SHY SHADOWS OF INFINITE-DIMENSIONAL PARTIALLY HYPERBOLIC INVARIANT SETS

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Abstract. Let $\mathcal{R}$ be a strongly compact C^2 map defined in an open subset of an infinite-dimensional Banach space such that the image of its derivative $D_F\mathcal{R}$ is dense for every F. Let Ω be a compact, forward invariant and partially hyperbolic set of $\mathcal{R}$ such that $\mathcal{R}\colon \Omega \to \Omega$ is onto. The δ-shadow $W^s_\delta(\Omega)$ of Ω is the union of the sets

$$W^s_\delta(G) = \{F\colon dist(\mathcal{R}^iF, \mathcal{R}^iG) \leq \delta, \ for\ every\ i \geq 0\},$$

where $G \in \Omega$. Suppose that $W^s_\delta(\Omega)$ has transversal empty interior, that is, for every C^{1+Lip} n-dimensional manifold M transversal to the distribution of dominated directions of Ω and sufficiently close to $W^s_\delta(\Omega)$ we have that $M \cap W^s_\delta(\Omega)$ has empty interior in M. Here n is the finite dimension of the strong unstable direction. We show that if δ' is small enough then

$$\cup_{i\geq 0}\mathcal{R}^{-i}W^s_{\delta'}(\Omega)$$

intercepts a C^k-generic finite dimensional curve inside the Banach space in a set of parameters with zero Lebesgue measure, for every $k \geq 0$. This extends to infinite-dimensional dynamical systems previous studies on the Lebesgue measure of stable laminations of invariants sets.

Contents

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1. Introduction.

In many areas of Mathematics, when we need to classify/study objects in a large family, it is quite often the case that one can not understand *all* objects in this family, but just *most* of them. If this family of objects is an open subset of a Banach space (or a Banach manifold), as for instance in the study of the typical behaviour of a large family of smooth discrete dynamical systems, the meaning of "most" it is not at all obvious, since infinite dimensional spaces do not carry a natural class of regular measures (in finite dimensional linear spaces the class of Lebesgue measures is such class). Sometimes by "most" one mean either an open and dense set or a residual set. But even in the finite dimensional case those are not equivalent with full Lebesgue measure sets. Moreover it may happens in the study of dynamical systems that the topological generic behaviour does not coincide with the typical measure-theoretical behaviour. See Hunt, Sauer and Yorke [11] and Hunt and Kaloshin [10] for a large number of examples.

In dynamical systems it came into favor the idea suggested by Kolmogorov [14] that it is good enough to understand the dynamical behaviour at *almost every parameter* in a smooth finite-dimensional family of dynamical systems that belongs to a *residual set of families.* This approach was very successful, specially in one-dimensional dynamics. See for instance the Palis' conjectures [25] [24], the results in Avila, Lyubich and de Melo [2] and Avila and Moreira [3] and the survey by Hunt and Kaloshin [10] on prevalence. Kaloshin [12](see also [10]) defined a *non-linear prevalence* on $C^r(M,N)$, where M and N are manifolds, in the spirit of Kolmogorov's suggestion.

It turns out that to understand the behaviour of most one-dimensional dynamical systems it is often necessary (Avila, Lyubich and de Melo [2]) to understand the dynamics of a highly non-linear, complex analytic compact operator acting on a Banach space of complex analytic dynamical systems, the *renormalization operator*. See Lyubich [20] and de Faria, de Melo and Pinto[9] for the unimodal case, and [27] for the definition of renormalization in the multimodal case. The action of this operator acting on unimodal maps is hyperbolic on its omega limit set [20] [9]. The stable lamination of this omega limit set consists of all infinitely renormalizable maps. A family of dynamical systems is a finite-dimensional smooth curve inside this Banach space, so if we want to know how large is the set of parameters in this family that corresponds to infinitely renormalizable maps, we need to know how such curve intercepts the stable lamination. So the typical behaviour in the *parameter* space of one-dimensional multimodal maps is closely connected with the typical behaviour in the infinite-dimensional *phase* space of the renormalization operator.

In the case of renormalization theory of unimodal maps, it was proven by Avila, Lyubich and de Melo [2] that for typical 1-dimensional real-analytic family of unimodal maps the set of parameters corresponding to infinitely renormalizable maps has zero Lebesgue measure. An essential step of the proof is to show that the stable lamination extends to a codimension-one complex analytic lamination of the whole space of of maps (except maybe for a few maps with very specific combinatorics) and as a consequence the holonomy of this lamination is a quasisymmetric map. This quite special regularity can be exploited to conclude the result. Indeed, using that the stable lamination has codimension one de Faria, de Melo and Pinto[9] proved that the holonomy of the stable lamination is $C^{1+\epsilon}$. The renormalization operator of multimodal maps has a stable lamination with higher codimension and such approach does not seem to be applicable in this case, so we need new tools.

Questions about the measure of invariant laminations started with the work of Bowen [5] (see also Bowen and Ruelle [6] for a similar result for flows), that proved that the stable lamination of a basic hyperbolic set of a C^2 diffeomorphism in a finite-dimensional manifold has zero Lebesgue measure if and only if it has empty interior. That result was generalized by Alves and Pinheiro [1] for horseshoe-like partially hyperbolic invariant sets. To the best of our knowledge, there is no result of this kind in the literature for infinite dimensional dynamical systems. The literature for measure-theoretical/ergodic theory of infinite-dimensional maps as partially hyperbolic maps considered here is indeed scarce. We should mention the work of Mañe [21], where he extends Oseledec theorem to compact smooth maps acting on Banach spaces and the works of Lian and Young [15] [16] and Lian, Young and Zeng [17] on infinite-dimensional dynamical systems.

We will show that for certain compact partially hyperbolic set on infinite dimensional Banach spaces a generic curve intercepts the stable lamination in a set of zero Lebesgue measure. In [28] we use this result to show that in a generic finite-dimensional family of real-analytic multimodal maps the set of parameters of maps that are infinitely renormalizable with bounded combinatorics has zero Lebesgue measure. However this does not exhaust the applications of our results. Indeed, one can expect their application in other fields, as the study of dissipative PDEs.

We adopted an elementary approach, both in methods and statements of the results. In particular we did not need to prove (or to use previous results on) the existence of invariant laminations for a partially hyperbolic invariant set.

There are many difficulties dealing with the general Banach space setting. Firstly, we need to conceive a notion of "thin" set that could be suitable and useful in this setting, since the notion of "zero Lebesgue measure" does not make sense anymore. There are many notions for "thin" sets in Banach spaces. Haar null sets (shy sets) were introduced by Christensen [7] and reintroduced by Hunt, Sauer and Yorke [11]. J. Lindenstrauss and D. Preiss [18] introduced the Γ-null sets in the study of the generalization to the Banach space setting of Rademacher's Theorem on the almost everywhere differentiability of Lipchitz functions. There are also stronger notions of null sets, as cubic, Gaussian and Aronszajn null sets. Those stronger notions coincide in separable Banach spaces, see Csörnyei [8]. We are going to introduce the concept of Γ^k-null set (see Section 3), that somehow unifies Kolmogorov's idea and Lindenstrauss & Preiss's approaches in the abstract setting of Banach spaces rather than $C^r(M,N)$. Indeed even when $C^r(M,N)$ is a Banach space the set of smooth curves in $C^r(M,N)$ considered by Kaloshin, as well the topology on this set, are somehow different of those used here.

Secondly, we need of a "strong compactness" assumption (see Section 2) to prove the pre-compactness of a sequence of curves transversal to the central stable direction obtained iterating our dynamical system (and cutting) on a given transversal curve. This is similar to the graph method approach for the proof of the Stable Manifold Theorem and the λ-Lemma. The lack of contraction in the center-stable direction does not allow us to prove the contraction of this "graph map"-like process, however the pre-compactness will be enough to our purposes. Such pre-compactness is obtained in Section 5 with careful estimates using both "strong" $|\cdot|_1$ and "weak" $|\cdot|_0$ norms that appears in the "strong compactness" assumption. Such a difficulty does not appear in the previous finite dimensional results, since in this setting every C^2 map is obviously strongly compact.

Finally, as usual in smooth ergodic theory, it is essential to be able to control how measures on finite-dimensional curves are perturbed by the iteration of the dynamics. This is sometimes called "bounded distortion control" and in the finite-dimensional setting the jacobian and the classical formulas for change of variables of integrals plays a crucial role to obtain these estimates. In the Banach space setting we need to consider Lebesgue measures on finite-dimensional smooth curves inside the Banach space. There is not a canonical way to do this. One may prefer either the full-dimensional Haussdorf measure induced by the norm of the Banach space restricted to such curve in certain situations, that has the advantage of being coordinate-free, or a Lebesgue measure induced by some ad-hoc inner product on the (finite-dimensional) parameter space of the curve, that could be more handy for bounded distortion estimates. All these measures will play a role in the proof of the main result and we need to be careful when we change the measures on consideration along our arguments. This is done in Section 6 and Section 7.

2. Statement of the main results.

Let $\mathcal{B}_0$ be a Banach space, either real or complex, with norm $|\cdot|_0$. We say that a C^2 Fréchet differentiable map

$$\mathcal{R}\colon V \to \mathcal{B}_0,$$

where $V \subset \mathcal{B}_0$ is an open set, is a **strongly compact C^2 map** if

A. There is a subspace $\mathcal{B}_1 \subset \mathcal{B}_0$, that endowed with a norm $|\cdot|_1$ is a Banach space, such that the inclusion

$$i\colon (\mathcal{B}_1, |\cdot|_1) \mapsto (\mathcal{B}_0, |\cdot|_0)$$

is a compact linear map, and

B. We can write

$$\mathcal{R} = i \circ \tilde{\mathcal{R}},$$

where $\tilde{\mathcal{R}}$ is a C^2 Fréchet differentiable map

$$\tilde{\mathcal{R}}\colon (V, |\cdot|_0) \to (\mathcal{B}_1, |\cdot|_1).$$

Note that if $\mathcal{B}_0$ has infinite dimension then $\mathcal{R}$ is not a diffeomorphism.

We will say that a forward $\mathcal{R}$-invariant compact subset $\Omega \subset \mathcal{B}_1$ is a **partially hyperbolic set** of $\mathcal{R}$ if there are two continuous $\mathcal{R}$-invariant distributions of subspaces of $\mathcal{B}_0$, the **horizontal directions**

$$G \in \Omega \mapsto E^h_G,$$

and the **unstable directions**

$$G \in \Omega \mapsto E^u_G,$$

such that $\mathcal{B}_0 = E^h_G \oplus E^u_G$ and moreover

A1. There exist n_0, $C > 0$, $\delta_1 > 0$ and $\lambda_G > 0$ such that if $v \in E^h_G$ then

$$|D_F\mathcal{R}^{n_0} \cdot v|_0 \leq \lambda_G |v|_0.$$

A2. There exists $\theta > 1$ and for every $G \in \Omega$ there exists $\theta_G > \theta \max\{1, \lambda_G\}$

$$|D_F\mathcal{R}^{n_0} \cdot v|_0 \geq \theta_G |v|_0.$$

For each $G \in \Omega$, define the δ-shadow of G as

$$W^s_\delta(G) = \{F \in V\colon\ dist_{\mathcal{B}_0}(\mathcal{R}^k F, \mathcal{R}^k G) \leq \delta,\ for\ every\ k \geq 0\}.$$

and the δ-shadow of Ω as

$$W^s_\delta(\Omega) = \cup_{G\in\Omega} W^s_\delta(G).$$

It is easy to see that $W^s_\delta(\Omega)$ is a closed subset of V (in $\mathcal{B}_0$ topology).

Since $\mathcal{R}$ is a compact map we have that E^u_F has finite dimension. Without loss of generality we are going to assume that $n = \dim_{\mathbb{R}} E^u_G$ does not depend on $G \in \Omega$. We say that $W^s_\delta(\Omega)$ has **real transversal empty interior**, if there is $\delta' > 0$ with the following property. For every embedded *real* C^{1+Lip} n-dimensional manifold $M \subset V$ satisfying

$$diam\ M + dist_{\mathcal{B}_0}(M, G) < \delta'$$

for some $G \in \Omega$ and such that M is transversal to E^h_G ($T_x M \oplus E^h_G = \mathcal{B}_0$ for every $x \in M$) we have that $M \cap W^s_\delta(\Omega)$ has empty interior in M. If $\mathcal{B}_0$ is a complex Banach space and $\mathcal{R}$ is complex analytic with $n = \dim_{\mathbb{C}} E^u_G$ then we say that $W^s_\delta(\Omega)$ has **complex transversal empty interior** if there exists some $\delta' > 0$ such that for every embedded *complex analytic* n-dimensional manifold $M \subset V$ the above property holds.

Using a notation similar to J. Lindenstrauss and D. Preiss [18] [19], let $\mathcal{B}$ be a Banach space. Consider $I = [-1, 1]$ with the normalized Lebesgue measure $m(a, b) = (b - a)/2$. Endow $T = I^{\mathbb{N}}$ with the product topology and the product Lebesgue measure m. Let $\Gamma^k(\mathcal{B})$, $k \in \mathbb{N} \cup \{\infty\}$, be the set of all continuous functions

$$\gamma\colon T \mapsto \mathcal{B}$$

with continuous partial derivatives

$$\lambda \in T \mapsto \partial^j_{i_1 i_2 \cdots i_j} \gamma(\lambda)$$

for every $j \leq k$, with $i_1, \ldots, i_j \in \mathbb{N}$ and $\lambda = (\lambda_i)_{i\in\mathbb{N}} \in T$ satisfying $\lambda_{i_p} \in (-1, 1)$ for every $p \leq j$. Moreover the partial derivatives extends continuously to a function in T. The pseudo-norms given by the supremum of γ and its partial derivatives up to order k on T endow $\Gamma^k(\mathcal{B})$ with the structure of a Fréchet space. We have that $\Gamma^k(\mathcal{B})$ is a Polish space, and consequently a Baire Space. Note that $\Gamma^1(\mathcal{B})$ coincides with $\Gamma(\mathcal{B})$ as defined in [18].

If $\mathcal{B}$ is a complex Banach space, we will denote by $C^\omega(\overline{\mathbb{D}}^j, \mathcal{B})$ the set of complex analytic functions

$$\gamma\colon \mathbb{D}^j \to \mathcal{B},$$

that have a continuous extension to $\overline{\mathbb{D}}^j$. Endowed with the sup norm on $\overline{\mathbb{D}}^j$ we have that $C^\omega(\overline{\mathbb{D}}^j, \mathcal{B})$ is a complex Banach space.

If $\mathcal{B}$ is a complex Banach space, we want to consider $\Gamma^\omega(\mathcal{B})$. To do this, replace in the definition of T the interval $[-1, 1]$ by $\overline{\mathbb{D}} = \{z \in \mathbb{C}\colon\ |z| \leq 1\}$ endowed with the normalized Lebesgue measure, obtaining $T_{\mathbb{C}}$. Then $\Gamma^\omega(\mathcal{B})$ is the space of all continuous functions $\gamma\colon T_{\mathbb{C}} \mapsto \mathcal{B}$ that are holomorphic when we fix all except a finite number of entries of $\lambda \in T_{\mathbb{C}}$ and $|\lambda_i| < 1$ for the remaining ones. The sup norm on γ turns $\Gamma^\omega(\mathcal{B})$ into a complex Banach space.

Finally we would like to consider real analytic families into a real Banach space $\mathcal{B}$. Denote by $\mathcal{B}_{\mathbb{C}}$ the complex Banach space that is a complexification of $\mathcal{B}$ endowed with a *desirable* norm as defined in Kirwan [13] (see also Muñoz, Sarantopoulos and

Tonge [22]). We will denote by $C^{\omega_{\mathbb{R}}}([-1,1]^j,\mathcal{B})$ the set of real analytic functions (see Bochnak and Siciak [4]).

$$\gamma\colon (-1,1)^j \to \mathcal{B}$$

that can be extended to a complex analytic function

$$\gamma\colon \mathbb{D}^j \to \mathcal{B}_{\mathbb{C}},$$

and moreover γ has a continuous extension to $\overline{\mathbb{D}}^j$. Endowed with the sup norm on $\overline{\mathbb{D}}^j$ we have that $C^{\omega_{\mathbb{R}}}([-1,1]^j,\mathcal{B})$ is a real Banach space.

We define $\Gamma^{\omega_{\mathbb{R}}}(\mathcal{B})$ as the *real* Banach space that consists of the restrictions to $T=[-1,1]^{\mathbb{N}}$ of the functions $\gamma \in \Gamma^{\omega}(\mathcal{B}_{\mathbb{C}})$ that satisfy $\gamma(\overline{\lambda})=\overline{\gamma(\lambda)}$.

Our first main result is

Theorem 1. *Let $\mathcal{R}$ be a strongly compact C^2 map on a real Banach space $\mathcal{B}_0$ with a compact invariant partially hyperbolic set Ω such that $\mathcal{R}\colon \Omega \to \Omega$ is onto. If*

A. *There is $\delta > 0$ such that $W^s_\delta(\Omega)$ has real transversal empty interior,*
B. *For every $i \geq 0$ and every $x \in \mathcal{R}^{-i}W^s_\delta(\Omega)$ we have that $D_x\mathcal{R}^i$ has dense image,*

then there exists $\delta_5 > 0$ such that $\cup_{i\in\mathbb{N}}\mathcal{R}^{-i}W^s_{\delta_5}(\Omega)$ is a $\Gamma^k(\mathcal{B}_0)$-null set, for every $k \in \mathbb{N}\cup\{\infty,\omega_{\mathbb{R}}\}$. Indeed for every j and $k \in \mathbb{N}\cup\{\infty,\omega_{\mathbb{R}}\}$ there exists a residual set of C^k maps

$$\gamma\colon [-1,1]^j \to \mathcal{B}_0$$

such that

$$m(t \in [-1,1]^j\colon\ \gamma(t) \in \cup_{i\in\mathbb{N}}\mathcal{R}^{-i}W^s_{\delta_5}(\Omega))=0.$$

Here m denotes the normalized Lebesgue measure on $[-1,1]^j$.

If $\mathcal{R}$ is complex analytic and $W^s_\delta(\Omega)$ has *real* transversal empty interior we can apply Theorem 1. We do not know if *complex* transversal empty interior implies the *real* transversal empty interior property. This is the motivation to

Theorem 2. *Let $\mathcal{R}$ be a strongly compact, complex analytic map on a complex Banach space $\mathcal{B}_0$ with a compact invariant partially hyperbolic set Ω such that $\mathcal{R}\colon \Omega\to\Omega$ is onto. If*

A. *There is $\delta > 0$ such that $W^s_\delta(\Omega)$ has complex transversal empty interior,*
B. *For every $i \geq 0$ and every $x \in \mathcal{R}^{-i}W^s_\delta(\Omega)$ we have that $D_x\mathcal{R}^i$ has dense image,*

then there exists $\delta_5 > 0$ such that $\cup_{i\in\mathbb{N}}\mathcal{R}^{-i}W^s_{\delta_5}(\Omega)$ is a $\Gamma^\omega(\mathcal{B}_0)$-null set. Indeed for every j there exists a residual set of maps $\gamma \in C^\omega(\overline{\mathbb{D}}^j,\mathcal{B}_0)$ such that

$$m(t \in \overline{\mathbb{D}}^j\colon\ \gamma(t) \in \cup_{i\in\mathbb{N}}\mathcal{R}^{-i}W^s_{\delta_5}(\Omega))=0.$$

Here m denotes the Lebesgue measure on $\overline{\mathbb{D}}^j$. Moreover, if we consider $\mathcal{B}_0$ as a real Banach space then all conclusions of Theorem 1 holds.

Remark 2.1. *We will replace the transversal empty condition for a weaker assumption that is a little more technical. We postpone the description of this condition to the end of Section 4.*

Remark 2.2. *If Ω is a hyperbolic invariant set for $\mathcal{R}$ and δ_5 is small enough then $\cup_{i\in\mathbb{N}}\mathcal{R}^{-i}W^s_{\delta_5}(\Omega)$ is the global stable lamination $W^s(\Omega)$.*

One can ask if we could extend our results to more general contexts that already appeared in the finite-dimensional setting. For instance, we could certainly generalize our result assuming that $\mathcal{R}$ satisfies the nonuniformly expanding condition on the centre-unstable direction as in Alves and Pinheiro [1]. However we feel that the most crucial adaptations to the infinite dimensional setting appeared already in the present case and that the additional modifications of their methods to achieve these results would be minor.

3. Null sets in Banach spaces.

Define $\Gamma^k_j(\mathcal{B}) \subset \Gamma^k(\mathcal{B})$ as the subset of all functions γ where $\gamma(\lambda)$ depends only on the first j entries of λ. Then

$$\cup_{j\in\mathbb{N}}\Gamma^k_j$$

is dense in $\Gamma^k(\mathcal{B})$. The proof is left to the reader (see the case $k=1$ in [18]). If $\Theta \subset \mathcal{B}$ is a borelian set and $\gamma \in \Gamma^k(\mathcal{B})$, denote

$$m_\gamma(\Theta) = m(\lambda \in T\colon \gamma(\lambda) \in \Theta),$$

where m is the product Lebesgue measure on T. Note that m and m_γ are regular Borel measures. We say that Θ is a $\Gamma^k(\mathcal{B})$-*null set* is there exists a residual subset $\mathcal{F} \subset \Gamma^k(\mathcal{B})$ such that $m_\gamma(\Theta) = 0$ for every $\gamma \in \mathcal{F}$. J. Lindenstrauss and D. Preiss [18] observed that if $\mathcal{B}$ has finite dimension then a borelian Θ is a $\Gamma^1(\mathcal{B})$-null set if and only if Θ has zero Lebesgue measure.

All results of this section hold for every $k \in \mathbb{N} \cup \{\infty, \omega, \omega_\mathbb{R}\}$, but keep in mind that whenever we consider $k = \omega$ the space $\mathcal{B}$ is a complex Banach space and in the case $k = \omega_\mathbb{R}$ the real Banach space $\mathcal{B}$ is the real trace of a complex Banach space $\mathcal{B}_\mathbb{C}$ and the definition of $\Gamma^{\omega_\mathbb{R}}(\mathcal{B})$ depends on $\mathcal{B}_\mathbb{C}$. We omit the proofs of Lemma 3.1, Lemma 3.2 and Proposition 3.3 below for $k = \omega, \omega_\mathbb{R}$. The proofs in these cases are quite similar and indeed easier, since we do not need to deal with partial derivatives.

Lemma 3.1. *Let Θ be a countable union of closed subsets of the Banach space $\mathcal{B}$. If there is a dense subset $\mathcal{S} \subset \Gamma^k(\mathcal{B})$ such that $m_\gamma(\Theta) = 0$ for $\gamma \in \mathcal{S}$ then Θ is a $\Gamma^k(\mathcal{B})$-null set.*

Proof. Since a countable union of $\Gamma^k(\mathcal{B})$-null sets is a $\Gamma^k(\mathcal{B})$-null set, it is enough to prove the lemma assuming that Θ is a closed set. Let $\gamma \in \mathcal{S}$. Then

$$\Theta_\gamma = \{\lambda \in T\colon \gamma(\lambda) \in \Theta\}$$

is a compact set with zero product Lebesgue measure. Let $O \subset T$ be an open subset such that $\Theta_\gamma \subset O$ and $m(O) < \epsilon$. Let $\delta = dist_\mathcal{B}(\gamma(O^c), \Theta) > 0$. If $\tilde\gamma \in \Gamma^k(\mathcal{B})$ satisfies

$$\sup_{\lambda\in T} |\gamma(\lambda) - \tilde\gamma(\lambda)| < \delta/2$$

then $\Theta_{\tilde\gamma} \subset O$, so $m_{\tilde\gamma}(\Theta) < \epsilon$. In particular

$$\mathcal{S}_\epsilon = \{\tilde\gamma \in \Gamma^k(\mathcal{B})\colon\ m_{\tilde\gamma}(\Theta) < \epsilon\}$$

contains an open and dense subset of $\Gamma^k(\mathcal{B})$. Since

$$\mathcal{F} = \{\tilde\gamma \in \Gamma^k(\mathcal{B})\colon\ m_{\tilde\gamma}(\Theta) = 0\} = \cap_{n\in\mathbb{N}^\star}\mathcal{S}_{1/n}$$

we conclude that $\mathcal{F}$ is a residual subset of $\Gamma^k(\mathcal{B})$, so Θ is a $\Gamma^k(\mathcal{B})$-null set. □

Lemma 3.2. *Let Θ be $\Gamma^k(\mathcal{B})$-null borelian set. Then for every j there is dense subset $\mathcal{S}_j \subset \Gamma^k_j(\mathcal{B})$ such that $m_\gamma(\Theta) = 0$ for every $\gamma \in \mathcal{S}_j$.*

Proof. Let $A \subset \Gamma^k(\mathcal{B})$ be an open set such that $A \cap \Gamma^k_j(\mathcal{B}) \neq \emptyset$. Let $\gamma \in A \cap \Gamma^k_j(\mathcal{B})$. Then there exists n vectors $a_\ell = (i^\ell_1, i^\ell_2, \cdots, i^\ell_{j_\ell})$, with $j_\ell \leq k$, $\ell \leq n$, such that if

$$\partial_{a_\ell} = \partial^{j_\ell}_{i^\ell_1 i^\ell_2 \cdots i^\ell_{j_\ell}}$$

then

$$A' = \{\beta \in \Gamma^k(\mathcal{B}) \colon \ |\partial_{a_\ell}(\beta - \gamma)|_\infty < \epsilon, \ \textit{for every } \ell \leq n\} \subset A.$$

Since Θ is a $\Gamma^k(\mathcal{B})$-null set there exists $\alpha \in A'$ such that $m_\alpha(\Theta) = 0$. For every $\omega = (t_{j+1}, t_{j+2}, \dots) \in I^{\mathbb{N}}$ define the finite dimensional subfamily $\alpha_\omega \in \Gamma^k_j(\mathcal{B})$ as

$$\alpha_\omega(t_1, \dots, t_j) = \alpha(t_1, \dots, t_j, t_{j+1}, \dots).$$

Note that $\alpha_\omega \in A'$. By the Fubini's Theorem for almost every $\omega \in I^{\mathbb{N}}$ we have that $m_{\alpha_\omega}(\Theta) = 0$. This proves the lemma. □

It was noted by J. Lindenstrauss, D. Preiss, and J. Tišer [19] in the case $k = 1$ that if Θ is a countable union of closed subsets we have that various concepts of null-sets coincides. Indeed for arbitrary $k \in \mathbb{N} \cup \{\omega, \omega_{\mathbb{R}}\}$ we have

Proposition 3.3. *Let Θ be a countable union of closed subsets in the Banach space $\mathcal{B}$. The following statements are equivalent.*

A. *The set Θ is a $\Gamma^k(\mathcal{B})$-null set.*
B. *For every j there is a dense subset $\mathcal{S}_j \subset \Gamma^k_j(\mathcal{B})$ such that $m_\gamma(\Theta) = 0$ for every $\gamma \in \mathcal{S}_j$.*
C. *For every j there is a residual subset $\mathcal{S}_j \subset \Gamma^k_j(\mathcal{B})$ such that $m_\gamma(\Theta) = 0$ for every $\gamma \in \mathcal{S}_j$.*

Proof. The implication $(A) \Rightarrow (B)$ follows from Lemma 3.2. Using the same argument of the proof of Lemma 3.1 (replacing $\Gamma^k(\mathcal{B})$ by $\Gamma^k_j(\mathcal{B})$) one can easily show that $(B) \Rightarrow (C)$. To show that $(C) \Rightarrow (A)$, note that since $\cup_j \Gamma^k_j(\mathcal{B})$ is dense in $\Gamma^k(\mathcal{B})$, item C implies that there is a dense subset $\mathcal{S} \subset \Gamma^k(\mathcal{B})$ such that $m_\gamma(\Theta) = 0$ for $\gamma \in \mathcal{S}$. By Lemma 3.1 we get A.

□

Proposition 3.4. *Suppose that*

(H) *The set Θ is a borelian subset of the Banach space $\mathcal{B}$ such that for every $x \in \mathcal{B}$ there exists $\delta = \delta(x) > 0$ and a finite dimensional family $\alpha \in \cup_j \Gamma^k_j(\mathcal{B})$ with $\alpha(0) = x$ such that for every $z \in \mathcal{B}$ satisfying $|z| < \delta$ we have $m_{\alpha_z}(\Theta) = 0$, where*

$$\alpha_z(\lambda) = \alpha(\lambda) + z,$$

Then for every $j \geq 0$ and every $k \in \mathbb{N} \cup \{\infty, \omega, \omega_{\mathbb{R}}\}$ there is a dense subset $\mathcal{S}_j$ of $\Gamma^k_j(\mathcal{B})$ such that $m_\gamma(\Theta) = 0$ for every $\gamma \in \mathcal{S}_j$.

Proof. Suppose $k \neq \omega$. Let $A \subset \Gamma^k(\mathcal{B})$ be open subset such that there exists $\gamma \in A \cap \Gamma^k_j(\mathcal{B})$. By the compactness of $[-1,1]^j$ and continuity of γ there exists a finite subset $\{\lambda_1, \dots, \lambda_p\} \subset [-1,1]^j$ and $\epsilon_i > 0$, with $i \leq p$, satisfying

$$[-1,1]^j \subset \cup_i B(\lambda_i, \epsilon_i),$$

and such that $|\lambda - \lambda_i| < \epsilon_i$ implies $|\gamma(\lambda) - \gamma(\lambda_i)| < \delta(\gamma(\lambda_i))/2$. Moreover there are families $\alpha_i \in \Gamma^k_{j_i}(\mathcal{B})$ such that $\alpha_i(0) = \gamma(\lambda_i)$ and $m_{\alpha_{i,z}}(\Theta) = 0$ for every $|z| < \delta(\gamma(\lambda_i))$. Here

$$\alpha_{i,z}(t) = \alpha_i(t) + z,$$

Let $J = j + \sum_i j_i$. Define $\beta \in \Gamma^k_J(\mathcal{B})$ as

$$\beta(t, t_1, \dots, t_p) = \gamma(t) + \sum_{i=1}^{p} \alpha_i(t_i) - \gamma(\lambda_i).$$

Here $t_i \in [-1,1]^{j_i}$ and $t \in [-1,1]^j$. Choose $\eta > 0$ small enough such that if $|t_i| < \eta$ for every $i \leq p$ then

$$\sum_{i=1}^{p} |\alpha_i(t_i) - \gamma(\lambda_i)| < \min_i \delta(\gamma(\lambda_i))/2.$$

Given $i_0 \leq p$, fix the entries $t, t_1, \dots, t_{i_0-1}, t_{i_0+1}, \dots, t_p$, with $|t - \lambda_{i_0}| < \epsilon_{i_0}$ and $|t_i| < \eta$ and consider the family β restricted to the segment

$$L = \{(t, t_1, \dots, t_{i_0-1}, t_{i_0}, t_{i_0+1}, \dots, t_p) \in T \colon \ |t_{i_0}| < \eta\}$$

Note that in this line

$$\beta(t, t_1, \dots, t_p) = \alpha_{i_0}(t_{i_0}) + z,$$

where

$$z = \gamma(t) - \gamma(\lambda_{i_0}) + \sum_{i \neq i_0} \alpha_i(t_i) - \gamma(\lambda_i),$$

Since $|z| < \delta(\gamma(\lambda_{i_0}))$ it follows that $m_{\alpha_{i_0}+z}(\Theta) = 0$. By Fubini's Theorem we have that the set of parameters

$$u \in \{t \in [-1,1]^j \colon |t - \lambda_{i_0}| < \epsilon_{i_0}\} \times_i \{t_i \in [-1,1]^{j_i} \colon \ |t_i| < \eta\}$$

such that $\beta(u) \in \Theta$ has zero Lebesgue measure. This holds for every $i_0 \leq p$, so we conclude that the set of parameters

$$u \in [-1,1]^j \times_i \{t_i \in [-1,1]^{j_i} \colon \ |t_i| < \eta\}$$

such that $\beta(u) \in \Theta$ has zero Lebesgue measure. Now we can apply Fubini's Theorem once again to conclude that for almost every

$$(t_1, \dots, t_p) \in \times_i \{t_i \in [-1,1]^{j_i} \colon \ |t_i| < \eta\}$$

the family $\beta_{t_1,\dots,t_p} \colon [-1,1]^j \mapsto \mathcal{B}$ given by

$$\beta_{t_1,\dots,t_p}(t) = \beta(t, t_1, \dots, t_p)$$

satisfies $m_{\beta_{t_1,\dots,t_p}}(\Theta) = 0$. But $\beta_{t_1,\dots,t_p}$ is a translation of the family γ, that is

$$\beta_{t_1,\dots,t_p}(t) = \gamma(t) + z_{t_1,\dots,t_p},$$

where

$$\lim_{(t_1,\dots,t_p) \to 0} z_{t_1,\dots,t_p} = 0,$$

so we can choose $(t_1, \dots, t_p)$ sufficiently small such that

$$\beta_{t_1,\dots,t_p} \in A \textit{ and } m_{\beta_{t_1,\dots,t_p}}(\Theta) = 0.$$

The proof for the case $k = \omega$ is obtained replacing $[-1,1]$ by $\overline{\mathbb{D}}$ everywhere. $\square$

Corollary 3.5. *Under assumption (H) in Proposition 3.4, if Θ is also a countable union of closed sets then Θ is a $\Gamma^k(\mathcal{B})$-null set.*

Proof. Since $\cup_j \Gamma^k_j(\mathcal{B})$ is dense in $\Gamma^k(\mathcal{B})$, by Proposition 3.4 the set $\cup_j \mathcal{S}_j$ is dense in $\Gamma^k(\mathcal{B})$. Now we can apply Lemma 3.1 to conclude that Θ is a $\Gamma^k(\mathcal{B})$-null set. $\square$

Remark 3.6. *We can obtain similar results considering either the weak or strong Whitney topology on $C^k((-1,1)^j,\mathcal{B})$, the space of all C^k functions*

$$\gamma\colon (-1,1)^j \to \mathcal{B}.$$

Indeed if Θ is a countable union of closed sets satisfying assumption (H) in Proposition 3.4 then for a residual set of functions $\gamma \in C^k((-1,1)^j,\mathcal{B})$ we have $m_\gamma(\Theta)=0$.

4. Adapted norms, cones and transversal families.

Assume that $\mathcal{R}$ and Ω satisfy the assumptions of either Theorem 1 or Theorem 2, so we are going to deal with the real smooth and complex analytic cases at the same time. Replacing $|v|_1$ by the equivalent norm $|v|_1+|v|_0$, we can assume without loss of generality that $|v|_1 \geq |v|_0$. Denote by π^h_G and π^u_G the continuous linear projections on E^h_G and E^u_G such that

$$v = \pi^u_G(v)+\pi^h_G(v).$$

Since $E^u_G \subset \mathcal{B}_1$ we have that $\pi^h_G(\mathcal{B}_1)\subset \mathcal{B}_1$ and that

$$\pi^u_G, \pi^h_G\colon \mathcal{B}_i \to \mathcal{B}_i$$

are bounded operators. Define the adapted norms

$$|v|_{G,i} = |\pi^u_G(v)|_i + |\pi^h_G(v)|_i.$$

Here $i=0,1$. Replacing $\mathcal{R}$ by some iteration of it we have the following properties

i. We have that

$$G\in\Omega \mapsto \pi^u_G$$

and

$$G\in\Omega \mapsto \pi^h_G$$

are continuous with respect to the $\mathcal{B}_0$ topology.

ii. We have $|v|_{G,0}\leq |v|_{G,1}$ for every $v\in\mathcal{B}_1$ and

$$|\pi^u_G(v)|_{G,i} \leq |v|_{G,i} \textit{ and } |\pi^h_G(v)|_{G,i} \leq |v|_{G,i}.$$

iii. There exist $\delta_1>0$ and $\lambda_G>0$ such that if $v\in E^h_G$ then

$$|D_F\mathcal{R}\cdot v|_{\mathcal{R}G,1} \leq \lambda_G |v|_{G,0}$$

provided $|F-G|_{G,0}<\delta_1$.

iv. There exists $\theta>1$ such that for every $G\in\Omega$ there exists $\theta_G > \theta\max\{1,\lambda_G\}$ such that if $v\in E^u_G$

$$|D_F\mathcal{R}\cdot v|_{\mathcal{R}G,0} \geq \theta_G |v|_{G,0}.$$

provided $|F-G|_{G,0}<\delta_1$.

v. There exists $C_1>1$ such that for every $G\in\Omega$ and $v\in E^u_G$

$$|v|_{G,1}\leq C_1|v|_{G,0}.$$

vi. There exists $C_2>1$ such that for every $G\in\Omega$

$$\frac{1}{C_2}|v|_i \leq |v|_{G,i} \leq C_2|v|_i. \tag{1}$$

vii. There exists $C_3>1$ such that for every $G,F\in\Omega$ and $v\in\mathcal{B}_1$ we have

$$|\pi^h_G(v)|_{G,1}\leq C_3|v|_{F,1}. \tag{2}$$

Remark 4.1. *Since $\mathcal{R}$ is a strongly compact C^2 map, it follows that*

$$D_G\mathcal{R}\colon \mathcal{B}_0 \to \mathcal{B}_0$$

is a compact linear operator. Indeed $Im(D_G\mathcal{R}) \subset \mathcal{B}_1$ and

$$D_G\mathcal{R}\colon \mathcal{B}_0 \to \mathcal{B}_1$$

is a bounded linear transformation. If $G \in \Omega$ then $D_G\mathcal{R}$ is a compact linear isomorphism between E^u_G and $E^u_{\mathcal{R}G}$. In particular E^u_G has finite dimension.

Indeed (i) follows from the continuity of the distributions E^h_G and E^h_g and (ii) follows from the definition of the adapted norms. Properties (iii) and (iv) follows from the fact that Ω is a partially hyperbolic invariant set for $\mathcal{R}$ and that $\mathcal{R}$ is a strongly compact operator. To prove (v), recall that $\mathcal{R}\colon \Omega \to \Omega$ is onto, so there exists $F \in \Omega$ such that $\mathcal{R}F = G$. By (iii), since E^u_G is finite dimensional, $E^u_G \subset \mathcal{B}_1$ and $D_F\mathcal{R}(E^u_F) = E^u_G$ for every $v \in E^u_G$ let $w \in E^u_F$ be such that $D_F\mathcal{R}\cdot w = v$. By (iv) we have

$$|w|_{F,0} \leq \frac{1}{\theta}|v|_{G,0}.$$

Since $\mathcal{R}$ is C^1 and Ω is compact, there exists C_4 such that

$$\sup_{F\in\Omega} |D_G\tilde{\mathcal{R}}|_{(\mathcal{R}F,1),(F,0)} < C_4,$$

so

$$|v|_{G,1} = |D_F\tilde{\mathcal{R}}\cdot w|_{G,1} \leq C_4|w|_{F,0} \leq \frac{C_4}{\theta}|v|_{G,0}.$$

We are going to prove (vi). The case $i = 0$ follows from (i) and the compactness of Ω. Then by (v) and the case $i = 0$

$$|\pi^u_G(v)|_{G,1} = |\pi^u_G(v)|_1 \leq C_1|\pi^u_G(v)|_{G,0} \leq C_2C_1|v|_0 \leq C_2C_1|v|_1.$$

so

$$|\pi^h_G(v)|_{G,1} = |\pi^h_G(v)|_1 = |v - \pi^u_G(v)|_{G,1} \leq |v|_1 + |\pi^u_G(v)|_1 \leq (1 + C_2C_1)|v|_1.$$

Consequently $|v|_{G,1} \leq (1+2C_2C_1)|v|_1$. Of course $|v|_1 \leq |v|_{G,1}$. This proves (vi) for the case $i = 1$. It remains to show (vii). Indeed by (v)

$$|\pi^h_G(v)|_{G,1} \leq |v|_{G,1} \leq C_2|v|_1 \leq C_2^2|v|_{F,1}.$$

Now we can define the unstable cones $C^u_{\epsilon,i}(G)$, $i = 0, 1$, as the set of all $v \in \mathcal{B}_i$ such that $v = u + w$, with $u \in E^u_G$ and $w \in E^h_G$ (note that this implies $u, w \in \mathcal{B}_i$) and moreover

$$|w|_{G,i} \leq \epsilon|u|_{G,i}.$$

Define the cone $C^u_{\epsilon,(1,0)}(G)$ as the set of $v \in \mathcal{B}_1$ such that $v = u + w$, with $u \in E^u_G$ and $w \in E^h_G$ and

$$|w|_{G,1} \leq \epsilon|u|_{G,0}.$$

Of course

$$C^u_{\epsilon,(1,0)}(G) \subset C^u_{\epsilon,0}(G) \cap C^u_{\epsilon,1}(G).$$

and if $\epsilon' < \epsilon$ then

$$C^u_{\epsilon',i}(G) \subset C^u_{\epsilon,i}(G).$$

Moreover these cones are forward invariant. Indeed

$$D_G\mathcal{R}(C^u_{\epsilon,0}(G)) \subset C^u_{\frac{\lambda_G}{\theta_G}\epsilon,(1,0)}(\mathcal{R}G) \subset C^u_{\frac{\epsilon}{\theta},0}(\mathcal{R}G) \subset C^u_{\epsilon,0}(\mathcal{R}G). \tag{3}$$

Since we are going to deal with many norms, it is convenient to introduce the following notation. If

$$T\colon L \to \mathcal{B}_i$$

is a linear transformation, where L is a subspace of $\mathcal{B}_j$, with $i, j \in \{0,1\}$, and $G, G' \in \Omega$, we denote

$$|T|_{(G,j),(G',i)} = \sup\{|T(v)|_{G',i}\colon \ |v|_{G,j} \leq 1\}.$$

and if

$$Q\colon L \times L \to \mathcal{B}_i$$

is a symmetric bilinear transformation then

$$|Q|_{(G,j),(G',i)} = \sup\{|Q(v,v)|_{G',i}\colon \ |v|_{G,j} \leq 1\}.$$

Define

$$|D\mathcal{R}|_\infty = \sup\{|D_F\tilde{\mathcal{R}}|_{(G,0),(\mathcal{R}G,1)}\colon \ G \in \Omega, |F-G|_{G,0} \leq \delta_1\},$$

$$|D^2\mathcal{R}|_\infty = \sup\{|D^2_F\tilde{\mathcal{R}}|_{(G,0),(\mathcal{R}G,1)}\colon \ G \in \Omega, |F-G|_{G,0} \leq \delta_1\}.$$

Fix $\epsilon > 0$ small enough to satisfy

$$\theta_1 = \min\{\theta - 2\epsilon, \frac{\theta-\epsilon}{1+\epsilon}, \theta\frac{(1-\epsilon)^3}{(1+\epsilon)^2}, \big(\frac{(1+\epsilon)^2}{\theta^2(1-\epsilon)^6} + \frac{\epsilon(1+\epsilon)|D\mathcal{R}|_\infty(1+\epsilon)^3}{\theta^3(1-\epsilon)^9}\big)^{-1}\} > 1.$$

In particular $D_G\mathcal{R}$ is expanding on unstable cones. Indeed, if $v \in C^u_0(G) := C^u_{\epsilon,0}(G)$ then

$$|D_F\mathcal{R}\cdot v|_{\mathcal{R}G,0} \geq \frac{\theta_G(1-\epsilon/\theta)}{1+\epsilon}|v|_{G,0} \geq \theta_1|v|_{G,0}$$

provided $|F-G|_{G,0} \leq \delta_1$. Let

$$d_\gamma = \sup\{|D_F\mathcal{R} - D_G\mathcal{R}|_{(G,0),(\mathcal{R}G,1)}\colon |F-G|_{G,0} \leq \gamma, G \in \Omega\}.$$

Choose $\delta_2 \in (0,\delta_1)$ small enough such that

$$\lambda_1 = \frac{\frac{1}{\theta} + \frac{d_{\eta_k}}{\epsilon\theta_1}}{1 - \frac{\epsilon}{\theta} - \frac{d_{\delta_2}}{\theta_1}} < 1$$

and

$$\frac{\lambda_1}{1-\lambda_1\epsilon} < 1.$$

This is possible because $\theta - 2\epsilon > 1$. Then for every F such that $|F-G|_{G,0} \leq \delta_2$, with $G \in \Omega$, we have

$$D_F\mathcal{R}(C^u_{\epsilon,0}(G)) \subset C^u_{\lambda_1\epsilon,(1,0)}(\mathcal{R}G) \subset C^u_{\epsilon,0}(\mathcal{R}G). \tag{4}$$

Let $G \in \Omega$. Define

$$\mathbb{B}(F,\delta) = \{\tilde{F} \in \mathcal{B}_0\colon \ |\tilde{F} - F|_0 \leq \delta\},$$

$$\mathbb{B}^u_G(v_0,\delta) = \{v \in E^u_G\colon \ |v - v_0|_{G,0} \leq \delta\}$$

and

$$E^h_G + G = \{v + G\colon \ v \in E^h_G\}.$$

Choose $\delta_3 > 0$ small enough such that $(2+\epsilon)\delta_3 < \delta_2$ and

$$C_5 = \sup\{|\tilde{\mathcal{R}}(F)|_1 \ s.t. \ dist_{\mathcal{B}_0}(F,\Omega) \leq C_2(2+\epsilon)\delta_3\} < \infty. \tag{5}$$

Let $\mathcal{T}^k_i(G,\delta)$, $i = 0,1$, with $\delta \in (0,\delta_3)$, be the set of all C^k functions

$$\mathcal{H}\colon \mathbb{B}^u_G(v_0,\delta) \to E^h_G \cap \mathcal{B}_i + G$$

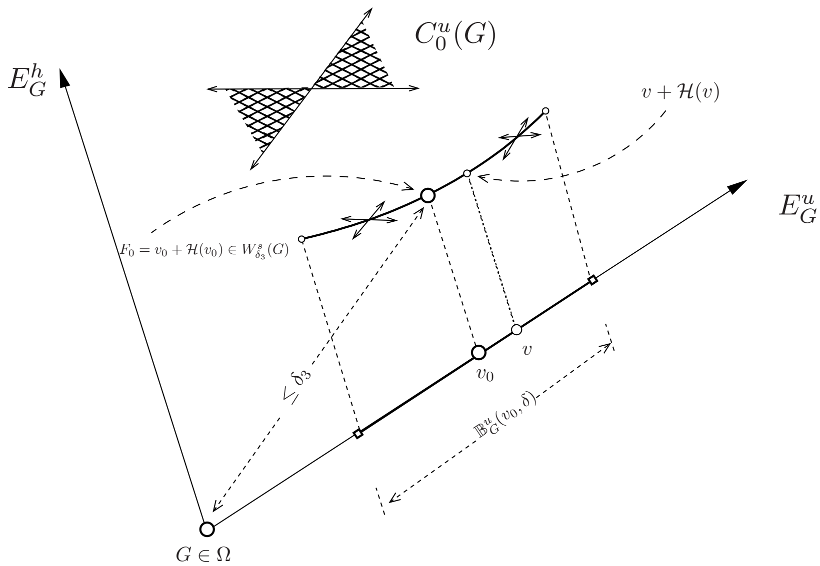

FIGURE 1. How the graph of $\mathcal{H} \in \mathcal{T}_i^k(G,\delta)$ looks like.

such that

$$|D\mathcal{H}|_{(G,0),(G,i)} \leq \epsilon$$

and

$$F_0 = v_0 + \mathcal{H}(v_0) \in W^s_{\delta_3}(G).$$

In particular

$$\sup_{v\in\mathbb{B}^u_G(v_0,\delta)} |v+\mathcal{H}(v)-G|_{G,0} \leq (2+\epsilon)\delta_3 < \delta_2. \tag{6}$$

We will call F_0 the *base point* of $\mathcal{H}$. We will denote the graph of $\mathcal{H}$ by $\hat{\mathcal{H}}$, that is,

$$\hat{\mathcal{H}} = \{v+\mathcal{H}(v)\colon\ v\in\mathbb{B}^u_G(v_0,\delta)\}.$$

The graph $\hat{\mathcal{H}}$ of a function $\mathcal{H} \in \mathcal{T}_i^k(G,\delta)$ will be called a **transversal family**. See Figure 1. Note that

$$\mathcal{T}_1^k(G,\delta) \subset \mathcal{T}_0^k(G,\delta).$$

In Theorems 1 and 2 we assume that $W^s_\delta(\Omega)$ has (either real or complex) transversal empty interior. But indeed we just need that the Transversal Empty Interior Assumption holds for n-dimensional manifolds of the form $\hat{\mathcal{H}}$. More precisely

Transversal Empty Interior Assumption: There is $\delta>0$ such that for every δ' small enough the following holds. For every $G\in\Omega$ and for every C^{1+Lip} function $\mathcal{H}\in\mathcal{T}_0^1(G,\delta')$ we have that $\hat{\mathcal{H}}\cap W^s_\delta(\Omega)$ has empty interior in $\hat{\mathcal{H}}$.

If $\mathcal{R}$ and Ω satisfy the assumptions of Theorem 2, then we interpret a C^{1+Lip} function as a C^{1+Lip} complex differentiable function, that is, a complex analytic function.

The definition of $\mathcal{T}_i^k(G,\delta)$ depends on $\delta_3 > 0$. However if $\mathcal{U}_i^k(G,\delta)$ is a similar collection of functions obtained replacing δ_3 by a smaller positive value $\gamma < \delta_3$ then $\mathcal{U}_i^k(G,\delta) \subset \mathcal{T}_i^k(G,\delta)$ and the transversal empty interior assumption remains true. So we will replace δ_3 by smaller positive values a few times along this work and we will keep the same notation $\mathcal{T}_i^k(G,\delta)$ for the new family of functions $\mathcal{U}_i^k(G,\delta)$.

5. Action of $\mathcal{R}$ on transversal families.

In this section we will see that not only the operator $\mathcal{R}$ keeps invariant the set of transversal families, but indeed, if we keep track only of the piece of the transversal family close to the Ω-limit set then the iterations of a transversal family consists in bounded subset in C^2 topology. This is similar to the well-know λ-Lemma for hyperbolic points due to Palis [23] (see also de Melo and Palis [26]).

Proposition 5.1. *There are $\theta_2 > 1$, $C_6 > 0$ and $\lambda_2, \lambda_3, \lambda_4 \in (0,1)$ with the following property. For every $\delta \in (0,\delta_3]$ and every $\mathcal{H}_0 \in \mathcal{T}_0^2(G,\delta)$, with $G \in \Omega$ and base point $F \in W^s_{\delta_3}(G)$ the following holds:*

A. *For every $x \in \mathbb{B}^u_{\mathcal{R}G}(v_1, \theta_2\delta)$, with $v_1 = \pi^u_{\mathcal{R}G}(\mathcal{R}F - \mathcal{R}G)$, there exists a unique $y \in E^h_{\mathcal{R}G}$ such that $\mathcal{R}G + x + y \in \mathcal{R}(\hat{\mathcal{H}}_0)$.*

B. *Define the function*

$$\mathcal{H}_1 \colon \mathbb{B}^u_{\mathcal{R}G}(v_1, \theta_2\delta) \to E^h_{\mathcal{R}G} + \mathcal{R}G$$

as $\mathcal{H}_1(x) = \mathcal{R}G + y$, where x, y are as in item A. Then $\mathcal{H}_1 \in \mathcal{T}_1^2(\mathcal{R}G, \theta_1\delta)$, where $\mathcal{R}F$ is a base point of $\mathcal{H}_1$. Moreover

$$|D\mathcal{H}_1|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq \lambda_2\epsilon.$$

C. *We have*

$$|D^2\mathcal{H}_1|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq C_6 + \lambda_3 |D^2\mathcal{H}_0|_{G,0}. \tag{7}$$

D. *We have that $w + \mathcal{H}_1(w) \in \mathcal{B}_1$ for every $w \in \mathbb{B}^u_{\mathcal{R}G}(v_1, \theta_1\delta)$ and*

$$|w + \mathcal{H}_1(w)|_1 \leq C_5. \tag{8}$$

E. *There is a function*

$$\mathcal{R}^{-1} \colon \hat{\mathcal{H}}_1 \mapsto \hat{\mathcal{H}}_0$$

such that $\mathcal{R} \circ \mathcal{R}^{-1}(Y) = Y$ for every $Y \in \hat{\mathcal{H}}_1$. Its image is an open subset of $\hat{\mathcal{H}}_0$ and $\mathcal{R}^{-1}$ is a diffeomorphism between $\hat{\mathcal{H}}_1$ and $\mathcal{R}^{-1}(\hat{\mathcal{H}}_1)$. Moreover

$$|\mathcal{R}^{-1}(F_1) - \mathcal{R}^{-1}(F_2)|_{G,0} \leq \lambda_4 |F_1 - F_2|_{\mathcal{R}G,0}. \tag{9}$$

and replacing $\mathcal{R}$ by a iteration of it

$$|\mathcal{R}^{-1}(F_1) - \mathcal{R}^{-1}(F_2)|_0 \leq \lambda_4 |F_1 - F_2|_0. \tag{10}$$

for every $F_1, F_2 \in \hat{\mathcal{H}}_1$.

Remark 5.2. *If $\mathcal{R}$ and Ω satisfy the assumptions of Theorem 2, one needs to consider $\mathcal{T}_0^2(G,\delta)$ as a set of C^2 complex differentiable functions, that is, complex analytic functions. In this case the estimates for the first and second derivatives of $\mathcal{H}_1$ are not necessary.*

Proof. Note that

$$(1-\epsilon)|v|_{G,0} \leq |v + D_x\mathcal{H}_0 \cdot v|_{G,0} \leq (1+\epsilon)|v|_{G,0}. \tag{11}$$

Let $\pi\colon \mathcal{B}_1 \mapsto \mathcal{B}_1$ be a linear transformation and let

$$\phi(x) = \pi(\tilde{\mathcal{R}}(x + \mathcal{H}_0(x)) - \tilde{\mathcal{R}}G).$$

Then

$$D_x\phi \cdot v = \pi[D_{x+\mathcal{H}_0(x)}\tilde{\mathcal{R}} \cdot (v + D_x\mathcal{H}_0 \cdot v)], \tag{12}$$

$$D_x^2\phi \cdot v^2 = \pi[D^2_{x+\mathcal{H}_0(x)}\tilde{\mathcal{R}} \cdot (v + D_x\mathcal{H}_0 \cdot v)^2 + D_{x+\mathcal{H}_0(x)}\tilde{\mathcal{R}} \cdot D_x^2\mathcal{H}_0 \cdot v^2], \tag{13}$$

Suppose from now on that $|\pi|_1 \leq 1$. We have

$$|D_x\phi|_{(G,0),(\mathcal{R}G,1)} \leq |D\tilde{\mathcal{R}}|_\infty(1+\epsilon). \tag{14}$$

Moreover since

$$D_x^2\mathcal{H}_0 \cdot v^2 \in E^h_G,$$

and

$$|D_{x+\mathcal{H}_0(x)}\mathcal{R} \cdot w|_{\mathcal{R}G,1} \leq \lambda_G |w|_{G,0} \tag{15}$$

for every $w \in E^h_G$, by (13) and (15) we have

$$\begin{aligned} |D_x^2\phi|_{(G,0),(\mathcal{R}G,1)} &\leq |D^2\tilde{\mathcal{R}}|_\infty(1+\epsilon)^2 + \lambda_G|D^2\mathcal{H}_0|_{G,0} \\ &\leq C_7 + \lambda_G|D^2\mathcal{H}_0|_{G,0}. \end{aligned} \tag{16}$$

Define

$$\phi_u(x) = \pi^u_{\mathcal{R}G}(\mathcal{R}(x + \mathcal{H}_0(x)) - \mathcal{R}(G)).$$
$$\phi_h(x) = \pi^h_{\mathcal{R}G}(\mathcal{R}(x + \mathcal{H}_0(x)) - \mathcal{R}(G)).$$

Note that for each $w \in C^u_{(1,0)}(\mathcal{R}G)$ we have

$$|\pi^u_{\mathcal{R}G}(w)|_{\mathcal{R}G,0} \geq (1-\epsilon)|w|_{\mathcal{R}G,0} \tag{17}$$

and since $v + D_x\mathcal{H}_0 \cdot v \in C^u_0(G)$ we have

$$|D_{x+\mathcal{H}_0(x)}\mathcal{R} \cdot (v + D_x\mathcal{H}_0 \cdot v)|_{\mathcal{R}G,0} \geq \frac{\theta_G(1-\epsilon)}{1+\epsilon}|v + D_x\mathcal{H}_0 \cdot v|_{G,0}. \tag{18}$$

It follows from (11), (17) and (18) that

$$|D_x\phi_u \cdot v|_{\mathcal{R}G,0} \geq \theta_G\frac{(1-\epsilon)^3}{1+\epsilon}|v|_{G,0} \geq \theta\frac{(1-\epsilon)^3}{1+\epsilon}|v|_{G,0},$$

so in particular ϕ_u is a diffeomorphism on $\mathbb{B}^u_G(v_0,\delta)$, its image contains $\mathbb{B}^u_{\mathcal{R}G}(v_1,\theta_2\delta)$, where $v_1 = \pi^u_{\mathcal{R}G}(\mathcal{R}F - \mathcal{R}G)$ and

$$\theta_2 = \theta\frac{(1-\epsilon)^3}{1+\epsilon} > 1.$$

Consequently

$$\phi_u^{-1}\colon \mathbb{B}^u_{\mathcal{R}G}(v_1,\theta_2\delta) \mapsto \mathbb{B}^u_G(v_0,\delta)$$

is well defined and it is a contraction on $\mathbb{B}^u_{\mathcal{R}G}(v_1,\theta_2\delta)$, since

$$|D_x\phi_u^{-1}|_{(\mathcal{R}G,0),(G,0)} \leq \frac{1+\epsilon}{\theta_G(1-\epsilon)^3} \leq \frac{1}{\theta_2} < 1. \tag{19}$$

Note that since $\phi_u^{-1}\circ\phi_u(x) = x$ we have

$$D_{\phi_u(x)}\phi_u^{-1} \cdot D_x\phi_u \cdot w = w,$$

so if $w = (D_x\phi_u)^{-1} \cdot v$

$$D_{\phi_u(x)}\phi_u^{-1} \cdot v = (D_x\phi_u)^{-1} \cdot v. \tag{20}$$

Moreover

$$D^2_{\phi_u(x)}\phi_u^{-1} \cdot (D_x\phi_u \cdot w)^2 + D_{\phi_u(x)}\phi_u^{-1} \cdot D^2_x\phi_u \cdot w^2 = 0, \tag{21}$$

so

$$D^2_{\phi_u(x)}\phi_u^{-1} \cdot v^2 = -D_{\phi_u(x)}\phi_u^{-1} \cdot D^2_x\phi_u \cdot ((D_x\phi_u)^{-1} \cdot v)^2. \tag{22}$$

By (16) and (19)

$$\begin{aligned} |D^2_{\phi_u(x)}\phi_u^{-1}|_{(\mathcal{R}G,0),(G,0)} &\leq \frac{(1+\epsilon)^3}{\theta_G^3(1-\epsilon)^9}(C_7 + \lambda_G|D^2\mathcal{H}_0|_{G,0}.) \\ &\leq C_8 + \frac{\lambda_G(1+\epsilon)^3}{\theta_G^3(1-\epsilon)^9}|D^2\mathcal{H}_0|_{G,0}. \\ &\leq C_8 + \lambda_5|D^2\mathcal{H}_0|_{G,0}, \qquad (23) \end{aligned}$$

where

$$\lambda_5 = \frac{(1+\epsilon)^3}{\theta^3(1-\epsilon)^9} < 1$$

Note that for each $w \in C^u_{(1,0)}(\mathcal{R}G)$ we have

$$|\pi^h_{\mathcal{R}G}(w)|_{\mathcal{R}G,1} \leq \epsilon|w|_{\mathcal{R}G,0}, \tag{24}$$

so since

$$D_{x+\mathcal{H}_0(x)}\mathcal{R} \cdot (v + D_x\mathcal{H}_0 \cdot v) \in C^u_{(1,0)}(\mathcal{R}G)$$

we obtain

$$|D\phi_h|_{(G,0),(\mathcal{R}G,1)} \leq \epsilon|D\mathcal{R}|_\infty(1+\epsilon).$$

Then

$$\mathcal{H}_1 = \phi_h \circ \phi_u^{-1} + \mathcal{R}G$$

is well defined on $\mathbb{B}^u_{\mathcal{R}G}(v_1, \theta_2\delta)$. Note that

$$D_x\mathcal{H}_1 \cdot v = D_{\phi_u^{-1}(x)}\phi_h \cdot D_x\phi_u^{-1} \cdot v. \tag{25}$$

$$D^2_x\mathcal{H}_1 \cdot v^2 = D^2_{\phi_u^{-1}(x)}\phi_h \cdot (D_x\phi_u^{-1} \cdot v)^2 + D_{\phi_u^{-1}(x)}\phi_h \cdot D^2_x\phi_u^{-1} \cdot v^2. \tag{26}$$

Moreover

$$|D\mathcal{H}_1|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq \epsilon. \tag{27}$$

Indeed, given v, let w be

$$w = D_x\phi_u^{-1} \cdot v,$$

that is, if

$$x = \phi_u(y)$$

then

$$v = \pi^u_{\mathcal{R}G}(D_{y+\mathcal{H}_0(y)}\mathcal{R} \cdot (w + D_y\mathcal{H}_0 \cdot w)).$$

Since

$$D_{y+\mathcal{H}_0(y)}\mathcal{R} \cdot (w + D_y\mathcal{H}_0 \cdot w) \in C^u_{\lambda_1\epsilon,(1,0)}(\mathcal{R}G) \tag{28}$$

we have that

$$|v|_{\mathcal{R}G,0} \geq (1-\lambda_1\epsilon)|D_{y+\mathcal{H}_0(y)}\mathcal{R}\cdot(w+D_y\mathcal{H}_0\cdot w)|_{\mathcal{R}G,0}. \tag{29}$$

Moreover

$$D_{\phi_u^{-1}(x)}\phi_h \cdot D_x\phi_u^{-1}\cdot v = \pi^h_{\mathcal{R}G}(D_{y+\mathcal{H}_0(y)}\mathcal{R}\cdot(w+D_y\mathcal{H}_0\cdot w))$$

Due (28) we got

$$|D_{\phi_u^{-1}(x)}\phi_h \cdot D_x\phi_u^{-1}\cdot v|_{\mathcal{R}G,1} \leq \epsilon\lambda_1|D_{y+\mathcal{H}_0(y)}\mathcal{R}\cdot(w+D_y\mathcal{H}_0\cdot w)|_{\mathcal{R}G,0}, \tag{30}$$

so due (29) and (30)

$$|D_{\phi_u^{-1}(x)}\phi_h \cdot D_x\phi_u^{-1}\cdot v|_{\mathcal{R}G,1} \leq \frac{\epsilon\lambda_1}{1-\epsilon\lambda_1}|v|_{G,0}$$

Recall we choose ϵ and δ_2 such that

$$\lambda_2 := \frac{\lambda_1}{1-\epsilon\lambda_1} < 1,$$

we obtain (27). Indeed

$$|D\mathcal{H}_1|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq \lambda_2\epsilon. \tag{31}$$

Note that $\mathcal{R}F \in W^s_{\delta_3}(\mathcal{R}G)$ and we have

$$\lambda_3 = \frac{(1+\epsilon)^2}{\theta^2(1-\epsilon)^6} + \frac{\epsilon(1+\epsilon)|D\mathcal{R}|_\infty(1+\epsilon)^3}{\theta^3(1-\epsilon)^9} < 1.$$

So by (26) we obtain

$$\begin{aligned}
|D^2_x\mathcal{H}_1|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} &\leq \frac{(1+\epsilon)^2}{\theta_G^2(1-\epsilon)^6}\big(C_7+\lambda_G|D^2\mathcal{H}_0|_{G,0}\big)\\
&+ \epsilon(1+\epsilon)|D\mathcal{R}|_\infty\big(C_8+\frac{(1+\epsilon)^3}{\theta^3(1-\epsilon)^9}|D^2\mathcal{H}_0|_{G,0}\big)\\
&\leq C_6+\Big(\frac{\lambda_G(1+\epsilon)^2}{\theta_G^2(1-\epsilon)^6}+\frac{\epsilon|D\mathcal{R}|_\infty(1+\epsilon)^4}{\theta^3(1-\epsilon)^9}\Big)|D^2\mathcal{H}_0|_{G,0}\\
&\leq C_6+\lambda_3|D^2\mathcal{H}_0|_{G,0}
\end{aligned} \tag{32}$$

This proves A, B and C. Property D follows from (5),(6) and that $\hat{\mathcal{H}}_1 \subset \mathcal{R}\hat{\mathcal{H}}_0$. To show property E define

$$\mathcal{R}^{-1}(Y) = \phi_u^{-1}\circ\pi^u_{\mathcal{R}G}(Y) + \mathcal{H}_0\circ\phi_u^{-1}\circ\pi^u_{\mathcal{R}G}(Y)$$

for every $Y \in \hat{\mathcal{H}}_1$. Note that

$$\mathcal{R}(X) = \phi_u\circ\pi^u_G(X) + \mathcal{H}_1\circ\phi_u\circ\pi^u_G(X)$$

for every $X \in \mathcal{R}^{-1}(\hat{\mathcal{H}}_1)$ so we have $\mathcal{R}\circ\mathcal{R}^{-1}(Y) = Y$ for every $Y \in \hat{\mathcal{H}}_1$. Since ϕ_u, ϕ_u^{-1}, $\mathcal{H}_0$ and $\mathcal{H}_1$ are C^1 functions we conclude that $\mathcal{R}^{-1}$ is a diffeomorphism on $\hat{\mathcal{H}}_1$. Moreover for every $F_1, F_2 \in \hat{\mathcal{H}}_1$ we have by (19)

$$\begin{aligned}
|\mathcal{R}^{-1}(F_1)-\mathcal{R}^{-1}(F_1)|_{G,0} &\leq (1+\epsilon)|\phi_u^{-1}\circ\pi^u_{\mathcal{R}G}(F_1)-\phi_u^{-1}\circ\pi^u_{\mathcal{R}G}(F_2)|_{G,0}\\
&\leq (1+\epsilon)|\phi_u^{-1}\circ\pi^u_{\mathcal{R}G}(F_1)-\phi_u^{-1}\circ\pi^u_{\mathcal{R}G}(F_2)|_{G,0}\\
&\leq \frac{1+\epsilon}{\theta_2}|\pi^u_{\mathcal{R}G}(F_1)-\pi^u_{\mathcal{R}G}(F_2)|_{\mathcal{R}G,0}\\
&\leq \frac{1+\epsilon}{\theta_2}|F_1-F_2|_{\mathcal{R}G,0}.
\end{aligned} \tag{33}$$

Choose

$$\lambda_4 = \frac{1+\epsilon}{\theta_2} = \frac{(1+\epsilon)^2}{\theta(1-\epsilon)^3} < 1.$$

Replacing $\mathcal{R}$ by an iteration of it, by (9) we have (10) for every $F_1, F_2 \in \hat{\mathcal{H}}_1$. $\square$

From now on replace $\mathcal{R}$ by an iteration of it such that (10) holds. Let $\mathcal{H}_0$, $\mathcal{H}_1$ be as in Proposition 5.1. We denote $\mathcal{H}_1 = \hat{\mathcal{R}}_G \mathcal{H}_0$.

Corollary 5.3. *Let $\theta_2 > 1$ be as in Proposition 5.1. There exists $C_9 > 0$ such that the following holds: Given $C_{10} > 0$ and $\delta \in (0, \delta_3]$ there exists $k_2 \geq 1$ such that for every $G \in \Omega$ and $\mathcal{H}_0 \in \mathcal{T}_0^2(G, \delta)$ with base point $F \in W^s_{\delta_3}(G)$ that satisfies*

$$|D^2\mathcal{H}_0|_{G,0} \leq C_{10}$$

then we have that

$$\mathcal{H}_k = \hat{\mathcal{R}}_{\mathcal{R}^k G}\mathcal{H}_{k-1} \in \mathcal{T}_1^2(\mathcal{R}^k G, \eta_k)$$

are well defined, with base point $\mathcal{R}^k F \in W^s_{\delta_3}(\mathcal{R}^k G)$, where $\delta_0 = \delta$, and

$$\eta_k \;=\; \min\{\theta_2 \eta_{k-1}, \delta_3\}$$

for $k > 0$. Moreover for every $k \geq 1$

$$|D\mathcal{H}_k|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq \lambda_2 \epsilon \tag{34}$$

and

$$|w + \mathcal{H}_k(w)|_{\mathcal{R}^k G,1} < C_5.$$

Here

$$v_k = \pi^u_{\mathcal{R}^k G}(\mathcal{R}^k F - \mathcal{R}^k G).$$

Furthermore for every $k \geq k_2$ we have

$$|D^2\mathcal{H}_k|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq \frac{C_9}{2}. \tag{35}$$

Proof. By Proposition 5.1.B it follows that $\mathcal{H}_k \in \mathcal{T}_1^2(\mathcal{R}^k G, \eta_k)$ for every $k \geq 1$. In particular if $k \geq k_0 = \min\{k \in \mathbb{N}\ s.t.\ \theta_1^k \delta > \delta_3\} + 1$ we have that $\mathcal{H}_k \in \mathcal{T}_1^2(\mathcal{R}^k G, \delta_3)$. By (7) we have

$$|D^2\mathcal{H}_k|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq C_6 + \lambda_3 |D^2\mathcal{H}_{k-1}|_{G,0},$$

for every k, so

$$|D^2\mathcal{H}_k|_{(\mathcal{R}G,0),(\mathcal{R}G,1)} \leq \sum_{j=0}^{k-1} C_6 {\lambda_3}^j + {\lambda_3}^k |D^2\mathcal{H}_0|_{G,0} \leq \frac{C_6}{1-\lambda_3} + {\lambda_3}^k C_{10},$$

Choose C_9 such that

$$\frac{C_6}{1-\lambda_3} < \frac{C_9}{2}$$

and $k_1 \geq k_0$ satisfying

$$\frac{C_6}{1-\lambda_3} + {\lambda_3}^{k_1} C_{10} < \frac{C_9}{2}.$$

Then (35) holds for every $k \geq k_1$. $\square$

Let k_2 given by Corollary 5.3 when we choose $C_{10} = C_9$. From now on we replace $\mathcal{R}$ by $\mathcal{R}^{k_2}$.

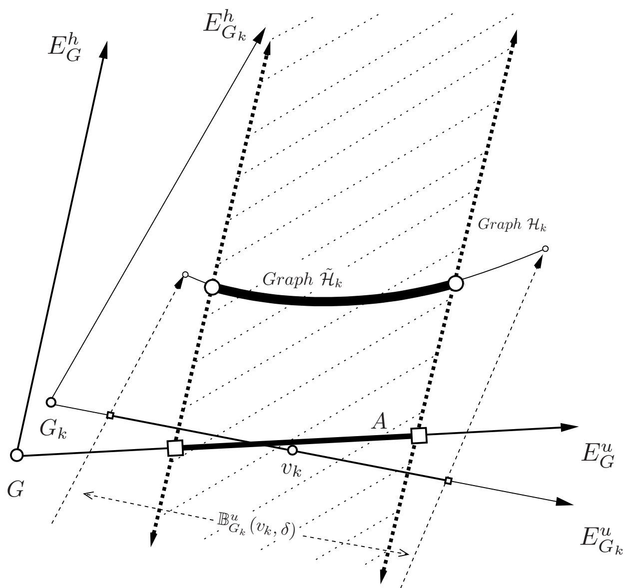

FIGURE 2. Convergence of transversal families.

6. LOWER BOUND TO THE MEASURE OF PARAMETERS OUTSIDE $W^s_{\tilde{\delta}_3}(\Omega)$.

We say that a sequence of C^j functions $\mathcal{H}_k \in \mathcal{T}^j_0(G_k,\delta)$, $\delta \in (0,\delta_3)$, with base point F_k converges to a C^j function $\mathcal{H} \in \mathcal{T}^j_0(G,\delta)$ with base point F if

$$\lim_k G_k = G, \lim_k F_k = F$$

and for every open set A compactly contained in $\mathbb{B}^u_G(v_\infty,\delta)$ there is k_0 such that for $k > k_0$ the set

$$\{w + \mathcal{H}_k(w)\colon\ w \in \mathbb{B}^u_{G_k}(v_k,\delta)\} \cap \{u + v + G\colon\ u \in A,\ v \in E^h_G\}$$

is the graph (See Figure 2) of a C^j function

$$\tilde{\mathcal{H}}_k\colon A \to E^h_G + G$$

and

$$\lim_k |\tilde{\mathcal{H}}_k - \mathcal{H}|_{C^j(A)} = 0.$$

Here

$$v_k = \pi^u_{G_k}(F_k - G_k) \text{ and } v_\infty = \pi^u_G(F - G).$$

Proposition 6.1. *Let $\mathcal{H}_k \in \mathcal{T}^2_1(G_k,\delta)$, $\delta \in (0,\delta_3)$, be a sequence of C^2 functions, for some $G_k \in \Omega$, base point F_k and satisfying*

$$|D\mathcal{H}_k|_{(G_k,0),(G_k,1)} \le \epsilon, |D^2\mathcal{H}_k|_{(G_k,0),(G_k,1)} \le C_9. \tag{36}$$

Moreover assume there exists C_{11} such that

$$|w + \mathcal{H}_k(w)|_{G_k,1} \le C_{11},$$

for every $w \in \mathbb{B}^u_{G_k}(v_k,\delta)$, with

$$v_k = \pi^u_{G_k}(F_k - G_k),$$

and $k \geq 0$. Then there exists a subsequence $\mathcal{H}_{k_i}$ that converges to a C^{1+Lip} function $\mathcal{H} \in \mathcal{T}_0^1(G,\delta)$, for some $G \in \Omega$.

Proof. Since Ω is compact, we can assume that the sequence G_k converges on $\mathcal{B}_0$ to some $G \in \Omega$. We claim that for k large enough the map

$$\pi_k \colon \mathbb{B}^u_{G_k}(v_k,\delta) \to E^u_G$$

defined as

$$\pi_k(w) = \pi^u_G(w + \mathcal{H}_k(w) - G). \tag{37}$$

is a homeomorphism on its image. It is enough to show that this map is injective. Indeed given $\gamma > 0$ then for k large enough we have

$$|\pi^u_{G_k} - \pi^u_G|_0,\ |\pi^h_{G_k} - \pi^h_G|_0 < \frac{\gamma}{C_2}.$$

Choose γ such that $\gamma(1+\epsilon) < 1$. Note that

$$\begin{aligned}\pi^u_G(w + \mathcal{H}_k(w) - G) &= \pi^u_{G_k}(w + \mathcal{H}_k(w)) - \pi^u_G(G) + (\pi^u_G - \pi^u_{G_k})(w + \mathcal{H}_k(w)) \\ &= w + \pi^u_G(G_k - G) + (\pi^u_G - \pi^u_{G_k})(w + \mathcal{H}_k(w)), \end{aligned} \tag{38}$$

so if

$$\pi^u_G(w + \mathcal{H}_k(w) - G) = \pi^u_G(w' + \mathcal{H}_k(w') - G)$$

we would have

$$\begin{aligned}|w - w'|_{G_k,0} &= |(\pi^u_G - \pi^u_{G_k})(w - w' + \mathcal{H}_k(w) - \mathcal{H}_k(w'))|_{G_k,0} \\ &\leq \gamma(1+\epsilon)|w - w'|_{G_k,0}, \end{aligned} \tag{39}$$

which implies $w = w'$. So the map defined in (37) is injective. Since F_k is a bounded sequence in $\mathcal{B}_1$, we can assume that F_k converges to some $F \in \mathcal{B}_0$. So

$$v_\infty = \lim_k v_k = \pi^u_G(F - G).$$

It is easy to see that $F \in W^s_{\delta_3}(G)$. Define

$$\pi_k \colon \mathbb{B}^u_{G_k}(v_k,\delta) \to E^u_G$$

as

$$\pi_k(w) = \pi^u_G(w + \mathcal{H}_k(w) - G).$$

Then by (38)

$$(1 - \tilde{\gamma}_k(1+\epsilon))|w - w'|_{G_k,0} \leq |\pi_k(w) - \pi_k(w')|_{G,0} \leq (1 + \tilde{\gamma}_k(1+\epsilon))|w - w'|_{G_k,0}, \tag{40}$$

where

$$\lim_k \tilde{\gamma}_k = 0.$$

Since π_k is a homeomorphism on its image, (40) implies that for every $\delta' < \delta$ there exists k_0 such that if $k > k_0$ then

$$\mathbb{B}^u_G(v_\infty,\delta') \subset \pi_k(\mathbb{B}^u_{G_k}(v_k,\delta)),$$

so for every convex open set A compactly contained in $\mathbb{B}^u_G(v_\infty,\delta)$ there is k_0 such that if $k > k_0$ then

$$A \subset \pi_k(\mathbb{B}^u_{G_k}(v_k,\delta)).$$

Define

$$\tilde{\mathcal{H}}_k \colon A \to E^h_G + G$$

as

$$\tilde{\mathcal{H}}_k(u) = \pi^h_G((Id + \mathcal{H}_k) \circ (\pi_k)^{-1}(u) - G).$$

Claim. *There exists C_{12} such that*

$$|D^b\pi^h_G \circ (Id + \mathcal{H}_k) \circ (\pi_k)^{-1}(u)|_{(G,0),(G,1)} \leq C_{12}$$

for every $u \in A$ and $k \geq k_0$, $b \in \{0,1,2\}$, that is,

$$|D^b\tilde{\mathcal{H}}_k|_{(G,0),(G,1)} \leq C_{12},$$

for every $b \in \{0,1,2\}$.

Indeed, note that

$$|\pi^h_G \circ (Id + \mathcal{H}_k) \circ (\pi_k)^{-1}(u)|_{G,1} \leq C_3C_{11}.$$

Moreover

$$D_w\pi_k \cdot v = \pi^u_G(v + D_w\mathcal{H}_k \cdot v).$$
$$D^2_w\pi_k \cdot v^2 = \pi^u_G(D^2_w\mathcal{H}_k \cdot v^2).$$

and

$$D_w\pi_k^{-1} \cdot v = (D_{\pi_k^{-1}(w)}\pi_k)^{-1} \cdot v,$$
$$D^2_w\pi_k^{-1} \cdot v^2 = -D_w\pi_k^{-1} \cdot \pi^u_G \cdot D^2_{\pi_k^{-1}(w)}\mathcal{H}_k \cdot (D_w\pi_k^{-1} \cdot v)^2$$

If k is large enough then

$$|D_w\pi_k \cdot v|_{G,0} \geq C_2^{-2}(1-\epsilon)^2|v|_{G_k,0},$$

so

$$|D_w\pi_k^{-1}|_{(G,0),(G_k,0)} \leq C_2^2(1-\epsilon)^{-2}.$$

and

$$|D^2_w\pi_k^{-1}|_{(G,0),(G_k,0)} \leq C_2^8(1-\epsilon)^{-6}C_9.$$

So using (v), (vii) and (36)

$$\begin{aligned} &|D\pi^h_G \circ (Id + \mathcal{H}_k) \circ \pi_k^{-1}(u)|_{(G,0),(G,1)} \\ &\leq |\pi^h_G|_{(G_k,1),(G,1)}|Id + D_{\pi_k^{-1}(u)}\mathcal{H}_k|_{(G_k,0),(G_k,1)}|D\pi_k^{-1}(u)|_{(G,0),(G_k,0)} \\ &\leq C_3(C_1+\epsilon)C_2^2(1-\epsilon)^{-2}. \end{aligned} \tag{41}$$

and

$$\begin{aligned} &|D^2\pi^h_G \circ (Id + \mathcal{H}_k) \circ \pi_k^{-1}(u)|_{(G,0),(G,1)} \\ \leq\ & |\pi^h_G \cdot D^2_u\pi_k^{-1}|_{(G,0),(G,1)} + |\pi^h_G \cdot D^2_{\pi_k^{-1}(u)}\mathcal{H}_k \cdot (D_u\pi_k^{-1})^2|_{(G,0),(G,1)} \\ +\ & |\pi^h_G \cdot D_{\pi_k^{-1}(u)}\mathcal{H}_k \cdot D^2_u\pi_k^{-1}|_{(G,0),(G,1)} \\ \leq\ & C_3C_1C_2^8(1-\epsilon)^{-6}C_9 + C_3C_9C_2^4(1-\epsilon)^{-4} + C_3\epsilon C_2^8(1-\epsilon)^{-6}C_9. \end{aligned} \tag{42}$$

This proves the claim. In particular, since A is convex, the maps

$$u \mapsto D^b\tilde{\mathcal{H}}_k(u) \ \textit{are uniformly Lipschitz on } A, \tag{43}$$

for $b \in \{0,1\}$, the set

$$\{\tilde{\mathcal{H}}_k(u):\ u \in A,\ k \geq k_1\}$$

is a relatively compact subset of $\mathcal{B}_0$, and

$$\{D\tilde{\mathcal{H}}_k(u):\ u \in A,\ k \geq k_1\}$$

is a relatively compact subset of the space of all continuous operators of $\mathcal{B}_0$. Let $\{u_i\}_{i\in\mathbb{N}}$ be a dense subset of A. Then using the Cantor's diagonal argument one can find a subsequence k_j such that the limits

$$\lim_j \ D^b\tilde{\mathcal{H}}_k(u_i),$$

exists for every $i \in \mathbb{N}$. Then (43) implies that

$$\mathcal{H}^b(u) = \lim_j \ D^b\tilde{\mathcal{H}}_k(u),$$

exists for every $u \in A$, $b \in \{0,1\}$. This convergence is uniform on $u \in A$. One can easily conclude that

$$D\mathcal{H}^0 = \mathcal{H}^1$$

So $\mathcal{H} = \mathcal{H}^0$ is C^{1+Lip} and $\tilde{\mathcal{H}}_k$ converges on A to $\mathcal{H}$ in C^1 topology. Since there exists an exhaustion of $\mathbb{B}^u_G(v_\infty, \delta)$ by convex, open and relatively compact sets A_k, we can use Cantor's diagonal argument once again to find a C^{1+Lip} map $\mathcal{H} \in \mathcal{T}^1_0(G,\delta)$ such that $\mathcal{H}_k$ converges to $\mathcal{H}$ in C^1 topology. □

For every $\mathcal{H} \in \mathcal{T}^1_0(G,\delta)$ we can define a borelian measure $m_\mathcal{H}$ on $\hat{\mathcal{H}}$ in the following way. The measure $m_\mathcal{H}$ is the n-dimensional *Hausdorff measure* on $\hat{\mathcal{H}}$ with respect to the metric induced in $\hat{\mathcal{H}}$ by the norm of $|\cdot|_{G,0}$.

Fix some $C_{13} > 0$. For each $G \in \Omega$, let $\gamma_G > 0$ be such that for every $\tilde{G} \in \mathbb{B}(G,\gamma_G)\cap\Omega$ the map

$$\pi^u_{\tilde{G}} \colon E^u_G \mapsto E^u_{\tilde{G}}$$

is a linear isomorphism whose norm and the norm of its inverse is bounded by some constant C_{13}, considering the norm $|\cdot|_{G,0}$ on E^u_G and the norm $|\cdot|_{\tilde{G},0}$ on $E^u_{\tilde{G}}$. This is possible since $G \mapsto \pi^u_G$ is continuous with respect to the $\mathcal{B}_0$ norm. By the compactness of Ω there is a finite set $G^\star_1, \dots, G^\star_m \in \Omega$ such that

$$\Omega \subset \cup_i \mathbb{B}(G^\star_i, \gamma_{G_i}).$$

Fix a basis $v^i_1, \dots v^i_n$ for $E^u_{G^\star_i}$. For every

$$G \in \mathbb{B}(G^\star_i, \gamma_{G^\star_i}) \cap \Omega$$

we have a basis $v^{G,i}_1, \dots, v^{G,i}_n$ for E^u_G given by $v^{G,i}_j = \pi^u_G(v^i_j)$. Let $|\cdot|_{G,G^\star_i}$ be the norm on E^u_G that turns E^u_G into a Hilbert space and $\mathcal{S}_{G,i} = (v^{G,i}_1, \dots, v^{G,i}_n)$ into a orthonormal basis of it. There is $C_{14} > 0$ such that

$$\frac{1}{C_{14}}|v|_{G^\star_i,G^\star_i} \leq |v|_{G^\star_i,0} \leq C_{14}|v|_{G^\star_i,G^\star_i}$$

for every i. This implies that

$$\frac{1}{C_{13}C_{14}}|v|_{G,G^\star_i} \leq |v|_{G,0} \leq C_{13}C_{14}|v|_{G,G^\star_i} \tag{44}$$

Together with (1), we conclude that the norms $|\cdot|_{G,G^\star_i}$, $|\cdot|_{G,0}$ and $|\cdot|_0$ are not only equivalent on E^u_G (which is obvious, once E^u_G has finite dimension), but also that there is a universal upper bound to the norm of the identity maps $Id \colon E^u_G \mapsto E^u_G$ that holds considering every $G \in \Omega$ and every one of these three norms on its domain and range.

Let $m_{G,i}$ be the Lebesgue measure on E^u_G such that

$$m_{G,i}\{v \in E^u_G \colon \ v = \sum_j \alpha_j v^{G,i}_j, \ with \ \alpha_j \in [0,1]\} = 1. \tag{45}$$

Note that the quotient of the measure $m_{G,i}$ by the n-dimensional Hausdorff measure induced by the norm $|\cdot|_{G,G_i^\star}$ is a constant that depends only on the dimension n of E^u. Of course by the uniqueness of the Haar measure on locally compact topological groups, if m_G is the n-dimensional Hausdorff measure induced by the norm $|\cdot|_{G,0}$ on E^u_G and $G \in \mathbb{B}(G_i^\star, \gamma_{G_i^\star})$ then there exists $K_{G,i} > 0$ such that $m_{G,i} = K_{G,i} m_G$. From (44) it easily follows that

$$\frac{\sigma(n)}{C_{13}^n C_{14}^n} \leq K_{G,i} \leq \sigma(n) C_{13}^n C_{14}^n. \tag{46}$$

where $\sigma(n)$ is a constant that depends only on n. Finally given some $\mathcal{H} \in \mathcal{T}_0^k(G,\delta)$ with base point F then

$$\Pi^u_G \colon \hat{\mathcal{H}} \mapsto \mathbb{B}^u_G(\pi^u_G(F-G), \delta)$$

given by $\Pi^u_G(y) = \pi^u_G(y-G)$ is a bilipchitz map and the Lipchitz constant of Π^u_G and its inverse is at most $1+\varepsilon$, considering the metric induced by $|\cdot|_{G,0}$ on $\hat{\mathcal{H}}$ and $\mathbb{B}^u_G(\pi^u_G(F-G),\delta)$. This implies that for every $A \subset \hat{\mathcal{H}}$ we have

$$(1+\epsilon)^{-n} \leq \frac{m_{\mathcal{H}}(A)}{m_G(\Pi^u_G(A))} \leq (1+\epsilon)^n. \tag{47}$$

Note also that if $m_{\mathcal{H},\mathcal{B}_0}$ is the n-dimensional Hausdorff measure on $\hat{\mathcal{H}}$ induced by the norm $|\cdot|_0$ then by (1) we have that

$$\frac{1}{C_2^n} \leq \frac{m_{\mathcal{H}}(A)}{m_{\mathcal{H},\mathcal{B}_0}(A)} \leq C_2^n.$$

for every borelian set $A \subset \hat{\mathcal{H}}$.

Lemma 6.2. *Given $\delta > 0$, there exists $C_{15}(\delta) > 0$ and $C_{16}(\delta) > 0$ with the following property. Suppose that $\mathcal{H} \in \mathcal{T}_0^1(G,\tilde{\delta})$, with base point $\tilde{F}$.*

A. *If $F \in \hat{\mathcal{H}}$ satisfies*

$$\mathbb{B}^u_G(\pi^u_G(F-G), C_2\delta) \subset \mathbb{B}^u_G(\pi^u_G(\tilde{F}-G), \tilde{\delta}).$$

then

$$m_{\mathcal{H}}\{F_1 \in \hat{\mathcal{H}} \colon\ |F_1 - F|_0 \leq \delta\} \leq C_{15}(\delta).$$

B. *If $F \in \hat{\mathcal{H}}$ satisfies*

$$\mathbb{B}^u_G(\pi^u_G(F-G), \frac{\delta}{C_2(1+\varepsilon)}) \subset \mathbb{B}^u_G(\pi^u_G(\tilde{F}-G), \tilde{\delta}).$$

then

$$m_{\mathcal{H}}\{F_1 \in \hat{\mathcal{H}} \colon\ |F_1 - F|_0 < \delta\} \geq C_{16}(\delta).$$

Proof of A. Note that if $F_1 \in \hat{\mathcal{H}}$ satisfies

$$|F_1 - F|_0 < \delta$$

then

$$\pi^u_G(F_1 - G) \in \mathbb{B}^u_G(\pi^u_G(F-G), C_2\delta).$$

Denote $\tilde{\gamma} = C_2\delta$ and $w_0 = \pi^u_G(F-G)$. We have that $G \in \mathbb{B}(G_i^\star, \gamma_{G_i^\star})$, for some i. So

$$\mathbb{B}^u_{G_k}(w_0, \tilde{\gamma}) \subset \{w_0 + w \in E^u_{G_k} \colon\ w = \sum_j \alpha_j v_j^{G,i_k},\ \textit{with}\ |\alpha_j| \leq C_{13}C_{14}\tilde{\gamma}\}$$

so by (45) and (46) we obtain

$$
\begin{aligned}
m_{\mathcal{H}}\{F_1 \in \hat{\mathcal{H}}\colon\ |F_1 - F|_0 < \delta\} &\leq m_{\mathcal{H}}\{v + \mathcal{H}(v)\colon\ v \in \mathbb{B}^u_G(w_0, \tilde{\gamma})\} \\
&\leq (1+\varepsilon)^n m_G(\mathbb{B}^u_G(w_0, \tilde{\gamma})) \\
&\leq \frac{C_{13}^n C_{14}^n (1+\varepsilon)^n}{\sigma(n)} m_{G,i}(\mathbb{B}^u_G(w_0, \tilde{\gamma})) \\
&\leq \frac{C_{13}^n C_{14}^n (1+\varepsilon)^n}{\sigma(n)} (2C_{13}C_{14}\tilde{\gamma})^n. \qquad (48)
\end{aligned}
$$

□

Proof of B. If

$$v \in \mathbb{B}^u_G(\pi^u_G(F-G), \frac{\delta}{2C_2(1+\varepsilon)})$$

then

$$v + \mathcal{H}(v) \in \{F_1 \in \hat{\mathcal{H}}\colon\ |F_1 - F|_0 < \delta\}.$$

Denote $\tilde{\gamma} = \frac{\delta}{C_2(1+\varepsilon)}$ and $w_0 = \pi^u_G(F-G)$. We have that $G \in \mathbb{B}(G_i^\star, \gamma_{G_i^\star})$, for some i. So

$$\{w_0 + w \in E^u_{G_k}\colon\ w = \sum_j \alpha_j v_j^{G,i_k},\ \mathit{with}\ \alpha_j \in [0, \frac{\tilde{\gamma}}{C_{13}C_{14}\sqrt{n}}]\} \subset \mathbb{B}^u_{G_k}(w_0, \tilde{\gamma})$$

so by (45) and (46) we obtain

$$
\begin{aligned}
m_{\mathcal{H}}\{F_1 \in \hat{\mathcal{H}}\colon\ |F_1 - F|_0 < \delta\} &\geq m_{\mathcal{H}}\{v + \mathcal{H}(v)\colon\ v \in \mathbb{B}^u_G(w_0, \tilde{\gamma})\} \\
&\geq \frac{1}{(1+\varepsilon)^n} m_G(\mathbb{B}^u_G(w_0, \tilde{\gamma})) \\
&\geq \frac{1}{\sigma(n) C_{13}^n C_{14}^n (1+\varepsilon)^n} m_{G,i}(\mathbb{B}^u_G(w_0, \tilde{\gamma})) \\
&\geq \frac{1}{\sigma(n) C_{13}^n C_{14}^n (1+\varepsilon)^n} \Big(\frac{\tilde{\gamma}}{C_{13}C_{14}\sqrt{n}}\Big)^n. \qquad (49)
\end{aligned}
$$

□

Proposition 6.3. *Let $C_9 > 0$ as in Corollary 5.3. If $\delta \in (0, \delta_3)$ and $\lambda_6 \in (0,1)$ then there exists $C_{17} = C_{17}(\delta, \lambda_6) > 0$ such that the following holds. Let $\mathcal{H} \in \mathcal{T}_1^2(G, \delta)$ be a C^2 function*

$$\mathcal{H}\colon \mathbb{B}^u_G(v_0, \delta) \to E^h_G + G$$

with $G \in \Omega$, base point F and satisfying

$$|D\mathcal{H}|_{(G,1),(G,0)} \leq \epsilon, |D^2\mathcal{H}|_{(G,1),(G,0)} \leq C_9. \qquad (50)$$

Moreover assume there exists C_{18} such that

$$|w + \mathcal{H}_k(w)|_1 \leq C_{18},$$

for every $w \in \mathbb{B}^u_G(v_0, \delta)$. Then we have that

$$m_{\mathcal{H}}(\hat{\mathcal{H}} \cap W^s_{\delta_3}(\Omega)^c \cap \mathbb{B}(F, \lambda_6\delta)) \geq C_{17}.$$

Proof. Suppose that there exists a sequence of maps $G_k \in \Omega$ and C^2 functions $\mathcal{H}_k$

$$\mathcal{H}_k\colon \mathbb{B}^u_{G_k}(\pi^u_{G_k}(F_k - G_k), \delta) \to E^s_{G_k} + G_k$$

such that $\mathcal{H}_k \in \mathcal{T}_1^2(G_k, \delta)$, with base point F_k and moreover

$$\lim_k m_{\mathcal{H}_k}(\hat{\mathcal{H}} \cap W^s_{\delta_3}(\Omega)^c \cap \mathbb{B}(F_k, \lambda_6 \delta)) = 0. \tag{51}$$

Since Ω is compact we can replace G_k by a subsequence such that

$$\lim_k G_k = G \in \Omega.$$

Since $\mathcal{H}_k$ satisfies (50), by Proposition 6.1 we can assume without loss of generality that the functions $\mathcal{H}_k$ converges in C^1 topology to a C^{1+Lip} function $\mathcal{H} \in \mathcal{T}_0^1(G, \delta)$ with a base point F. Moreover F_k converges to F in $\mathcal{B}_0$. By the transversal empty interior assumption the transversal family $u \mapsto u + \mathcal{H}(u)$ has a parameter

$$u_\infty \in \mathbb{B}^u_G(\pi^u_G(F - G), \frac{1}{3}\frac{\lambda_6 \delta}{C_2(1+\epsilon)})$$

such that $u_\infty + \mathcal{H}(u_\infty) \in W^s_{\delta_3}(\Omega)^c \cap \mathbb{B}(F, \lambda_6\delta/3)$. Since $W^s_{\delta_3}(\Omega)$ is a closed set there is $\gamma > 0$ be such that $\mathbb{B}(u_\infty + \mathcal{H}(u_\infty), \gamma) \subset W^s_{\delta_3}(\Omega)^c \cap \mathbb{B}(F, \lambda_6\delta/3)$.

Since the families $u \mapsto u + \mathcal{H}_k(u)$ converges to this family in C^1 topology we can easily conclude that for large k there exists

$$u_k \in \mathbb{B}^u_{G_k}(\pi^u_{G_k}(F_k - G_k), \frac{2}{3}\frac{\lambda_6\delta}{C_2(1+\epsilon)})$$

such that

$$|u_\infty + \mathcal{H}(u_\infty) - u_k - \mathcal{H}_k(u_k)|_0 < \frac{\gamma}{2}.$$

Let

$$\tilde{\gamma} = \min\{\frac{1}{3}\frac{\lambda_6\delta}{C_2(1+\epsilon)}, \frac{\gamma}{2C_2(1+\varepsilon)}\}.$$

If

$$v \in \mathbb{B}^u_{G_k}(u_k, \tilde{\gamma}) \subset \mathbb{B}^u_{G_k}(\pi^u_{G_k}(F_k - G_k), \frac{\lambda_6\delta}{C_2(1+\epsilon)})$$

then

$$v + \mathcal{H}_k(v) \in \mathbb{B}(u_\infty + \mathcal{H}(u_\infty), \gamma),$$

so $v + \mathcal{H}_k(v) \in W^s_{\delta_3}(\Omega)^c \cap \mathbb{B}(F_k, \lambda_6\delta)$. We have that $G_k \in \mathbb{B}(G^\star_{i_k}, \gamma_{G^\star_{i_k}})$, for some i_k. Note that (44) implies

$$\{u_k + v \in E^u_{G_k}\colon\ v = \sum_j \alpha_j v_j^{G_k, i_k},\ with\ \alpha_j \in [0, \frac{\tilde{\gamma}}{C_{13}C_{14}\sqrt{n}}]\} \subset \mathbb{B}^u_{G_k}(u_k, \tilde{\gamma})$$

so by (45) and (46) we obtain

$$\begin{aligned} m_{\mathcal{H}_k}(\hat{\mathcal{H}} \cap W^s_{\delta_3}(\Omega)^c \cap \mathbb{B}(F, \lambda_6\delta)) &\geq m_{\mathcal{H}_k}\{v + \mathcal{H}_k(v)\colon\ v \in \mathbb{B}^u_{G_k}(u_k, \tilde{\gamma})\} \\ &\geq \frac{1}{(1+\varepsilon)^n} m_{G_k}(\mathbb{B}^u_{G_k}(u_k, \tilde{\gamma})) \\ &\geq \frac{1}{\sigma(n)C_{13}^n C_{14}^n (1+\varepsilon)^n} m_{G, i_k}(\mathbb{B}^u_{G_k}(u_k, \tilde{\gamma})) \\ &\geq \frac{1}{\sigma(n)C_{13}^n C_{14}^n (1+\varepsilon)^n}\Big(\frac{\tilde{\gamma}}{C_{13}C_{14}\sqrt{n}}\Big)^n. \end{aligned} \tag{52}$$

So

$$\limsup_k m_{\mathcal{H}_k}(W^s_{\delta_3}(\Omega)^c \cap \mathbb{B}(F, \lambda_6\delta)) > 0,$$

which is a contradiction with (51). □

7. Dynamical balls in transversal families.

Given $F \in W^s_{\delta_3}(\Omega)$, let G_F be an element of Ω such that $F \in W^s_{\delta_3}(G_F)$. Choose $\delta_4 < \delta_3$ in such way that if $G_1, G_2 \in \Omega$ and $|G_1 - G_2|_{G_1} \leq 3C_2\delta_4$ then

$$\frac{1}{2}|v|_{G_2,0} \leq |v|_{G_1,0} \leq 2|v|_{G_2,0} \tag{53}$$

Fix $G \in \Omega$. In this section $\mathcal{H}$ is a C^2 function

$$\mathcal{H} \colon W \to E^h_G + G,$$

where $W \subset E^u_G$ is an convex open set. We can choose $\delta_5 < \delta_4$ such that if

A1. $|D\mathcal{H}|_{G,0} \leq \lambda_2 \epsilon$ and
A2. $\inf_{x\in W} |x + \mathcal{H} - G|_0 + (1+\epsilon) diam_G W < 2C_2\delta_5$,

then for every

$$F \in \hat{\mathcal{H}} \cap W^s_{\delta_5}(\Omega),$$

and for every $\delta \in (0, \delta_5/(3C_2))$ satisfying

$$\mathbb{B}^u_G(\pi^u_G(F-G), 3C_2\delta) \subset W$$

we have that

$$\{w + \mathcal{H}(w) \colon\ w \in W\} \cap \{u + v + G_F \colon\ u \in \mathbb{B}^u_{G_F}(\pi^u_{G_F}(F - G_F), 2C_2\delta),\ v \in E^s_{G_F}\}$$

is the graph $\hat{\mathcal{H}}_F$ of a C^2 function

$$\mathcal{H}_F \in \mathcal{T}(G_F, 2C_2\delta)$$

with base point F and

$$|D\mathcal{H}_F|_{G_F,0} \leq \epsilon$$

and additionally if we assume

A3. $|D^2\mathcal{H}|_{G,0} \leq C_9/2$.

then

$$|D^2\mathcal{H}|_{G_F,0} \leq C_9.$$

The goal of this section is to show

Proposition 7.1. *Let $\mathcal{H}$ be a C^2 function satisfying A1-A3. Then*

$$m_{\mathcal{H}}(W^s_{\delta_5}(\Omega) \cap \hat{\mathcal{H}}) = 0. \tag{54}$$

7.1. Dynamical balls.

Let $\mathcal{H}$ be as in Proposition 7.1. Define the **dynamical ball**

$$B_{\mathcal{H}}(F, \delta, k) = \{F_1 \in \hat{\mathcal{H}} \colon |\mathcal{R}^i F_1 - \mathcal{R}^i F|_0 < \delta, \text{ for every } 0 \leq i \leq k\}.$$

We can choose $\delta < \delta_5$ small enough such that

$$B_{\mathcal{H}}(F, \delta, 0) \subset \hat{\mathcal{H}}_F, \tag{55}$$

Consider the map ϕ_u as in the proof of Propostition 5.1

$$x \in E^u_G \mapsto \phi_u(x) = \pi^u_{\mathcal{R}G}(\mathcal{R}(x + \mathcal{H}(x) + G) - \mathcal{R}G) \in E^u_{\mathcal{R}G}.$$

Lemma 7.2. *Let C_9 be as in Corollary 5.3. There exists C_{20} such that for every $G \in \Omega$ and $\mathcal{H} \in \mathcal{T}_0^2(G, 2C_2\delta)$ with base point $F = v_0 + \mathcal{H}(v_0) \in W^s_{\delta_5}(G)$ that satisfies*

$$|D^2\mathcal{H}|_{G,0} \leq C_9$$

then the following holds. Consider the map ϕ_u as in the proof of Propostition 5.1

$$\phi_u \colon \mathbb{B}^u_G(v_0, 2C_2\delta) \mapsto E^u_{\mathcal{R}G}$$

defined by

$$\phi_u(x) = \pi^u_{\mathcal{R}G}(\mathcal{R}(x + \mathcal{H}(x)) - \mathcal{R}G).$$

If S_1 and S_2 are borelian subsets of E^u_G, with

$$S_1 \subset S_2 \subset \mathbb{B}^u_G(v_0, 2C_2\delta)$$

then

$$\frac{m_{G,i}(S_1)}{m_{G,i}(S_2)} e^{-2C_{20} diam_{G,0} S_2} \leq \frac{m_{\mathcal{R}G,j}(\phi_u(S_1))}{m_{\mathcal{R}G,j}(\phi_u(S_2))} \leq \frac{m_{G,i}(S_1)}{m_{G,i}(S_2)} e^{2C_{20} diam_{G,0} S_2}. \tag{56}$$

Here $G \in \mathbb{B}(G_i, \gamma_{G_i})$ and $\mathcal{R}G \in \mathbb{B}(G_j, \gamma_{G_j})$.

Proof. Due (44) and the estimates for ϕ^u in the proof of Proposition 5.1 there is C_{19} such that

$$\max\{|D\phi_u|, |D^2\phi_u|, |D\phi_u^{-1}|, |D^2\phi_u^{-1}|\} \leq C_{19}. \tag{57}$$

where here we are considering the norms $|\cdot|_{G,G_i^\star}$ in E^u_G, and $|\cdot|_{\mathcal{R}G,G_j^\star}$ in $E^u_{\mathcal{R}G}$. If M_x is the matrix representation of $D\phi_u(x)$ with respect to the basis $\mathcal{S}_{G,i}$ and $\mathcal{S}_{\mathcal{R}G,j}$, define

$$J\phi_u(x) = Det\ M_x.$$

It is easy to see that (57) implies that there exists C_{20} satisfying

$$\ln\left|\frac{J\phi^u(x)}{J\phi^u(y)}\right| \leq C_{20}|x - y|_G. \tag{58}$$

Since

$$m_{\mathcal{R}G,j}(\phi_u(S)) = \int_S J\phi^u\ dm_{G,i},$$

we have that (56) follows easily from (58). ∎

Proposition 7.3. *If $\delta > 0$ is small enough there exists $C_{21} > 0$ such that the following holds. We have that*

$$\mathcal{R}^i \colon B_{\mathcal{H}}(F, \delta, i) \to B_{\hat{\mathcal{R}}^i(\mathcal{H}_F)}(\mathcal{R}^i F, \delta, 0)$$

is a diffeomorphism. Furthermore for every borelian set $A \subset B_{\mathcal{H}}(F, \delta, i)$ we have

$$\frac{1}{C_{21}} \frac{m_{\mathcal{H}_F}(A)}{m_{\mathcal{H}_F}(B_{\mathcal{H}}(F, \delta, i))} \leq \frac{m_{\mathcal{R}^i(\mathcal{H}_F)}(\mathcal{R}^i(A))}{m_{\mathcal{R}^i(\mathcal{H}_F)}(B_{\hat{\mathcal{R}}^i(\mathcal{H}_F)}(\mathcal{R}^i F, \delta, 0))} \leq C_{21} \frac{m_{\mathcal{H}_F}(A)}{m_{\mathcal{H}_F}(B_{\mathcal{H}}(F, \delta, i))}. \tag{59}$$

Proof. First we will prove by induction on i that

$$\mathcal{R}^i \colon B_{\mathcal{H}}(F, \delta, i) \to B_{\hat{\mathcal{R}}^i(\mathcal{H}_F)}(\mathcal{R}^i F, \delta, 0) \tag{60}$$

is a diffeomorphism. Indeed, for $i = 0$ we have by (55) that

$$B_{\mathcal{H}}(F, \delta, k) = B_{\mathcal{H}_F}(F, \delta, k).$$

Now assume by induction that (60) holds for some i. Denote $\mathcal{H}_i := \hat{\mathcal{R}}^i(\mathcal{H}_F)$. By Proposition 5.1 we have that

$$\mathcal{R}\colon \hat{\mathcal{H}}_i \mapsto \mathcal{R}(\hat{\mathcal{H}}_i)$$

is invertible and its inverse $\mathcal{I}$ satisfies

$$|\mathcal{I}(F_1) - \mathcal{I}(F_2)|_0 \leq \lambda_4 |F_1 - F_2|_0.$$

so if

$$z \in B_{\hat{\mathcal{R}}^{i+1}(\mathcal{H}_F)}(\mathcal{R}^{i+1}F, \delta, 0)$$

we have $z \in \hat{\mathcal{H}}_{i+1} \subset \mathcal{R}(\hat{\mathcal{H}}_i)$ and consequently

$$|\mathcal{I}(z) - \mathcal{R}^i(F)|_0 \leq \lambda_4 |z - \mathcal{R}^{i+1}(F)|_0 \leq \lambda_4 \delta < \delta,$$

so

$$\mathcal{I}(B_{\hat{\mathcal{R}}^{i+1}(\mathcal{H}_F)}(\mathcal{R}^{i+1}F, \delta, 0)) \subset B_{\hat{\mathcal{R}}^{i}(\mathcal{H}_F)}(\mathcal{R}^{i}F, \delta, 0).$$

By the induction assumption there exists an open set $\mathbb{W} \subset B_{\mathcal{H}}(F, \delta, i)$ such that

$$\mathcal{R}^i(\mathbb{W}) = \mathcal{I}(B_{\hat{\mathcal{R}}^{i+1}(\mathcal{H}_F)}(\mathcal{R}^{i+1}F, \delta, 0)).$$

We conclude that

$$\mathcal{R}^{i+1}\colon \mathbb{W} \mapsto B_{\hat{\mathcal{R}}^{i+1}(\mathcal{H}_F)}(\mathcal{R}^{i+1}F, \delta, 0)$$

is a diffeomorphism. Of course $\mathbb{W} \subset B_{\mathcal{H}}(F, \delta, i+1)$. We claim that

$$\mathbb{W} = B_{\mathcal{H}}(F, \delta, i+1).$$

Notice that

$$\mathcal{H}_i := \hat{\mathcal{R}}^i(\mathcal{H}_F) \in \mathcal{T}(\mathcal{R}^i G_F, 2C_2\delta).$$

By the proof of Propostition 5.1 there exists an open set $W \subset E^u_{\mathcal{R}^{i+1}G_F}$ such that $\mathcal{R}(\hat{\mathcal{H}}_i)$ is the graph of a function

$$\mathcal{H}_{i+1}\colon W \mapsto E^s_{\mathcal{R}^{i+1}G_F} + G_F$$

such that

$$|\mathcal{H}_{i+1}(x) - \mathcal{H}_{i+1}(y)|_{\mathcal{R}^{i+1}G_F,0} \leq \varepsilon |x - y|_{\mathcal{R}^{i+1}G_F,0}$$

and

$$\mathbb{B}^u_{\mathcal{R}^{i+1}G_F}(\pi^u(\mathcal{R}^{i+1}F), 2\theta_1 C_2 \delta) \subset W.$$

In particular

$$\hat{\mathcal{R}}^{i+1}(\mathcal{H}_F) \in \mathcal{T}(\mathcal{R}^{i+1}G_F, \theta_1 2 C_2 \delta).$$

Note that if

$$x \in W \setminus \mathbb{B}^u_{\mathcal{R}^{i+1}G_F}(\pi^u(\mathcal{R}^{i+1}F), 2C_2\delta)$$

then

$$\begin{aligned}
& |x + \mathcal{H}_{i+1}(x) - \mathcal{R}^{i+1}F|_{\mathcal{B}_0} \\
\geq\ & \frac{1}{C_2} |x + \mathcal{H}_{i+1}(x) - \mathcal{R}^{i+1}F|_{\mathcal{R}^{i+1}G_F,0} \\
\geq\ & \frac{1}{C_2} |\pi^u_{\mathcal{R}^{i+1}G_F}(x + \mathcal{H}_{i+1}(x) - \mathcal{R}^{i+1}F)|_{\mathcal{R}^{i+1}G_F,0} \\
\geq\ & \frac{1}{C_2} |\pi^u_{\mathcal{R}^{i+1}G_F}(x) - \pi^u_{\mathcal{R}^{i+1}G_F}(\mathcal{R}^{i+1}F)|_{\mathcal{R}^{i+1}G_F,0} \\
\geq\ & \theta_1 2\delta > \delta. \qquad (61)
\end{aligned}$$

So if $y \in B_{\mathcal{H}}(F, \delta, i+1) \subset B_{\mathcal{H}}(F, \delta, i)$ then by induction assumption we have

$$\mathcal{R}^i(y) \in B_{\hat{\mathcal{H}}_i}(\mathcal{R}^i F, \delta, 0)$$

and

$$\mathcal{R}^{i+1}(y) \in \mathcal{R}(\hat{\mathcal{H}}_i)$$

and by (61)

$$\pi^u_{\mathcal{R}^{i+1}G_F}(\mathcal{R}^{i+1}(y)) \in \mathbb{B}^u_{\mathcal{R}^{i+1}G_F}(\pi^u(\mathcal{R}^{i+1}F), 2C_2\delta)$$

and of course

$$\mathcal{R}^{i+1}(y) \in B_{\hat{\mathcal{R}}^{i+1}(\mathcal{H}_F)}(\mathcal{R}^{i+1}F, \delta, 0).$$

There is $\tilde{y} \in \mathbb{W}$ such that

$$\mathcal{R}^i(\tilde{y}) \in B_{\hat{\mathcal{H}}_i}(\mathcal{R}^i F, \delta, 0)$$

and

$$\mathcal{R}^{i+1}(\tilde{y}) = \mathcal{R}^{i+1}(\tilde{y}).$$

Since $\mathcal{R}$ is injective on $\hat{\mathcal{H}}_i$ we conclude that $\mathcal{R}^i(\tilde{y}) = \mathcal{R}^i(y)$. Since by induction assumption $\mathcal{R}^i$ is injective on $B_{\mathcal{H}}(F, \delta, i)$ we conclude that $y = \tilde{y}$. So $\mathbb{W} = B_{\mathcal{H}}(F, \delta, i+1)$ and we conclude that proof that the map in (60) is a diffeomorphism.

It remains to prove the inequality in the statement of Proposition 7.3. Define, for every $i \leq k$

$$S^i_1 = \pi^u_{\mathcal{R}^i G_F}(\mathcal{R}^i(A) - \mathcal{R}^i(G_F)),$$

$$S^i_2 = \pi^u_{\mathcal{R}^i G_F}(\mathcal{R}^i(B_{\mathcal{H}}(F, \delta, k)) - \mathcal{R}^i(G_F)),$$

Note that

$$S^i_1 \subset S^i_2 \subset \mathbb{B}^u_{\mathcal{R}^i G_F}(\pi^u(\mathcal{R}^i F), 2C_2\delta).$$

If

$$\phi^i_u(x) = \pi^u_{\mathcal{R}^{i+1}G_F}(\mathcal{R}(x + \mathcal{H}_i(x)) - \mathcal{R}^{i+1}G_F)$$

then $\phi^i_u(S^i_a) = S^{i+1}_a$, $a = 1, 2$. Choosing j_i such that $\mathcal{R}^i G \in \mathbb{B}(G_{j_i}, \gamma_{G_{j_i}})$, Lemma 7.2 implies

$$\frac{m_{G_F,j_0}(S^0_1)}{m_{G_F,j_0}(S^0_2)} e^{-2C_{20}\sum_{i\leq k}\delta_i} \leq \frac{m_{\mathcal{R}^k G_F, j_k}(S^k_1)}{m_{\mathcal{R}^k G_F, j_k}(S^k_2)} \leq \frac{m_{G_F,j_0}(S^0_1)}{m_{G_F,j_0}(S^0_2)} e^{2C_{20}\sum_{i\leq k}\delta_i}. \tag{62}$$

with $\delta_i = diam\ S^i_2$. By (9) we have

$$\delta_i = diam_{\mathcal{R}^i G_F, 0}\ S^i_2 \leq 2C_2\lambda_4^{k-i}\delta \tag{63}$$

for every $i \leq k$. Let $C_{22} = 2C_{20}\sum_{k=0}^{\infty} 2C_2\lambda_4^k$. Then by (46)

$$\frac{m_{G_F}(S^0_1)}{m_{G_F}(S^0_2)} e^{-C_{22}} \leq \frac{m_{\mathcal{R}^k G_F}(S^k_1)}{m_{\mathcal{R}^k G_F}(S^k_2)} \leq \frac{m_{G_F}(S^0_1)}{m_{G_F}(S^0_2)} e^{C_{22}}. \tag{64}$$

So by (47)

$$\frac{1}{C_{23}} \frac{m_{\mathcal{H}_F}(A)}{m_{\mathcal{H}_F}(B_{\mathcal{H}}(F, \delta, k))} \leq \frac{m_{\mathcal{H}_k}(\mathcal{R}^k(A))}{m_{\mathcal{H}_k}(B_{\hat{\mathcal{R}}^k(\mathcal{H}_F)}(\mathcal{R}^k F, \delta, 0))} \leq C_{23} \frac{m_{\mathcal{H}_F}(A)}{m_{\mathcal{H}_F}(B_{\mathcal{H}}(F, \delta, k))} \tag{65}$$

where $C_{23} = e^{C_{22}}(1+\epsilon)^{4n}$. Now we need to deal with the fact that $m_{\mathcal{H}}$ may not coincide with $m_{\mathcal{H}_F}$, since $m_{\mathcal{H}}$ is the n-dimensional Hausdorff measure induced by the norm $|\cdot|_{G,0}$ and $m_{\mathcal{H}_F}$ is is the n-dimensional Hausdorff measure induced by the norm $|\cdot|_{G_F,0}$. But by (1) we have

$$\frac{1}{C_2^{2n}} \leq \frac{m_{\mathcal{H}}(A)}{m_{\mathcal{H}_F}(A)} \leq C_2^{2n}. \tag{66}$$

From (65) and (66) we can easily obtain the inequality in the statement of Proposition 7.3 with $C_{21} = C_2^{2n} C_{23}$. □

Proof of Proposition 7.1. It is enough to show that for every compact subset $K \subset W^s_{\delta_5}(\Omega) \cap \hat{\mathcal{H}}$ we have $m_{\mathcal{H}}(K) = 0$. Choose $\delta > 0$ small enough such that $\mathcal{H}_F$, as defined in Section 7, is well-defined for every $F \in K$. Following the notation of Bowen [5], we say that a subset $S \subset K$ is (k,δ)-separated is for every $F_1, F_2 \in S$, with $F_1 \neq F_2$, there exists $i \leq k$ such that

$$|\mathcal{R}^i F_1 - \mathcal{R}^i F_2|_0 > \delta.$$

Fixed k and δ, the family $\mathcal{F}_{k,\delta}$ of (k,δ)-separated subsets of $K \neq \emptyset$ is a non empty family, ordered by the inclusion relation. One can easily see that we can apply Zorn's Lemma to this family to show the it has a maximal element $S_{k,\delta}$. The maximality of $S_{k,\delta}$ implies that

$$K \subset \cup_{F \in S_{k,\delta}} B_{\mathcal{H}}(F, \delta, k). \tag{67}$$

Since *diam* $B_{\mathcal{H}}(F,\delta,k) \leq 2C_2\lambda_4^k \delta$ we get that

$$\lim_k m_{\mathcal{H}}(\cup_{F \in S_{k,\delta}} B_{\mathcal{H}}(F,\delta,k)) = m_{\mathcal{H}}(K). \tag{68}$$

In particular

$$\lim_k m_{\mathcal{H}}(K^c \cap \cup_{F \in S_{k,\delta}} B_{\mathcal{H}}(F,\delta,k)) = 0. \tag{69}$$

Note that for every $F \in S_{k,\delta}$ we have

$$\hat{\mathcal{R}}^k \mathcal{H}_F \in \mathcal{T}_1^2(\mathcal{R}^k G_F, \frac{\delta}{2}) \subset \mathcal{T}_1^2(\mathcal{R}^k G_F, \frac{\delta}{2C_2(1+\varepsilon)})$$

Applying Proposition 6.3 to the family

$$\hat{\mathcal{H}}_k = \{v + (\hat{\mathcal{R}}^k \mathcal{H}_F)(v) \colon \ v \in \mathbb{B}^u_{\mathcal{R}^k G_F}(\pi^u_{\mathcal{R}^k G_F}(\mathcal{R}^k F - \mathcal{R}^k G_F), \frac{\delta}{2C_2(1+\varepsilon)})\}$$

we conclude that there exists $C_{17} > 0$, that does not depend on k and $F \in S_{k,\delta}$ such that

$$m_{\hat{\mathcal{R}}^k \mathcal{H}_F}(W^s_{\delta_5}(\Omega)^c \cap \hat{\mathcal{H}}_k \cap \mathbb{B}(\mathcal{R}^k F, \frac{\delta}{2})) > C_{17}.$$

Note that

$$W^s_{\delta_5}(\Omega)^c \cap \hat{\mathcal{H}}_k \cap \mathbb{B}(\mathcal{R}^k F, \frac{\delta}{2}) \subset B_{\hat{\mathcal{R}}^k \mathcal{H}_F}(\mathcal{R}^k F, \frac{\delta}{2}, 0).$$

So by Lemma 6.2 we have

$$\frac{m_{\hat{\mathcal{R}}^k \mathcal{H}_F}(W^s_{\delta_5}(\Omega)^c \cap B_{\hat{\mathcal{R}}^k \mathcal{H}_F}(\mathcal{R}^k F, \frac{\delta}{2}, 0))}{m_{\hat{\mathcal{R}}^k \mathcal{H}_F}(B_{\hat{\mathcal{R}}^k \mathcal{H}_F}(\mathcal{R}^k F, \frac{\delta}{2}, 0))} \geq \frac{C_{17}}{C_{15}(\delta/2)}$$

By Proposition 7.3 there exists a set $A_F \subset B_{\mathcal{H}}(F, \delta/2, k)$ such that

$$\mathcal{R}^k(A_F) = W^s_{\delta_5}(\Omega)^c \cap B_{\hat{\mathcal{R}}^k \mathcal{H}_F}(\mathcal{R}^k F, \frac{\delta}{2}, 0).$$

Since $W^s_{\delta_5}(\Omega)$ is forward invariant, we have that $W^s_{\delta_5}(\Omega)^c$ is backward invariant and consequently $A_F \subset W^s_{\delta_5}(\Omega)^c \subset K^c$. Moreover

$$\frac{m_{\mathcal{H}}(A_F)}{m_{\mathcal{H}}(B_{\mathcal{H}}(F,\frac{\delta}{2},k))} \geq \frac{C_{17}}{C_{21}C_{15}(\delta/2)}.$$

Lemma 6.2 also implies

$$\frac{m_{\hat{\mathcal{R}}^k\mathcal{H}_F}(B_{\hat{\mathcal{R}}^k\mathcal{H}_F}(\mathcal{R}^kF,\frac{\delta}{2},0))}{m_{\hat{\mathcal{R}}^k\mathcal{H}_F}(B_{\hat{\mathcal{R}}^k\mathcal{H}_F}(\mathcal{R}^kF,\delta,0))} \geq \frac{C_{16}(\delta/2)}{C_{15}(\delta)},$$

so applying Proposition 7.3 again we get

$$\frac{m_{\mathcal{H}}(B_{\mathcal{H}}(F,\delta/2,k))}{m_{\mathcal{H}}(B_{\mathcal{H}}(F,\delta,k))} \geq \frac{C_{16}(\delta/2)}{C_{21}C_{15}(\delta)}.$$

Because $S_{k,\delta} \in \mathcal{F}_{k,\delta}$ we have that

$$\{B_{\mathcal{H}}(F,\delta/2,k)\}_{F\in S_{k,\delta}}$$

is a family of pairwise disjoint dynamical balls. So

$$\begin{aligned}
m_{\mathcal{H}}(K^c\cap \cup_{F\in S_{k,\delta}}B_{\mathcal{H}}(F,\delta,k)) &\geq m_{\mathcal{H}}(\cup_{F\in S_{k,\delta}}B_{\mathcal{H}}(F,\delta,k)\cap K^c)\\
&\geq m_{\mathcal{H}}(\cup_{F\in S_{k,\delta}}B_{\mathcal{H}}(F,\delta/2,k)\cap K^c)\\
&\geq \sum_{F\in S_{k,\delta}} m_{\mathcal{H}}(B_{\mathcal{H}}(F,\delta/2,k)\cap K^c)\\
&\geq \sum_{F\in S_{k,\delta}} m_{\mathcal{H}}(A_F)\\
&\geq \frac{C_{17}}{C_{21}C_{15}(\delta/2)}\sum_{F\in S_{k,\delta}} m_{\mathcal{H}}(B_{\mathcal{H}}(F,\delta/2,k))\\
&\geq \frac{C_{17}C_{16}(\delta/2)}{C_{21}^2C_{15}(\delta/2)C_{15}(\delta)}\sum_{F\in S_{k,\delta}} m_{\mathcal{H}}(B_{\mathcal{H}}(F,\delta,k))\\
&\geq \frac{C_{17}C_{16}(\delta/2)}{C_{21}^2C_{15}(\delta/2)C_{15}(\delta)} m_{\mathcal{H}}(\cup_{F\in S_{k,\delta}}B_{\mathcal{H}}(F,\delta,k))\\
&\geq \frac{C_{17}C_{16}(\delta/2)}{C_{21}^2C_{15}(\delta/2)C_{15}(\delta)} m_{\mathcal{H}}(K). \qquad (70)
\end{aligned}$$

It follows easily from (67), (69) and (70) that $m_{\mathcal{H}}(K)=0$. □

Corollary 7.4. *Let $\mathcal{H}$ be a C^2 function satisfying A1-A2, for some $G\in\Omega$. Then*

$$m_{\mathcal{H}}(W^s_{\delta_5}(\Omega)\cap\hat{\mathcal{H}})=0.$$

Proof. Suppose that $m_{\mathcal{H}}(W^s_{\delta_5}(\Omega)\cap\hat{\mathcal{H}})>0$. Then by the Lebesgue's density Theorem there exists a point $F\in W^s_{\delta_5}(\Omega)\cap\hat{\mathcal{H}}$ such that for every open subset A of $\hat{\mathcal{H}}$ such that $F\in A$ we have

$$m_{\mathcal{H}}(W^s_{\delta_5}(\Omega)\cap A)>0.$$

In particular

$$m_{\mathcal{H}_F}(W^s_{\delta_5}(\Omega)\cap A)>0$$

for every open subset A of $\hat{\mathcal{H}}_F$ such that $F\in A$. Let

$$\mathcal{H}_i=\hat{\mathcal{R}}^i_{G_F}\mathcal{H}_F.$$

Consequently

$$m_{\mathcal{H}_i}(W^s_{\delta_5}(\Omega) \cap A) > 0$$

for every open subset O of $\mathcal{H}_i$ such that $\mathcal{R}^i F \in O$. By Corollary 5.3 if i is large enough then $\mathcal{H}_i$ satisfies $A1$-$A3$. That contradicts Proposition 7.1. □

8. Proof of the main results.

Proposition 8.1. *Let*

$$\mathcal{M}\colon P \to \mathcal{B}_0$$

be a C^2 map, where P is an open subset of a Banach space $\mathcal{B}_2$ and moreover suppose that $D_x\mathcal{M}$ has dense image for every $x \in \mathcal{M}^{-1}W^s_{\delta_5}(\Omega)$. If $\mathcal{R}$ and Ω satisfy the assumptions of Theorem 1 then $\mathcal{M}^{-1}W^s_{\delta_5}(\Omega)$ is a $\Gamma^k(\mathcal{B}_2)$-null set, for every $k \in \mathbb{N} \cup \{\infty, \omega_{\mathbb{R}}\}$.

Proof. It is enough to prove that $\mathcal{M}^{-1}W^s_{\delta_5}(\Omega)$ satisfies assumption (H) in Proposition 3.4 for every $k \in \mathbb{N} \cup \{\infty, \omega_{\mathbb{R}}\}$. There are two cases.

Case I. If $x \notin \mathcal{M}^{-1}W^s_{\delta_5}(\Omega)$. Let $\alpha\colon T \to \mathcal{B}_2$ be the constant family $\alpha(\lambda) = x$, for every $\lambda \in T$. Since $\mathcal{M}^{-1}W^s_{\delta_5}(\Omega)$ is a closed set, if $z \in \mathcal{B}_2$ is small enough we have $m_{\alpha_z}(\mathcal{M}^{-1}W^s_{\delta_5}(\Omega)) = 0$.

Case II. If $x \in \mathcal{M}^{-1}W^s_{\delta_5}(\Omega)$. Then $\mathcal{M}x \in W^s_{\delta_5}(G)$, for some $G \in \Omega$. Since the image of $D_x\mathcal{M}$ is dense, there exists $v_1, \dots, v_n \in \mathcal{B}_2$ such that

$$\{\pi^u_G \cdot D_x\mathcal{M} \cdot v_j\}_{j \leq n}$$

is a basis of E^u_G and moreover

$$\{\sum_j \lambda_j D_x\mathcal{M} \cdot v_j,\ \lambda_j \in \mathbb{R}\} \subset C^u_{\frac{\lambda_2}{2}\epsilon,0}(G).$$

Consider the function

$$Q\colon U_1 \times U_2 \to \mathcal{B}_0$$

where $U_1 \subset \mathcal{B}_2$ is a small neighborhood of 0 and $U_2 \subset \mathbb{R}^n$ is a small neighborhood of 0, defined by

$$Q(z, \lambda_1, \dots, \lambda_n) = \mathcal{M}(x + z + \sum_j \lambda_j v_j).$$

Define the C^2 map

$$\tilde{Q}\colon U_1 \times U_2 \to U_1 \times E^u_G$$

as

$$\tilde{Q}(z, \lambda_1, \dots, \lambda_n) = (z, \pi^u_G \circ Q(z, \lambda_1, \dots, \lambda_n) - \pi^u_G(G))$$

Note that $D\tilde{Q}_{(0,0,\dots,0)}$ is a continuous linear isomorphism. By the Inverse Function Theorem there is a neighborhood $O = O_1 \times O_2 \subset \mathcal{B}_0 \times E^u_G$ of

$$\tilde{Q}(0, 0, \dots, 0) = (0, \pi^u_G \circ \mathcal{M}(x) - \pi^u_G(G))$$

and a C^2 diffeomorphism on its image $S\colon O \to S(O)$, where $S(O)$ is a neighborhood of $(0, \dots, 0)$ such that $\tilde{Q} \circ S = Id$. In particular for each $z \in O_1$ the map

$$t \in O_2 \mapsto S(z, t)$$

is a C^2 diffeomorphism whose inverse is

$$(\lambda_1, \dots, \lambda_n) \mapsto H_z(\lambda_1, \dots, \lambda_n) = \pi^u_G \circ Q(z, \lambda_1, \dots, \lambda_n) - \pi^u_G(G). \tag{71}$$

For each $z \in O_1$ define the C^2 function

$$\beta_z \colon O_2 \to E^h_G + G$$

as

$$\beta_z(t) = \pi^h_G \circ Q \circ S(z,t) - \pi^h_G(G) + G.$$

Of course $t + \beta_z(t) = Q \circ S(z,t)$. Reducing O_2 to a very small convex open neighboohood of $\pi^u_G(\mathcal{R}^i(x)) - \pi^u_G(G)$ in E^u_G and O_1 to a very small neighborhood of 0 in $\mathcal{B}_2$ we have that β_z satisfies $A1$-$A2$ for every $z \in O_1$, so by Corollary 7.4 we conclude that

$$m_{\beta_z}(W^s_{\delta_5}(\Omega)) = 0. \tag{72}$$

Choose $\gamma > 0$ and $\delta > 0$ such that

$$\{z \in \mathcal{B}_2 \colon\ |z| < \delta\} \times [-\gamma, \gamma]^n \subset S(O_1 \times O_2),$$

and define

$$\alpha \colon [-1,1]^n \mapsto \mathcal{B}_2$$

as

$$\alpha(\lambda_1, \dots, \lambda_n) = x + \sum_i \gamma \lambda_i v_i \tag{73}$$

Notice that if $|z| < \delta$

$$\begin{aligned}
\mathcal{M}(\alpha_z(\lambda_1, \dots, \lambda_n)) &= \mathcal{M}(x + z + \sum_i \gamma\lambda_i v_i) \\
&= Q(z, \gamma\lambda_1, \dots, \gamma\lambda_n) \\
&= \pi^u_G \circ Q(z, \gamma\lambda_1, \dots, \gamma\lambda_n) - \pi^u_G(G) \\
&+ \pi^h_G \circ Q(z, \gamma\lambda_1, \dots, \gamma\lambda_n) - \pi^h_G(G) + G \\
&= t + \beta_z(t), \qquad (74)
\end{aligned}$$

where $t = \pi^u_G \circ Q(z, \gamma\lambda_1, \dots, \gamma\lambda_n) - \pi^u_G(G) = H_z(\gamma\lambda_1, \dots, \gamma\lambda_n)$. Since $t \mapsto S(z,t)$ is a diffeomorphism whose inverse is H_z, by (72) we have that

$$m_{\alpha_z}(\mathcal{M}^{-1} W^s_{\delta_5}(\Omega)) = 0.$$

Note that $\alpha \in \Gamma^k(\mathcal{B})$, for every $k \in \mathbb{N} \cup \{\infty\}$. To finish the proof in the case $k = \omega_{\mathbb{R}}$ everything we need to do is to extend α to a complex affine function

$$\alpha \colon \mathbb{C}^j \to \mathcal{B}_{\mathbb{C}}$$

using (73) with $\lambda_i \in \mathbb{C}$. □

Proposition 8.2. *Let*

$$\mathcal{M} \colon P \to \mathcal{B}_0$$

be a complex analytic map, where P is an open subset of a complex Banach space $\mathcal{B}_2$ and moreover suppose that $D_x\mathcal{M}$ has dense image for every $x \in \mathcal{M}^{-1} W^s_{\delta_5}(\Omega)$. If $\mathcal{R}$ is a complex analytic map satisfying the assumptions of Theorem 2 then $\mathcal{M}^{-1} W^s_{\delta_5}(\Omega)$ is $\Gamma^\omega(\mathcal{B}_2)$-null set and, considering $\mathcal{B}_2$ as a real Banach space, all conclusions of Proposition 8.1 holds.

Proof. We can easily replace $[-1,1]$ by $\overline{\mathbb{D}}$ in the proof of Proposition 8.1 and construct for each $x \in \mathcal{B}_2$ a complex affine function $\alpha \in \Gamma^\omega(\mathcal{B}_2)$ that satisfies the assumption (H) in Proposition 3.4 for $k = \omega$. Considering $\mathcal{B}_2$ as a *real* Banach space we can see α as a real affine transformation

$$\alpha\colon \mathbb{R}^{2n} \to \mathcal{B}_2$$

defined by

$$\alpha(\lambda_1^a, \lambda_1^b, \dots, \lambda_n^a, \lambda_n^b) = x + z + \sum_m \gamma(\lambda_m^a + i\lambda_m^b) v_m.$$

If $\mathcal{B}_\mathbb{C}$ is a Banach space complexification of $\mathcal{B}_2$, every point $w \in \mathcal{B}_\mathbb{C}$ is a pair $(u,v) \in \mathcal{B}_2^2$, and $\mathcal{B}_\mathbb{C}$ is endowed with the obvious sum and the scalar multiplication

$$(\lambda^a + i\lambda^b)(u,v) = (\lambda^a u - \lambda^b v, \lambda^a v + \lambda^b u)$$

for every $\lambda^a, \lambda^b \in \mathbb{R}$. We identify $u \in \mathcal{B}_2$ with $(u,0) \in \mathcal{B}_\mathbb{C}$. We can extend α to an affine complex map

$$\alpha_z\colon \mathbb{C}^{2n} \to \mathcal{B}_\mathbb{C}$$

taking $\lambda_m^a, \lambda_m^b \in \mathbb{C}$ and defining

$$\alpha(\lambda_1^a, \lambda_1^b, \dots, \lambda_n^a, \lambda_n^b) = (x,0) + \sum_m \gamma(\lambda_m^a + i\lambda_m^b)(v_m, 0).$$

Note that

$$m\{w \in \mathbb{R}^{2n}\colon \alpha_z(w) \in \mathcal{M}^{-1} W^s_{\delta_5}(\Omega), \ \textit{and}\ |w_i| < \frac{1}{\sqrt{2}}\} = 0.$$

for z small enough and $\alpha_z(w) = z + \alpha(w)$. Define the affine complex map

$$\beta\colon \overline{\mathbb{D}}^{2n} \mapsto \mathcal{B}_\mathbb{C}$$

as $\beta(w) = \alpha(w/\sqrt{2})$. Then $\beta \in \Gamma^k(\mathcal{B}_2)$ for every $k \in \mathbb{N} \cup \{\infty, \omega_\mathbb{R}\}$, $\beta(0) = x$ and

$$m\{w \in [-1,1]^{2n}\colon \beta_z(w) \in \mathcal{M}^{-1} W^s_{\delta_5}(\Omega)\} = 0,$$

for every z small. So for each $x \in \mathcal{B}_2$ the corresponding β satisfies the assumption (H) in Proposition 3.4 for every $k \in \mathbb{N} \cup \{\infty, \omega_\mathbb{R}\}$. This complete the proof. □

Proof of Theorems 1 and 2. Since a countable union of $\Gamma^k(\mathcal{B}_0)$-null sets is a $\Gamma^k(\mathcal{B}_0)$-null set, it is enough to prove that $\mathcal{R}^{-i} W^s_{\delta_5}(\Omega)$ is a $\Gamma^k(\mathcal{B}_0)$-null set for every $i \geq 0$. This follows immediately from Proposition 8.1 and Proposition 8.2 taking $\mathcal{M} = \mathcal{R}^i$. □

9. The shadow is shy.

Note that if a set Θ satisfies the assumptions of Proposition 3.4 then Θ is a locally shy set in the sense of Hunt, Sauer and Yorke [11], and a shy set if $\mathcal{B}$ is separable. Shy sets are the same as Haar null sets in abelian polish groups as defined by Christensen [7] in the case of separable Banach spaces. Consequently if $\mathcal{B}$ is separable and under the assumptions of either Proposition 8.1, Proposition 8.2, Theorem 1 or Theorem 2 the sets under consideration are also shy sets (Haar null sets).

10. Curves transversal to a global stable lamination.

The following results are useful to prove that *specific* families γ satisfy

$$m_\gamma(\cup_{i\geq 0}\mathcal{R}^{-i}W^s_{\delta_5}(\Omega)) = 0.$$

Proposition 10.1. *Let $\mathcal{R}$, Ω, $\mathcal{B}_0$ be as in Theorem 1 and*

$$\mathcal{M}_i\colon P_i \to \mathcal{B}_0,$$

where $i \in \Lambda \subset \mathbb{N}$ and P_i are open subsets of a Banach space $\mathcal{B}_2$, be C^2 maps. Let $\gamma \in \Gamma^2(\mathcal{B}_2)$ and K_i, $i \in \Lambda$, be the subset of parameters $\lambda \in (-1,1)^n$ such that

i. *We have that $\gamma(\lambda) \in P_i$,*
ii. *We have $\mathcal{M}_i \circ \gamma(\lambda) \in W^s_{\delta_5}(G)$ for some $G \in \Omega$,*
iii. *The derivative $D_\lambda(\mathcal{M}_i \circ \gamma)$ is injective and*
iv. *The image of $D_\lambda(\mathcal{M}_i \circ \gamma)$ is contained in $C^u_{\frac{\lambda_2}{2}\epsilon,0}(G)$.*

Then

$$m(\cup_{i\in\Lambda}K_i) = 0.$$

The same holds if we replace the assumptions of Theorem 1 by the assumptions of Theorem 2, $(-1,1)^n$ by $\mathbb{D}^n$ and $\gamma \in \Gamma^\omega(\mathcal{B}_2)$.

Proof. Suppose that $m(K_i) > 0$ for some $i \in \Lambda$. Let $\lambda_0 \in K_i$ be a Lesbegue density point of K_i. Due iii. and iv. the derivative $D_{\lambda_0}(\pi^u_G \circ \mathcal{M}_i \circ \gamma)$ is injective so we can use the same method in the proof of Proposition 8.1 to show that there exists a small neighboorhood O of λ_0 in $(-1,1)^n$ such that $\mathcal{M}_i \circ \gamma(O)$ is the graph of a C^2 function $\mathcal{H}\colon U \mapsto E^h_G + G$, where U is an open subset of E^u_G, and there is a C^2 diffeomorphism $S\colon U \mapsto O$ such that $F + \mathcal{H}(F) = \mathcal{M}_i \circ \gamma(S(F))$. Because the image of $D_\lambda(\mathcal{M}_i\circ\gamma)$ is contained in $C^u_{\frac{\lambda_2}{2}\epsilon,0}(G)$ we can reduce O such that $\mathcal{H}$ satisfies conditions $A1$-$A2$ in Corollary 7.4. Since λ_0 is a Lesbegue density point of K_i we conclude that $m_{\mathcal{H}}(W^s_{\delta_5}(G)) > 0$. That contradicts the conclusion of Corollary 7.4. A similar proof also works in the second case. □

An important case is when we are under the assumptions of Theorem 1, Ω is hyperbolic and its stable directions are the horizontal directions E^h. Suppose that

$$\mathcal{M}\colon P \to \mathcal{B}_0$$

is a C^2 map, where P is an open subset of a Banach space $\mathcal{B}_2$. For every $G \in \Omega$ define the set

$$W^{st}(\mathcal{M}, G) = \{F \in P\colon \lim_i |\mathcal{R}^i \circ \mathcal{M}(F) - \mathcal{R}^i G|_0 = 0\}$$

and the *global stable lamination*

$$W^{st}(\mathcal{M}, \Omega) = \cup_{G\in\Omega} W^{st}(\mathcal{M}, G).$$

Moreover suppose that $D_F\mathcal{R}^i \circ \mathcal{M}$, $i \in \mathbb{N}$, is injective and it has dense image for every

$$F \in W^{st}(\mathcal{M}, \Omega).$$

For each $F \in W^{st}(\mathcal{M}, G)$ define the subspace

$$E^h_F = \{v \in \mathcal{B}_2\colon \sup_i |D_F(\mathcal{R}^i \circ \mathcal{M}) \cdot v|_0 < \infty\}.$$

Note that

i. If $v \notin E^h_F$ then for every $\epsilon' > 0$ there exists $i_{(F,v)}$ such that
$$D_F(\mathcal{R}^i \circ \mathcal{M}) \cdot v \in C^u_{\epsilon',0}(\mathcal{R}^i G)$$
for every $i > i_{(F,v)}$,
ii. The subspace E^h_F is closed,
iii. *codim* $E^h_F = n$.
iv. There exists $C \geq 0$ and $\lambda \in (0,1)$ such that
$$|D_F(\mathcal{R}^i \circ \mathcal{M}) \cdot v|_0 \leq C\lambda^i |v|_0$$
for every $v \in E^h_F$.

Indeed (i) and (iv) follows from the hyperbolicity of Ω. Property (ii) follows from (i), since $D_{\mathcal{R}^i \circ \mathcal{M}(F)}\mathcal{R}$ maps $C^u_{\epsilon,0}(\mathcal{R}^i G) \setminus \{0\}$ strictly inside
$$C^u_{\epsilon,0}(\mathcal{R}^{i+1} G) \setminus \{0\}$$
for every large i and it expands vectors in that cone. Since $D_F\mathcal{R}^i \circ \mathcal{M}$, $i \in \mathbb{N}$ is injective and it has dense image, it is easy to prove (iii).

Corollary 10.2. *Under the above conditions we have* $m_\gamma(W^{st}(\mathcal{M}, \Omega)) = 0$ *for every* $\gamma \in \Gamma^2_n(\mathcal{B}_0)$ *such that* $\gamma \pitchfork W^{st}(\mathcal{M}, \Omega)$*, that is, if* $\gamma(\lambda) \in W^{st}(\mathcal{M}, \Omega)$ *then* $Im\ D_\lambda \gamma \pitchfork E^h_{\gamma(\lambda)}$*. The same conclusion holds if we are under the conditions of Theorem 2 and* $\gamma \in \Gamma^\omega_n(\mathcal{B}_0)$*.*

Proof. If $\gamma(\lambda_0) = F \in W^{st}(\mathcal{M}, G)$ for some $G \in \Omega$ we have that for i large enough $\mathcal{R}^i \circ \mathcal{M}(F) \in W^s_{\delta_5}(\mathcal{R}^i(G))$,
$$Im\ D_{\lambda_0}(\mathcal{R}^i \circ \mathcal{M} \circ \gamma) \subset C^u_{\frac{\lambda_2}{2}\epsilon,0}(\mathcal{R}^i(G)),$$
and moreover
$$D_{\lambda_0}(\pi^u_{\mathcal{R}^i G} \circ \mathcal{R}^i \circ \mathcal{M} \circ \gamma)$$
is injective. Take $\mathcal{M}_i = \mathcal{R}^i \circ \mathcal{M}$ and apply Proposition 10.1. □

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