

TOPOLOGICAL CLASSIFICATION OF SIMPLE MORSE BOTT FUNCTIONS ON SURFACES

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ABSTRACT. We present a global topological classification of Morse Bott functions on orientable closed surfaces. The invariant is based on the Reeb graph of the function and the topological type of the singular level sets. Connection with other known invariants it is shown. We also prove a realization Theorem and some stability properties of the Morse Bott functions.

1. INTRODUCTION

The characterization of a set of functions that verifies a determined property involves two major questions. To get an invariant for particular functions of the set and how the entire set is organized in the space of smooth functions.

With respect to the first question, the classification problem of singular points of smooth map germs is an important problem in Singularity Theory, nevertheless not local results are not abundant. The classification of functions on surfaces up different types of equivalence can be found, for instance, in [3, 23] considering smooth equivalence and in [14, 26, 27] considering topological equivalence.

In particular, Arnold, [2], Kulinich [14] and Sharko [27] classified Morse functions on surfaces using Reeb graphs with some additional information and Prishlyak [21] classified smooth functions with isolated critical points on closed surfaces.

In this paper we present a topological classification of Morse Bott functions on orientable closed surfaces. In order to obtain this classification we construct an invariant that is based on the Reeb graph of the function and on the topological type of the singular level sets. This is a kind of additional information that, as far as we know, it was not used until now. In the section 3 of the paper we will show how the topological type of the singular level sets of a function is related with the order of the vertex of the Reeb graph associated to f and induced by the values of f .

Given a Morse Bott function f and denoting by $\mathcal{R}^\eta(f)$ the ordered Reeb graph associated to f the first main result of this paper is:

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Theorem 1. *Two simple Morse Bott functions $f, g : \Sigma \rightarrow \mathbb{R}$ are conjugate if and only if $\mathcal{R}^\eta(f)$ and $\mathcal{R}^\eta(g)$ are isomorphic.*

The definition of conjugated functions is given in Section 2.

The second main result of this paper is a realization theorem for Morse Bott functions.

Theorem 2. *Let Σ be an orientable closed surface. A connected directed, finite graph G can be the Reeb graph associated to a Morse Bott function $f : \Sigma \rightarrow \mathbb{R}$ on Σ if and only if the following conditions are satisfied:*

- (a) **local conditions:** the sink and source vertices of G have degree 1 or 2; the interior vertices has degree 3.
- (b) **global conditions:**
 - (b.1) the cycle rank of G coincides with the genus of the surface Σ ,
 - (b.2) the direction of the graph defines a linear ordering of the vertices.
- (c) **Conditions for orientability:** G possesses no loops nor oriented cycles

With respect to the second problem stated at the beginning of the introduction, the natural approach will be to consider a stratification of the space of smooth functions on surfaces and to locate the space of Morse Bott functions in this stratification. The case of Morse function is considered in the paper [8]. But as the Morse Bott functions have not isolated singularities, in a neighborhood of one particular function one must expect to have another type of singularities. A stability property is studied in the last part of the paper. We present a new topology on the set $C^\infty(\Sigma)$ of the smooth functions defined from Σ to \mathbb{R} , such that endowed with this topology some Morse Bott functions on the sphere are stable, as stated in Theorem 28 of Section 6.

A final remark is that the classification of functions is related with some topological questions as homology theory and, at the same time, with problems pertaining to areas outside the topology since Morse Bott functions or Morse functions are associated with other concepts such as polynomials, flows and computer graphic recognition. See for instance [2, 16, 20].

2. BASIC CONCEPTS

Let $f : \mathbb{M}^n \rightarrow \mathbb{M}^m$ be a twice continuously differentiable function. A point $p \in \mathbb{M}^n$ is called a *singular point* if $\text{rank}(\text{df}(p))$ is not maximum. Otherwise it is a *regular point*. A point $b \in \mathbb{R}$ is called a *singular value* of f if $f^{-1}(b)$ contains a singular point of f . The *singular set* of f , denoted by $\text{Sing}(f)(\mathbb{M}^n)$, is the set of all singular points of f on \mathbb{M}^n . In this paper we assume that \mathbb{M}^m is \mathbb{R} unless otherwise indicated.

For each $a \in \mathbb{R}$ consider the level set $I_a(f) = f^{-1}(a)$. $I_a(f)$ is a union of connected components, $i_a^k(f)$, $k = 1, \dots, m(a)$, called *fibers*. A *singular*

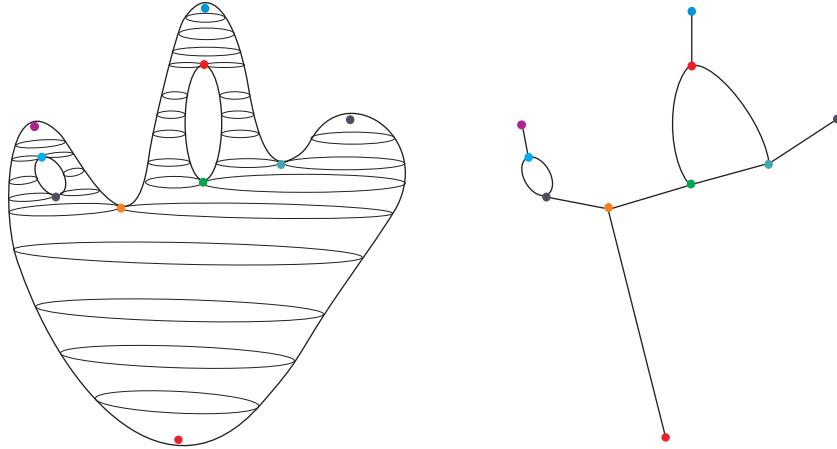


FIGURE 1. Left: the 2-torus; Right: the Reeb graph of the height function.

fiber is a connected component of a level set $I_a(f)$ which contains a singular point of f and it is denoted by $s_a(f)$.

If all nearby fibers around a singular fiber are homeomorphic to it then this fiber is *reducible*. See [1] for details.

From now on, we assume that f is a *simple* or *non resonant* function, what means that there is a unique connected component of singular points in the singular level. It is contained in a singular fiber $s_a(f) \subset I_a(f)$ for each $a \in \mathbb{R}$.

If $f : M^n \rightarrow \mathbb{R}$ has isolated singular points then the *Reeb graph* of f , denoted by \mathcal{R}_f , is the graph obtained by contracting each fiber to a point, the vertices correspond to the singular fibers of f (see [22]). The Reeb graph is also known as the Kronrod-Reeb graph [27]. See Figure 1 for an example where f is the height function on the 2-torus.

As the target of f is \mathbb{R} , the linear order in \mathbb{R} defines a unique direction on the Reeb graph. This direction, in turn, induces an order in the set of singular fibers.

Associated with the quotient map $\pi : M^n \rightarrow \mathcal{R}_f$ and the inclusion map $\iota : \mathcal{R}_f \rightarrow M^n$ there are two known functions

- (i) Stein factor of f (\bar{f}), such that $f = \bar{f} \circ \pi$ (see [23] for instance).
- (ii) The restriction of f to \mathcal{R}_f ($f_{\mathcal{R}}$), given by $f_{\mathcal{R}} = f \circ \iota$.

Both functions are closely related since the Reeb graph can be considered as a sub-complex of the surface, (see [12]).

Definition 3 ([6]). *Let $f : M^n \rightarrow \mathbb{R}$ be a smooth function on a n -dimensional manifold. A smooth submanifold $S \subset \text{Sing}(f)(M^n)$ is a nondegenerate singular submanifold of f if:*

- $\partial S = \emptyset$

- S is compact and connected
- $\forall s \in S$, we have $T_s S = \ker (Hess_s f)$.

The function f is a Morse Bott function (\mathcal{MB} function from now on) if the set $Sing(f)(\mathbb{M}^n)$ consists of isolated points and nondegenerate singular submanifolds.

Let $p \in Sing(f)(\mathbb{M}^n)$. By the Morse Bott Lemma([4]) there exists a local chart of \mathbb{M}^n around p and a local splitting of the normal bundle of S , $N_p(S) = N_p^+(S) \oplus N_p^-(S)$ so that if $p = (s, x, y)$, $s \in S$, $x \in N_p^+(S)$, $y \in N_p^-(S)$:

$$T_p(\mathbb{M}^n) = T_p(S) \oplus N_p^+(S) \oplus N_p^-(S) \quad \text{and}$$

$$f(p) = f(S) + |x|^2 - |y|^2.$$

The dimension of $N_p^-(S)$ is the index of S and if p is not an isolated singularity of f then f is locally a Morse function on $N_p(S)$.

It also follows from the Morse Bott Lemma that Morse functions are \mathcal{MB} functions with isolated singular points. Moreover, if \mathbb{M}^n is compact then the function has a finite number of isolated singular points.

Another particular case of \mathcal{MB} functions consist of the Round Bott functions ([13]) where all singular submanifolds are circles.

Let \mathbb{M}^n be a compact connected orientable surface Σ of genus $g \geq 0$, denoted by $\Sigma(g, 0)$. Considering the dimension of the singular submanifolds and its index, the singular set of f , $Sing(f)(\Sigma(g, 0))$ can be subdivided in three subsets

- (i) $Cir(f)(\Sigma(g, 0))$: points in singular submanifolds that are homeomorphic to S^1 . On these circles the function assumes extremal values. We call such singular submanifolds singular circles.
- (ii) $Cen(f)(\Sigma(g, 0))$: isolated singular points which are extremum points of f called center points.
- (iii) $Sad(f)(\Sigma(g, 0))$: isolated singularities of index 1 of f called saddle points.

2.1. Morse Bott foliations and its singularities. Let $\mathcal{F}(\Sigma, f)$ be the foliation on $\Sigma(g, 0)$ defined by the level sets of f and let $Sing(\mathcal{F}(\Sigma, f))$ be the set of singularities of $\mathcal{F}(\Sigma, f)$. Some arguments in these notations may be omitted if are irrelevant or can be determined from context. A Morse Bott foliation (\mathcal{MB} foliation from now on) is a foliation defined by the level sets of a \mathcal{MB} function.

As f is simple then $\mathcal{F}(\Sigma, f)$ is also simple.

Two \mathcal{MB} foliations on Σ are *topologically equivalent* if it exists a homeomorphism on Σ that sends leaves of one foliation to leaves of the other.

As Σ is an orientable surface, the singular set of a \mathcal{MB} foliation $Sing(\mathcal{F}(\Sigma))$ consist of (see [24], [25] for more details): $Cen(\mathcal{F}(\Sigma))$ center points and

$Sad(\mathcal{F}(\Sigma))$ saddle points. The singular circles of f are not singularities of the foliation.

2.2. Equivalence of Morse Bott functions. Let $\Sigma(g, 0)$ be a compact connected orientable surface of genus g and $f_1, f_2: \Sigma(g, 0) \rightarrow \mathbb{R}$, smooth functions. Functions f_1, f_2 are *topologically equivalent* if there are homeomorphisms $k: \Sigma(g, 0) \rightarrow \Sigma(g, 0)$, $l: \mathbb{R} \rightarrow \mathbb{R}$ such that $f_1 \circ k = l \circ f_2$. The choice of k and l is not unique and k sends level sets of f_1 to level sets of f_2 since they are related by the following equality $I_{l(a)}(f_2) = k(I_a(f_1))$.

We say that f_1 and f_2 are topologically conjugated if they are topologically equivalent and l preserves the orientation. See for instance [28, 21, 27, 26]. In this case if f_1 and f_2 are topologically conjugated then f_1 and $-f_2$ are not necessarily topologically conjugated.

In the case of a \mathcal{MB} function the homeomorphism k could send a singular circle and its neighborhood to a regular cylinder, i.e. it does not preserve the singular circle, so the definition of topologically equivalence must be adapted here for the case of \mathcal{MB} functions.

Definition 4. *Two \mathcal{MB} functions f_1 and f_2 from Σ to \mathbb{R} are conjugated if there exist homeomorphisms $k: \Sigma(g, 0) \rightarrow \Sigma(g, 0)$, $l: \mathbb{R} \rightarrow \mathbb{R}$ such that l preserves orientation, $f_2 = l \circ f_1 \circ k^{-1}$ and k sends singular fibers of f_1 to singular fibers of f_2 .*

As f is simple we get the following result.

Proposition 5. *If f_1 and f_2 are simple conjugated \mathcal{MB} functions such that $f_2 = l \circ f_1 \circ k^{-1}$, then the sets $s_a(f_1)$ and $s_{l(a)}(f_2)$ are homeomorphic.*

An *essential singular value* of a \mathcal{MB} function is a value $a \in \mathbb{R}$ such that $I_a(f)$ is topologically distinct to $I_b(f)$ for any b in a neighborhood of a . In the case of Morse functions all singular values are essential.

We will use the notion of *slicing of a Morse function* defined in [20, 19] and adapted here for the case of \mathcal{MB} functions. This definition will be useful in this paper. An *slicing* of a \mathcal{MB} function f with n isolated singular fibers is an increasing sequence of real numbers

$$-\infty = a_0 < a_1 < \cdots < a_n < \infty$$

such that for every $i = 1, \dots, n$ the intervals (a_{i-1}, a_i) contain precisely one singular value of f .

The definition of topologically equivalent Morse functions on $\Sigma(g, 0)$ given in [20] uses the idea of the slicing as follows. Two Morse functions f_1 and f_2 on $\Sigma(g, 0)$ are topologically equivalent if there exist a slicing of f_1 , $a_0 < a_1 < \cdots < a_n$, a slicing of f_2 , $b_0 < b_1 < \cdots < b_n$ and homeomorphisms Φ_i , $i = 1, \dots, n$ preserving orientation

$$\Phi_i : M_{a_i}(f_1) \rightarrow M_{b_i}(f_2), \quad i = 1, \dots, n,$$

where $M_c(f_j) = \{p \in \Sigma(g) : f_j(p) \leq c\}$, $j = 1, 2$ (see [18]).

Then, if f_1 and f_2 are conjugated \mathcal{MB} functions on $\Sigma(g, 0)$, the existence of the homeomorphisms k from $\Sigma(g, 0)$ to $\Sigma(g, 0)$ and l from \mathbb{R} to \mathbb{R} such that k sends singular fibers of f_1 to singular fibers of f_2 gives us the existence of homeomorphisms Φ_i , $i = 1, \dots, n$. In fact, each set $M_c(f_j)$ is the union of level sets of f_1 and f_2 respectively, and k generates a bijection between these sets.

3. CONSTRUCTION OF THE INVARIANT

The main objective of this section is to define a topological invariant to classify Morse Bott functions on compact, connected and orientable surfaces. To do this we will use the classification of the singular level sets of Morse Bott functions that are closed curves and eights. This classification is done in the paper [16].

3.1. Classification of circles and separatrix eights.

Definition 6. *An embedded circle on $\Sigma(g, 0)$ will be the image of an embedding $\phi : S^1 \rightarrow \Sigma(g, 0)$.*

Definition 7. *A separatrix eight \mathfrak{B} , or in short an eight, is the image of an immersion of S^1 into Σ , $\psi : S^1 \rightarrow \Sigma(g, 0)$, homeomorphic to two circumferences glued by a point p . A component s_i , will be any of the two circumferences.*

Denote by $\mathfrak{N}\mathfrak{B}$ a closed regular neighborhood of $\mathfrak{B} = s_1 \cup_{p} s_2$. (for details about regular neighborhoods see [9], [11]). Then:

Lemma 8 ([16]). *A closed regular neighborhood of \mathfrak{B} is homeomorphic to $\Sigma(0, 3)$ or $\Sigma(1, 1)$. The \mathfrak{B} whose regular neighborhood is $\Sigma(1, 1)$ is not an admissible singular fiber of a \mathcal{MB} function.*

Definition 9 ([16]). *We will say that \mathfrak{B} is a toroidal eight, if $\mathfrak{N}\mathfrak{B}$ is homeomorphic to $\Sigma(1, 1)$ and a planar eight, if $\mathfrak{N}\mathfrak{B}$ is homeomorphic to $\Sigma(0, 3)$.*

Definition 10. *We will say that two circles (or two eights) are topologically equivalent if there is an homeomorphism on $\Sigma(g, 0)$ that sends one of them in the other.*

Let $E(a)$ be the largest integer not greater than a and $C(a)$ is the smallest integer not less than a .

Theorem 11 ([16]). *Let $\Sigma(g, 0)$ be an orientable closed surface. The number of non-equivalent embedding of S^1 on Σ is*

- (i) 1 if $g = 0$,
- (ii) $E\left(\frac{g}{2}\right) + 2$ with representant $l_0, l_1, \dots, l_{E\left(\frac{g}{2}\right)}, l_K$ if $g > 0$.

Theorem 12 ([16]). *Let $\Sigma(g, 0)$ be an orientable, closed surface with $g \geq 0$. Then, the number of topological types of eights on $\Sigma(g, 0)$ is*

- (1) $3g + 1$, if $g = 0, 1$,

$$(2) \quad E\left(\frac{g}{2}\right)C\left(\frac{g}{2}\right) + E\left(\frac{g}{2}\right) + 2g + 3, \text{ if } g \geq 2.$$

If f is a \mathcal{MB} function then \mathfrak{B} is a planar eight so $\mathfrak{N}\mathfrak{B}$, the closed regular neighborhood of \mathfrak{B} , has three boundary curves. Two boundary curves J_1 and J_2 of $\mathfrak{N}\mathfrak{B}$ are contractible to s_1 and s_2 respectively and we will note these type of curves by J_s . The third boundary curve, J_3 , is contractible to \mathfrak{B} and will be noted by $J_{\mathfrak{B}}$. Considering the Reeb graph of f , in a saddle singularity the edge containing $J_{\mathfrak{B}}$ curves bifurcates in two edges of J_s circles.

Foliations defined by Morse functions may differ from foliations defined by a \mathcal{MB} functions:

Proposition 13. *Let f be a Morse function on Σ and $\mathcal{F}(\Sigma, f)$ the Morse foliation induced by the level sets of f . Then two components of an eight \mathfrak{B} of $\mathcal{F}(\Sigma, f)$ cannot be connected by a family of closed invariant curves. Moreover, two regular cylinders connecting two eights only contain circles of the type J_s .*

Proof. Assume that two components of an eight \mathfrak{B} are connected by a family of closed invariant curves. This family can be parameterized by an open interval $]a, b[$ with a and b assigned to the components of the eight. As $f(a) = f(b)$, f must have a least a singular value on $]a, b[$. The singular level set will be a circle but this is not an admissible singularity in Morse foliations.

Given an eight \mathfrak{B} , in $\mathfrak{N}\mathfrak{B}$ the signs of $f(\mathfrak{B}) - f(J_s)$ and $f(\mathfrak{B}) - f(J_{\mathfrak{B}})$ are always opposite. This sign induces an order in the level sets of $\mathfrak{N}\mathfrak{B}$. Suppose that there are two regular cylinders connecting \mathfrak{B}_1 and \mathfrak{B}_2 and one of the cylinders are bounded by all \mathfrak{B}_2 ; therefore it contains $J_{\mathfrak{B}}$ circles. Consider a parametrization on the second connecting cylinder as in the last case and $f(\mathfrak{B}_1) = f(a) < f(\mathfrak{B}_2) = f(b)$. Let c_2 be a J_s invariant circle near \mathfrak{B}_2 ; assume that it corresponds to the parameter $b + \epsilon$.

Assume that by the ordering in $\mathfrak{N}\mathfrak{B}_2$, $f(c_2) > f(b)$. Then f decreases from $f(b + \epsilon)$ to $f(b)$, but f must also be an increasing function in part of the cylinder since $f(a) < f(b)$. Therefore f must have at least one singularity along this second cylinder. Other assumptions on the ordering on $\mathfrak{N}\mathfrak{B}_1$ and $\mathfrak{N}\mathfrak{B}_2$ conclude similarly. \square

3.2. Invariant for topologically equivalent \mathcal{MB} functions. First of all we recall some basic terminology from topological graph theory (more details in [10]). A directed graph G (or *digraph*) consists of a finite nonempty set V of points together with a prescribed collection E of ordered pairs of distinct points. The elements of V are called *vertices* and the elements of E are *directed edges or arcs*. By definition, a directed graph is simple if it has no loops or multiple edges. Denote by $e = (u, v)$ an edge of G . Then, the edge goes from u to v and it is *incident* with u and v . We also say that e is adjacent to v and v is adjacent from u . The *outdegree* of a vertex v , denoted

by e^- , is the number of vertices adjacent from it, and the *indegree*, denoted by e^+ , is the number of vertices adjacent to it. The sum of the indegree and the outdegree of v is called the *degree* of v . A *source vertex* is a vertex with indegree 0, a *sink vertex* has outdegree 0 and an *interior vertex* has nonzero indegree and nonzero outdegree.

Let $\nu > 0$ be the number of critical values of a simple \mathcal{MB} function f and \mathcal{R}_f the Reeb graph of f . Let η be the function that associates to each vertex a natural number between 1 and ν following the order induced by the slicing of f . From now on, $\mathcal{R}^\eta(f)$ denotes the pair (\mathcal{R}_f, η) , i.e. the ordered Reeb graph of f . A vertex of \mathcal{R}_f is a *saddle vertex* if it is a vertex associated to a saddle point of f .

Definition 14. *Let ξ be the function that associates to each saddle vertex of \mathcal{R}_f the edge that contains the J_B circle. See Figure 2.*

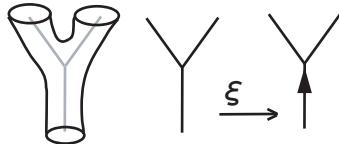


FIGURE 2. The function ξ .

In [16] the authors proved:

Proposition 15. *The Reeb graph of a \mathcal{MB} function and ξ determines the topological type of the eights.*

Proposition 16. *Given a \mathcal{MB} function f , η determines the function ξ on the Reeb graph \mathcal{R}_f .*

Proof. Given a saddle point s there exists an interval (a_i, a_{i+1}) in the slicing of f such that s corresponds to a real value in the interval (a_i, a_{i+1}) and this real number splits the interval (a_i, a_{i+1}) in two components. Each point of an edge bounded by s and near s is associated with a real value in the interval (a_i, a_{i+1}) . Therefore given p_1, p_2 and p_3 points on different edges, two of them have associated real numbers in the same component of the interval (a_i, a_{i+1}) . The other component is associated to the edge bounded by s which contains the curves of type J_B . \square

The values of f on the vertices make \mathcal{R}_f a directed graph. Conversely, given a directed graph, we can not compare final vertices as the values of a function are not specified.

Proposition 17. *Let Γ be a connected subgraph of the Reeb graph associated to a \mathcal{MB} function f having only saddle vertices. The function ξ and the direction of one edge of Γ determines the direction in all Γ . In other words,*

the function ξ determines a linear ordering of the vertices of Γ that coincides with the order defined by η or with the inverse order, depending on the choice of the direction of one edge of Γ .

Proof. Given a saddle vertex s , the function ξ distinguish what is the s -adjacent edge that contains curves J_B . So the function ξ and the direction of an edge adjacent to a saddle vertex determines the direction on the other two adjacent edges. If Γ is a subgraph having only saddle vertices, chosen one of the two possible directions to one edge of Γ and fixing the function ξ we obtain a directed subgraph which has the same *direction* (order) of η or the inverse order, depending on the chosen direction. This concludes the proof of this proposition. \square

Two graphs are *isomorphic* if there exists a one-to-one correspondence between their vertices and edges which preserves adjacency.

The graph of a \mathcal{MB} foliation on orientable closed surface Σ is defined in [16] as follows.

Definition 18. Let \mathcal{F} be a \mathcal{MB} foliation and f such that $\mathcal{F} = \mathcal{F}(f)$. Then the Graph $\Theta(\mathcal{F})$ of the \mathcal{MB} foliation is:

a) A circle, in the case of a regular foliation by circles on the torus.

b) The graph obtained from the Reeb graph of f transforming the union of each vertex v associated to singular circles of f and the two incident edges in a new edge.

$\Theta(\mathcal{F})$ does not depend on the particular function f such that $\mathcal{F} = \mathcal{F}(f)$. This construction is related to the construction described in [7], section 1.3 and page 13. This graph $\Theta(\mathcal{F})$ carries the information about the surface Σ since the number of independent cycles in $\Theta(\mathcal{F})$ is the genus of Σ .

Proposition 19 ([16]). Let ξ be a function on $\Theta(\mathcal{F})$ that associates to each saddle vertex the edge that contains the J_B circles. The graph and ξ determines the topological type of the eight.

By $\Theta_\xi(\mathcal{F})$ we denoted the pair formed by $\Theta(\mathcal{F})$ and the function ξ introduced in Proposition 19.

We assume here that $\Theta_\xi(\mathcal{F}_1)$ and $\Theta_\xi(\mathcal{F}_2)$ are isomorphic if there exists an isomorphism from $\Theta(\mathcal{F}_1)$ onto $\Theta(\mathcal{F}_2)$ that preserves the assignments of the functions ξ .

Theorem 20 ([16]). $\Theta_\xi(\mathcal{F})$ is a complete topological invariant for \mathcal{MB} foliations on orientable closed surfaces.

Definition 21. We say that $\mathcal{R}^\eta(f)$ and $\mathcal{R}^\eta(g)$ are isomorphic if \mathcal{R}_f and \mathcal{R}_g are isomorphic and the assignments of the function η are preserved.

Theorem 22 (Completeness Theorem). Two simple \mathcal{MB} function $f : \Sigma \rightarrow \mathbb{R}$ and $g : \Sigma \rightarrow \mathbb{R}$ are conjugate if and only if $\mathcal{R}^\eta(f)$ and $\mathcal{R}^\eta(g)$ are isomorphic.

Proof. (Necessity). Let f and g be simple \mathcal{MB} function conjugated and $a_0 < a_1 < \dots < a_n$ is a slicing of f and $b_0 < b_1 < \dots < b_n$ a slicing of g then there exist a homeomorphism preserving the orientation $\Phi_i : M_{a_i}(f) \rightarrow M_{b_i}(g)$, $i = 1, \dots, n$. and $\mathcal{R}^\eta(f)$ and $\mathcal{R}^\eta(g)$ are isomorphic.

(Sufficiency). The functions f and g induces Morse Bott foliations $\mathcal{F}(f)$ and $\mathcal{F}(g)$ respectively on Σ . The isomorphism from $\mathcal{R}^\eta(f)$ onto $\mathcal{R}^\eta(g)$ defines a isomorphism between $\Theta_\xi(\mathcal{F}(f))$ and $\Theta_\xi(\mathcal{F}(g))$. By Theorem 20, we conclude that there is a homeomorphism k from Σ into Σ that conjugates $\mathcal{F}(f)$ and $\mathcal{F}(g)$. Since the singular circles are reducible singularities we can choose k in such a way that it sends singular circles of one foliation to singular circles of the other foliation.

The function $f \circ k^{-1}$ is conjugate to f and defines on the surface the same foliation than g . Therefore $f \circ k^{-1}$ and g are conjugate and by transitivity, g and f also. \square

4. EXAMPLES

In this section we present several example of Morse Bott functions.

Example 1. Let $\Sigma = S^2$ and embed S^2 in \mathbb{R}^3 as the unit sphere. Consider $f : S^2 \rightarrow \mathbb{R}$, $f(x, y, z) = z$ and $g : S^2 \rightarrow \mathbb{R}$, $g(x, y, z) = -z^2$.

The functions f and g are Morse Bott functions. The non degenerate critical sub manifolds of f are two critical points the north pole and the south pole, and the non degenerate critical sub manifolds of g are the north pole, the south pole and the equator points $z = 0$. As they have non isomorphic Reeb graphs, from Theorem 22, f and g are not conjugated.

The Reeb graph is not enough to characterize \mathcal{MB} functions on orientable connected and compact surfaces.

Example 2. In [2] there is an example (see Example 1 and 3 in [2]) of two Morse functions with the same ordered Reeb graph which are non conjugated.

There are other special examples of Morse Bott functions, for instance, Wigner's functions. These functions are quasi-probability distribution functions in phase-space. See [15], [29]. According to [29] Wigner's functions have been useful in describing transport in quantum optics; nuclear physics; and quantum computing, decoherence and chaos. We present here some particular cases.

Example 3. Let us consider

$$f_n(x, p) = \frac{(-1)^n}{\pi \cdot h} e^{-2H/h} \cdot L_n(4H/h),$$

where $L_n = \frac{1}{n!} e^z \partial_z^n (e^{-z} z^n)$ are Laguerre polynomials, for $n = 0, 1, 2$, so that $L_0 = 1$, $L_1 = 1 - \frac{4H}{h}$, $L_2 = \frac{8H^2}{h^2} - \frac{8H}{h} + 1$. When we suppose $H = \frac{p^2 + x^2}{2}$ and $h = 1$ the Wigner's functions are:

- (i) $f_0(x, p) = (2p^2 + 2x^2)/\pi \cdot e^{(-p^2 - x^2)}$;
- (ii) $f_1(x, p) = (-2p^2 - 2x^2 + 4p^4 + 8x^2p^2 + 4x^4)/\pi \cdot e^{(-p^2 - x^2)}$;
- (iii) $f_2(x, p) = (4p^6 + 12x^2p^4 + 12x^4p^2 - 8p^4 - 16x^2p^2 + 2p^2 + 4x^6 - 8x^4 + 2x^2)/\pi \cdot e^{(-p^2 - x^2)}$.

These function are Morse Bott functions. All these functions have the origin as non degenerate critical point. Also, they have one, two and three non degenerate critical sub manifolds homeomorphic to circles, respectively. See Figures 3, 4 and 5.

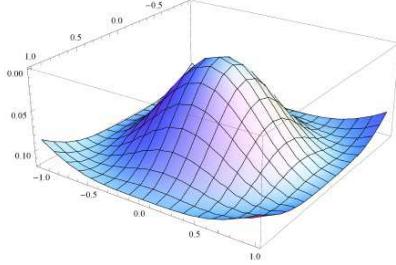


FIGURE 3. The function $f_0(x, p)$.

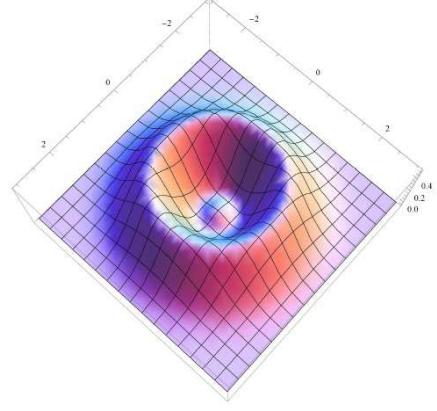
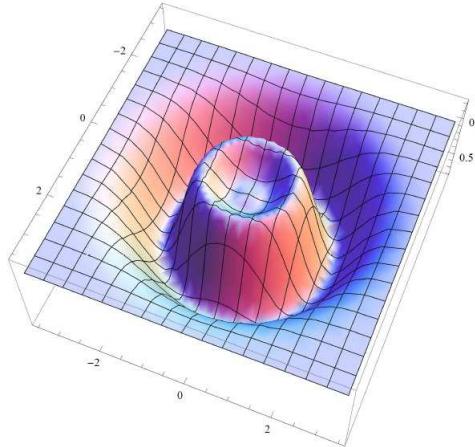


FIGURE 4. The function $f_1(x, p)$.

5. REALIZATION THEOREM

In this section we describe sufficient and necessary conditions for a connected oriented and finite graph G to be associated with a Morse Bott function defined on an orientable closed surface Σ .

A *walk* in a graph is an alternating sequence of vertices and edges, $v_0, e_1, v_1, e_2, \dots, x_n, v_n$ in which each edge is either $e_i = (v_{i-1}, v_i)$ or $e_i = (v_i, v_{i-1})$. A walk is a *path* if its vertices (and thus necessarily all the edges) are distinct. A walk is a *cycle* if $v_0 = v_n$ and its edges are distinct. The first vertex of the first edge of a path is the *origin* and the second vertex of the last edge is the *final vertex*. Both origin and final vertex are called *endpoints* of the path. A graph is *connected* if every pair of vertices are joined by a walk. An oriented cycle is a cycle with all the edges being oriented in the same direction. If G is a connected graph, then the *cycle rank* is $m(G) = \#E - \#V + 1$.

FIGURE 5. The function $f_2(x, p)$.

Let be a f Morse function, we will say that \mathcal{R}_f is canonical (see [8]) if:

- it has exactly a maximal and a minimal vertex.
- the cycles, if any, have length two.

A *path graph* or *linear graph* is a connected simple graph that contains only vertices of degree 2 and 1.

Let Σ be an orientable closed surface and f be a \mathcal{MB} function defined on it. Then, the Reeb graph \mathcal{R}_f associated to f is a finite graph, G , with labels at the vertices. Suppose that G has vertices of degree 1, 2 or 3. Each vertex of degree 1 is associated with the basin of a maximum or minimum value of f . Each vertex of degree 2 is associated with a neighborhood of a singular circle (a cylinder) and each vertex of degree 3, which is not a endpoint, with a neighborhood of a saddle point. Moreover, we endow each edge of the graph with a f -orientation given by the direction in which the function f increases (see [28]). Thus, each vertex of a saddle point has one (respectively two) incoming edge and two (respectively one) outgoing edges and a vertex of a singular circle has two incoming edges or two outgoing edges.

5.1. Prove of Theorem 2. We prove here Theorem 2 that gives necessary and sufficient conditions for a connected oriented and finite graph G to be associated to Morse Bott function on an orientable closed surface with values in \mathbb{R} . To prove this result, we recall the following theorem.

Theorem 23 ([17]). *Let G be a finite graph without loops. Then, there exists a smooth function $f : \Sigma \rightarrow \mathbb{R}$ on a closed surface Σ with finitely many critical values such that its Reeb graph \mathcal{R}_f is homeomorphic to G .*

From Theorem 23, any finite graph satisfying the conditions (b) and (c) of Theorem 2 can be realized as the Reeb graph of a smooth function with finitely many critical values on a compact surface without boundary.

The proof of this theorem is constructive, Masumoto and Saeki [17] showed how to construct a smooth function f by means of conditions (a)-(c) of Theorem 2. At first, they construct a continuous function f_1 from G to \mathbb{R} that is an embedding on each edge. Such a function can be constructed considering first any injective map $V(G) \rightarrow \mathbb{R}$ and then by extending it to the edges so that it is linear on each edge, where $V(G)$ denote the set of vertices of G . Then, for each vertex $v \in V(G)$, its neighborhood satisfies one of the conditions in (a) of Theorem 2 and for each cases in (a) Masumoto and Saeki construct a smooth function $g_v : N_v \rightarrow \mathbb{R}$ on a compact surface with boundary where N_v is a neighborhood of v .

Finally, it is performed a gluing operation of the smooth function g_v for each vertex v of G in order to get the smooth function f from Σ to \mathbb{R} , such that $f|_{N_v} = g_v$ for each vertex v of G and f_1 can be identified with the function $\bar{f} : \mathcal{R}_f \rightarrow \mathbb{R}$.

Proof. (Necessity) Let \mathcal{R}_f be the Reeb graph associated to a \mathcal{MB} function f on Σ . Condition (a) of Theorem 2 is a consequence of the type of singularities in a \mathcal{MB} function and from the definition of the Reeb graph. Condition (b-1) is necessary since the surface can be contracted to the Reeb graph. Moreover, since f is a monotonic function, we can provide \mathcal{R}_f with an orientation given by the sense of growth of f . Then f determines an order on the vertices of \mathcal{R}_f by the following relation: $v_1 < v_2$ if $f(v_1) < f(v_2)$. Then the conditions (b-2) and (c) are satisfied.

(Sufficiency) We must guarantee the existence of a \mathcal{MB} function $f : \Sigma \rightarrow \mathbb{R}$ such that the Reeb graph associated to f is isomorphic to the abstract graph G satisfying the conditions (a)-(c) of Theorem 2.

We begin by enumerating the vertices of G , v_1, \dots, v_n in such a way that this order will be consistent with the order defined in G . Consider a neighborhood N_i for the vertex v_i in G that contains v_i but not any other vertex of G . According to the degree of v_i we associate a \mathcal{MB} function to N_i for $i = 1, \dots, n$. These functions are constructed using the same arguments and ideas by Matsumoto and Saeki (see more details in the proof of the Theorem 23 in [17]). \square

Given an ordered Reeb graph associated to a Morse function f on a fixed surface Σ it is possible to transform the graph into another one corresponding to another Morse function g on Σ by means of a set of elementary deformations. These deformations are listed in [8] and we call them *Fabio-Landi transformations*. To obtain a similar result in the case of Morse-Bott functions a new elementary deformation must be added:

Definition 24. Consider a Reeb graph, $\mathcal{R}_1^\eta(f)$, a saddle vertex v with three adjacent edges u_1, u_2, u_3 where u_3 connects v with a final center vertex c .

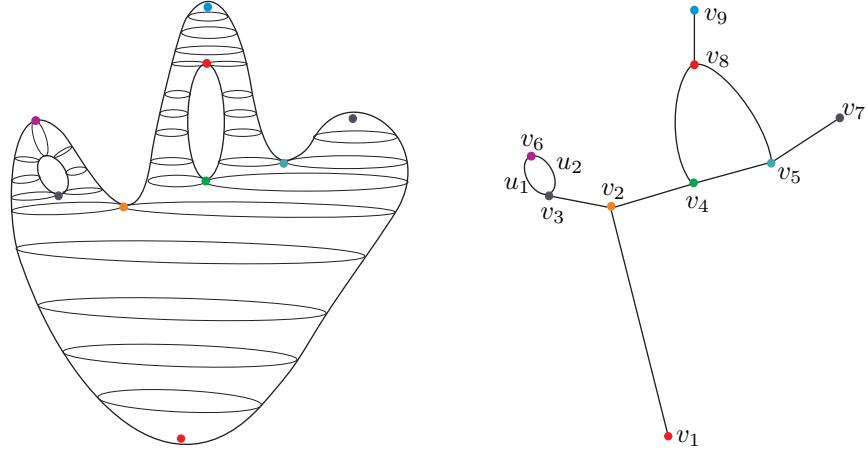


FIGURE 6. On the left a Morse Bott foliation on the 2-torus; on the right the associated Reeb graph.

A direct MB elementary deformation applied to $\mathcal{R}_1^\eta(f)$ yields another Reeb graph without the edge u_3 and the center c . The vertex v is now associated with a singular circle.

Definition 25. Given a Reeb graph $\mathcal{R}_1^\eta(f)$, f a Morse Bott function, v a vertex associated with a singular circle, with u_1, u_2 adjacent edges, an inverse MB elementary deformation on a $\mathcal{R}_1^\eta(f)$ consist of adding a new adjacent edge to v connecting v with a new vertex c associated to a center singular point. Moreover, $\min(f(x), x \in u_1 \cup u_2) < f(c) < \max(f(x), x \in u_1 \cup u_2)$.

In the Figure 6 we consider a Morse Bott foliation on the 2-torus associated to a particular \mathcal{MB} function g . In the Reeb graph $\mathcal{R}_1^\eta(g)$ of g , the vertex v_6 is associated with a singular circle of g with u_1, u_2 adjacent edges. Applying an inverse MB elementary deformation on $\mathcal{R}_1^\eta(g)$, we add a new adjacent edge to v_6 connecting v_6 with a new vertex c associated to a center singular point. Then the Reeb graph obtained is associated to the foliation of the height function in the Figure 1.

Proposition 26. Every $\mathcal{R}^\eta(f)$ can be transformed into a canonical one through a finite sequence of elementary deformations.

Proof. An inverse MB elementary deformation does not change the graph genus or cycles. A sequence of inverse MB elementary deformations transforms a Reeb graph associated with a \mathcal{MB} function in a graph associated with a Morse function in the same surface. By applying Fabio-Landi transformations it is possible to transform the graph into a canonical graph. \square

A realization theorem based in transformations of the graph is important to define a topology in the set of Reeb graphs of Morse Bott functions (see

[5, 8]). When Reeb graphs are used to recognize images, to have stability of the graph against small perturbations is a very convenient property.

6. STABILITY OF MORSE-BOTT FUNCTIONS ON Σ

Let $C^\infty(\Sigma)$ be the set of all smooth functions from Σ to \mathbb{R} and endowed the Whitney topology. Although the set of the Morse Bott functions on Σ is a dense set of $C^\infty(\Sigma)$ (as it contains the set of the Morse function defined in the same space) it is not an open set.

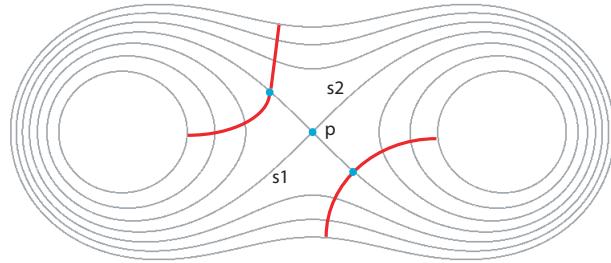
In this section we define a new topology on the set of all smooth functions defined on Σ to \mathbb{R} in order to obtain a stability result for some \mathcal{MB} functions .

Given a \mathcal{MB} function f , each regular point p and each point in a critical circumference is contained in a fiber of f homeomorphic to S^1 that will be denoted $o(p)$ and called a *circled component*. When p is a saddle critical point it is contained in 2 circles $o_1(p), o_2(p)$.

A path or a cycle \mathbf{c}_f in the Reeb graph of f can be embedded in the surface and represented by this embedding $i(\mathbf{c}_f)$. We will say that $i(\mathbf{c}_f)$ is properly embedded if:

- (i) The representative point in the surface is not a singular point if p is not a center nor a point in a critical circumference
- (ii) The embedding is smooth and cuts transversally one-dimensional fibers.

The set of all properly embeddings $i(\mathbf{c}_f)$ will denoted by \mathcal{C}_f . Thus a component \mathbf{c}_f of \mathcal{C}_f is either a line or a circle.



The restriction of f to an $i(\mathbf{c}_f)$ defines a new function $f_{i(\mathbf{c}_f)}$

Theorem 27. *Given a \mathcal{MB} function f , $f_{i(\mathbf{c}_f)}$ is a Morse function.*

Proof. The singularities of $f_{i(\mathbf{c}_f)}$ on $i(\mathbf{c}_f)$ are local non degenerate maximums or minimums since they correspond to centers or circle singularities. \square

Given two smooth functions on a compact surface $f, g : \Sigma \rightarrow \mathcal{W}$ and a point $p \in \Sigma$ define the t^0 distance between f and g at p as

$$t^0(f, g)(p) = |f(p) - g(p)|$$

The t^r distance at p is given by

$$t^r(f, g)(p) = \max(|d^r f(q) - d^r g(r)|, q \in I_{f(p)}(f), r \in I_{g(p)}(g))$$

The T^r distance between f and g at p is

$$\max(t^0(f, g)(p), t^1(f, g)(p), \dots, t^r(f, g)(p))$$

Finally we define the T^r distance between f and g as

$$\max(T^r(f, g)(p), p \in \Sigma(g, 0))$$

Theorem 28. *In the space of smooth functions from a compact orientable surface Σ to \mathbb{R} such that the Reeb Graph is a path graph, the set of \mathcal{MB} functions is open with the topology induced by the T^r , $r \geq 2$ distance.*

Proof. Since the T^r topology is more restrictive than the Whitney topology, saddle and center singularities are preserved in a neighborhood of a \mathcal{MB} function .

Given a \mathfrak{c}_f that do not intersect saddle levels of f , it can be considered as the quotient space under the level equivalence. On \mathfrak{c}_f the quotient topology of the T^r topology is the Whitney topology. As $f_{\mathfrak{c}_f}$ is a Morse function, and Morse functions are generic, from the preservation of critical points of $f_{\mathfrak{c}_f}$ for close enough functions, derives the preservation of critical circles of f .

Since the number of singularities is finite, the last two properties imply the Theorem. \square

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REFERENCES

1. V. I. Arnold, *Singularities of smooth mappings.*, Uspehi Mat. Nauk **23** (1968), 1–44.
2. V. I. Arnold, *Topological classification of Morse functions and generalizations of Hilbert’s 16-th problem*, Math. Phys. Anal. Geom. **10** (2007), 227–236.
3. V.I. Arnold, V.V. Goryunov, O.V. Lyashko, and V.A. Vasilév, *Singularity Theory I*, Encyclopedia of Mathematical Sciences, vol 6, 1998.

4. A. Banyaga and D. Hurtubise, *A proof of the Morse-Bott Lemma*, *Expo. Math.* **22** (2004), 365–373.
5. U. Bauer, Xiaoyin Ge, and Yusu Wang, *Measuring distance between Reeb graphs*, arXiv:1307.2839 (2013), 1–21.
6. R. Bott, *Nondegenerate critical manifolds*, *Ann. of Math.(2)* **60** (1954), 248–261.
7. Alberto Candel and Lawrence Conlon, *Foliations II*, American Mathematical Society, 2003.
8. Barbara Di Fabio and Claudia Landi, *The Edit Distance for Reeb Graphs*, arXiv:1411.1544 (2014), 1–30.
9. P. Giblin, *Graphs, Surfaces and Homology*, Cambridge University Press, New York, 2010.
10. F. Harary, *Graphs Theory*, Perseus Books, 1994.
11. M. W. Hirsch, *Differential Topology*, Springer-Verlag, New York, 1994.
12. M. Kaluba, W. Marzantowicz, and N. Silva, *On representation of the Reeb graph as a sub-complex of manifold.*, arXiv:1405.4597v1 [math. GT] (2014), 1–18.
13. G. Khimshiashvili and D. Siersma, *Remarks on minimal round funtions*, *Banach Center Publications* **62** (2004), 159–172.
14. E. V. Kulinich, *On topological equivalence of Morse functions on surfaces.*, *Methods Functional Anal. Topology* **1** (1998), 22–28.
15. M. A. Marchioli, *Mecânica Quântica no Espaço de fase: I. Formulação de Weyl-Wigner*, *Revista Brasileira de Ensino de Física* **24** (2002), 421–436.
16. J. Martinez-Alfaro, I. S. Meza-Sarmiento, and R. D. S. Oliveira, *Singular levels and topological invariants of Morse Bott systems on surfaces*, *Notas da Série Matemática do ICMC-USP.* **394** (2014), 1–23, Available in http://www.icmc.usp.br/CMS/Arquivos/arquivos_enviados/ESTAGIO-BIBLIO_171_Serie%20Mat%20394.pdf.
17. Y. Masumoto and O. Saeki, *A smooth function on a manifold with given Reeb graph*, *Kyushu J. Math.* **65** (2011), 75–84.
18. Y. Matsumoto, *An introduction to Morse theory*, American Mathematical Society, 2002.
19. L. I. Nicolaescu, *An invitation to Morse theory*, New York : Springer, London., 2007.
20. _____, *Counting Morse functions on the 2-sphere.*, *Compositio Math.* **144** (2008), 1081–1108.
21. A. O Prishlyak, *Topological equivalence of smooth functions with isolated critical points on a closed surface*, *Topology and its Applications* **119** (2002), 257–267.
22. G. Reeb, *Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique*, *C. R Acad. Sci. Paris* **222** (1946), 847–849.
23. O. Saeki, *Topology of Singular Fibers of Differentiable Maps*, Springer, 2004.

24. B. Scárdua and J. Seade, *Codimension one foliations with Bott-Morse singularities I*, J. Differential Geometry **83** (2009), 189–212.
25. _____, *Codimension 1 foliations with Bott-Morse singularities II*, J. Topology **4** (2011), 343–382.
26. V. V. Sharko, *On topological equivalence Morse functions on surfaces*, Internat. Conference at Chelyabinsk State Univ., Low-Dimensional Topology and Combinatorial Group Theory (1996), 19–23.
27. _____, *Smooth and topological equivalence of functions on surfaces*, Ukrainian Mathematical Journal **55** (2003), 832–846.
28. V. V. Sharko, *About Krondrod-Reeb Graph of a Function on a Manifold*, Methods of Functional Analysis and Topology **12** (2006), 389–396.
29. C. K. Zachos, D. B. Fairlie, and T. L. Curtright, *Quantum Mechanics in phase space. An Overview with Selected Papers*, World Scientific Series in 20th Century Physics **34** (2012), 1–55.

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