-1, 1] \times [-1, 1] : |m_i| \leq c\epsilon^{1/4}, i = 1, 2\}. \quad (9.6)$$

Proof Using (9.4) and (9.3), we easily see that

$$\hat{f}_\epsilon(m_1, m_2) \geq \hat{f}_\epsilon(0, 0) + c(m_1^4 + m_2^4) - 2\epsilon,$$

so that $\hat{f}_\epsilon(m_1, m_2) \geq \hat{f}_\epsilon(0, 0)$ if $(m_1, m_2) \notin \mathcal{G}_{c'}$ with c' large enough, hence (9.5). $\square$

We denote by $f_\epsilon(m)$, $m = (m_+, m_-)$, the function $\hat{f}_\epsilon(m_1, m_2)$ when m_1, m_2 are written in terms of $m_\pm$ as in (8.6). In the sequel $m = (m_+, m_-)$, $h = (h_+, h_-)$ and

$$\langle h, m \rangle := h_+m_+ + h_-m_- = 2(m_1h_1 + m_2h_2). \quad (9.7)$$

Given m and taking h as in (8.8) we then have

$$f_\epsilon(m) = -\frac{1}{4}(m_+^2 + m_-^2) + \phi_\epsilon(m) = -\frac{1}{4}\langle m, m \rangle + \frac{1}{2}\langle h, m \rangle - \pi_\epsilon(h). \quad (9.8)$$

By (9.5) the infimum of $f_\epsilon(m)$ is achieved in the set $\mathcal{G}_c$ for c large enough and the minimizers are critical points in such a set. Denoting by D the gradients, the critical points satisfy

$$Df_\epsilon(m) = -\frac{1}{2}m + D\phi_\epsilon(m) = 0. \quad (9.9)$$

Then by Lemma 2 with $\theta = m$,

$$\begin{aligned} m_+ &= 2e^\epsilon \frac{\sinh(m_+)}{e^\epsilon \cosh(m_+) + e^{-\epsilon} \cosh(m_-)}, \\ m_- &= 2e^{-\epsilon} \frac{\sinh(m_-)}{e^\epsilon \cosh(m_+) + e^{-\epsilon} \cosh(m_-)}. \end{aligned} \quad (9.10)$$

Of course $m_+ = m_- = 0$ is a solution. In the next lemma we shall prove that any solution has $m_- = 0$.

Lemma 6 *For any $x \in \mathbb{R}$ and any $\epsilon \geq 0$ the equation*

$$y = 2e^{-\epsilon} \frac{\sinh(y)}{e^\epsilon \cosh(x) + e^{-\epsilon} \cosh(y)}, \quad y \in \mathbb{R} \quad (9.11)$$

has a unique solution: $y = 0$.

Proof If y solves (9.11) then so does $-y$. Therefore we only need to prove that there is no solution with $y > 0$. Define

$$U(y) := \frac{2e^{-\epsilon}}{e^\epsilon + e^{-\epsilon} \cosh(y)} \sinh(y).$$

Since $U(y)$ is not smaller than the right hand side of (9.11), the lemma will be proved once we show that $U(y) < y$ for all $y > 0$. Suppose by contradiction that there is $y > 0$ so that $y \leq U(y)$. Then

$$e^\epsilon + e^{-\epsilon} \cosh(y) \leq 2e^{-\epsilon} \frac{\sinh(y)}{y},$$

which yields:

$$\{e^\epsilon + e^{-\epsilon}\} + e^{-\epsilon} \sum_{n \geq 1} \frac{y^{2n}}{(2n)!} \leq 2e^{-\epsilon} + e^{-\epsilon} \sum_{n \geq 1} \frac{2y^{2n}}{(2n+1)!}.$$

But this last inequality is not true, since $e^\epsilon + e^{-\epsilon} \geq 2e^{-\epsilon}$ and $(2n+1)! > 2(2n)!$ for $n \geq 1$. $\square$

We can thus put $m_- = 0$ in the first equation of (9.10) which then becomes an equation for m_+ alone. The proof of Proposition 3 is then a consequence of the following lemma:

Lemma 7 *There is $\delta > 0$ so that for all $\epsilon > 0$ small enough the equation*

$$x = 2e^\epsilon \frac{\sinh(x)}{e^\epsilon \cosh(x) + e^{-\epsilon}}, \quad x \in (0, \delta) \quad (9.12)$$

has a unique solution x_ϵ and

$$|x_\epsilon - \sqrt{12\epsilon}| \leq c\epsilon^{3/2}. \quad (9.13)$$

Proof (9.12) can be rewritten

$$e^\epsilon \cosh(x) + e^{-\epsilon} = 2e^\epsilon \frac{\sinh(x)}{x}, \quad x \in (0, \delta) \quad (9.14)$$

and therefore as $F(x^2) = 0$, where for $z \geq 0$

$$F(z) = \left\{ e^\epsilon + e^{-\epsilon} + \frac{z}{2} e^\epsilon + z g_1(z) \right\} - \left\{ 2e^\epsilon + \frac{z}{3} e^\epsilon + z g_2(z) \right\} \quad (9.15)$$

and,

$$g_1(z) = e^\epsilon \sum_{n \geq 2} \frac{z^{n-1}}{(2n)!}, \quad g_2(z) = 2e^\epsilon \sum_{n \geq 2} \frac{z^{n-1}}{(2n+1)!}.$$

We have:

$$\begin{aligned} F(12\epsilon) &= \{e^\epsilon + e^{-\epsilon} + 6\epsilon e^\epsilon + (12\epsilon)g_1(12\epsilon)\} - \{2e^\epsilon + 4\epsilon e^\epsilon + (12\epsilon)g_2(12\epsilon)\} \\ &= c\epsilon^2. \end{aligned} \quad (9.16)$$

On the other hand for any $a < 1/6$ there are δ_0 and ϵ_0 positive so that for any $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$ we have

$$\frac{dF}{dz} \geq a \text{ on } (0, \delta).$$

Hence there is a unique $z^* \in (0, \delta)$ so that $F(z^*) = 0$. Moreover

$$|z^* - 12\epsilon| \leq \frac{c\epsilon^2}{a},$$

proving the lemma. $\square$

Theorem 6 *For any $\epsilon > 0$ small enough the Hessian $D^2 f_\epsilon$ of the free energy f_ϵ is positive definite at the minimizers $\pm m^{(\epsilon)}$.*

Proof Since we shall be dealing with functions in the m and h domain, to prevent confusion we write D_m and D_h for the corresponding gradients (and for the Jacobian matrices). Analogously for the corresponding Hessian matrices. From (9.9) and (8.11), by differentiating we have: $D_m^2 f_\epsilon = -\frac{1}{2}\mathbb{I} + D_m^2 \phi_\epsilon$ and

$$D_m \phi_\epsilon = \frac{h}{2}, \quad D_m^2 \phi_\epsilon = \frac{D_m h}{2}.$$

Since $h(m)$ is the inverse of $m(h)$:

$$D_h m D_m h = \mathbb{I}.$$

On the other hand, by (8.10), $m = 2D_h \pi$, so that

$$D_h m = 2G, \quad G = D_h^2 \pi$$

and therefore

$$2GD_m h = \mathbb{I}, \quad D_m h = \frac{1}{2}G^{-1}$$

and, in conclusion,

$$D_m^2 \phi_\epsilon = \frac{1}{4}G^{-1}.$$

Thus

$$D_m^2 f_\epsilon = -\frac{1}{2}\mathbb{I} + \frac{1}{4}G^{-1} = \frac{1}{4}G^{-1}(\mathbb{I} - 2G).$$

The elements of G are given in the equations which follow (8.2) and they must be computed at $m^{(\epsilon)}$, so that $m_- = 0$. Then G is diagonal and its diagonal elements, denoted by G_{++} and G_{--} , are:

$$\begin{aligned} 2G_{++} &= 2e^\epsilon \frac{\cosh(h_+)[e^\epsilon \cosh(h_+) + e^{-\epsilon}] - e^\epsilon \sinh^2(h_+)}{[e^\epsilon \cosh(h_+) + e^{-\epsilon}]^2} \\ &= 2e^\epsilon \frac{e^\epsilon + e^{-\epsilon} \cosh(h_+)}{[e^\epsilon \cosh(h_+) + e^{-\epsilon}]^2} \leq \frac{2e^\epsilon}{e^\epsilon \cosh(h_+) + e^{-\epsilon}} \leq 1 - \frac{\epsilon}{2}. \end{aligned} \quad (9.17)$$

The last inequality holds for ϵ small enough and it is proved as follows. We develop in Taylor series all terms up to first order in ϵ , thus the equality below are meant modulo terms in ϵ^2 . Recalling (9.13) we also bound from below $h_+^2 = m_\epsilon^2 > 8\epsilon$. The last fraction in (9.17) is then bounded by

$$\frac{2 + 2\epsilon}{(1 + \epsilon)(1 + h_+^2/2) + 1 - \epsilon} < \frac{2 + 2\epsilon}{2 + 4\epsilon} = \frac{1 + \epsilon}{1 + 2\epsilon} \leq 1 - \epsilon,$$

hence (9.17) for ϵ small enough. We have

$$2G_{--} = \frac{2e^{-\epsilon}}{e^\epsilon \cosh(h_+) + e^{-\epsilon}} \leq 1 - \frac{\epsilon}{2} \quad (9.18)$$

for ϵ small enough (and for any value of h_+). We have thus seen that $\mathbb{I} - 2G$ is diagonal and its diagonal elements are $\geq \frac{\epsilon}{2}$ for ϵ small. $\square$

Corollary 4 For any $\epsilon > 0$ small enough there is $c > 0$ so that for any $\zeta > 0$ small enough:

$$\left| f_\epsilon(m) - f_\epsilon(m^{(\epsilon)}) \right| \geq c\zeta^2, \quad \text{for all } m \text{ such that } |m \mp m^{(\epsilon)}| \geq \zeta. \quad (9.19)$$

Proof From what was already seen, the inequality holds if $|m| \geq c\epsilon^{1/4}$ for suitable $c > 0$. The infimum of $f_\epsilon(m)$ in $|m \mp m^{(\epsilon)}| \geq \zeta$ must then be achieved in the set

$$\{|m \mp m^{(\epsilon)}| \geq \zeta\} \cap \{|m| \leq c\epsilon^{1/4}\}.$$

In such a set $D_m f_\epsilon \neq 0$, thus the infimum must be achieved at the boundaries, hence (9.19). $\square$

Appendix 3: Multi-Canonical Constraints

The setup is the following: $I = C_0^{\ell_-} = [0, \ell_-) \cap \mathbb{Z}$ where, recalling (2.2), $\ell_- = \gamma^{-(1-\alpha)}$, $\alpha > 0$ and small. Let $(m_1(x), m_2(x)) \in [-1, 1]^2$ for $x \in I$, $(\bar{m}_1(x), \bar{m}_2(x)) \in [-1, 1]^2$ for $x \in \mathbb{Z} \setminus I$. Dropping the dependence on γ and I , let

$$\begin{aligned} \mathcal{F}(m | \bar{m}) = & \sum_{x \in I} \hat{\phi}_\epsilon(m_1(x), m_2(x)) - \sum_{i=1,2} \left\{ \frac{1}{2} \sum_{x \neq y \in I} J_\gamma(x, y) m_i(x) m_i(y) \right. \\ & \left. + \sum_{x \in I, y \notin I} J_\gamma(x, y) m_i(x) \bar{m}_i(y) \right\} \end{aligned} \quad (10.1)$$

where $\hat{\phi}_\epsilon$ is the canonical free energy in Proposition 1.

Proposition 2 follows at once from the result below, which is the analogue for two layers of Theorem 6.4.1.1 of [10], after the vertical interaction is added in.

Proposition 5 For all γ small enough and all $u = (u_1, u_2)$, $u_i \in [-1, 1]$, there is a unique $\hat{m}$ such that $\sum_{x \in I} \hat{m}_i(x) = |I| u_i$, $i = 1, 2$, and

$$\mathcal{F}(m | \bar{m}) \geq \mathcal{F}(\hat{m} | \bar{m}), \quad \text{for all } m \text{ such that } \sum_{x \in I} m_i(x) = |I| u_i, \quad i = 1, 2. \quad (10.2)$$

Moreover the minimizer $\hat{m}$ is smooth in the sense that there is a constant c so that

$$\max_{i=1,2} \max_{x \in I} |\hat{m}_i(x) - u_i| \leq c\gamma^\alpha.$$

Proof The statement is trivially true when $|u_1| \vee |u_2| = 1$. We therefore assume in the sequel that $|u_1| \vee |u_2| < 1$.

Remark As in Appendix 1, for the free energy computation it is sometimes convenient to use the variables $m_\pm(x)$, $x \in I$ as in (8.6); we then write $m(x) = (m_+(x), m_-(x))$ and analogously $\bar{m}(x) = (\bar{m}_+(x), \bar{m}_-(x))$, and write $\hat{\phi}_\epsilon(m_1(x), m_2(x)) = \phi_\epsilon(m(x))$, with the function ϕ_ϵ given by (8.8).

To find the minimizers under the above constraint, we introduce the Lagrange multipliers $\lambda = (\lambda_+, \lambda_-)$ and define

$$\begin{aligned} \mathcal{F}_\lambda(m | \bar{m}) = & \sum_{x \in I} \left(\phi_\epsilon(m(x)) - \frac{1}{2} \langle m(x), \lambda \rangle - \frac{1}{2} \langle \kappa(x), m(x) \rangle \right) \\ & - \frac{1}{4} \sum_{x \neq y \in I} J_\gamma(x, y) \langle m(x), m(y) \rangle + \frac{1}{2} \langle \lambda, u \rangle |I| \end{aligned} \quad (10.3)$$

with $u_\pm = u_2 \pm u_1$, $\langle a, b \rangle = a_+ b_+ + a_- b_-$ and

$$\kappa_\pm(x) := \sum_{y \notin I} J_\gamma(x, y) \bar{m}_\pm(y), \quad x \in I. \quad (10.4)$$

Observe that $\mathcal{F}_\lambda(m | \bar{m}) = \mathcal{F}(m | \bar{m})$ for all m under the constraint in (10.2). Let $\bar{\kappa} = \frac{1}{|I|} \sum_{x \in I} \kappa(x)$. We introduce an interpolating parameter $s \in [0, 1]$ and define

$$\begin{aligned} \mathcal{F}_{\lambda,s}(m | \bar{m}) = & \sum_{x \in I} \left\{ \phi_\epsilon(m(x)) - \frac{1}{2} \langle m(x), [\lambda + \bar{\kappa}] \rangle \right\} - s \left(\sum_{x \in I} \left\{ \frac{1}{2} \langle [\kappa(x) - \bar{\kappa}], m(x) \rangle \right\} \right. \\ & \left. - \frac{1}{4} \sum_{x \neq y \in I} J_\gamma(x, y) \langle m(x), m(y) \rangle \right) + \frac{1}{2} \langle \lambda, u \rangle |I| \end{aligned} \quad (10.5)$$

so that $\mathcal{F}_{\lambda,1} = \mathcal{F}_\lambda$. To find the minimizer of $\mathcal{F}_{\lambda,s}(m | \bar{m})$ we need to find its critical points, namely the solutions of

$$\frac{\partial \mathcal{F}_{\lambda,s}(m | \bar{m})}{\partial m(x)} = D\phi_\epsilon(m(x)) - \frac{1}{2} \theta(x) = 0, \quad (10.6)$$

where

$$\theta(x) = \frac{1}{2} (\lambda + \bar{\kappa}) + \frac{s}{2} (\kappa(x) - \bar{\kappa}) + \frac{s}{2} \sum_{y \neq x, y \in I} J_\gamma(x, y) m(y). \quad (10.7)$$

By an explicit computation:

$$\frac{\partial^2 \mathcal{F}_{\lambda,s}(m | \bar{m})}{\partial m(x) \partial m(y)} = D^2 \phi_\epsilon(m(x)) \mathbf{1}_{x=y} - s J_\gamma(x, y) \mathbf{1}_{x \neq y \in I}$$

which for γ small enough is a positive symmetric operator, namely there is $c > 0$ so that

$$\sum_{x \in I} \langle \psi(x), D^2 \phi_\epsilon(m(x)) \psi(x) \rangle - \sum_{x \neq y \in I} s J_\gamma(x, y) \langle \psi(x), \psi(y) \rangle \geq c \sum_{x \in I} \langle \psi(x), \psi(x) \rangle. \quad (10.8)$$

This shows that if there is a critical point of $\mathcal{F}_{\lambda,s}$ it is unique and it minimizes $\mathcal{F}_{\lambda,s}$. By (10.6) m is a critical point if $D\phi_\epsilon(m(x)) = \frac{\theta(x)}{2}$ for all x . By Lemma 2, $m(x) = 2D\pi_\epsilon(\theta(x))$ which is (8.10) namely

$$\begin{aligned} m_+(x) &= 2e^\epsilon \frac{\sinh(\theta_+(x))}{e^\epsilon \cosh(\theta_+(x)) + e^{-\epsilon} \cosh(\theta_-(x))}, \\ m_-(x) &= 2e^{-\epsilon} \frac{\sinh(\theta_-(x))}{e^\epsilon \cosh(\theta_+(x)) + e^{-\epsilon} \cosh(\theta_-(x))}. \end{aligned} \quad (10.9)$$

Observing that the right hand side depends weakly on m as $|\sum_{y \in I} J_\gamma(x, y)m(y)| \leq \gamma^\alpha$, we then get, as in the proof of Theorem 6.4.1.1 in [10], that there is a unique solution $m_{\lambda, s}$ of (10.9) which is the unique minimizer of $\mathcal{F}_{\lambda, s}$ and whose fluctuations are of order $O(\gamma^\alpha)$. To conclude the proof of the theorem it suffices to show that there exists a function $\lambda(s)$ such that

$$\sum_{x \in I} m_{\lambda(s), s}(x) = u |I|, \quad \text{for all } s \in [0, 1]. \quad (10.10)$$

There is obviously a unique solution $\lambda(0)$ of (10.10) when $s = 0$. We will find $\lambda(s)$ by solving the evolution equation

$$\sum_{x \in I} \frac{d}{ds} \Psi(\theta_{\lambda(s), s}(x)) = \sum_{x \in I} D_\theta \Psi(\theta_{\lambda(s), s}(x)) \frac{d}{ds} \theta_{\lambda(s), s}(x) = 0, \quad s \in [0, 1] \quad (10.11)$$

obtained by differentiating (10.10) and recalling that $m = 2D\pi_\epsilon(\theta) =: 2\Psi(\theta)$, where the explicit expression of $2\Psi = 2(\Psi_+(\theta), \Psi_-(\theta))$ is given by the r.h.s. of (10.9).

We will now prove that $\lambda(s)$ is differentiable and its derivative has order γ^α . We proceed by supposing that λ is differentiable and get a formula for its derivative. We will then check that the primitive for such an expression is indeed $\lambda(s)$. The formula will also show that $\lambda'(s)$ has order γ^α .

By (10.11) we first need to show that $\theta_{\lambda(s), s}$ is differentiable in s . But $\theta_{\lambda(s), s}(x)$ defined through (10.7) can be expressed as

$$\theta_{\lambda(s), s}(x) = \theta_{\lambda(s)}^{(0)} + s \theta_{\lambda(s)}^{(1)}(x), \quad (10.12)$$

where $\theta_{\lambda(s)}^{(0)} = \frac{1}{2}(\lambda + \bar{\kappa})$ and $\theta_{\lambda(s)}^{(1)}(x) = \frac{1}{2} \left[(\kappa(x) - \bar{\kappa}) + \sum_{y \neq x, y \in I} J_\gamma(x, y)m_{\lambda(s)}(y) \right]$, so that

$$|\theta_{\lambda(s)}^{(1)}(x)| \leq c\gamma^\alpha. \quad (10.13)$$

We have:

$$\frac{d\theta_{\lambda(s), s}}{ds}(x) = \partial_\lambda \theta_{\lambda(s), s}(x) \frac{d\lambda}{ds} + \frac{\partial \theta_{\lambda(s), s}}{\partial s}(x) \quad (10.14)$$

with:

$$\frac{\partial \theta_{\lambda(s), s}}{\partial s}(x) = \theta_{\lambda(s)}^{(1)}(x), \quad (10.15)$$

$$\partial_\lambda \theta_{\lambda(s), s}(x) = \partial_\lambda \theta^{(0)} + s \partial_\lambda \theta^{(1)} = \frac{1}{2} \mathbb{I} + s \sum_{y \in I} J_\gamma(x, y) D_\lambda m_\lambda, \quad (10.16)$$

where $\partial_\lambda := \left(\frac{\partial}{\partial \lambda_+}, \frac{\partial}{\partial \lambda_-} \right)$, $\mathbb{I}$ is the identity matrix, and m_λ is defined by equation

$$m(x) = \Psi(\tilde{\theta}_{\lambda, m}(x)), \quad (10.17)$$

with $\tilde{\theta}_{\lambda, m}$ as the same expression of θ (10.7) but as a function of λ, m , namely:

$$\tilde{\theta}_{\lambda, m}(x) = \frac{1}{2}(\lambda + \bar{\kappa}) - \frac{s}{2} \left[(\kappa(x) - \bar{\kappa}) + \sum_{y \neq x, y \in I} J_\gamma(x, y)m(y) \right]. \quad (10.18)$$

We postpone the proof that m_λ is differentiable in λ and there is a constant c so that:

$$|D_\lambda m| < c\gamma^\alpha \quad (10.19)$$

which, recalling (10.16), implies that there is a constant c_J so that:

$$\frac{1}{2} - c_J \gamma^\alpha < \|\partial_\lambda \theta\| < \frac{1}{2} + c_J \gamma^\alpha. \quad (10.20)$$

Going back to (10.11) and denoting by Υ_θ the operator that acts on a vector $v(x)$ as $\Upsilon_\theta v(x) := \sum_{x \in I} D_\theta \Psi(\theta_{\lambda(s),s}(x))v(x)$, we write in a compact form:

$$[\Upsilon_\theta \partial_\lambda \theta_{\lambda(s),s}] \frac{d\lambda}{ds} = -\Upsilon_\theta \theta_{\lambda(s),s}^{(1)}. \quad (10.21)$$

There are positive constants $c_\epsilon^\pm$ (see (10.9)) so that:

$$|I|c_\epsilon^- < \|\Upsilon_\theta\| < |I|c_\epsilon^+. \quad (10.22)$$

Then, by (10.20), $[\Upsilon_\theta \partial_\lambda \theta_{\lambda(s),s}]$ is invertible if γ is small enough. Hence $\frac{d\lambda}{ds}$ is well defined and

$$\left| \frac{d\lambda}{ds} \right| = \left| [\Upsilon_\theta \partial_\lambda \theta_{\lambda(s),s}]^{-1} \Upsilon_\theta \theta_{\lambda(s),s}^{(1)} \right| < c\gamma^\alpha. \quad (10.23)$$

The last inequality follows by (10.15).

Let us now prove (10.19). Differentiating (10.17):

$$D_\lambda m = D_{\tilde{\theta}} \Psi \left[\partial_\lambda \tilde{\theta} + \partial_m \tilde{\theta} \cdot D_\lambda m \right] \quad (10.24)$$

and so

$$\left[1 - D_{\tilde{\theta}} \Psi \cdot \partial_m \tilde{\theta} \right] D_\lambda m = D_{\tilde{\theta}} \Psi \cdot \partial_\lambda \tilde{\theta}. \quad (10.25)$$

By (10.18) we see that $\partial_\lambda \tilde{\theta} = 1/2\mathbb{I}$, and $D_m \tilde{\theta}$ is a smooth function order $o(s\gamma^\alpha)$. Then for γ small enough $\|D_\theta \Psi D_m \theta\| < 1$ and $\left[1 - D_\theta \Psi D_m \theta \right]$ is invertible and

$$D_\lambda m = \left[1 - D_\theta \Psi D_m \theta \right]^{-1} D_\theta \Psi \cdot \partial_\lambda \tilde{\theta} \quad (10.26)$$

is well defined and there is a constant c so that:

$$|D_\lambda m| < c\gamma^\alpha. \quad (10.27)$$

Appendix 4: A Contraction Property of the Mean Field Free Energy

In this section we shall prove Proposition 4. The basic bound comes from the analysis of the previous section. As we use here the variables (m_1, m_2) instead of (m_+, m_-) we need to translate the results into the new variables. The minimizer $m^{(\epsilon)} = (m_+^{(\epsilon)}, 0)$ becomes (m_ϵ, m_ϵ) , $m_\epsilon = m_+^{(\epsilon)}/2$. We have

Lemma 8 *Let G_{++} and G_{--} be given by (9.17–9.18), then*

$$\frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_i^2}(m_\epsilon, m_\epsilon) = G_{++} + G_{--}, \quad i = 1, 2, \quad \frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_2 \partial h_1}(m_\epsilon, m_\epsilon) = G_{++} - G_{--}. \quad (11.1)$$

Both $G_{++} + G_{--}$ and $G_{++} - G_{--}$ are non negative, and for $i = 1, 2$

$$\left| \frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_i^2}(m_\epsilon, m_\epsilon) \right| + \left| \frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_2 \partial h_1}(m_\epsilon, m_\epsilon) \right| = 2G_{++} \leq 1 - \frac{\epsilon}{2}. \quad (11.2)$$

Proof

$$\begin{aligned} \frac{\partial \hat{\pi}_\epsilon}{\partial h_1} &= \frac{\partial \pi_\epsilon}{\partial h_+} - \frac{\partial \pi_\epsilon}{\partial h_-}, \quad \frac{\partial \hat{\pi}_\epsilon}{\partial h_2} = \frac{\partial \pi_\epsilon}{\partial h_+} + \frac{\partial \pi_\epsilon}{\partial h_-} \\ \frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_1^2} &= \frac{\partial^2 \pi_\epsilon}{\partial h_+^2} + \frac{\partial^2 \pi_\epsilon}{\partial h_-^2} - 2 \frac{\partial^2 \pi_\epsilon}{\partial h_+ \partial h_-}, \quad \frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_2 \partial h_1} = \frac{\partial^2 \pi_\epsilon}{\partial h_+^2} - \frac{\partial^2 \pi_\epsilon}{\partial h_-^2}, \end{aligned}$$

hence (11.1) because $\frac{\partial^2 \pi_\epsilon}{\partial h_+ \partial h_-}(2m_\epsilon, 0) = 0$. $\square$

It follows by continuity that:

Corollary 5 *There are $c_0 > 0$ and $r < 1$ so that the following holds. Call*

$$R_{i,j} = \sup_{h=(h_1, h_2): |h_i - m_\epsilon| \leq c_0, i=1,2} \left| \frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_i \partial h_j}(h) \right|. \quad (11.3)$$

Then

$$\sum_{j=1,2} R_{i,j} \leq r, \quad i = 1, 2 \quad (11.4)$$

and the matrix $(1 - R)$ is invertible.

We are now ready for the proof of Proposition 4. We fix throughout the sequel a pair (x, i) and (x, i') of vertically interacting sites in Δ_{in} , thus $(x, i') = v_{x,i}$ and use the above properties to study the function $g_\epsilon(m)$ introduced in (5.4). We write $m = (m_i, m_{i'})$ and write $a_i = a_{x,i}$, $a_{i'} = a_{x,i'}$, the latter defined in (4.8). We also write λ_i and $\lambda_{i'}$ dropping the superscript u on which they depend via (5.5). We finally shorthand a_j for $a_{x,j}$.

Lemma 9 *There is a constant c so that for any $u \in \mathcal{N}_{x,i,i'}$, see (5.3),*

$$\frac{|\lambda_j - \lambda_j^{\text{eq}}|}{1 - a_j} \leq \zeta + c\gamma^\alpha, \quad j = i, i', \quad (11.5)$$

where λ_j^{eq} is defined in Proposition 4.

Proof By (5.5)

$$\begin{aligned} \lambda_j - \lambda_j^{\text{eq}} &= \sum_{y \neq x: (y,j) \in C_x^{\ell-,j}} J_\gamma(x, y)(u(y, j) - m_\epsilon) \\ &\quad + \sum_{y: (y,j) \notin \{C_x^{\ell-,j} \cup \Delta_0\}} J_\gamma(x, y)(u(y, j) - m_\epsilon). \end{aligned}$$

We add and subtract $\hat{J}_\gamma(x, y)$ to $J_\gamma(x, y)$ where $\hat{J}_\gamma(x, y)$ is obtained by averaging $J_\gamma(x, y')$ over $C_y^{\ell-,j}$. In the term with $\hat{J}_\gamma(x, y)$ we can replace $u(y, j)$ by its average and use (5.3), (4.8) and that J_γ is a probability kernel to get the bound $(1 - a_j)\zeta$. The sum over the terms with $\hat{J}_\gamma(x, y) - J_\gamma(x, y)$ is bounded by $c'\gamma^\alpha$, by the smoothness of J_γ . $\square$

In the sequel we shall only use the bound (11.5) and not the specific form of the λ_j .

By differentiating (5.4) we get

$$\frac{\partial^2 g_\epsilon}{\partial m_j \partial m_{j'}} = \frac{\partial^2 \hat{\phi}_\epsilon}{\partial m_j \partial m_{j'}} - a_j \mathbf{1}_{j=j'}, \quad j, j' \in \{i, i'\}.$$

At $\epsilon = 0$ $\frac{\partial^2 g_0}{\partial m_j \partial m_{j'}}$ is diagonal with entries $-I''(m_j) - a_j$, $j = i, i'$. The minimum of $-I''$ is at 0 and $-I''(0) = 1$. Since $a_j \leq 1/2$ (this follows from the choice of x and the symmetry of J ; see (4.8)), we then conclude that:

Lemma 10 *There is $c_1 > 0$ so that g_ϵ is strictly convex for $\epsilon \leq c_1$ and for any such ϵ it has a unique minimizer $\tilde{m}$ (called $m^{(u)}$ in Proposition 4).*

In the sequel we tacitly suppose $\epsilon \leq c_1$. The critical point m of g_ϵ satisfies

$$\hat{D}\hat{\phi}_\epsilon(m) = \hat{\theta} := (a_i m_i + \lambda_i, a_{i'} m_{i'} + \lambda_{i'}). \quad (11.6)$$

Then, by Lemma 3,

$$m = \hat{D}_h \hat{\pi}_\epsilon(\hat{\theta}), \quad (11.7)$$

where $\hat{D}_h \hat{\pi}_\epsilon(\hat{\theta})$ is the gradient of $\hat{\pi}_\epsilon(h_1, h_2)$ computed at $(h_1, h_2) = \hat{\theta}$. We shall study (11.7) distinguishing among the possible values of a_i and $a_{i'}$ which depend on the horizontal distance of (x, i) and respectively (x, i') from Δ_0 . Because of the geometric properties of Δ_{in} , only three cases can occur: (i) $a_i = a_{i'} = 0$; (ii) $a_i = a_{i'} \in (0, \frac{1}{2}]$; (iii) $a_i \in (0, \frac{1}{2}]$ and $a_{i'} = 0$ or viceversa. Case (i) occurs when the horizontal distances of (x, i) and (x, i') from Δ_0 are both $> \gamma^{-1}$. Case (ii) is when the distances of (x, i) and (x, i') from Δ_0 are both $\leq \gamma^{-1}$ and case (iii) is when one is $\leq \gamma^{-1}$ and the other $> \gamma^{-1}$. We start from case (i) which is the easiest.

Case (i). By (11.5) for γ small enough $|\hat{\theta}_j - m_\epsilon| < c_0$, c_0 as in Corollary 5, hence by (11.5)

$$|m_j - m_\epsilon| \leq \sum_{j'} R_{j,j'} \left| \lambda_{j'} - \lambda_{j'}^{\text{eq}} \right|, \quad \sum_{j'} R_{j,j'} \leq r < 1$$

in agreement with in Proposition 4 after setting $C_{x,i,i'}(j, j') = R_{j,j'}$ and recalling that in case (i) $a_j = 0$, $j = i, i'$.

Case (ii). $\hat{\theta}$ in (11.7) is now (after adding and subtracting m_ϵ)

$$\hat{\theta}_j = m_\epsilon + a(m_j - m_\epsilon) + \lambda_j - (1 - a)m_\epsilon, \quad j = i, i'. \quad (11.8)$$

Since $\hat{\theta}$ depends on m , (11.7) is an equation in m and not a formula for m as in case (i). We introduce an interpolating parameter $t \in [0, 1]$ and define

$$\hat{\theta}_j(t) = m_\epsilon + a(m_j - m_\epsilon) + t(\lambda_j - (1 - a)m_\epsilon), \quad j = 1, 2 \quad (11.9)$$

calling $m(t)$ the solution of (11.7) with $\hat{\theta}$ replaced by $\hat{\theta}(t)$. Observe that $m(0) = m_\epsilon$ is the solution at $t = 0$ while the solution at $t = 1$ is what we want to find because $\hat{\theta}(1) = \hat{\theta}$.

Supposing that $m(t)$ and its derivative $\dot{m}(t)$ exist we can then differentiate (11.7) to get

$$\dot{m}_j = \sum_{p=i,i'} K_{jp} \{a \dot{m}_p + (\lambda_p - (1 - a)m_\epsilon)\}, \quad K_{jp} = \frac{\partial^2 \hat{\pi}_\epsilon}{\partial h_j \partial h_p}(\hat{\theta}(t)), \quad (11.10)$$

where $\hat{\theta}(t)$ is computed at $m = m(t)$. If moreover $|m(t) - m_\epsilon| \leq 2\zeta$ then $|\hat{\theta}_j(t) - m_\epsilon| \leq 2\zeta$ and $1 - aK$ is invertible and we have

$$\dot{m} = V(m, t) := (1 - aK)^{-1} K(\lambda - (1 - a)u^{\text{eq}}), \quad u^{\text{eq}} = (m_\epsilon, m_\epsilon), \quad \lambda = (\lambda_i, \lambda_{i'}). \quad (11.11)$$

The evolution equation (11.11) starting from $m(0) = m_\epsilon$ has a unique solution till the first time T when $|m_j(T) - m_\epsilon| = c_0$, because by Corollary 5, $(1 - K(t))$ is invertible and smooth for $t \leq T$ and we have

$$|m_j(t) - m_\epsilon| \leq t \sum_{n=0}^{\infty} a^n \sum_{p=i, i'} (R^{n+1})_{jp} |\lambda_p - (1 - a)m_\epsilon|. \quad (11.12)$$

Set

$$C_{x, i, i'}(j, j') = (1 - a) \sum_{n=0}^{\infty} a^n (R^{n+1})_{j, j'}.$$

Then, by (11.4), we get

$$\sum_{j'=i, i'} C_{x, i, i'}(j, j') \leq (1 - a) \frac{r}{1 - ar} < r, \quad j = i, i'$$

Thus $|m_j(t) - m_\epsilon| \leq 2\zeta$ for $t \leq \min\{T, 1\}$, hence the above holds till $t = 1$ and Proposition 4 is proved in case (ii).

Case (iii) with $a_i = a > 0$ and $a_{i'} = 0$ (same proof applies when $a_i = 0$ and $a_{i'} > 0$). Here

$$\hat{\theta} = (m_\epsilon + a(m_i - m_\epsilon) + \lambda_i - (1 - a)m_\epsilon, m_\epsilon + (\lambda_{i'} - m_\epsilon))$$

and proceeding as in case (ii) we set

$$\hat{\theta}(t) = (m_\epsilon + a(m_i - m_\epsilon) + t[\lambda_i - (1 - a)m_\epsilon], m_\epsilon + t(\lambda_{i'} - m_\epsilon)). \quad (11.13)$$

Analogously to (11.10),

$$\dot{m}_i = K_{i,i} a \dot{m}_i + \{K_{i,i}(\lambda_i - (1 - a)m_\epsilon) + K_{i,i'}(\lambda_{i'} - m_\epsilon)\}.$$

Since $K_{i,i} a < 1$ till when $|m_i(t) - m_\epsilon| < 2\zeta$ proceeding as in case (ii) we get that the evolution equation has solution till time $t = 1$ and

$$|m_i(1) - m_\epsilon| \leq \sum_{n=0}^{\infty} (aR_{i,i})^n \{R_{i,i} |\lambda_i - (1 - a)m_\epsilon| + R_{i,i'} |\lambda_{i'} - m_\epsilon|\}. \quad (11.14)$$

We then set:

$$C_{x, i, i'}(i, i) = \frac{R_{i,i}(1 - a)}{1 - aR_{i,i}}, \quad C_{x, i, i'}(i, i') = \frac{R_{i,i'}}{1 - aR_{i,i}} \quad (11.15)$$

which verifies the condition in Proposition 4 because

$$\sum_{j=i, i'} C_{x, i, i'}(i, j) = \frac{R_{i,i}(1 - a)}{1 - aR_{i,i}} + \frac{R_{i,i'}}{1 - aR_{i,i}} \leq \frac{r - aR_{i,i}}{1 - aR_{i,i}} < r.$$

Since $m_{i'} = \frac{\partial \hat{\pi}_\epsilon}{\partial h_{i'}}(\hat{\theta})$,

$$|m_{i'} - m_\epsilon| \leq R_{i', i'} |\lambda_{i'} - m_\epsilon| + R_{i', i} (a|m_i - m_\epsilon| + |\lambda_i - (1 - a)m_\epsilon|).$$

We then set

$$\begin{aligned}
C_{x,i,i'}(i', i') &= R_{i',i'} + a R_{i',i} \frac{R_{i,i'}}{1 - a R_{i,i}} \\
C_{x,i,i'}(i', i) &= R_{i',i} \left(a \frac{R_{i,i}(1-a)}{1 - a R_{i,i}} + 1 - a \right)
\end{aligned} \tag{11.16}$$

and

$$\begin{aligned}
\sum_{j=i,i'} C_{x,i,i'}(i', j) &= R_{i',i'} + \frac{a R_{i',i} R_{i,i'}}{1 - a R_{i,i}} + R_{i',i} \left(\frac{a R_{i,i}(1-a)}{1 - a R_{i,i}} + (1-a) \right) \\
&\leq R_{i',i'} + R_{i',i} - a R_{i',i} \left(1 - \frac{R_{i,i'}}{1 - a R_{i,i}} - \frac{R_{i,i}(1-a)}{1 - a R_{i,i}} \right) \\
&\leq r - a R_{i',i} \left(1 - \frac{R_{i,i'} + R_{i,i}(1-a)}{1 - a R_{i,i}} \right) < r
\end{aligned}$$

having bounded in the last bracket $R_{i,i'} + R_{i,i} < 1$.

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