Transposed Poisson ultra algebras

Aline Jaqueline de Oliveira Andrade* ![ORCID]
Bruno Leonardo Macedo Ferreira ![ORCID]

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Abstract

In this paper, we introduce transposed Poisson structures on the Heisenberg-Virasoro ultra algebra. We obtain a result that tells us that there are no non-trivial $\frac{1}{2}$-ultra derivations on the Heisenberg-Virasoro ultra algebra and, as a consequence, we show that there are no non-trivial transposed Poisson ultra algebra structures defined on the Heisenberg-Virasoro ultra algebra.

Introduction

The study of transposed Poisson algebra has been of interest to many researchers, in 2023, Bai, Bai, Guo, and Wu [3] introduced a innovate extension of Poisson algebras termed transposed Poisson algebras, wherein the roles of the two binary operations in the Leibniz rule defining Poisson algebras are exchanged. Their research demonstrated that transposed Poisson algebras, thus defined, not only possess the common properties of Poisson algebras, such as closure under tensor products and Koszul self-duality as an operad, but also exhibit a diverse array of identities. Notably, a transposed Poisson algebra naturally emerges from a Novikov-Poisson algebra through the derivation of the commutator Lie algebra of the Novikov algebra, as elucidated by Beites, Ferreira and Kaygorodov [4]. Furthermore, any unital transposed Poisson algebra is identified as a specific instance of both a "contact bracket" algebra and a quasi-Poisson algebra.

According to a recent study conducted by Ferreira, Kaygorodov, and Lopatkin [8], the connection between $\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras was established. Leveraging these insights, the authors provided a comprehensive description of transposed Poisson structures across various algebraic contexts, including Witt and Virasoro algebras [8], twisted Heisenberg-Virasoro, Schrödinger-Virasoro, and extended Schrödinger-Virasoro algebras by Yuang and Hua [14], as well as oscillator algebras [4].

Chen and Dai [5], in 2020, introduced a class of super Heisenberg-Virasoro algebras based on conformal modules of Lie conformal superalgebras. They constructed a series of simple super Heisenberg-Virasoro modules, derived from simple modules of finite-dimensional solvable Lie superalgebras. These modules encompassed various structures, including highest weight modules, Whittaker modules, and high-order Whittaker modules.

Subsequently, in 2022, Chen, Dai and Hong [6] presented the Heisenberg-Virasoro Lie conformal superalgebra s, utilizing a class of Heisenberg-Virasoro Lie conformal modules. By defining the Heisenberg-Virasoro super algebra of Ramond type G from the formal distribution Lie superalgebra of s, the authors established a set of simple G-modules, isomorphic to simple restricted G-modules, and inclusive of highest weight modules, Whittaker modules, and high-order Whittaker modules. Additionally, they identified a subalgebra of G isomorphic to the super Heisenberg-Virasoro algebra of Neveu-Schwarz type.

In the same year, Li and Sun [11] conducted a comprehensive investigation into super-biderivations on the super Heisenberg-Virasoro algebra, unveiling the existence of non-inner super-skewsymmetric super-biderivations of the algebra.

Recent research by Yang and Tang [12] delved into transposed Poisson structures on the super case of the twisted Heisenberg-Virasoro algebra. Their findings established that the super Heisenberg-Virasoro algebra lacks non-trivial $\frac{1}{2}$-superderivations, precluding the existence of non-trivial transposed Poisson structures. These inquiries, rooted in the framework of graded $\mathbb{Z}_2$ algebras, mirror the dualistic nature of elementary particles in particle physics, categorized as bosons

and fermions, each subcategory associated with distinct parts of the graded algebra.

The study of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded structures holds significant importance in physics, with Aizawa, Kuznetsova, and Toppan [1, 2] introduced the notion of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical mechanics and also the notion fo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantic mechanics. Kuznetsova, and Toppan [10] presented the classification over the fields of real and complex numbers, of the minimal $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras spanned by four generators and with no empty graded sector.

Inspired by Kac [9], Ferreira [7] coined the term "ultra algebra" to denote $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebras, emphasizing their distinction from superalgebras.

Motivated by the pivotal role of ultra-algebras in physics, this article explores transposed Poisson structures within a broader context than the super Heisenberg-Virasoro algebra. Specifically, we introduce the concepts of ultra Heisenberg-Virasoro algebras and transposed Poisson ultra algebras, establishing a fundamental relationship between $\frac{1}{2}$-ultra derivations of Lie ultra algebras and transposed Poisson ultra algebras.

This paper is structured as follows: in section 2 we give the definition of Transposed Poisson ultra algebra, for short TPUA, and other definitions and results necessary for the development of this work. In section 3 we define the Heisenberg-Virassoro ultra algebra, for short $\mathfrak{u}\mathcal{HV}$, and prove the lemmas that form the proof of our main theorems.

1 Preliminares

Definition 1. *(see [7]) Let vector space A over a field $\mathbb{F}$ be a direct sum of four subspaces A_{ij}, $i,j \in \{0,1\}$, where $A = A_{00} \oplus A_{10} \oplus A_{01} \oplus A_{11}$. For any two elements in A, we define a product $\cdot$ satisfying the following rule*

(i) Closure: For all $a, b \in A$ we have $a \cdot b \in A$.

(ii) Bilinearity: For all $a, b, c \in A$, and $\alpha, \beta \in \mathbb{F}$, we have

$$(\alpha a + \beta b) \cdot c = \alpha a \cdot c + \beta b \cdot c, \qquad \text{and} \qquad c \cdot (\alpha a + \beta b) = \alpha c \cdot a + \beta c \cdot b.$$

(iii) Grading: For all $a \in A_{ij}, b \in A_{mn}$, with $i, j, m, n \in \{0, 1\}$, we have

$$a \cdot b \in A_{(i+m)\mathrm{mod2},(j+n)\mathrm{mod2}}.$$

A linear space satisfying the above conditions is called a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra that Ferreira in [7] appointed as ultra algebra.

Below is an example of a ultra algebra.

Example 2. *Consider the matrix representation of the quaternions (see [10]) which are generated by e_0, e_1, e_2 and e_3 such that*

$$e_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra or ultra algebra is $A = A_{00} \oplus A_{01} \oplus A \oplus A_{11}$, where $A_{00} = \mathrm{span}_{\mathbb{R}}\{e_0\}$, $A_{10} = \mathrm{span}_{\mathbb{R}}\{e_1\}$, $A_{01} = \mathrm{span}_{\mathbb{R}}\{e_2\}$ and $A_{11} = \mathrm{span}_{\mathbb{R}}\{e_3\}$.

Definition 3. *Let A be an ultra algebra and δ an element of the field. A homogeneous endomorphism ψ of ultra space of endomorphisms is called δ-ultra derivation if satisfies*

$$\psi([x,y]) = \delta([\psi(x), y] + (-1)^{\deg(x)\ \deg(\psi)}\ [x, \psi(y)]).$$

Example 4. *Let $f(x) = \lambda\ x = \lambda Id(x)$, where $\lambda \in \mathbb{F}$. Observe that f is a δ-ultra derivation. These δ-ultra derivation are called trivial.*

Lemma 5. *Let ψ_1 be a δ_1-ultra derivation of (i,j) type and let ψ_2 be δ_2-ultra derivation of (k,t) type. Then the ultra commutator*

$$[\psi_1, \psi_2]_{\mathfrak{u}} = \psi_1\psi_2 - (-1)^{\deg(\psi_1)\deg(\psi_2)}\ \psi_2\psi_1$$

is a $\delta_1\delta_2$-ultra derivation of $(i+k, j+t)$ mod 2 type. In fact, let $x \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11}$, such that $\deg(x) = (m,n)\mathrm{mod2}$. Thus, $[\psi_1, \psi_2](x) = \left(\psi_1\psi_2 - (-1)^{\deg(\psi_1)\deg(\psi_2)}\psi_2\psi_1\right)(x)$. Obseve that $\psi_2(x) \in A_{(m+k,n+t)\mathrm{mod2}}$ and $\psi_1(\psi_2(x)) \in A_{(m+i+k,j+n+t)\mathrm{mod2}}$.

On the other hand, $\psi_1(x) \in A_{(m+i,n+j)\mathrm{mod2}}$ and $\psi_2(\psi_1(x)) \in A_{(k+m+i,t+n+j)\mathrm{mod2}}$. Then $[\psi_1, \psi_2](x) \in A_{(i+k+m,j+t+n)\mathrm{mod2}}$. Therefore $[\psi_1, \psi_2] \in A_{(i+k,j+t)\mathrm{mod2}}$.

Let G be a group. A G-graded algebra is a algebra $(A, \cdot)$ equipped with a G-graded given by

$$A = \bigoplus_{g \in G} A_g,$$

such that $A_g \cdot A_h \subseteq A_{g+h}$ for all $g, h \in G$.

Definition 6. *Let A be an ultra algebra. If A satisfies:*

$$[a, b] = -(-1)^{\bar{a}\bar{b}}[b, a] \qquad (ultra\ bracket\ identity)$$

and

$$[a, [b, c]](-1)^{\bar{a}\bar{c}} + [b, [c, a]](-1)^{\bar{a}\bar{b}} + [c, [a, b]](-1)^{\bar{b}\bar{c}} = 0 \qquad (ultra\ Jacobi\ identity),$$

where $a, b, c \in A_{00} \cup A_{10} \cup A_{01} \cup A_{11}$ then A is called a Lie ultra algebra. And we will say that an Lie ultra algebra A is a perfect if $[A, A] = A$.

We will now introduce the concept of transposed Poisson ultra algebra, for short TPUA, which is a natural extension of the concept of transposed Poisson algebra (TPA) and it is also an extension of a transposed Poisson superalgebra.

Definition 7. *Let $A = A_{00} \oplus A_{10} \oplus A_{01} \oplus A_{11}$ be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graduated vector space equipped with two bilinear ultra operations $\cdot$ and $[\,\cdot, \cdot\,]$. The tern $(A, \cdot, [\,\cdot, \cdot\,])$ is called a TPUA, if $(A, \cdot)$ is a commutative associative ultra algebra and $(A, [\,\cdot, \cdot\,])$ is a Lie ultra algebra which satisfies the compatibility condition*

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{\deg(x)\deg(y)} [x, z \cdot y], \quad \text{for all } x, y, z \in A_{00} \cup A_{10} \cup A_{01} \cup A_{11}.$$

Remark 8. *The authors provide the following result [3, Proposition 2.2]: Let $(L, \cdot)$ be a commutative algebra and let D be a derivation. Define the multiplication*

$$[x, y] =: [x, y]_D =: xD(y) - D(x)y, \quad \forall\ x, y \in L.$$

Then $(L, \cdot, [\,\cdot, \cdot\,])$ is a transposed Poisson algebra. This result in the ultra algebra version would be as follows: Let $(L, \cdot)$ be a non-trivial commutative ultra algebra and let D be a ultra derivation. Define the multiplication

$$[x, y] =: [x, y]_D =: xD(y) - (-1)^{\deg(y)\deg(D)} D(x)y, \quad \forall\ x, y \in L.$$

Then $(L, \cdot, [\,\cdot, \cdot\,])$ is a transposed Poisson ultra algebra. But with simple calculations we can see that this result is not true. In fact, considerer $\deg(D) = (1, 0)$ and $\deg(x) = \deg(y) = \deg(z) = (0, 1)$, note that ultra Jacobi identity is not satisfied. This is because

$$(-1)^{\deg(x)\deg(z)}[x, [y, z]] + (-1)^{\deg(y)\deg(x)}[y, [z, x]] + (-1)^{\deg(z)\deg(y)}[z, [x, y]]$$
$$= -2xyD^2(z) - 2D(x)D(y)z + 2xD^2(y)z - 2yzD^2(x) \neq 0.$$

Definition 9. *Let $(A, [\ ,\ ])$ be a Lie ultra algebra. A transposed Poisson ultra algebra structure on $(A, [\ ,\ ])$ is a ultra commutative associative multiplication $\cdot$ on A which makes $(A, \cdot, [\ ,\ ])$ a TPUA. A transposed Poisson structure $\cdot$ on A is called trivial, if $x \cdot y = 0$ for all $x, y \in A$.*

Lemma 10. *Let $(A, \cdot, [\ ,\ ])$ be a TPUA and $z \in A_{00} \cup A_{10} \cup A_{01} \cup A_{11}$. Then the left multiplication L_z in the ultra commutative associative ultra algebra $(A, \cdot)$ gives a $\frac{1}{2}$-ultra derivation of the Lie ultra algebra $(A, [\ ,\ ])$ and $\deg(L_z) = \deg(z)$.*

Proof. From the Definitions 7 and 9, we have that L_z is a $\frac{1}{2}$-ultra derivation of the Lie ultra algebra $(A, [\ ,\ ])$. Let L_z be the homogeneous left multiplication of the $\deg(L_z) = (i, j)$. Considerer $\deg(z) = (q_1, q_2)$ and the homogeneous element $u \in A$, with $\deg(u) = (k, t)$. Thus, $z \cdot u \in A_{(q_1+k, q_2+t)mod2}$ and $z \cdot u = L_z(u) \in A_{(i+k, j+t)mod2}$. Therefore $(q_1, q_2) = (i, j)$, i.e., $\deg(z) = \deg(L_z)$. $\square$

In this sense, our next result is a generalization of the [8, Theorem 8], whose proof follows the same reasoning as Theorem 8, but for the convenience of the reader we will give the proof for ultra algebras.

Lemma 11. *Let A be a Lie ultra algebra of* $\dim > 1$ *without non-trivial $\frac{1}{2}$-derivations. Then every TPUA defined over A is a trivial.*

Proof. Let $(A, \cdot, [\ ,\ ])$ be a TPUA. Consider $\psi_y(x) = y \cdot x$, from the Lemma 10, we have that ψ_y is a $\frac{1}{2}$-ultra derivation of $(A, [\ ,\ ])$. As $(A, \cdot)$ is a ultra commutative associative ultra algebra, thus $\psi_y(x) = y \cdot x = x \cdot y = \psi_x(y)$ and ψ_x is a $\frac{1}{2}$-ultra derivation of $(A, [\ ,\ ])$. By the hypothesis, there are $k_x, k_y \in \mathbb{C}$ such that $\psi_x(z) = k_x z$ and $\psi_y(z) = k_y z$, for all homogeneous element $z \in A_{00} \cup A_{01} \cup A_{10} \cup A_{11}$ and extend it by linearity to all elements of A.

If $(A, \cdot, [\ ,\ ])$ is a non-trivial, then $\dim A > 1$ and we can choose $x, y \in A$ such that x and y are linear independent elements. Thus, $0 = \psi_x(y) - \psi_y(x) = k_x y - k_y x$, implies that $k_x = k_y = 0$. Therefore, $x \cdot y = \psi_x(y) = k_x y = 0$. $\square$

2 Heisenberg-Virasoro ultra Algebra ($\mathfrak{u}\mathcal{HV}$)

In this section, we will define $\mathfrak{u}\mathcal{HV}$, as well as present other basic concepts and results related to these structures. As mentioned in the introduction in 2020 Chen, Dai and Hong [5] and in 2022 the same authors [6] introduced a new class of algebraic structures namely super Heisenberg-Virassoro algebra, and provided a characterization of simple restricted modules for it. So we give here a broader definition in the sense of ultra algebras in order to extend this concept, but first we recall the definition of a Lie ultra algebra.

Now we present the definition of ultra Heisenberg-Virassoro algebra $\mathfrak{u}\mathcal{HV}$. The definition of $\mathfrak{u}\mathcal{HV}$ is given in such a way that there exists a subalgebra of $\mathfrak{u}\mathcal{HV}$ which is isomorphic to $s\mathcal{HV}$. The $\mathfrak{u}\mathcal{HV}$ is a Lie ultra algebra $\mathcal{U}$ of infinite dimensional

$$\mathcal{U} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}L_m \bigoplus_{m \in \mathbb{Z}} \mathbb{C}G_{m+i} \bigoplus_{m \in \mathbb{Z}} \mathbb{C}I_m \bigoplus_{m \in \mathbb{Z}} \mathbb{C}P_m \bigoplus_{m \in \mathbb{Z}} \mathbb{C}Q_m \bigoplus \mathbb{C}c,$$

where $i = 0$ (the Ramond case) and if $i = \frac{1}{2}$ (the Neveu-Schwarz case), in both cases, c is central in ultra algebra and the ultra-brackets are given by

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\ c, \qquad\qquad [Q_{s_1}, Q_{s_2}] = 2I_{s_1+s_2}$$
$$[L_m, I_n] = nI_{m+n} \qquad\qquad [L_m, P_t] = tP_{t+m}$$
$$[L_m, G_r] = rG_{m+r} \qquad\qquad [L_m, Q_s] = sQ_{m+s}$$
$$[G_r, G_s] = 2I_{r+s} \qquad\qquad [Q_s, G_r] = rP_{s+r}$$
$$[P_{t_1}, P_{t_2}] = 2I_{t_1+t_2} \qquad\qquad [I_m, Q_s] = sQ_{m+s}$$
$$[P_t, Q_s] = sG_{t+s}$$

By definition, we consider the following decomposition:

$$\mathcal{U} = \mathcal{U}_{00} \oplus \mathcal{U}_{10} \oplus \mathcal{U}_{01} \oplus \mathcal{U}_{11},$$

where $\mathcal{U}_{00} = \text{span}_{\mathbb{C}}\{L_m, I_m, c\}_{m\in\mathbb{Z}}$, $\mathcal{U}_{10} = \text{span}_{\mathbb{C}}\{P_t\}_{t\in\mathbb{Z}}$, $\mathcal{U}_{01} = \text{span}_{\mathbb{C}}\{Q_s\}_{s\in\mathbb{Z}}$ and $\mathcal{U}_{11} = \text{span}_{\mathbb{C}}\{G_r\}_{r\in i+\mathbb{Z}}$. It easy to see that $\mathcal{U}_{00} \oplus \mathcal{U}_{11}$ is isomorphic to a subalgebra of $s\mathcal{HV}$.

Remark 12. *Every* $\mathfrak{u}\mathcal{HV}$ *algebra is a perfect.*

Our objective is to investigate the $\frac{1}{2}$-derivations of Heisenberg-Virassoro ultra algebra.

Question 13. *There are non-trivial TPUA structures defined in* $\mathfrak{u}\mathcal{HV}$ *algebras?*

Let $((\mathfrak{u}\mathcal{HV})_{00})_j = \text{span}_{\mathbb{C}}\{L_j, I_j\}$, for $0 \neq j \in \mathbb{Z}$ and $((\mathfrak{u}\mathcal{HV})_{00})_0 = \text{span}_{\mathbb{C}}\{L_0, I_0, c\}$. Thus $(\mathfrak{u}\mathcal{HV})_{00} = \bigoplus_{j\in\mathbb{Z}} ((\mathfrak{u}\mathcal{HV})_{00})_j$ is a $\mathbb{Z}$-graded algebra. We easily conclude that $(\mathfrak{u}\mathcal{HV})_{00}$ is finitely generated.

The [13, Lemma 2.4] show that the $\mathbb{Z}$-graduate of $(\mathfrak{u}\mathcal{HV})_{00}$ induce the decomposition $\varphi = \sum_{j\in\mathbb{Z}} \varphi_j$, where φ_j is a $\frac{1}{2}$-derivation on $(\mathfrak{u}\mathcal{HV})_{00}$ of degree j, that is, $\varphi_j(((\mathfrak{u}\mathcal{HV})_{00})_k) \subseteq ((\mathfrak{u}\mathcal{HV})_{00})_{j+k}$, for all $k \in \mathbb{Z}$.

We know that there is a subalgebra of $\mathfrak{u}\mathcal{HV}$ (say U) such that $s\mathcal{HV} \cong U = \mathcal{U}_{00} \oplus \mathcal{U}_{11}$, such a way that $s\mathcal{HV}_{\bar{0}} \cong \mathcal{U}_{00}$ and $s\mathcal{HV}_{\bar{1}} \cong \mathcal{U}_{11}$. Note that a $\frac{1}{2}$-ultra derivation on $(\mathfrak{u}\mathcal{HV})_{00}$ is a $\frac{1}{2}$-derivation em $(\mathfrak{u}\mathcal{HV})_{00}$. From the [12, Lemma 3.1] we get the following result.

Lemma 14. *Every* $\frac{1}{2}$-*ultra derivation on* $(\mathfrak{u}\mathcal{HV})_{00}$ *is trivial.*

The following lemma is a generalization of [12, Lemma 3.2] and for this reason we have used the same ideas as the authors. In this way, the proof of the Lemma 15 is analogous to [12, Lemma 3.2], but in order to make the article self-contained, we've done the proof below.

Lemma 15. *Every homogeneous* $\frac{1}{2}$-*ultra derivation* $(0,0)$ *type is trivial.*

Proof. It's easy to see that $\varphi_{00}|_{(\mathfrak{u}\mathcal{HV})_{00}}$ is a derivation of $(\mathfrak{u}\mathcal{HV})_{00}$ and by the Lemma 14, we know $\varphi_{00}|_{(\mathfrak{u}\mathcal{HV})_{00}}$ is trivial. Therefore, we can consider $\varphi_{00}|_{(\mathfrak{u}\mathcal{HV})_{00}} = 0$.

Let $X_l \in \operatorname{span}_{\mathbb{C}}\{P_t\}_{t \in \mathbb{Z}} \cup \operatorname{span}_{\mathbb{C}}\{Q_s\}_{s \in \mathbb{Z}} \cup \operatorname{span}_{\mathbb{C}}\{G_r\}_{r \in i+\mathbb{Z}}$. If $\varphi_{00}(X_l) = \sum_{t \in i+\mathbb{Z}} \Gamma_t^l X_t$, $\Gamma_t^l \in \mathbb{C}$, then

$$(1) \qquad 2\varphi_{00}([L_0, X_l]) = 2l\varphi_{00}(X_l) = 2l \sum_{t \in i+\mathbb{Z}} \Gamma_t^l X_t.$$

On the other hand,

$$2\varphi_{00}([L_0, X_l]) = [\varphi_{00}(L_0), X_l] + [L_0, \varphi_{00}(X_l)]$$

$$(2) \qquad = [L_0, \varphi_{00}(X_l)] = \left[L_0, \sum_{t \in i+\mathbb{Z}} \Gamma_t^l X_t\right] = \sum_{t \in i+\mathbb{Z}} t\Gamma_t^l X_t.$$

By comparing the Equations 1 and 2, we obtain

$$2l \sum_{t \in i+\mathbb{Z}} \Gamma_t^l X_t = \sum_{t \in i+\mathbb{Z}} t\Gamma_t^l X_t.$$

Thus, $\Gamma_t^l = 0$, $t \neq 2l$. Follows that

$$(3) \qquad \varphi_{00}(X_l) = 0, \quad \forall l \in \frac{1}{2} + \mathbb{Z} \quad \text{and} \quad \varphi_{00}(X_l) = \Gamma_{2l}^l X_{2l}, \ \forall l \in \mathbb{Z}.$$

We consider only the case $i = 0$. Now,

$$0 = 2\varphi_{00}(2I_{r+s}) = 2\varphi_{00}([X_r, X_s]) = [\varphi_{00}(X_r), X_s] + [X_r, \varphi_{00}(X_s)]$$
$$= [\Gamma_{2r}^r X_{2r}, X_s] + [X_r, \Gamma_{2s}^s X_{2s}] = 2\Gamma_{2r}^r I_{2r+s} + 2\Gamma_{2s}^s I_{r+2s}.$$

For $r \neq s$, we have $\Gamma_{2r}^r = 0$, for all $r \in \mathbb{Z}$. Thus,

$$(4) \qquad \varphi_{00}(X_l) = 0, \quad \forall l \in \mathbb{Z}.$$

From the equations 3 and 4, we obtain $\varphi_{00} = 0$.

Therefore, we conclude that every homogeneous $\frac{1}{2}$-ultra derivation of $(0,0)$ type is a trivial. $\qquad \square$

Lemma 16. *Every homogeneous $\frac{1}{2}$-ultra derivation $(1,0)$ type φ_{10} is zero.*

Proof. Let φ_{10} be an $\frac{1}{2}$-ultra derivation $(1,0)$ type and ad_{P_r} inner ultra derivation $(1,0)$ type. From the Lemma 15, $[\varphi_{10}, \operatorname{ad}_{P_r}]_{\mathfrak{u}}$ is a $\frac{1}{2}$-ultra derivation $(0,0)$ type of $\mathfrak{u}\mathcal{HV}$ trivial, in other words, $[\varphi_{10}, \operatorname{ad}_{P_r}]_{\mathfrak{u}} = \beta_r Id$. If $\varphi_{10}(c) = \sum_{t \in \mathbb{Z}} \alpha_t P_t$, then $\alpha_t = 0$ for all $t \in \mathbb{Z}$ and $\beta_r = 0$ for all $r \in \mathbb{Z}$.

In fact, as $[\varphi_{10}, \operatorname{ad}_{P_r}]_{\mathfrak{u}} = \beta_r Id$, it follows that,

$$\beta_r c = [\varphi_{10}, \operatorname{ad}_{P_r}](c) = (\varphi_{10}\operatorname{ad}_{P_r} - \operatorname{ad}_{P_r}\varphi_{10})(c) = 0 - [P_r, \varphi_{10}(c)]$$

$$= -\left[P_r, \sum_{t \in \mathbb{Z}} \alpha_t P_t\right] = -\sum_{t \in \mathbb{Z}} \alpha_t [P_r, P_t] = -2\sum_{t \in \mathbb{Z}} \alpha_t I_{r+t},$$

which it follows that $\left(2\sum_{t\in\mathbb{Z}}\alpha_t I_{r+t}+\beta_r c\right)=0$. Now, as $\{I_m,c\}$ is a linearly independent set, we have $\alpha_t=0$, for all $t\in\mathbb{Z}$ and $\beta_r=0$, for all $r\in\mathbb{Z}$.

Thus,

(5) $\varphi_{10}(c)=0,$ and $[\varphi_{10},\mathrm{ad}_{P_r}]_{\mathfrak{u}}=0,\ \ \forall r\in\mathbb{Z}.$

And more,

$$0=2[\varphi_{10},\mathrm{ad}_{P_r}](L_m)=2\left(\varphi_{10}\mathrm{ad}_{P_r}-\mathrm{ad}_{P_r}\varphi_{10}\right)(L_m)=2\varphi_{10}([P_r,L_m])-2[P_r,\varphi_{10}(L_m)]$$
$$=[\varphi_{10}(P_r),L_m]+[P_r,\varphi_{10}(L_m)]-2[P_r,\varphi_{10}(L_m)]=[\varphi_{10}(P_r),L_m]-[P_r,\varphi_{10}(L_m)].$$

Thus,

(6) $[L_m,\varphi_{10}(P_r)]=[P_r,\varphi_{10}(L_m)]$
(7) $2\varphi_{10}([L_m,P_r])=[\varphi_{10}(L_m),P_r]+[L_m,\varphi_{10}(P_r)].$

By Equations 6 and 7, we have

(8) $\varphi_{10}([L_m,P_r])=[\varphi_{10}(L_m),P_r]=[L_m,\varphi_{10}(P_r)].$

Assume $\varphi_{10}(P_r)=\sum_{k\in\mathbb{Z}}\theta_k^r L_k+\sum_{k\in\mathbb{Z}}\varepsilon_k^r I_k+\sigma^r c$. Then

$$[L_0,\varphi_{10}(P_r)]=\left[L_0,\sum_{k\in\mathbb{Z}}\theta_k^r L_k+\sum_{k\in\mathbb{Z}}\varepsilon_k^r I_k+\sigma^r c\right]=\sum_{k\in\mathbb{Z}}\theta_k^r[L_0,L_k]+\sum_{k\in\mathbb{Z}}\varepsilon_k^r[L_0,I_k]+\sigma^r[L_0,c]$$
$$=\sum_{k\in\mathbb{Z}}k\theta_k^r L_k+\sum_{k\in\mathbb{Z}}k\varepsilon_k^r I_k.$$

On the other hand,

$$\varphi_{10}([L_0,P_r])=\varphi_{10}(rP_r)=r\varphi_{10}(P_r)=r\sum_{k\in\mathbb{Z}}\theta_k^r L_k+r\sum_{k\in\mathbb{Z}}\varepsilon_k^r I_k+r\sigma^r c,$$
$$\varphi_{10}(P_r)=\theta_r^r L_r+\varepsilon_r^r I_r,\quad r\neq 0,$$
$$\varphi_{10}(P_0)=\theta_0^0 L_0+\varepsilon_0^0 I_0+\sigma^0 c.$$

We assume that $\varphi_{10}(L_m)=\sum_{t\in\mathbb{Z}}w_t^m P_t$, $w_t^m\in\mathbb{C}$. Then,

$$0=\varphi_{10}([L_m,P_0])=[L_m,\varphi_{10}(P_0)]=[L_m,\theta_0^0 L_0+\varepsilon_0^0 I_0+\sigma^0 c]$$
$$=-m\theta_0^0 L_0+\frac{\theta_0^0(m^3-m)}{12}\delta_{m,0}c.$$

If $m\neq 0$, then $\theta_0^0=0$ and $\varphi_{10}(P_0)=\varepsilon_0^0 I_0+\sigma^0 c$. If $r\neq 0$, then

$$[L_{-r},\varphi_{10}(P_r)]=[L_{-r},\theta_r^r L_r+\varepsilon_r^r I_r]=2r\theta_r^r L_0+\frac{\theta_r^r(r-r^3)}{12}c+r\varepsilon_r^r I_0.$$

and
$$[L_{-r}, \varphi_{10}(P_r)] = \varphi_{10}[L_{-r}, P_r] = r\varphi_{10}(P_0) = r\varepsilon_0^0 I_0 + r\sigma^0 c.$$

We have that $\theta_r^r = 0$, $\varepsilon_r^r = \varepsilon_0^0$, $r \neq 0$ and $\sigma^0 = 0$ which implies

$$\varphi_{10}(P_r) = \varepsilon_0^0 I_r, \quad \forall r \in \mathbb{Z}.$$

Thus,

$$r\varepsilon_0^0 I_{m+r} = r\varphi_{10}(P_{m+t}) = \varphi_{10}([L_m, P_r]) = [\varphi_{10}(L_m), P_r] = \left[\sum_{t\in\mathbb{Z}} w_t^m P_t, P_r\right] = 2\sum_{t\in\mathbb{Z}} w_t^m I_{t+r}.$$

Therefore, $w_t^m = 0$, for all $t \neq m$ and $2w_m^m = r\varepsilon_0^0$. if $r = 0$, then $w_m^m = 0$. If $r \neq 0$, then $\varepsilon_0^0 = 0$. Thus, $\varphi_{10}(L_m) = 0$, $\varphi_{10}(P_r) = 0$ and $4\varphi_{10}(I_{r+s}) = 2\varphi_{10}([P_r, P_s]) = [\varphi_{10}(P_r), P_s] - [P_r, \varphi_{10}(P_s)] = 0$. By the arbitrariness of r and s, we have $\varphi_{10}(I_m) = 0$.

Now, as $\varphi_{10}(Q_s) \in \mathcal{U}_{11}$, we have $\varphi_{10}(Q_s) = \sum_{r\in i+\mathbb{Z}} \eta_r^s G_r$. From the product $[I_m, Q_s] = sQ_{m+s}$, we have $[I_0, Q_s] = sQ_s$. Thus

$$2s\sum_{r\in i+\mathbb{Z}} \eta_r^s G_r = 2s\varphi_{10}(Q_s) = 2\varphi_{10}([I_0, Q_s]) = [\varphi_{10}(I_0), Q_s] + [I_0, \varphi_{10}(Q_s)]$$

$$= \left[I_0, \sum_{r\in i+\mathbb{Z}} \eta_r^s G_r\right] = \sum_{r\in i+\mathbb{Z}} \eta_r^s[I_0, G_r] = 0.$$

We have $\eta_r^s = 0$, for all $r \in i + \mathbb{Z}$.

Thus $\varphi_{10}(Q_s) = 0$. Now, from the product $[P_t, Q_s] = sG_{t+s}$, we have $[P_0, Q_s] = sG_s$. Thus

$$2s\varphi_{10}(G_s) = 2\varphi_{10}([P_0, Q_s]) = [\varphi_{10}(P_0), Q_s] + [P_0, \varphi_{10}(Q_s)] = 0.$$

If $s \neq 0$, then $\varphi_{10}(G_s) = 0$. If $s = 0$, then from the product $[L_m, G_r] = rG_{m+r}$, we obtain $[L_{-r}, G_r] = rG_0$, for all $r \neq 0$. Thus

$$2r\varphi_{10}(G_0) = 2\varphi_{10}([L_{-r}, G_r]) = [\varphi_{10}(L_{-r}), G_r] + [L_{-r}, \varphi_{10}(G_r)] = 0.$$

Hence $r \neq 0$, we have $\varphi_{10}(G_0) = 0$. Thus, $\varphi_{10}(G_s) = 0$. Therefore, we conclude that $\varphi_{10} = 0$.

$\square$

Lemma 17. *Every homogeneous $\frac{1}{2}$-ultra derivation $(0, 1)$ type φ_{01} is trivial or zero.*

Proof. Let φ_{01} be a $\frac{1}{2}$-ultra derivation $(0,1)$ type. Consider ad_{Q_r} a inner ultra derivation $(0,1)$ type. From the Lemma 15, we have $[\varphi_{01}, \mathrm{ad}_{Q_r}]_{\mathbf{A}}$ is a $\frac{1}{2}$-ultra derivation $(0,0)$ type of $\mathfrak{u}\mathcal{H}\mathcal{V}$. And more, By Lemma 15, it follows that $[\varphi_{01}, \mathrm{ad}_{Q_r}]_{\mathfrak{u}}$ is trivial, that is, $[\varphi_{01}, \mathrm{ad}_{Q_r}]_{\mathfrak{u}} = \gamma_r Id$. We assume that $\varphi_{01}(c) = \sum_{t \in \mathbb{Z}} \alpha_t Q_t$. Thus

(9)

$$\gamma_r c = [\varphi_{01}, \mathrm{ad}_{Q_r}](c) = [Q_r, \varphi_{01}(c)] = -\left[Q_r, \sum_{t \in \mathbb{Z}} \alpha_t Q_t\right] = -\sum_{t \in \mathbb{Z}} \alpha_t [Q_r, Q_t] = -2 \sum_{t \in \mathbb{Z}} \alpha_t I_{r+t}$$

thus $\gamma_r = \alpha_t = 0$, for all $r, t \in \mathbb{Z}$. Then, $\varphi_{01}(c) = 0$ and $[\varphi_{01}, \mathrm{ad}_{Q_r}]_{\mathfrak{u}} = 0$, for all $r \in \mathbb{Z}$. Moreover,

$$0 = 2[\varphi_{01}, \mathrm{ad}_{Q_r}](L_m) = 2(\varphi_{01}\mathrm{ad}_{Q_r} - \mathrm{ad}_{Q_r}\varphi_{01})(L_m) = 2\varphi_{01}(\mathrm{ad}_{Q_r}(L_m)) - 2\mathrm{ad}_{Q_r}(\varphi_{01}(L_m))$$

$$= 2\varphi_{01}([Q_r, L_m]) - 2[Q_r, \varphi_{01}(L_m)] = [\varphi_{01}(Q_r), L_m] + [Q_r, \varphi_{01}(L_m)] - 2[Q_r, \varphi_{01}(L_m)]$$

$$= [\varphi_{01}(Q_r), L_m] - [Q_r, \varphi_{01}(L_m)] = -[L_m, \varphi_{01}(Q_r)] + [\varphi_{01}(L_m), Q_r]$$

(10)

$$\Rightarrow [L_m, \varphi_{01}(Q_r)] = [\varphi_{01}(L_m), Q_r]$$

Even more,

(11)

$$2\varphi_{10}([L_m, Q_r]) = 2\left(\frac{1}{2}[\varphi_{01}(L_m), Q_r] + \frac{1}{2}[L_m, \varphi_{01}(Q_r)]\right) = [\varphi_{01}(L_m), Q_r] + [L_m, \varphi_{01}(Q_r)].$$

From the Equations 10 and 11, we have

(12)

$$2\varphi_{01}([L_m, Q_r]) = 2[\varphi_{01}(L_m), Q_r] \Rightarrow \varphi_{01}([L_m, Q_r]) = [\varphi_{01}(L_m), Q_r] = [L_m, \varphi_{01}(Q_r)].$$

We assume that $\varphi_{01}(Q_r) = \sum_{k \in \mathbb{Z}} \theta_k^r L_k + \sum_{k \in \mathbb{Z}} \varepsilon_k^r I_k + \delta^r c$, with $\theta_k^r, \varepsilon_k^r, \delta^r \in \mathbb{C}$. Thus,

$$[L_0, \varphi_{01}(Q_r)] = \left[L_0, \sum_{k \in \mathbb{Z}} \theta_k^r L_k + \sum_{k \in \mathbb{Z}} \varepsilon_k^r I_k + \delta^r c\right] = \sum_{k \in \mathbb{Z}} \theta_k^r [L_0, L_k] + \sum_{k \in \mathbb{Z}} \varepsilon_k^r [L_0, I_k] + \delta^r [L_0, c]$$

(13)

$$= \sum_{k \in \mathbb{Z}} \theta_k^r k L_k + \sum_{k \in \mathbb{Z}} \varepsilon_k^r k I_k.$$

On the other hand,

(14) $\quad \varphi_{01}([L_0, Q_r]) = \varphi_{01}(rQ_r) = r\varphi_{01}(Q_r) = r\sum_{k \in \mathbb{Z}} \theta_l^r L_k + r\sum_{k \in \mathbb{Z}} \varepsilon_k^r I_k + r\delta^r c.$

Equating the equations 13 and 14, we obtain

(15)
$$\begin{aligned} k \neq r &\Rightarrow \theta_k^r = 0, \\ k \neq r &\Rightarrow \varepsilon_k^r = 0, \\ r \neq 0 &\Rightarrow \delta^r = 0. \end{aligned}$$

From the Equation 15 we have

(16) $\quad \varphi_{01}(Q_0) = \theta_0^0 L_0 + \varepsilon_0^0 I_0 + \delta^0 c \quad$ and $\quad \varphi_{01}(Q_r) = \theta_r^r L_r + \varepsilon_r^r I_r.$

Assume that

(17) $$\varphi_{01}(L_m) = \sum_{t \in \mathbb{Z}} w_t^m Q_t, \quad w_t^m \in \mathbb{C}.$$

Thus

$$0 = \varphi_{01}([L_m, Q_0]) = [L_m, \varphi_{01}(Q_0)] = [L_m, \theta_0^0 L_0 + \varepsilon_0^0 I_0 + \delta^0 c] = \theta_0^0 [L_m, L_0] + \varphi_0^0 [L_m, I_0]$$

$$= -m\theta_0^0 L_m + \frac{m^3 - m}{12} \theta_0^0 \delta_{m,0} c.$$

If $m \neq 0$, then $\theta_0^0 = 0$. Thus, $\varphi_{01}(Q_0) = \varepsilon_0^0 I_0 + \delta^0 c$. Se $r \neq 0$ and using the formula 16, then

(18) $\quad [L_{-r}, \varphi_{01}(Q_r)] = [L_{-r}, \theta_r^r L_r + \varepsilon_r^r I_r] = 2r\theta_r^r L_0 + \dfrac{(r - r^3)}{12} \theta_r^r c + \varepsilon_r^r I_0.$

On the other hand,

(19) $\quad [L_{-r}, \varphi_{01}(Q_r)] = \varphi_{01}([L_{-r}, Q_r]) = \varphi_{01}(rQ_0) = r\varphi_{01}(Q_0) = r\varepsilon_0^0 I_0 + r\delta^0 c.$

Equating the equations 18 and 19, we obtain

$$\begin{aligned} \theta_r^r &= 0, \quad r \neq 0, \\ \varepsilon_r^r &= \varepsilon_0^0, \quad r \neq 0, \\ \delta^0 &= 0. \end{aligned}$$

From the equations 12 and 17, we have

$$\begin{aligned} r\varepsilon_0^0 I_{m+r} = r\varphi_{01}(Q_{r+m}) = \varphi_{01}([L_m, Q_r]) &= [\varphi_{01}(L_m), Q_r] \\ &= \left[\sum_{t \in \mathbb{Z}} w_t^m Q_t, Q_r\right] = \sum_{t \in \mathbb{Z}} w_t^m [Q_t, Q_r] = 2\sum_{t \in \mathbb{Z}} w_t^m I_{t+r}. \end{aligned}$$

If $t \neq m$, then

(20) $$w_t^m = 0$$

and if $m = t$, then

(21) $$2w_m^m = r\varepsilon_0^0.$$

By replacing $r = 0$, in equation 21, we have

(22) $$w_m^m = 0.$$

If $r = 0$ in equation 21, then

(23) $\varepsilon_0^0 = 0.$

From the equations 20 and 22, we have

(24) $\varphi_{01}(L_m) = 0.$

Now, from the equations 19 and 23, we obtain

(25) $\varphi_{01}(Q_r) = 0.$

Thus

$$4\varphi_{01}(I_{r+s}) = 2\varphi_{01}([Q_r, Q_s]) = [\varphi_{01}(Q_r), Q_s] + [Q_r, \varphi_{01}(Q_s)] = 0.$$
(26) $\Rightarrow \varphi_{01}(I_m) = 0.$

Now, we show that $\varphi_{01}(P_t) = 0$. Observe that $\varphi_{01}(P_t) \in \mathcal{U}_{11}$, that is, $\varphi_{01}(P_t) = \sum_{s \in \mathbb{Z}} \alpha_s^t G_s$. We start by considering the product $[L_m, P_t] = tP_{m+t}$, where $[L_0, P_t] = tP_t$. Thus,

$$2\varphi_{01}(tP_t) = [\varphi_{01}(L_0), P_t] + [L_0, \varphi_{01}(P_t)] = [L_0, \varphi_{01}(P_t)].$$

Then,

$$2t \sum_{s \in \mathbb{Z}} \alpha_s^t G_s = \sum_{s \in \mathbb{Z}} \alpha_s^t [L_0, G_s] = \sum_{s \in \mathbb{Z}} \alpha_s^t s G_s \;\Rightarrow\; (2t - s) \sum_{s \in \mathbb{Z}} \alpha_s^t G_s = 0.$$

If $s \neq 2t$, then $\alpha_s^t = 0$, for all $s \in \mathbb{Z}$. Otherwise, we have $\varphi_{01}(P_t) = \alpha_{2t}^t G_{2t}$. Consider the product $[P_{t_1}, P_{t_2}] = 2I_{t_1+t_2}$, where $(t_1 \neq t_2)$. Thus,

$$4\varphi_{01}(I_{s_1+s_2}) = [\varphi_{01}(P_{t_1}), P_{t_2}] + [P_{t_1}, \varphi_{01}(P_{t_2})] = \alpha_{2t_1}^{t_1}[G_{t_1}, P_{t_2}] + \alpha_{2t_2}^{t_2}[P_{t_1}, G_{2t_2}].$$

From the Equation 26, we have

$$0 = -2t_1 \alpha_{2t_1}^{t_1} Q_{2t_1+t_2} + 2t_2 \alpha_{2t_2}^{t_2} Q_{t_1+2t_2} \Rightarrow 2t_1 \alpha_{2t_1}^{t_1} Q_{2t_1+t_2} = 2t_2 \alpha_{2t_2}^{t_2} Q_{t_1+2t_2}.$$

As $t_1 \neq t_2$, we obtain $\alpha_{2t_1}^{t_1} = \alpha_{2t_2}^{t_2} = 0$. Thus, $\varphi_{01}(P_r) = 0$, for all $r \in \mathbb{Z}$.

All that remains is to show that $\varphi_{01}(\mathcal{U}_{11}) = 0$. From the product $[P_t, Q_s] = sG_{t+s}$, we obtain

$$2\varphi_{01}(rG_r) = [\varphi_{01}(P_0), Q_r] + [P_0, \varphi_{01}(Q_r)] = 0.$$

If $r \neq 0$, then $\varphi_{01}(G_r) = 0$. On the other hand, from the product $[L_m, G_r] = rG_{r+m}$, we have $2\varphi_{01}(sG_0) = [\varphi_{01}(P_{-s}), Q_s] + [P_{-s}, \varphi_{01}(Q_s)] = 0$, the arbitrariness of s, we obtain $\varphi_{01}(G_0) = 0$. Thus, $\varphi_{01}(G_r) = 0$, for all $r \in i + \mathbb{Z}$.

From this and from the Equation 24, 25 and 26, we have $\varphi_{01} = 0$.

$\square$

Remark 18. *Observe that φ_{11} is a odd $\frac{1}{2}$-super derivation in $U \cong \mathcal{U}_{00} \oplus \mathcal{U}_{11}$. In fact, from the $\frac{1}{2}$-ultra derivation definition, if $x \in \mathcal{U}_{00}$, then $\varphi_{11}(x) \in \mathcal{U}_{11}$ and if $x \in \mathcal{U}_{11}$, then $\varphi_{11}(x) \in \mathcal{U}_{00}$. From this we conclude that φ_{11} plays the same role as a odd $\frac{1}{2}$-superderivation in $U \cong \mathcal{U}_{00} \oplus \mathcal{U}_{11}$. Moreover from the [12, Lemma 3.3], we have $\varphi_{11}|_U = 0$ because φ_{11} plays the same role as a odd $\frac{1}{2}$-superderivation in U.*

Lemma 19. *Every homogeneous $\frac{1}{2}$-ultra derivation $(1,1)$ type φ_{11} is zero.*

Proof. From the Remark 18, as $\varphi_{11}(Q_s) \in \mathbf{A}_{10}$, we have $\varphi_{11}(Q_s) = \sum_{t \in \mathbb{Z}} \alpha_t^s P_t$ and as $sQ_s = [L_0, Q_s]$, we have

$$2s \sum_{t \in \mathbb{Z}} \alpha_t^s P_t = 2s\varphi_{11}(Q_s) = 2\varphi_{11}([L_0, Q_s]) = [\varphi_{11}(L_0), Q_s] + [L_0, \varphi_{11}(Q_s)]$$

$$= \left[L_0, \sum_{t \in \mathbb{Z}} \alpha_t^s P_t \right] = \sum_{t \in \mathbb{Z}} \alpha_t^s [L_0, P_t] = \sum_{t \in \mathbb{Z}} \alpha_t^s t P_t.$$

Thus, $\sum_{t \in \mathbb{Z}} (t - 2s)\alpha_t^s P_t = 0$. If $t \neq 2s$, we have $\alpha_t^s = 0$. Now, if $t = 2s$, then $\varphi_{11}(Q_s) = \alpha_{2s}^s P_{2s}$. Thus,

$$0 = 4\varphi_{11}(I_{s_1+s_2}) = 2\varphi_{11}([Q_{s_1}, Q_{s_2}]) = [\varphi_{11}(Q_{s_1}, Q_{s_2})] - [Q_{s_1}, \varphi_{11}(Q_{s_2})].$$

Which implies $[\varphi_{11}(Q_{s_1}, Q_{s_2})] = [Q_{s_1}, \varphi_{11}(Q_{s_2})]$. Thus

$$\alpha_{2s_1}^{s_1}[P_{2s_1}, Q_{s_2}] = \alpha_{2s_2}^{s_2}[Q_{s_1}, P_{2s_2}] \Rightarrow s_2 \alpha_{2s_1}^{s_1} G_{2s_1+s_2} = s_1 \alpha_{2s_2}^{s_2} G_{s_1+2s_2}.$$

Thus, $\alpha_{2s}^s = 0$, for all $s \in \mathbb{Z}$. Then, $\varphi_{11}(Q_s) = 0$. Moreover, from the product $[Q_s, G_r] = sP_{s+r}$, we have $2r\varphi_{11}(P_{s+r}) = [\varphi_{11}(Q_s), G_r] + [Q_s, \varphi_{11}(G_r)] = 0$. By the arbitrariness of $r, s \in \mathbb{Z}$, we have $\varphi_{11}(P_t) = 0$, for all $t \in \mathbb{Z}$. Therefore, $\varphi_{11} = 0$. $\square$

From the Lemmas 14-19 we obtain the following theorems,

Theorem 20. *There are no non-trivial $\frac{1}{2}$-ultra derivations of Heisenberg-Virassoro ultra algebra $\mathfrak{u}\mathcal{HV}$.*

Proof. Let φ a $\frac{1}{2}$-ultra derivation of $\mathfrak{u}\mathcal{HV}$. From the [13, Lemma 2.4], we have $\varphi = \varphi_{00} + \varphi_{10} + \varphi_{01} + \varphi_{11}$. Thus, from the Lemmas 4-7, follows that $\varphi = \varphi_{00} + \varphi_{10} + \varphi_{01}$ or $\varphi = \varphi_{00}$. Then, φ is trivial. $\square$

As a corollary to Theorem 20, we have that there is no non-trivial $\frac{1}{2}$-derivation in $s\mathcal{HV}$.

Theorem 21. *There are no non-trivial TPUA structures defined on $\mathfrak{u}\mathcal{HV}$.*

Proof. The result follows from Theorem 20 and Lemma 11 $\square$

References

[1] Aizawa, N., Kuznetsova, Z., and Toppan, F. (2020). $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: The classical theory. *The European Physical Journal C*, 80(7):668.

[2] Aizawa, N., Kuznetsova, Z., and Toppan, F. (2021). $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: the quantization. *Nuclear Physics B*, 967:115426.

[3] Bai, C., Bai, R., Guo, L., and Wu, Y. (2023). Transposed Poisson algebras, Novikov-Poisson algebras and 3-Lie algebras. *Journal of Algebra*, 632:535–566.

[4] Beites, P. D., Ferreira, B. L. M., and Kaygorodov, I. (2024). Transposed Poisson structures. *Results in Mathematics*, 79(93).

[5] Chen, H., Dai, X., and Hong, Y. (2020). A class of super Heisenberg-Virasoro algebras. *arXiv: Representation Theory*.

[6] Chen, H., Dai, X., and Hong, Y. (2022). The Heisenberg-Virasoro Lie conformal superalgebra. *Journal of Geometry and Physics*, 178:104560.

[7] Ferreira, B. L. M. (2022). On ultra algebras. *Proyecciones (Antofagasta)*, 41(6):1321–1332.

[8] Ferreira, B. L. M., Kaygorodov, I., and Lopatkin, V. (2021). $\frac{1}{2}$-derivations of Lie algebras and transposed poisson algebras. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 115(3):142.

[9] Kac, V. (1977). Lie superalgebras. *Advances in mathematics*, 26(1):8–96.

[10] Kuznetsova, Z. and Toppan, F. (2021). Classification of minimal $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie (super) algebras and some applications. *Journal of Mathematical Physics*, 62(6).

[11] Li, J. and Sun, J. (2022). Super-biderivations on the super Heisenberg-Virasoro algebra. *Linear and Multilinear Algebra*, 70(21):6500–6510.

[12] Yang, Y. and Tang, X. (2023). Transposed Poisson structures on the super Heisenberg-Virasoro algebra.

[13] Yang, Y., Tang, X., and Khudoyberdiyev, A. (2023). Transposed Poisson structures on Schrodinger algebra in (n+ 1)-dimensional space-time. *arXiv preprint arXiv:2303.08180*.

[14] Yuan, L. and Hua, Q. (2022). δ-(bi)derivations and transposed Poisson algebra structures on Lie algebras. *Linear and Multilinear Algebra*, 70(22):7672–7701.

ADDRESSES OF ALINE JAQUELINE DE OLIVEIRA ANDRADE[1] ⓘ
University of ABC,
dos Estados Avenue, 5001,
09210-580, Santo André, Brazil
E-mail address: aline.jaqueline@ufabc.edu.br or alinejaqueline.o.a@gmail.com

ADDRESSES OF BRUNO LEONARDO MACEDO FERREIRA ⓘ
Federal University of Technology-Paraná-Brazil,
Avenida Professora Laura Pacheco Bastos, 800,
85053-510, Guarapuava, Paraná, Brazil
Federal University of ABC and São Paulo University,
Brazil, Av. dos Estados, 5001 - Bangú,
Santo André - SP, 09280-560 and Rua Matão, 1010,
CEP 05508-090, São Paulo, Brazil
E-mail address: brunoferreira@utfpr.edu.br or brunolmfalg@gmail.com