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CHAPTER 11

Dimension Walks on Generalized Spaces

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This paper is dedicated to Daryl J. Daley and Robert Schaback. Such beautiful minds.

There has been an increasing interest in stochastic processes that are defined over product spaces in many branches of applied sciences, including climate modeling, atmospheric sciences, geophysical science, and even finance. Our work provides the mathematical foundations for certain classes of stochastic processes that are defined over special classes of product spaces. Specifically, let $\boldsymbol{d}, \boldsymbol{k}$ be positive integers. We call generalized spaces the Cartesian product of the $\boldsymbol{d}$-dimensional sphere, $\mathbb{S}^{\boldsymbol{d}}$, with the $\boldsymbol{k}$-dimensional Euclidean space, $\mathbb{R}^{\boldsymbol{k}}$. We consider the class $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ of continuous functions $\varphi : [\mathbf{-1}, \mathbf{1}] \times [\mathbf{0}, \infty) \to \mathbb{R}$ such that the mapping $C : \left(\mathbb{S}^d \times \mathbb{R}^k\right)^2 \to \mathbb{R}$, defined as $C\left((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')\right) = \varphi\left(\cos\theta\left(x, x'\right), \|y - y'\|\right), (x, y), \ \left(x', y'\right) \in \mathbb{S}^d \times \mathbb{R}^k$, is positive definite. We propose linear operators that allow for walks through dimensions within generalized spaces while preserving positive definiteness.

Keywords: Positive Definite Functions, Montée Operators, Descente Operators, Spheres, Euclidean Spaces, Generalized Product.

11.1. Introduction

11.1.1. *Context*

The paper deals with positive definite functions over what we term *generalized spaces*, that is, product spaces that involve manifolds of different natures. In fact, those are defined here as the Cartesian product of the k-dimensional Euclidean space with a d-dimensional unit sphere. Albeit this work is of clear mathematical essence, it provides the foundations of stochastic processes over product spaces as certified by the increasing interest in several branches of applied sciences. Some motivations to consider the present framework are listed below.

(a) There has been an increasing interest from several branches of statistics, machine learning, and finance, for positive definite functions defined over these product spaces, and the reader is referred to the recent review by Porcu et al. [1]. The applications to real cases are ubiquitous, ranging from climate and atmospheric sciences to deep learning on manifolds.

As far as finance is concerned, Gaussian processes play a central role in financial modeling. The field of econometrics has devoted much effort to the modeling of financial time series [2]. The Brownian motion is essential to the pricing of financial derivatives [3]. The Ornstein–Uhlenbeck process is often used to develop investment and trading strategies [4]. Gaussian processes are one-dimensional applications of the general concept of Gaussian random field (GRF). We motivate some of the uses of GRFs in finance. A key feature of financial datasets is time and spatial dependence. Coetaneous observations from variables in close proximity tend to be more similar. For example, returns from US stocks last month are more similar to returns from US stocks this month than returns from US stocks 1 year ago or returns from Chinese stocks last month. Willinger et al. [5] noted that a Brownian motion with drift does not replicate the time dependence observed in asset returns. Not surprisingly, GRFs have attracted considerable interest among researchers interested in modeling the joint time-space dynamics of financial processes. To cite a few examples, Refs. [6] and [7] modeled the term structure of interest rates as a two-dimensional random field. In their models, time increments are independent, while the correlation structure between bond yields of different maturities can be modeled with great flexibility. Kimmel [8] enhanced this approach by adding a state-dependent volatility. Albeverio et al. [9] introduced Lévy fields to the modeling of yield curves. Özkan and Schmidt [10] applied random fields to incorporate credit risk into the modeling of yield curves. As important as the term structure of interest rates is, it is not the only financial application of Gaussian fields. At least two further applications stand out: option pricing and actuarial modeling. For example, Hainaut [11] proposes an alternative model for asset prices with sub-exponential, exponential, and hyper-exponential autocovariance structures. Hainaut sees price processes as conditional Gaussian fields indexed by time. Under this framework, option prices can be computed using the technique of the change of numeraire. Biffis and Millossovich [12] applied random fields to modeling the intensity of mortality in an attempt to incorporate cross-generation effects. Biagini et al. [13] built on that work to price and hedge life insurance liabilities.

(b) Several branches of spatial statistics and computer sciences are interested in the simulation of random processes defined over generalized spaces, and we refer the reader to Ref. [14]. It turns out that the use of these operators becomes crucial when associated with turning band techniques [15], which allow for simulation of a given random process from projections on lower dimensional spaces.

(c) Projection operators for radial positive definite functions allowed to build positive definite functions that are compactly supported over balls embedded in k-dimensional Euclidean spaces. This inspired a fertile literature from spatial statistics with the goal of achieving accurate estimates while allowing for computational scalability. For instance, the tapering approach [16] is substantially based on this idea.

(d) There is a fertile literature from projection operators for symmetric (or radially symmetric) distributions, where radial symmetry is intended with respect to the composition of a given candidate function with the classical α-norms [17].

11.1.2. *Literature review*

Let $\mathbb{R}^k$ denote the k-dimensional Euclidean space, and let $\mathbb{S}^d$ be the d-dimensional unit sphere embedded in $\mathbb{R}^{d+1}$. Let $\|\cdot\|$ denote Euclidean distance and $\theta(x, y) := \arccos(\langle x, y\rangle)$ denote the geodesic distance in $\mathbb{S}^d$, with $\langle .,.\rangle$ denoting the dot product in $\mathbb{R}^{d+1}$. A continuous function $C : \mathbb{R}^k \to \mathbb{R}$ is called radially symmetric if there exists a continuous function $f : [0, \infty) \to \mathbb{R}$ such that $C(x) = f\circ\|x\|, x \in \mathbb{R}^k$, with $\circ$ denoting composition. The function f is called the radial part of C. Radial symmetry is known as *isotropy* in spatial statistics [18]. A function $C : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ is called geodesically isotropic if $C(x, y) = g \circ \theta(x, y)$ for some continuos function $g : [0, \pi] \to \mathbb{R}$.

Positive definite functions that are radially symmetric over k-dimensional Euclidean spaces have a long history that can be traced back to Ref. [19]. Projection operators that map a positive definite radial mapping from $\mathbb{R}^k$ into $\mathbb{R}^{k\pm h}$, for h a positive integer, have been considered in Matheron's clavier spherique [15, 20]. Matheron coined the terms *descente* and *montée* to define special operators that will be described throughout. The terms originate from an appealing physical interpretation in a mining context. These projection operators have then been investigated by Ref. [21], and subsequently by Refs. [22–24] in the context of positive definite radial functions that are additionally compactly supported on balls embedded in $\mathbb{R}^k$ with given radii. The work by Daley and Porcu [18] provides a general perspective of such operators, in concert with some generalizations of the previously mentioned works. These linear operators have turned out to be very useful to establish criteria of the Pólya type for radially symmetric positive definite functions [25], as well as in the definition of multiradial positive definite functions [26]. In probability theory, similar projection operators turned useful in the seminal paper by Ref. [17] and in Ref. [27].

Positive definite functions that are geodesically isotropic on d-dimensional spheres have been characterized in Ref. [28]. Projection operators for this class of functions have been studied to a limited extent only, and we refer to the recent papers by Ref. [29] and more recently to the same authors [30, 31]. Properties of these operators have then been inspected in Ref. [32].

11.1.3. *The problem and our contribution*

The characterization of projection operators on product spaces of the type $\mathbb{S}^d\times\mathbb{R}^k$ has been elusive so far. The only exception is Ref. [33], who consider the product space $\mathbb{S}^d \times \mathbb{R}$, and projections that are defined marginally for the sphere only.

Our paper contributes to the literature as follows. In Section 11.2, we provide the notations and basic literature. In Section 11.3, we define the Descente and Montée operators on the generalized space $\mathbb{S}^d \times \mathbb{R}^k$. The main results are statement in Section 11.4, and their proofs are in Appendix A.

11.2. Notations and Background

Let X, Y be nonempty sets. A function $C : (X \times Y)^2 \to \mathbb{R}$ is called positive definite if, for any finite system $\{a_k\}_{k=1}^N \subset \mathbb{R}$ and points $\{(x_k, y_k)\}_{k=1}^N \subset X \times Y$, the following inequality is preserved:

$$\sum_{k=1}^{N} \sum_{h=1}^{N} a_k C\left((x_k, y_k), (x_h, y_h)\right) a_h \geq 0.$$

We deal with the case $X = \mathbb{S}^d$ and $Y = \mathbb{R}^k$, for d and k being positive integers. Additionally, we suppose C to be continuous and that there exists a continuous function $\varphi : [-1, 1] \times [0, \infty) \to \mathbb{R}$ such that

$$C\left((x, y), (x', y')\right) = \varphi\left(\cos\theta\left(x, x'\right), \|y - y'\|\right), (x, y), \left(x', y'\right) \in \mathbb{S}^d \times \mathbb{R}^k. \tag{11.1}$$

We call $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ the class of such functions, φ. Analogously, we call $\mathcal{P}\left(\mathbb{S}^d\right)$ the class of continuous functions $\psi : [-1, 1] \to \mathbb{R}$ such that for the function C in Eq. (11.1) it is true that, for $y = y'$, $C\left((x, y), \left(x', y\right)\right) = \varphi\left(\cos\theta\left(x, x'\right), 0\right) = \psi\left(\cos\theta\left(x, x'\right)\right)$. The class $\mathcal{P}\left(\mathbb{R}^k\right)$ is defined analogously. The classes $\mathcal{P}\left(\mathbb{R}^k\right)$ and $\mathcal{P}\left(\mathbb{S}^d\right)$ have been characterized by Refs. [19] and [28], respectively. The class $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ has been characterized by Ref. [34] through a uniquely determined expansion of the type

$$\varphi(x, t) = \sum_{n=0}^{\infty} f_n^d(t) C_n^{(d-1)/2}(x), (x, t) \in [-1, 1] \times [0, \infty),$$

where the functions f_n^d belong to $\mathcal{P}\left(\mathbb{R}^k\right)$, $n \in \mathbb{Z}_+$, and

$$\sum_{n=0}^{\infty} f_n^d\,(0)\, C_n^{(d-1)/2}\,(1) < \infty. \tag{11.2}$$

The expansion above is uniformly convergent on $[-1, 1] \times [0, \infty)$. The coefficients functions f_n^d are called d-Schoenberg functions of φ. The functions $C_n^{(d-1)/2}$ are the Gegenbauer polynomials of degree n associated with the index $(d-1)/2$ [35].

Proposition 3.8 in Ref. [34] shows that if φ belongs to the class $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$, then it is continuously differentiable with respect to the first variable.

It is also important to note that a continuous function $x \in [-1, 1] \mapsto \varphi(x, t)$ has an Abel summable expansion for each $t \in [0, \infty)$ in the form [see the proof of Theorem 3.3 in Ref. [34]

$$\varphi\,(x, t) \sim \sum_{n=0}^{\infty} f_n^d\,(t)\, C_n^{(d-1)/2}\,(x), \tag{11.3}$$

where

$$f_n^d\,(t) = \varsigma_n^d \int_{-1}^{1} \varphi\,(x, t)\, C_n^{(d-1)/2}\,(x)\left(1 - x^2\right)^{d/2-1} dx, \tag{11.4}$$

and ς_n^d are positive constants.

11.2.1. *Some useful facts*

Arguments in Ref. [19] prove that, for every $n = 0, 1, \ldots$, each function $f_n^d \in \mathcal{P}\left(\mathbb{R}^k\right)$ in Eq. (11.2) admits a uniquely determined Riemann–Stieltjes integral representation of the form

$$f_n^d(t) = \int_0^\infty \Omega_k(tr)\, dF_n(r), \quad t \in [0, \infty), \tag{11.5}$$

where F_n is a non-negative bounded measure on $[0, \infty)$. The function $\Omega_k : [0, \infty) \to \mathbb{R}$ is given by

$$\Omega_k(t) = \Gamma\left(\frac{k}{2}\right)\left(\frac{2}{t}\right)^{(k-2)/2} J_{(k-2)/2}(t), \tag{11.6}$$

where J_ν is the Bessel function of the first kind of order ν given by

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{t}{2}\right)^{2m}.$$

We follow Ref. [18], and we call F_n the k-Schoenberg measure of f_n^d. We also note that we are abusing notation when writing F_n instead of F_n^d. This last notation will not be used unless explicitly needed.

Some technicalities will be exposed here to allow for a neater exposition. The derivative function of the function Ω_k is uniformly bounded, and it is given by (see Refs. [18, 24, 26]).

$$\frac{d\Omega_k}{dt}(t) = \Omega_k'(t) = -\frac{1}{k} t\Omega_{k+2}(t), \quad t \geq 0. \tag{11.7}$$

Also,

$$\left|\Omega_k(t)\right| < 1 = \Omega_k(0), \quad t > 0. \tag{11.8}$$

Since $\lim_{t\to\infty}\Omega_k(t) = 0$ for $k > 0$ (see Ref. [18]), we have

$$\int_t^\infty u\Omega_k(u)\, du = (k-2)\,\Omega_{k-2}(t), \quad t \geq 0. \tag{11.9}$$

Some properties of Gegenbauer polynomials will turn out to be useful throughout. For instance, we can invoke 4.7.14 in Ref. [35] to infer that

$$\frac{dC_n^\lambda}{dx}(x) = \left(C_n^\lambda\right)'(x) = \delta_\lambda C_{n-1}^{\lambda+1}(x), \quad -1 \leq x \leq 1, \tag{11.10}$$

and, as a consequence

$$\int_{-1}^x C_n^\lambda(x)\, dx = \frac{1}{\delta_\lambda}\left(C_{n+1}^{\lambda-1}(x) - C_{n+1}^{\lambda-1}(-1)\right), \tag{11.11}$$

where

$$\delta_\lambda = \begin{cases} 2\lambda, & \lambda > -1/2(\lambda \neq 0), \\ 2, & \lambda = 0. \end{cases} \tag{11.12}$$

Theorem 7.32.1 and Equation 4.7.3 in Ref. [35] show that, for $\lambda > -1/2$,

$$\left|C_n^{\lambda}(x)\right| \leq C_n^{\lambda}(1) = \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)\Gamma(2\lambda)}, \quad x \in [-1,1]. \tag{11.13}$$

Also, it is true that

$$\frac{C_{n+j}^{\lambda-k}(1)}{C_n^{\lambda}(1)} \leq \frac{\Gamma(2\lambda)}{\Gamma(2\lambda-2k)} := \varrho_{\lambda,k}, \quad \forall n \in \mathbb{Z}_+. \tag{11.14}$$

The following inequality (see Ref. [36]) will be repeatedly used in the manuscript:

$$|f(t)| \leq f(0), \quad t \in [0,\infty), \quad f \in \mathcal{P}\left(\mathbb{R}^k\right).$$

We will also make use of the following fact: if $\varphi : [-1,1] \times \mathbb{R} \to \mathbb{R}$ has a derivative φ_x with respect to the first variable for each $t \in [0,\infty)$ and if both functions have Gegenbauer expansions of the form

$$\varphi_x(x,t) \sim \sum_{n=0}^{\infty} f_n^{\lambda}(t)\, C_n^{\lambda}(x), \quad \varphi_x(x,t) \sim \sum_{n=0}^{\infty} \tilde{f}_n^{\lambda+1}(t)\, C_n^{\lambda+1}(x), \tag{11.15}$$

$(x,t) \in [-1,1] \times [0,\infty)$, then

$$\tilde{f}_{n-1}^{\lambda+1}(t) = \delta_{\lambda} f_n^{\lambda}(t), \qquad n \in \mathbb{Z}_+^*, \quad \lambda > 0. \tag{11.16}$$

The proof is very similar to the Proof of Lemma 2.4 in Ref. [30] and we omit it for the sake of brevity.

11.3. An Historical Account on *Montée* and *Descente* Operators

Beatson and zu Castell [31] defined the Descente and Montée operators for the class $\mathcal{P}\left(\mathbb{S}^d\right)$. Specifically, the Descente $\mathcal{D}$ is defined as

$$(\mathcal{D}f)(x) = \frac{d}{dx} f(x) = f'(x), \quad x \in [-1,1],$$

provided such a derivative exists. The Montée $\mathcal{I}$ is instead defined as

$$(\mathcal{I}f)(x) = \int_{-1}^{x} f(u)\, du, \quad x \in [-1,1].$$

Beatson and zu Castell [31] has shown that $f \in \mathcal{P}\left(\mathbb{S}^{d+2}\right)$ implies that there exists a constant, $\varkappa$, such that $\varkappa + \mathcal{I}f \in \mathcal{P}\left(\mathbb{S}^d\right)$. Also, $f \in \mathcal{P}\left(\mathbb{S}^d\right)$ implies $\mathcal{D}f \in \mathcal{P}\left(\mathbb{S}^{d+2}\right)$. The implications in terms of differentiability at $x = 1$ are nicely summarized therein.

The *tour de force* by Ref. [31] has then been generalized by Ref. [33]: let $d \in \mathbb{N}$ and $\varphi : [-1,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. The Montée $\mathcal{I}$ and Descente $\mathcal{D}$ operators are defined, respectively, by

$$\mathcal{I}(\varphi)(x,t) := \int_{-1}^{x} \varphi(u,t)\, du, \quad (x,t) \in [-1,1] \times [0,\infty), \tag{11.17}$$

when *f* is integrable with respect to the first variable, and

$$\mathcal{D}\left(\varphi\right)(x,t) := \frac{\partial \varphi}{\partial x}(x,t), \quad (x,t) \in [-1,1] \times [0,\infty). \tag{11.18}$$

They prove that if $\varphi \in \mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}\right)$, then $\mathcal{D}\varphi \in \mathcal{P}\left(\mathbb{S}^{d+2} \times \mathbb{R}\right)$, and in their correction of Theorem 2.1, they provided conditions under $\varphi \in \mathcal{P}\left(\mathbb{S}^{d+2} \times \mathbb{R}\right)$ such that $\mathcal{I}\varphi \in \mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}\right)$.

Montée and Descente operators with the class $\mathcal{P}\left(\mathbb{R}^k\right)$ have been defined much earlier, and we follow Ref. [24] to summarize them here. The Descente and Montée operators are respectively defined as

$$\mathcal{D}\varphi(t) = \begin{cases} 1, & t = 0 \\ \dfrac{\varphi'(t)}{t\varphi''(0)}, & t > 0, \end{cases} \tag{11.19}$$

where $\varphi''(0)$ denotes the second derivative of φ evaluated at $t = 0$, and

$$\widetilde{\mathcal{I}}\varphi(t) = \int_t^\infty u\varphi(u)\,du \left(\int_0^\infty u\varphi(u)\,du\right)^{-1}. \tag{11.20}$$

Gneiting [24] proved that if $\varphi \in \mathcal{P}\left(\mathbb{R}^k\right), k \geq 3$, and $u\varphi(u)$ is integrable over $[0,\infty)$, then $\widetilde{\mathcal{I}}\varphi \in \mathcal{P}\left(\mathbb{R}^{k-2}\right)$. Invoking standard properties of Bessel functions in concert with direct inspection, Ref. [24] proved that, if $\varphi \in \mathcal{P}\left(\mathbb{R}^k\right)$ and $\varphi''(0)$ exists, then $\mathcal{D}\varphi \in \mathcal{P}\left(\mathbb{R}^{k+2}\right)$. Under mild regularity conditions, the operators $\mathcal{D}$ and $\widetilde{\mathcal{I}}$ are inverse operators:

$$\widetilde{\mathcal{I}}\left(\mathcal{D}\varphi\right) = \mathcal{D}\left(\widetilde{\mathcal{I}}\varphi\right) = \varphi.$$

11.3.1. *Descente and Montée operators on generalized spaces*

We start by defining the following Descente and Montée operators. The first is actually taken from Ref. [33]: we define the derivate operator D_1 by

$$D_1\varphi(x,t) := \varphi_x(x,t) = \frac{\partial \varphi}{\partial x}(x,t), \quad (x,t) \in [-1,1] \times [0,\infty). \tag{11.21}$$

The integral operator I_1 is given by

$$I_1\varphi(x,t) := \int_{-1}^{x} \varphi(u,t)\,du, \quad (x,t) \in [-1,1] \times [0,\infty), \tag{11.22}$$

when $\varphi(u,t)$ is integrable over $[-1,1]$ for each $t \in [0,\infty)$.

We define

$$D_2\varphi(x,t) := \begin{cases} 1, & (x,t) = (1,0) \\ \dfrac{\varphi_t(x,t)}{t\varphi_{tt}(1,0)}, & (x,t) \in [-1,1) \times (0,\infty) \end{cases} \tag{11.23}$$

whenever $\varphi_{tt}(1,0) := \frac{\partial^2\varphi}{\partial t^2}(1,0)$ exists, and

$$I_2\varphi(x,t) := \frac{\int_t^\infty v\varphi(x,v)\,dv}{\int_0^\infty v\varphi(1,v)\,dv}, \quad (x,t) \in [-1,1]\times[0,\infty), \tag{11.24}$$

when $v\varphi(1,v)$ is integrable over $[0,\infty)$ and provided the denominator is not identically equal to zero.

The composition between the operators defined in Refs. [33] and [24] provides a new operator, which we define here as

$$I_3\varphi(x,t) := \int_t^\infty \int_{-1}^x v\varphi(u,v)\,dudv, \qquad (x,t) \in [-1,1]\times[0,\infty), \tag{11.25}$$

when $v\varphi(u,v)$ is integrable over $[-1,1]\times[0,\infty)$.

Given $\kappa \in \mathbb{Z}_+$, we define the operator I_j^κ by recurrence as:

$$I_j^0\varphi := \varphi, \quad I_j^1\varphi := I_j\varphi, \quad \text{and} \quad I_j^\kappa\varphi := I_j\left(I_j^{\kappa-1}\varphi\right), \quad j = 1,2,3.$$

11.4. Dimension Walks within the Class $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$

This section contains our original findings. Proofs are deferred to the Appendix.

11.4.1. *Descente operators*

We start with a simple result, which is an extension of Theorem 2.3 in Ref. [30]. In Appendix A, we provide a quick sketch of the main steps.

Theorem 11.4.1. *If $\varphi : [-1,1]\times[0,\infty) \to \mathbb{R}$ belongs to $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$, then $D_1\varphi$ belongs to $\mathcal{P}\left(\mathbb{S}^{d+2} \times \mathbb{R}^k\right)$.*

The next result requires instead a lengthy proof and relates about the operator D_2.

Theorem 11.4.2. *Let $d,k \in \mathbb{Z}_+^*$, $\varphi : [-1,1]\times[0,\infty) \to \mathbb{R}$ be a function in $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ and let F_n be the k-Schoenberg measures associated with the d-Schoenberg functions of φ. If*

(1) $\int_0^\infty r^2 dF_n(r) < \infty$, *for all $n \in \mathbb{Z}_+$;*

(2) $0 < \frac{\partial^2\varphi}{\partial t^2}(1,0) = \sum_{n=0}^\infty \int_0^\infty r^2 dF_n(r) < \infty$,

then $D_2\varphi$ belongs to $\mathcal{P}\left(\mathbb{S}^{d+2} \times \mathbb{R}^k\right)$.

11.4.2. *Montée operators*

In this section we consider functions $\varphi : [-1,1] \times [0,\infty) \to \mathbb{R}$ belonging to $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ as in Eq. (11.2) such that

$$\int_0^\infty (1/r^{2\kappa}) dF_n(r) < \infty \quad \left(\kappa \in \mathbb{Z}_+^*\right). \tag{11.26}$$

Thus, the functions defined by

$$g_n^\kappa(t) := \int_0^\infty \Omega_{k-2\kappa}(tr) \frac{1}{r^{2\kappa}} dF_n(r), \quad t \in [-1,1], \quad n,\kappa \in \mathbb{Z}_+, \tag{11.27}$$

belong to the class $\mathcal{P}\left(\mathbb{R}^{k-2\kappa}\right)$.

The first finding relates to the operator I_1. Again, the proof is deferred to the Appendix.

Theorem 11.4.3. *Let $k,\kappa \in \mathbb{Z}_+^*$ and d be an integer such that $d > 2\kappa$. If $\varphi : [-1,1] \times [0,\infty) \to \mathbb{R}$ is a function in $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ such that $u \mapsto I_1^{\kappa-1}\varphi(u,t)$ is integrable over $[-1,1]$ for each $t \in [0,\infty)$. Then, the function $I_1^\kappa \varphi$ has a representation in the form of a Gegenbauer series:*

$$I_1^\kappa(x,t) = \sum_{n=0}^\infty \tilde{f}_n^{d,\kappa}(t)\, C_n^{(d-2\kappa-1)/2}(x), \quad (x,t) \in [-1,1] \times [0,\infty), \tag{11.28}$$

where

$$\tilde{f}_n^{d,\kappa}(t) := \begin{cases} \tau^{d,\kappa} \sum_{i=0}^\infty (-1)^i \chi_i^{n,d,\kappa} f_i^d(t), & n = 0,1,\ldots,\kappa-1, \\ \tau^{d,\kappa} f_{n-\kappa}^d(t), & n \geq \kappa. \end{cases} \tag{11.29}$$

The functions f_n^d are the d-Schoenberg functions of φ as in Eq. (11.2), the positive constant $\tau^{d,\kappa} := \left(\prod_{j=1}^\kappa \delta_{(d-2j+1)/2}\right)^{-1}$ and the coefficients

$$\begin{cases} \chi_i^{0,d,\kappa} := \sum_{j=1}^{\kappa-1} (-1)^{j+1} \chi_i^{j-1,d,\kappa-1} C_j^{(d-2\kappa-1)/2}(1) - (-1)^{\kappa+1} C_{i+\kappa}^{(d-2\kappa-1)/2}(1), \\ \chi_i^{n,d,\kappa} := \chi_i^{n-1,d,\kappa-1}, \quad n = 1,2,\ldots,\kappa-1, \end{cases} \tag{11.30}$$

satisfy

$$\left|\chi_i^{n,d,\kappa}\right| \leq \Upsilon^{n,d,\kappa} C_i^{(d-1)/2}(1), \quad n = 0,1,\ldots,\kappa-1, \quad i \in \mathbb{Z}_+, \tag{11.31}$$

where, for each $n = 0,1,\ldots,\kappa-1$ and $\kappa \in \mathbb{Z}_+^*$, $\Upsilon^{n,d,\kappa}$ is a positive constant that depends only on d. Moreover, $\sum_{n=0}^\infty \tilde{f}_n^{d,\kappa}(0) C_n^{(d-2\kappa-1)/2}(1) < \infty$.

Corollary 11.4.4. *Under the conditions of Theorem 11.4.3, there exists a bounded function H^κ on $[-1,1] \times [0,\infty)$ such that $H^\kappa + I_1^\kappa \varphi$ belongs to $\mathcal{P}\left(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^k\right)$.*

Remark 11.4.5. Direct inspection shows that $\chi_i^{0,d,\varkappa} \geq 0$, for $\varkappa = 1, 2$. Therefore, $\chi_i^{1,d,\varkappa}, \chi_i^{2,d,\varkappa} \ldots, \chi_i^{\varkappa-1,d,\varkappa} \geq 0$ for all $\varkappa \geq 2$ and $i \in \mathbb{Z}_+$.

Remark 11.4.6. By Remark 11.4.5, if $f_{2n+1}^d \equiv 0$ for all n, then $I_1^\varkappa \varphi$, for $\varkappa = 1{,}2$, belongs to the class $\mathcal{P}\left(\mathbb{S}^{d-2\varkappa} \times \mathbb{R}^k\right)$. Therefore, our result generalizes the corrected version of Theorem 11.2.1 in Ref. [33].

We can modify the functions $\tilde{f}_n^{d,\varkappa}, n = 0, 1, \ldots, \varkappa - 1$, in Eq. (11.28) so that the new *quasi* Montée operator belongs to $\mathcal{P}\left(\mathbb{S}^{d-2\varkappa} \times \mathbb{R}^k\right)$. Theorem 11.4.7 sheds some light in this direction.

Theorem 11.4.7. *Let the functions $\tilde{f}_n^{d,\varkappa} \in \mathcal{P}\left(\mathbb{R}^k\right), n \geq \varkappa$, and $h_{1,n}^\varkappa, h_{2,n}^\varkappa \in \mathcal{P}\left(\mathbb{R}^k\right)$ be as, respectively, defined at Eqs. (11.29) and (A.5).*

Let $k, \varkappa \in \mathbb{Z}_+^*$ and let d be an integer such that $d > 2\varkappa$. Let $\varphi : [-1, 1] \times [0, \infty) \to \mathbb{R}$ be a function in $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ such that $u \mapsto I_1^{\varkappa-1}\varphi(u, t)$ is integrable over $[-1, 1]$ for each $t \in [0, \infty)$. If

(1) $\sum_{n=0}^{\infty} C_n^{(d-1)/2}(1) \int_0^\infty dF_n(r) < \infty$;

(2) *There exists a constant $K > 0$ such that $\sum_{n=0}^\infty C_n^{(d-1)/2}(1) dF_n(r) \leq K, 0 \leq r < \infty$, then there exist $2\varkappa$ constants A^n and $B^n, n = 0, \ldots, \varkappa - 1$, such that*

$$
\begin{aligned}
I_1^{\varkappa,A^0..A^{\varkappa-1},B^0..B^{\varkappa-1}}\varphi(x,t) := &\sum_{n=0}^{\varkappa-1} (A^n h_{1,n}^\varkappa(t) - B^n h_{2,n}^\varkappa(t)) C_n^{(d-2\varkappa-1)/2}(x) \\
&+ \sum_{n=\varkappa}^{\infty} \tilde{f}_n^{d,\varkappa}(t) C_n^{(d-2k-1)/2}(x),
\end{aligned}
\tag{11.32}
$$

belongs to $\mathcal{P}\left(\mathbb{S}^{d-2\varkappa} \times \mathbb{R}^k\right)$.

Remark 11.4.8. For any $A^n \geq 0, n = 1, \ldots, \varkappa - 1$ the function $I_1^{\varkappa,0A^1..A^{\varkappa-1},0..0}\varphi$ belongs to $\mathcal{P}\left(\mathbb{S}^{d-2\varkappa} \times \mathbb{R}^k\right)$. This also can be seen as a generalization of the correction of Theorem 2.1 in Ref. [33] (to appear).

The next result is related to the operator I_2.

Theorem 11.4.9. *Let $d, \varkappa \in \mathbb{Z}_+^*$ and k be an integer such that $k > 2\varkappa$. If $\varphi : [-1, 1] \times [0, \infty) \to \mathbb{R}$ is a function in $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ such that*

(1) $g_n^\nu(0) = \int_0^\infty \left(1/r^{2\nu}\right) dF_n(r) < \infty$, *for all $n \in \mathbb{Z}_+$ and $\nu \in \{1, 2, \ldots, \varkappa\}$.*

(2) $0 \neq \sum_{n=0}^\infty g_n^{(\nu)}(0) C_n^{(d-1)/2}(1) < \infty$, *for $\nu \in \{1, 2, \ldots, \varkappa\}$,* *then the function* $I_2^\kappa \varphi$ *has a representation in Gegenbauer series in the form*

$$I_2^\kappa \varphi\,(x,t) = \frac{1}{\sum_{n=0}^\infty \; g_n^\kappa\,(0)\, C_n^{(d-1)/2}\,(1)} \sum_{n=0}^\infty \; g_n^\kappa\,(t)\, C_n^{(d-1)/2}\,(x)\,. \tag{11.33}$$

The functions g_n^κ *are defined in Eq. (11.27), and* F_n *are the k-Schoenberg measures of the d-Schoenberg functions of* φ.

Moreover, $I_2^\kappa \varphi$ *belongs to* $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^{k-2\kappa}\right)$.

We finish this part with the Montée operator I_3.

Theorem 11.4.10. *Let* $\kappa \in \mathbb{Z}_+^*$, *d, and k be integers such that* $d, k > 2\kappa$. *If* $\varphi : [-1,1] \times [0,\infty) \to \mathbb{R}$ *is a function in* $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ *such that*

(1) $g_n^\nu(0) = \int_0^\infty \frac{1}{r^{2\nu}} dF_n(r) < \infty$ *for all* $n \in \mathbb{Z}_+$ *and* $\nu \in \{1,2,\ldots,\kappa\}$;

(2) $\sum_{n=0}^\infty g_n^\nu(0) = \sum_{n=0}^\infty \int_0^\infty \frac{1}{r^{2\nu}} dF_n(r) < \infty$, *for and* $\nu \in \{1,2,\ldots,\kappa\}$,

(3) $\sum_{n=0}^\infty g_n^\nu(0) C_{n+\nu}^{(d-2\nu-1)/2}(1) < \infty$, *for and* $\nu \in \{1,2,\ldots,\kappa\}$,

then the function $I_3^\kappa \varphi$ *has a representation in Gegenbauer series in the form*

$$I_3^\kappa \varphi\,(x,t) = \sum_{n=0}^\infty \; h_n^\kappa\,(t)\, C_n^{(d-2\kappa-1)/2}\,(x), \;\; (x,t) \in [-1,1] \times [0,\infty], \tag{11.34}$$

where

$$h_n^\kappa\,(t) := \begin{cases} \gamma^{d,k,\kappa} \sum_{i=0}^\infty (-1)^i \chi_i^{n,d,\kappa} g_i^\kappa(t), & n = 0,1,\ldots,\kappa-1, \\ \gamma^{d,k,\kappa} g_{n-\kappa}^\kappa(t), & n \geq \kappa, \end{cases} \tag{11.35}$$

with $\gamma^{d,k,\kappa} := \prod_{j=1}^\kappa \frac{k-2j}{\delta_{(d-2j+1)/2}} > 0$ *and* $\sum_{n=0}^\infty h_n^\kappa(0) C_n^{(d-2\kappa-1)/2}(1) < \infty$. *The functions* g_n^κ *are defined in Eq. (11.27) and belong to the class* $\mathcal{P}\left(\mathbb{R}^k\right)$ *and* $\chi_i^{n,d,\kappa}$ *are given in Eq. (11.30).*

Remark 11.4.11. By Remark 11.4.5, if $g_{2n+1}^\kappa \equiv 0$ for all n, then $I_3^\kappa \varphi$, for $\kappa = 1,2$, belongs to the class $\mathcal{P}\left(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa}\right)$.

Corollary 11.4.12. *Under the conditions of Theorem 11.4.10, there exists a bounded function* H^κ *on* $[-1,1] \times [0,\infty)$ *such that* $H^\kappa + I_3^\kappa \varphi$ *belongs to* $\mathcal{P}\left(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa}\right)$.

As previously mentioned, we can replace the functions $h_n^\kappa, n = 0,1,\ldots,\kappa-1$, with others such that the new *quasi* Montée operator belongs to $\mathcal{P}\left(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa}\right)$. Theorem 11.4.13 provides a construction in this sense.

Theorem 11.4.13. *Let* $\kappa \in \mathbb{Z}_+^*$, *d, and k be integers such that* $d, k > 2\kappa$. *Let* $\varphi : [-1,1] \times [0,\infty) \to \mathbb{R}$ *be a function that belongs to the class* $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$ *satisfying the hypotheses of Theorem 11.4.10. If additionally, the k-Schoenberg measures* F_n *of the d-Schoenberg functions of* φ *satisfy*

(1) $\sum_{n=0}^{\infty} g_n^{\nu}(0) C_n^{(d-1)/2}(1) = \sum_{n=0}^{\infty} C_n^{(d-1)/2}(1) \int_0^{\infty} \frac{1}{r^{2\nu}} dF_n(r) < \infty$, *for and* $\nu \in \{1, 2, \ldots, \kappa\}$,

(2) *There exists a constant* $K > 0$ *such that* $\sum_{n=0}^{\infty} C_n^{(d-1)/2}(1) dF_n(r) \leq K, 0 \leq r < \infty$;

then there exist 2κ *constants* A^n *and* $B^n, n = 0, \ldots, \kappa - 1$, *such that*

$$I_3^{\kappa, A^0..A^{\kappa-1}, B^0..B^{\kappa-1}} \varphi(x,t) := \sum_{n=0}^{\kappa-1} (A^n \tilde{h}_{1,n}^{\kappa}(t) - B^n \tilde{h}_{2,n}^{\kappa}(t)) C_n^{(d-2\kappa-1)/2}(x) + \sum_{n=\kappa}^{\infty} \tilde{h}_n^{\kappa}(t) C_n^{(d-2\kappa-1)/2}(x). \tag{11.36}$$

The functions $h_n^{\kappa} \in \mathcal{P}\left(\mathbb{R}^k\right), n \geq \kappa$ *and* $\tilde{h}_{1,n}^{\kappa}, \tilde{h}_{2,n}^{\kappa} \in \mathcal{P}\left(\mathbb{R}^k\right)$ *are defined, respectively, in Eqs. (11.35) and (A.10).*

Remark 11.4.14. For any $A^n \geq 0, n = 1, \ldots, \kappa - 1$, the function $I_3^{\kappa, 0A^1..A^{\kappa-1}, 0..0} \varphi$ belongs to $\mathcal{P}\left(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^{k-2\kappa}\right)$.

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APPENDIX A

Proof of Theorem 11.4.1. Since φ belongs to $\mathcal{P}\left(\mathbb{S}^d \times \mathbb{R}^k\right)$, then φ is continuously differentiable with respect to the first variable (see Ref. [34, Proposition 3.8]) and has a Gegenbauer expansion as Eq. (11.3). Also φ_x has a Gegenbauer expansion in the form

$$\varphi_x(x,t) \sim \sum_{n=0}^{\infty} \tilde{f}_n^{d+1}(t) C_n^{(d+1)/2}(x).$$

Using Eqs. (11.15)–(11.16), the remainder of the proof follows as in Ref. [30, Theorem 11.2.3]. ■

Proof of Theorem 11.4.2. Let φ be a function as in Eq. (11.2). By Eq. (11.7),

$$\frac{df_n^d}{dt}(t) = \int_0^{\infty} -\frac{1}{k} tr^2 \Omega_{k+2}(tr)\, dF_n(r).$$

Deriving term by term, we obtain

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(x,t) &= \sum_{n=0}^{\infty} \frac{df_n^d}{dt}(t) C_n^{(d-1)/2}(x)\\ &= -\frac{1}{k}\sum_{n=0}^{\infty}\left(\int_0^{\infty} \Omega_{k+2}(tr) r^2 dF_n(r)\right) C_n^{(d-1)/2}(x).\end{aligned} \tag{A.1}$$

By Lemma 3 in Ref. [37], we have

$$\frac{d^2 f_n^d}{dt^2}(0) = -\frac{1}{k}\int_0^{\infty} r^2 dF_n(r).$$

Thus,

$$\frac{\partial^2 \varphi}{\partial t^2}(x,0) = -\frac{1}{k}\sum_{n=0}^{\infty}\left(\int_0^{\infty} r^2 dF_n(r)\right) C_n^{(d-1)/2}(x), \quad x \in [-1,1]. \tag{A.2}$$

In particular,

$$\frac{\partial^2 \varphi}{\partial t^2}(1,0) = -\frac{1}{k}\sum_{n=0}^{\infty}\left(\int_0^{\infty} r^2 dF_n(r)\right) C_n^{(d-1)/2}(1). \tag{A.3}$$

Thus, by Eqs. (A.1) and (A.3), for $x \in [-1, 1\,)$ and $t > 0$, we have

$$D_2\varphi(x,t) = \frac{\varphi_t(x,t)}{t\varphi_{tt}(1,0)} = \frac{1}{\sum_{n=0}^{\infty} \int_0^{\infty} r^2 dF_n(r)} \sum_{n=0}^{\infty} g_n^d(t)\, C_n^{(d-1)/2}(x), \tag{A.4}$$

where the functions $g_n^d(t) : [0,\infty) \to \mathbb{R}$ are defined by

$$g_n^d(t) := \int_0^{\infty} \Omega_{k+2}(tr)\, r^2 dF_n(r).$$

We invoke Hypothesis (1) to imply that $g_n^d \in \mathcal{P}\left(\mathbb{R}^{k+2}\right)$. Thus, the series in Eq. (A.4) converges absolutely and uniformly on $[-1,1]\times[0,\infty)$. Hence, letting $x = 1$ and $t = 0$ in the expression of the series in Eq. (A.4) provides

$$\frac{1}{\sum_{n=0}^{\infty} \int_0^{\infty} r^2 dF_n(r)} \sum_{n=0}^{\infty} g_n^d(0)\, C_n^{(d-1)/2}(1) = 1 = D_2\varphi(1,0).$$

Therefore, $D_2\varphi$ is a continuous function on $[-1,1]\times[0,\infty)$ having a representation series as in Eq. (11.2) with d-Schoenberg functions $g_n^d \in \mathcal{P}\left(\mathbb{R}^{k+2}\right)$. Since, by (2),

$$\sum_{n=0}^{\infty} g_n^d(0) = \sum_{n=0}^{\infty} \int_0^{\infty} r^2 dF_n(r) < \infty,$$

we can conclude that $D_2\varphi$ belongs to $\mathcal{P}\left(S^d \times \mathbb{R}^{k+2}\right)$. ∎

Proof of Theorem 11.4.3. We prove the statement by induction on $\varkappa \in \mathbb{Z}_+^*$.
Step $\varkappa = 1$: We have

$$I_1^1\varphi(x,t) = I_1\varphi(x,t) = \int_{-1}^{x} \varphi(u,t)du.$$

By Eqs. (11.2) and (11.11), integrating term by term, we obtain

$$I_1^1\varphi(x,t) = \sum_{n=0}^{\infty} f_n^d(t)\frac{1}{\delta_{(d-1)/2}}\left(C_{n+1}^{(d-3)/2}(x) - C_{n+1}^{(d-3)/2}(-1)\right).$$

Since $C_{n+1}^{(d-3)/2}(-1) = (-1)^{n+1}C_{n+1}^{(d-3)/2}(1)$, we have

$$I_1^1\varphi(x,t) = \sum_{n=0}^{\infty} \widetilde{f}_n^{d,1}(t)C_n^{(d-3)/2}(x)$$

where

$$\widetilde{f}_n^{d,1}(t) := \begin{cases} \dfrac{1}{\delta_{(d-1)/2}} \displaystyle\sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,1} f_i^d(t), & n = 0 \\ \dfrac{1}{\delta_{(d-1)/2}} f_{n-1}^d(t), & n \geq 1, \end{cases}$$

where $\chi_i^{0,d,1} = C_{i+1}^{(d-3)/2}(1)$ and, by Eq. (11.14),

$$0 \leq \chi_i^{0,d,1} = \left| C_i^{(d-1)/2}(1) \frac{C_{i+1}^{(d-3)/2}(1)}{C_i^{(d-1)/2}(1)} \right| \leq \underbrace{\varrho_{(d-1)/2,1}}_{\Upsilon^{0,d,1}} C_i^{(d-1)/2}(1),$$

which implies

$$\left|(-1)^i \chi_i^{0,d,1} f_i^d(t)\right| \leq \Upsilon^{0,d,1} f_i(0) C_i^{(d-1)/2}(1)$$

and by Eq. (11.2), the series in the definition of $\tilde{f}_n^{d,1}$ is uniformly convergent on $[0,\infty)$. Again by Eq. (11.14), for $n \geq 1$,

$$\left|\widetilde{f}_n^{d,1}(0)C_n^{(d-3)/2}(1)\right| \leq \frac{1}{\delta_{(d-1)/2}}\Upsilon^{0,d,1}f_{n-1}^d(0)C_{n-1}^{(d-1)/2}(1).$$

Thus, $\sum_{n=0}^{\infty} \tilde{f}_n^{d,1}(0)C_n^{(d-3)/2}(1) < \infty$.

Step $\varkappa = 2$: By algebraic manipulation we have

$$I_1^2\varphi(x,t) = \sum_{n=0}^{\infty} \tilde{f}_n^{d,2}(t)C_n^{(d-5)/2}(x)$$

where

$$\widetilde{f}_n^{d,2}(t) := \begin{cases} \tau^{d,2}\sum_{i=0}^{\infty}(-1)^i\chi_i^{n,d,2}f_i^d(t), & n = 0, 1, \\ \tau^{d,2}f_{n-\varkappa}^d(t), & n \geq 2, \end{cases}$$

with $\tau^{d,2} = \left(\delta_{(d-1)/2}\delta_{(d-3)/2}\right)^{-1}$ and

$$\chi_i^{0,d,2} := C_{i+1}^{(d-3)/2}(1)C_1^{(d-5)/2}(1) - C_{i+2}^{(d-5)/2}(1),$$

$$\chi_i^{1,d,2} := C_{i+1}^{(d-3)/2}(1) = \chi_i^{0,d,1}.$$

It is clear that $0 \leq \chi_i^{1,d,2} \leq \Upsilon^{1,d,2}C_i^{(d-1)/2}(1)$, with $\Upsilon^{1,d,2} = \Upsilon^{0,d,1}$. It is not difficult to see that

$$\chi_i^{0,d,2} = \frac{(d-5)\Gamma(i+d-1)(i+1)}{\Gamma(d-3)(i+d-3)\Gamma(i+3)} \geq 0$$

and, by Eq. (11.14),

$$\begin{aligned}\left|\chi_i^{0,d,2}\right| &\leq \left(\left|\frac{C_{i+1}^{(d-3)/2}(1)}{C_i^{(d-1)/2}(1)}C_1^{(d-5)/2}(1)\right| + \left|\frac{C_{i+2}^{(d-5)/2}(1)}{C_i^{(d-1)/2}(1)}\right|\right)C_i^{(d-1)/2}(1) \\ &\leq \left(\varrho_{(d-1)/2,1}\frac{\Gamma(1+d-5)}{\Gamma(2)\Gamma(d-5)} + \varrho_{(d-1)/2,2}\right)C_i^{(d-1)/2}(1) \\ &= \underbrace{\left(\varrho_{(d-1)/2,1}(d-5) + \varrho_{(d-1)/2,2}\right)}_{\Upsilon^{0,d,2}}C_i^{(d-1)/2}(1)\end{aligned}$$

By the same argument of the step $\varkappa = 1$, we can conclude that the series in the definition of $\tilde{f}_n^{d,2}$ for $n = 0, 1$ is uniformly convergent on $[0, \infty)$ and also $\sum_{n=0}^{\infty} \tilde{f}_n^{d,2}(0) C_n^{(d-5)/2}(1) < \infty$.

Induction step: Let us assume that the expression in Eq. (11.28) of $I_1^{\varkappa}\varphi$ holds up to $\varkappa$, and let us prove it holds for $I_1^{\varkappa+1}\varphi$. We have

$$I_1^{\varkappa+1}\varphi(x,t) = I_1\left(I_1^{\varkappa}\varphi\right)(x,t) = \int_{-1}^{x} I_1^{\varkappa}\varphi(u,t)\,du.$$

Using the induction hypothesis and integrating term by term, for $(x,t) \in [-1,1] \times [0,\infty)$, we obtain

$$\begin{aligned} I_1^{\varkappa+1}\varphi(x,t) &= \sum_{n=0}^{\varkappa-1} \tilde{f}_n^{d,\varkappa}(t) \int_{-1}^{x} C_n^{(d-2\varkappa-1)/2}(u)du + \sum_{n=\varkappa}^{\infty} \tilde{f}_n^{d,\varkappa}(t) \int_{-1}^{x} C_n^{(d-2\varkappa-1)/2}(u)du \\ &= \sum_{n=0}^{\varkappa-1} \tau^{d,\varkappa} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\varkappa} f_i^d(t) \frac{1}{\delta_{(d-2\varkappa-1)/2}} \left(C_{n+1}^{(d-2\varkappa-3)/2}(x) - C_{n+1}^{(d-2\varkappa-3)/2}(-1)\right) \\ &\quad + \sum_{n=\varkappa}^{\infty} \tau^{d,\varkappa} f_{n-\varkappa}^d(t) \frac{1}{\delta_{(d-2\varkappa-1)/2}} \left(C_{n+1}^{(d-2\varkappa-3)/2}(x) - C_{n+1}^{(d-2\varkappa-3)/2}(-1)\right). \end{aligned}$$

Thus,

$$\begin{aligned} I_1^{\varkappa+1}\varphi(x,t) &= \tau^{d,\varkappa+1} \sum_{n=0}^{\varkappa-1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\varkappa} f_i^d(t) \left(C_{n+1}^{(d-2\varkappa-3)/2}(x) - C_{n+1}^{(d-2\varkappa-3)/2}(-1)\right) \\ &\quad + \tau^{d,\varkappa+1} \sum_{n=\varkappa}^{\infty} f_{n-\varkappa}^d(t) \left(C_{n+1}^{(d-2\varkappa-3)/2}(x) - C_{n+1}^{(d-2\varkappa-3)/2}(-1)\right). \end{aligned}$$

After some algebraic manipulation,

$$I_1^{\varkappa+1}\varphi(x,t) = \sum_{n=0}^{\infty} \tilde{f}_n^{d,\varkappa+1}(t)\, C_n^{(d-2(k+1)-1)/2}(x),$$

where

$$\tilde{f}_n^{d,\varkappa+1}(t) = \begin{cases} \tau^{d,\varkappa+1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\varkappa+1} f_i^d(t), & n = 0, 1, \ldots, \varkappa, \\ \tau^{d,\varkappa+1} f_{n-(\varkappa+1)}^d(t), & n \geq \varkappa + 1, \end{cases}$$

with

$$\chi_i^{0,d,\varkappa+1} := \sum_{j=1}^{\varkappa} (-1)^{j+1} \chi_i^{j-1,d,\varkappa} C_j^{(d-2(\varkappa+1)-1)/2}(1) - (-1)^{\varkappa+1} C_{i+\varkappa+1}^{(d-2(\varkappa+1)-1)/2}(1)$$

and

$$\chi_i^{n,d,\varkappa+1} := \chi_i^{n-1,d,\varkappa},\ n = 1, 2, \ldots, \varkappa.$$

It is clear that $\left|\chi_i^{n,d,\kappa+1}\right| \leq \Upsilon^{n,d,\kappa+1} C_i^{(d-1)/2}(1)$, with $\Upsilon^{n,d,\kappa+1} = \Upsilon^{n-1,d,\kappa}$. Now, by Eq. (11.14),

$$\left|\chi_i^{0,d,\kappa+1}\right| \leq \left(\sum_{j=1}^{\kappa} \frac{\left|\chi_i^{j-1,d,\kappa}\right|}{C_i^{(d-1)/2}(1)} C_j^{(d-2\kappa-3)/2}(1) + \frac{C_{i+\kappa+1}^{(d-2\kappa-3)/2}(1)}{C_i^{(d-1)/2}(1)} \right) C_i^{(d-1)/2}(1).$$

By induction hypothesis (11.31) and by Eq. (11.14) we obtain

$$\left|\chi_i^{0,d,\kappa+1}\right| \leq \underbrace{\left(\sum_{j=1}^{\kappa} \Upsilon^{j-1,d,\kappa}\,(d-2\kappa-3) + \varrho_{(d-1)/2,2(\kappa+1)} \right)}_{\Upsilon^{0,d,\kappa+1}} C_i^{(d-1)/2}\,(1)\,.$$

The convergence of the series in the definition of $\tilde{f}_n^{d,\kappa+1}$ for $n = 0, 1, \ldots, \kappa$ and $\sum_{n=0}^{\infty} \tilde{f}_n^{d,\kappa+1}(0) C_n^{(d-2(\kappa+1)-1)/2}(1)$ follows as in the previous steps. ■

Proof of Corollary 11.4.4. Note that, for $n = 0, 1, \ldots, \kappa - 1$, we can rewrite $\tilde{f}_n^{d,\kappa}$ as

$$\tilde{f}_n^{d,\kappa}(t) = h_{1,n}^{\kappa}(t) - h_{2,n}^{\kappa}(t),$$

where

$$h_{1,n}^{\kappa}(t) := \tau^{d,\kappa} \sum_{i=0}^{\infty} \chi_{2i}^{n,d,\kappa} f_{2i}^{d}(t), \quad \text{and} \tag{A.5}$$

$$h_{2,n}^{\kappa}\,(t) := \tau^{d,\kappa} \sum_{i=0}^{\infty} \chi_{2i+1}^{n,d,\kappa} f_{2i+1}^{d}\,(t) \tag{A.6}$$

Define the function H^{κ} on $[-1, 1] \times [0, \infty)$ by

$$H^{\kappa}(x,t) := \sum_{n=0}^{\kappa-1} h_{2,n}^{\kappa}(t) C_n^{(d-2\kappa-1)/2}(x) - h_{1,0}^{\kappa}(t) C_0^{(d-2\kappa-1)/2}(x)$$

which is bounded on $[-1, 1] \times [0, \infty)$ because, by Eqs. (11.13), (11.31), and (11.2),

$$\left|H^{\kappa}(x,t)\right| \leq \tau^{d,\kappa} \sum_{n=0}^{\kappa-1} \Upsilon^{n,d,\kappa} \left(\sum_{i=0}^{\infty} f_{2i+1}^{d}(0) C_{2i+1}^{(d-1)/2}(1) \right) C_n^{(d-1)/2}(1)$$

$$+\tau^{d,\kappa} \Upsilon^{0,d,\kappa} \left(\sum_{i=0}^{\infty} f_{2i}^{d}\,(0)\, C_{2i}^{(d-1/2)}\,(1) \right) C_0^{(d-2\kappa-1/2)}\,(1) < \infty,$$

for all $(x, t) \in [-1, 1] \times [0, \infty)$.

By Remark 11.4.5, it is clear that $h^{\varkappa}_{1,n} \in \mathcal{P}\left(\mathbb{R}^k\right), n = 1, 2, \ldots, \varkappa - 1$, and also $\tilde{f}^{d,\varkappa}_n \in \mathcal{P}\left(\mathbb{R}^k\right)$ for $n \geq \varkappa$. Therefore,

$$H^{\varkappa}(x,t) + I_1^{\varkappa}\varphi(x,t) = \sum_{n=1}^{\varkappa-1} h^{\varkappa}_{1,n}(t) C_n^{(d-2\varkappa-1)/2}(x) + \tau^{d,\varkappa} \sum_{n=\varkappa}^{\infty} \tilde{f}^{d,\varkappa}_n(t) C_n^{(d-2\varkappa-1)/2}(x)$$

has an expansion uniformly convergent as (11.2) due to Theorem 11.4.3. By Theorem 11.3.3 of Ref. [34] (see Eq. (11.2)), we can conclude that the function $H^{\varkappa} + I_1^{\varkappa}\varphi$ belongs to the class $\mathcal{P}\left(\mathbb{S}^{d-2\varkappa} \times \mathbb{R}^k\right)$. ■

We observe that the function $H^{\varkappa}$ is not unique and that the construction presented allows us to highlight the properties of the coefficient functions and consider the maximum of the non-zero d-Schoenberg functions of φ.

Proof of Theorem 11.4.7. By Eq. (11.5), for any constants A and B,

$$Ah^{\varkappa}_{1,n}(t) - Bh^{\varkappa}_{2,n}(t) = \tau^{d,\varkappa} \sum_{i=0}^{\infty} \int_0^{\infty} \Omega_k(tr) \left(A\chi^{n,d,\varkappa}_{2i} dF_{2i}(r) - B\chi^{n,d,\varkappa}_{2i+1} dF_{2i+1}(r)\right). \quad \text{(A.7)}$$

Since, by Eq. (11.31),

$$\int_0^{\infty} A\left|\chi^{n,d,\varkappa}_i\right| dF_i(r) \leq A\Upsilon^{n,d,\varkappa} C_i^{(d-1)/2}(1) \int_0^{\infty} dF_i(r),$$

we have

$$\int_0^{\infty} \left(A\chi^{n,d,\varkappa}_{2i} dF_{2i}(r) - B\chi^{n,d,\varkappa}_{2i+1} dF_{2i+1}(r)\right) < \infty.$$

By Eq. (11.31) and (1) the series in Eq. (A.7) converges absolutely and uniformly on $[0, \infty)$.

Thus,

$$Ah^{\varkappa}_{1,n}(t) - Bh^{\varkappa}_{2,n}(t) = \tau^{d,\varkappa} \int_0^{\infty} \Omega_k(tr) d\left(\sum_{i=0}^{\infty} A\chi^{n,d,\varkappa}_{2i} F_{2i}(r) - B\chi^{n,d,\varkappa}_{2i+1} F_{2i+1}(r)\right).$$

By Eq. (11.31) and (2), the series $\sum_{i=0}^{\infty} \chi^{n,d,\varkappa}_{2i} F_{2i}$ and $\sum_{i=0}^{\infty} \chi^{n,d,\varkappa}_{2i+1} F_{2i+1}$ are uniformly bounded on $[0, \infty)$. Then we can choose A^n, B^n such that the series $\sum_{i=0}^{\infty} A^n \chi^{n,d,\varkappa}_{2i} F_{2i} - B^n \chi^{n,d,\varkappa}_{2i+1} F_{2i+1}$ is non-negative, which allows us to conclude that $A^n h^{\varkappa}_{1,n} - B^n h^{\varkappa}_{2,n} \in \mathcal{P}\left(\mathbb{R}^k\right)$. The convergence uniform of the series (11.32) follows by Theorem 4.3 and the result by Theorem 3.3 of Ref. [34] (see Eq. (11.2)). ■

Proof of Theorem 11.4.9. We will prove Eq. (11.33) by mathematical induction on $\varkappa$.

Step $\varkappa = 1$: We have

$$I_2\varphi(x,t) = \frac{1}{\int_0^{\infty} v\varphi(1,v)\, dv} \int_t^{\infty} v\varphi(x,v)\, dv.$$

By Eq. (11.2), integrating term by term, we obtain

$$\int_t^\infty v\varphi(x,v)\,dv = \sum_{n=0}^\infty \left(\int_t^\infty v \int_0^\infty \Omega_k(vr)\,dF_n(r)\,dv \right) C_n^{(d-1)/2}(x).$$

Using Fubini Theorem, we have

$$\int_t^\infty v\varphi(x,v)\,dv = \sum_{n=0}^\infty \left[\int_0^\infty \left(\int_{tr}^\infty \frac{w}{r^2}\Omega_k(w)\,dw \right) dF_n(r) \right] C_n^{(d-1)/2}(x).$$

By Eq. (11.9), for $(x,t) \in [-1,1] \times [0,\infty)$,

$$\int_{tr}^\infty \frac{w}{r^2}\Omega_k(w)\,dw = \frac{(k-2)}{r^2}\Omega_{k-2}(tr).$$

Hence, for $(x,t) \in [-1,1] \times [0,\infty)$,

$$\int_t^\infty v\varphi(x,v)\,dv = (k-2)\sum_{n=0}^\infty g_n^1(t)\,C_n^{(d-1)/2}(x), \tag{A.8}$$

where g_n^1 is defined in Eq. (11.27). In particular,

$$\int_0^\infty v\varphi(1,v)\,dv = (k-2)\sum_{n=0}^\infty g_n^1(0)\,C_n^{(d-1)/2}(1), \tag{A.9}$$

which is nonzero and finite. By Eqs. (A.8) and (A.9), $I_2\varphi$ has the representation given in Eq. (11.33).

Induction step: Let us assume the expression in Eq. (11.33) of $I_2^\kappa\varphi$ holds up to κ, and let us prove it holds for $I_2^{\kappa+1}\varphi$.

We have

$$I_2^{\kappa+1}\varphi(x,t) = I_2\left(I_2^\kappa\varphi\right)(x,t) = \frac{1}{\int_0^\infty vI_2^\kappa\varphi(1,v)\,dv}\int_t^\infty vI_2^\kappa\varphi(x,v)\,dv.$$

Note that the Hypothesis (1) guarantees that $g_n^\kappa \in \mathcal{P}\left(\mathbb{R}^{k-2\kappa}\right)$ and consequently the series in (11.33) converges absolutely and uniformly.

Using the induction hypothesis, integrating term by term, using the Fubini theorem and Eq. (11.9), for $(x,t) \in [-1,1] \times [0,\infty)$, we obtain:

$$\begin{aligned}
\int_t^\infty vI_2^\kappa\varphi(x,v)dv &= \sum_{n=0}^\infty \left[\int_0^\infty \left(\int_t^\infty v\Omega_{k-2\kappa}(vr)dv \right) \frac{1}{r^{2\kappa}}dF_n(r) \right] C_n^{(d-1)/2}(x) \\
&= (k-2\kappa-2)\sum_{n=0}^\infty \left[\int_0^\infty \Omega_{k-2(\kappa+1)}(tr)\frac{1}{r^{2(\kappa+1)}}dF_n(r) \right] C_n^{(d-1)/2}(x) \\
&= (k-2\kappa-2)\sum_{n=0}^\infty g_n^{\kappa+1}(t)C_n^{(d-1)/2}(x).
\end{aligned}$$

In particular,

$$\int_0^\infty vI_2^\kappa\varphi(1,v)\,dv = (k-2\kappa-2)\sum_{n=0}^\infty g_n^{\kappa+1}(0)\,C_n^{(d-1)/2}(1),$$

which is nonzero and finite by (2). Therefore,

$$I_2^{\kappa+1}\varphi(x,t) = \frac{1}{\sum_{n=0}^\infty g_n^{\kappa+1}(0)C_n^{(d-1)/2}(1)}\sum_{n=0}^\infty g_n^{\kappa+1}(t)C_n^{(d-1)/2}(x)$$

and Eq. (11.33) is proved.

Finally, given $\kappa \in \mathbb{Z}_+^*$, by (1) the d-Schoenberg functions g_n^κ of $I_2^\kappa\varphi$ belong to the class $\mathcal{P}\left(\mathbb{R}^{k-2\kappa}\right)$ and together with (2) we can conclude $0 < \sum_{n=0}^\infty g_n^\kappa(0)C_n^{(d-1)/2}(1) < \infty$. Therefore, Theorem 3.3 of Ref. [34] (see Eq. (11.2)) allows us to infer that $I_2^\kappa\varphi$ belongs to $\mathcal{P}\left(\mathbb{S}^d\times\mathbb{R}^{k-2\kappa}\right)$. ■

Proof of Theorem 11.4.10. We will prove Eq. (11.34) by mathematical induction on κ.

Step $\kappa = 1$: For each $(x,t)\in[-1,1]\times[0,\infty)$,

$$I_3^1\varphi(x,t) = \int_t^\infty\int_{-1}^x v\varphi(u,v)\,dudv.$$

Using Eqs. (11.2) and (11.5),

$$\int_t^\infty\int_{-1}^x v\varphi(u,v)\,dudv = \int_t^\infty\int_{-1}^x v\sum_{n=0}^\infty\left(\int_0^\infty \Omega_k(vr)\,dF_n(r)\right)C_n^{(d-1)/2}(u)\,dudv.$$

Integrating term by term and by the Fubini theorem, we have

$$\int_t^\infty\int_{-1}^x v\varphi(u,v)dudv = \sum_{n=0}^\infty\left[\int_0^\infty\left(\int_t^\infty v\Omega_k(vr)dv\right)dF_n(r)\right]\left[\int_{-1}^x C_n^{(d-1)/2}(u)du\right].$$

By Eqs. (11.9) and (11.11), we obtain

$$\begin{aligned}\int_t^\infty\int_{-1}^x v\varphi(u,v)dudv = \sum_{n=0}^\infty&\left[\int_0^\infty\frac{(k-2)}{r^2}\Omega_{k-2}(tr)dF_n(r)\right]\times\\ &\times\left[\frac{1}{\delta_{(d-1)/2}}\left(C_{n+1}^{(d-3)/2}(x)-C_{n+1}^{(d-3)/2}(-1)\right)\right]\end{aligned}$$

Since $C_{n+1}^{(d-3)/2}(-1) = (-1)^{n+1}C_{n+1}^{(d-3)/2}(1)$,

$$\begin{aligned}\int_t^\infty\int_{-1}^x v\varphi(u,v)dudv = \frac{(k-2)}{\delta_{(d-1)/2}}&\left[\left(\sum_{n=0}^\infty(-1)^nC_{n+1}^{(d-3)/2}(1)g_n^1(t)\right)C_0^{(d-3)/2}(x)\right.\\ &\left.+\sum_{n=0}^\infty g_n^1(t)C_{n+1}^{(d-3)/2}(x)\right],\end{aligned}$$

where g_n^1 is given in Eq. (11.27).

Therefore,

$$I_3^1\varphi(x,t) = \sum_{n=0}^{\infty} h_n^1(t) C_n^{(d-3)/2}(x),$$

where

$$h_n^1(t) = \begin{cases} \gamma^{d,k,1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{0,d,1} g_i^1(t), & n = 0 \\ \gamma^{d,k,1} g_{n-1}^1(t), & n \geq 1, \end{cases}$$

where $\gamma^{d,k,1} = \frac{(k-2)}{\delta_{(d-1)/2}} > 0$. Moreover $\sum_{n=0}^{\infty} h_n^1(0) C_n^{(d-3)/2}(1) < \infty$ because, by Eq. (11.31) and (2)–(3), we have

$$\sum_{n=0}^{\infty} \left| h_n^1(0) C_n^{(d-3)/2}(1) \right| \leq \Upsilon^{0,d,1} C_0^{(d-3)/2}(1) \sum_{i=0}^{\infty} g_i^1(0) + \sum_{n=1}^{\infty} g_{n-1}^1(0) C_n^{(d-3)/2}(1) < \infty$$

Induction step: Let us assume the expression in Eq. (11.34) of $I_3^{\kappa}\varphi$ holds up to κ, and let us prove it holds for $I_3^{\kappa+1}\varphi$.

We have

$$I_3^{\kappa+1}\varphi\,(x,t) = I_3\left(I_3^{\kappa}\varphi\right)(x,t) = \int_t^{\infty} \int_{-1}^{x} v I_3^{\kappa}\varphi\,(u,v)\,du dv.$$

Using the induction hypothesis, integrating term by term, using the Fubini theorem, and Eqs. (11.27), (11.9), (11.11), and making algebraic manipulations similar to the previous ones, for $(x,t) \in [-1,1] \times [0,\infty)$, we obtain

$$\begin{aligned} \int_t^{\infty} \int_{-1}^{x} v I_3^{\kappa}\varphi(u,v) du dv &= \gamma^{d,k,\kappa} \sum_{n=0}^{\kappa-1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\kappa} \int_t^{\infty} v g_i^{\kappa}(v)\, dv \int_{-1}^{x} C_n^{(d-2\kappa-1)/2}(u) du \\ &\quad + \gamma^{d,k,\kappa} \sum_{n=\kappa}^{\infty} \int_t^{\infty} v g_{n-\kappa}^{\kappa}(v)\, dv \int_{-1}^{x} C_n^{(d-2\kappa-1)/2}(u) du \\ &= \gamma^{d,k,\kappa} \frac{(k-2\kappa-2)}{\delta_{(d-2\kappa-1)/2}} \times \\ &\quad \left[\sum_{n=0}^{\kappa-1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\kappa} g_i^{\kappa+1}(t) \left(C_{n+1}^{(d-2\kappa-3)/2}(x) - C_{n+1}^{(d-2\kappa-3)/2}(-1) \right) \right. \\ &\quad \left. + \sum_{n=\kappa}^{\infty} g_{n-\kappa}^{\kappa+1}(t) \left(C_{n+1}^{(d-2\kappa-3)/2}(x) - C_{n+1}^{(d-2\kappa-3)/2}(-1) \right) \right] \end{aligned}$$

Thus, as in the Proof of Theorem 11.4.3,

$$I_3^{\kappa+1}\varphi(x,t) = \gamma^{d,k,\kappa+1} \sum_{n=0}^{\infty} h_n^{\kappa+1}(t) C_n^{(d-2(\kappa+1)-1)/2}(x),$$

where

$$h_n^{\kappa+1}(t) := \begin{cases} \gamma^{d,k,\kappa+1} \sum_{i=0}^{\infty} (-1)^i \chi_i^{n,d,\kappa+1} g_i^{\kappa+1}(t), & n = 0, 1, \ldots, \kappa \\ \gamma^{d,k,\kappa+1} g_{n-(\kappa+1)}^{\kappa+1}(t), & n \geq \kappa + 1 \end{cases}$$

with $\gamma^{d,k,\kappa+1} > 0$, By (11.31) and (2)–(3), $\sum_{n=0}^{\infty} h_n^{\kappa+1}(0) C_n^{(d-2(\kappa+1)-1)/2}(1) < \infty$ ■

Proof of Corollary 11.4.12. We can proceed as in the Proof of Corollary 11.4.4 and rewrite $h_n^{\kappa}, n = 0, 1, \ldots, \kappa - 1$, as

$$h_n^{\kappa}(t) = \tilde{h}_{1,n}^{\kappa}(t) - \tilde{h}_{2,n}^{\kappa}(t),$$

where

$$\tilde{h}_{1,n}^{\kappa}(t) := \gamma^{d,k,\kappa} \sum_{i=0}^{\infty} \chi_{2i}^{n,d,\kappa} g_{2i}^{\kappa}(t), \quad \text{and}$$

$$\tilde{h}_{2,n}^{\kappa}(t) := \gamma^{d,k,\kappa} \sum_{i=0}^{\infty} \chi_{2i+1}^{n,d,\kappa} g_{2i+1}^{\kappa}(t). \tag{A.10}$$

Define the bounded function H^{κ} on $[-1, 1] \times [0, \infty)$ by

$$H^{\kappa}(x, t) := \sum_{n=0}^{\kappa-1} \tilde{h}_{2,n}^{\kappa}(t) C_n^{(d-2\kappa-1)/2}(x) - \tilde{h}_{1,0}^{\kappa}(t) C_0^{(d-2\kappa-1)/2}(x)$$

By Remark 11.4.5, it is clear that $\tilde{h}_{1,n}^{\kappa} \in \mathcal{P}\left(\mathbb{R}^k\right), n = 1, 2, \ldots, \kappa - 1$, and also $h_n^{\kappa} \in \mathcal{P}\left(\mathbb{R}^k\right)$ for $n \geq \kappa$. Therefore,

$$H^{\kappa}(x, t) + I_3^{\kappa}\varphi(x, t) = \sum_{n=1}^{\kappa-1} \tilde{h}_{1,n}^{\kappa}(t) C_n^{(d-2\kappa-1)/2}(x) + \sum_{n=\kappa}^{\infty} h_n^{\kappa}(t) C_n^{(d-2\kappa-1)/2}(x)$$

has an expansion as Eq. (11.2) with the series uniformly convergent on $[-1, 1] \times [0, \infty)$ due to Theorem 11.4.10. By Theorem 11.3.3 of Ref. [34] (see Eq. (11.2)), we can conclude that the function $H^{\kappa} + I_1^{\kappa}\varphi$ belongs to the class $\mathcal{P}\left(\mathbb{S}^{d-2\kappa} \times \mathbb{R}^k\right)$. ■

Proof of Theorem 11.4.13. As in the Proof of Theorem 11.4.7, for any constants A and B, by Eq. (11.27),

$$\begin{aligned} &A\tilde{h}_{1,n}^{k}(t) - B\tilde{h}_{2,n}^{k}(t) \\ &\quad = \sum_{i=0}^{\infty} \int_0^{\infty} \Omega_{k-2\kappa}(tr) \left(A\chi_{2i}^{n,d,\kappa} \frac{1}{r^{2\kappa}} dF_{2i}(r) - B\chi_{2i+1}^{n,d,\kappa} \frac{1}{r^{2\kappa}} dF_{2i+1}(r) \right). \end{aligned} \tag{A.11}$$

Since, by Eq. (11.31),

$$\int_0^{\infty} A \left|\chi_i^{n,d,\kappa}\right| \frac{1}{r^{2\kappa}} dF_i(r) \leq A \Upsilon^{n,d,\kappa} C_i^{(d-1)/2}(1) \int_0^{\infty} \frac{1}{r^{2\kappa}} dF_i(r),$$

we have

$$\int_0^\infty \left(A\chi_{2i}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} dF_{2i}(r) - B\chi_{2i+1}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} dF_{2i+1}(r) \right) < \infty$$

and by Hypothesis (1) the series in (A.11) converges absolutely and uniformly on $[0, \infty)$.

Thus,

$$A\tilde{h}_{1,n}^k(t) - B\tilde{h}_{2,n}^k(t) = \int_0^\infty \Omega_k(tr) d \left(\sum_{i=0}^\infty A\chi_{2i}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} F_{2i}(r) - B\chi_{2i+1}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} F_{2i+1}(r) \right).$$

By (2), the series $\sum_{i=0}^\infty \chi_{2i}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} F_{2i}$ and $\sum_{i=0}^\infty \chi_{2i+1}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} F_{2i+1}$ are uniformly bounded on $[0, \infty)$. Then we can choose A^n, B^n such that $\sum_{i=0}^\infty A^n \chi_{2i}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} F_{2i} - B^n \chi_{2i+1}^{n,d,\varkappa} \frac{1}{r^{2\varkappa}} F_{2i+1}$ is non-negative, which allows us to conclude that $A^n \tilde{h}_{1,n}^\varkappa - B^n \tilde{h}_{2,n}^\varkappa \in \mathcal{P}\left(\mathbb{R}^k\right)$. The uniform convergence of the series (11.36) follows by Theorem 11.4.10 and the result by Theorem 11.3.3 of Ref. [34] (see Eq. (11.2)). ■

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