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Robust Regulator for Markov Jump Linear Systems with Random State Delays and Uncertain Transition Probabilities ⋆

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Abstract: The recursive robust regulation problem for Markov jump linear systems with state delay is investigated. The random delay belongs to a known interval and its maximum variation rate is considered. The transition probabilities matrix is subject to polytopic uncertainties. We model the delay by a Markov chain and obtain a delay-free Markovian system, through the augmented system approach. Based on combination of regularized least-squares with polytopic uncertainties and penalty function method, we propose a mode-dependent state-feedback control. The solution is given in terms of coupled Riccati equations presented in a symmetric matrix arrangement. With a numerical example, we compare our approach with other robust controllers from the literature.

Keywords: Markov jump linear system, state time-delay, discrete-time systems, recursive robust regulator, Riccati equations

1. INTRODUCTION

Dynamic systems that undergo sudden changes in their dynamics, due to sensor failures, packet losses, environmental changes and so forth, are often described by a set of linear sub-systems constantly changing (or jumping) from one model to another. In this case, it is possible to describe these jumps in terms of transitions probabilities and have an underlying Markov chain orchestrating the overall system. Markov jump linear systems (MJLS for short) have been subjects of intense research efforts and encompasses numerous real-world applications, such as fault-tolerant aircraft control, communication systems, and robotic manipulators (see, for instance, Costa et al. (2005), Fu and Li (2016)).

In this paper, we are interested in the class of discrete-time Markov jump linear systems with delay (MJLSD). The presence of time-delay makes the control problem more challenging, since they might induce undesirable behaviors such as degenerated performance and instability (Fridman, 2014). Recently, many theoretical results have been reported, such as robust and H_∞ control approaches (see Kang et al. (2013), Luan et al. (2014), Yang et al. (2014) and references therein). However, these results are investigated for the delayed Markovian systems with complete knowledge of transition probabilities, which is a costly process in practical situations (Zhang et al., 2008). In this way, investigating general Markovian jump processes with uncertain transition probabilities and state time-delay is significant. Particularly, Dzung and Hien (2017) proposed a mode-dependent state feedback controller for MJLS with mode-dependent delay. Li et al. (2021) treated the stochastic stabilization of MJLS with time-varying delay and two mutually uncorrelated homogeneous Markov processes. These approaches are based upon optimization procedures with linear matrix inequalities (LMI) constraints. To the best of our knowledge, robust approaches for MJLSD with uncertain transition probabilities are scarce in the specialized literature.

That said, we propose a recursive robust regulator for MJLSD with polytopic uncertain transition probabilities. We model the occurrence of delays by means of an additional independent Markov chain and the maximum delay variation rate by the probability matrix. Next, we define an augmented system where the exact knowledge of time-delays is no longer required. It serves as basis for us to formulate a min-max problem combining regularized least-squares with polytopic data and penalty functions. The solution for this optimization problem recursively yields the mode-dependent state-feedback gains capable of stabilizing the closed-loop MJLS despite the random state delays and polytopic uncertain probabilities. It is worth emphasizing that, (Bueno et al., 2022) considered the simpler case without delays. In contrast, we propose a robust solution for DMJLS subject to uncertain probabilities and random state delays.

We organize this paper as follows: in Section 2 we describe the MJLS with random delays, its augmented form, and formulate the robust control problem; in Section 3 we

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provide some useful preliminary results, which are the starting point of our solution; in Section 4 we present the main results; a numerical example is provided in Section 5 to assess the performance of our solution; and in Section 6 we close the paper with concluding remarks.

Notations: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the $n-$dimensional Euclidean space and $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices. I_n is the identity matrix of dimensions $n\times n$. The transpose of the matrix A is represented by A^T. The matrix $\mathbf{diag}(A,B)$ is a block-diagonal matrix constituted by A,B. $\mathbb{E}\{x\}$ is the expected value of a random variable x. The probability of a event occurring is denoted by Pr. The Kronecker product operator is defined by $\otimes$. Let P be a real matrix, then $P\succ 0$ ($P\succeq 0$) represents a symmetric positive (semi)definite matrix. The weighted squared Euclidean norm of x is denoted by $\|x\|_P^2 = x^T P x$. Let $x\in\mathbb{R}^n$ and $e\in\mathbb{R}$, then we use $x^T P(\bullet)$ and $\sqrt{e}P(\bullet)$ to denote $x^T P x$ and $\sqrt{e}P\sqrt{e}$, respectively, whenever convenient. The superscript $\circ 1/2$ denotes the element-wise square root of a matrix. Let $\mathbf{1}_s$ represent the vector $[1\ \ldots\ 1]^T\in\mathbb{R}^s$.

2. PROBLEM FORMULATION

2.1 System Model

Consider the dynamical model of a discrete-time MJLSD described by

$$\begin{aligned} x_{k+1} &= A_{\theta_k,k}x_k + A_{d,\theta_k,k}x_{k-d_k} + B_{\theta_k,k}u_k,\\ x_k &= \varphi_0(k),\ k\in[-d_{max},0], \end{aligned} \tag{1}$$

for $k=0,1,\ldots,N$, with state vector $x_k\in\mathbb{R}^n$, state delayed $x_{k-d_k}\in\mathbb{R}^n$, control input $u_k\in\mathbb{R}^m$, and initial condition $\varphi_0(k)$ for $k=-d_{max},-d_{max}+1,\ldots,0$. $A_{\theta_k,k}$, $A_{d,\theta_k,k}\in\mathbb{R}^{n\times n}$ and $B_{\theta_k,k}\in\mathbb{R}^{n\times m}$ are known nominal parameter matrices.

The stochastic process $\{\theta_k, k\ge 0\}$ is described by a discrete-time Markov chain $\theta_k\in\mathcal{S}=\{1,\ldots,s\}$, with the uncertain transition probability matrix $\mathbb{P}\in\mathbb{R}^{s\times s}$, $i,j\in\mathcal{S}$, defined by $\mathbb{P}=\mathbb{P}_0+\delta\mathbb{P}$, satisfying the following conditions

$$\begin{aligned} \mathbb{P} &= \Pr(\theta_{k+1}=j\,|\,\theta_k=i)=[p_{ij}^{(0)}+\delta p_{ij}],\\ &\sum_{j=1}^{s}(p_{ij}^{(0)}+\delta p_{ij})=1,\ \ 0\le p_{ij}^{(0)}+\delta p_{ij}\le 1,\\ &\text{and}\ \ \Pr(\theta_0=i)=\pi_i+\delta\pi_i, \end{aligned} \tag{2}$$

where the uncertainty $\delta\mathbb{P}\in\mathbb{R}^{s\times s}$ belongs to the polyhedral domain

$$\mathbb{V}_k := \left\{\delta\mathbb{P}=\sum_{\ell=1}^{\nu}\alpha_{\ell,k}\mathbb{P}^{(\ell)}=\sum_{\ell=1}^{\nu}\alpha_{\ell,k}p_{ij}^{(\ell)}=\delta p_{ij}\right\}, \tag{3}$$

with $\mathbb{P}^{(\ell)}\in\mathbb{R}^{s\times s}$ being the known polytope vertices, $p_{ij}^{(\ell)}$ the (i,j) element of $\mathbb{P}^{(\ell)}$, and $\alpha_k := [\alpha_{1,k}\ldots\alpha_{\nu,k}]^T$ are unknown coefficients belonging to the unitary simplex

$$\Lambda_\nu := \left\{\alpha_k\in\mathbb{R}^\nu:\ \sum_{\ell=1}^{\nu}\alpha_{\ell,k}=1,\ \alpha_{\ell,k}\ge 0\right\}. \tag{4}$$

We consider the random time-delay d_k to be described by a second Markov chain $\bar\theta_k\in\bar{\mathcal{S}}=\{1,\ldots,\bar s\}$, with $\bar s = d_{max}-d_{min}+1$, and has upper and lower bounds

$$0\le d_{min}\le d_k\le d_{max}\ \ \text{and}\ \ \|d_{k+1}-d_k\|\le\triangle d\le d_{max},$$

where $d_{min}=\min\{d_k\}$, $d_{max}=\max\{d_k\}$ and $\triangle d$ is the maximum delay variation rate between two consecutive instants, in which the future sample d_{k+1} can assume any values belonging to the interval $[\bar d_{min},\ \bar d_{max}]$, with $\bar d_{min}=max(d_{min},d_k-\triangle d)$, $\bar d_{max}=min(d_{max},d_k+\triangle d)$.

The Markov chain $\bar\theta_k$ is associated with a known transition probability matrix $\bar{\mathbb{P}}=\bar p_{\bar i\bar j}\in\mathbb{R}^{\bar s\times\bar s}$, $\bar i,\bar j\in\bar{\mathcal{S}}$, with

$$\begin{aligned} \bar{\mathbb{P}} &= \Pr(\bar\theta_{k+1}=\bar j\,|\,\bar\theta_k=\bar i)=\bar p_{\bar i\bar j},\\ \Pr(\bar\theta_0=\bar i) &= \bar\pi_{\bar i},\ \sum_{\bar j=1}^{s}\bar p_{\bar i\bar j}=1,\ \ 0\le\bar p_{\bar i\bar j}\le 1, \end{aligned} \tag{5}$$

where, the transition probability matrix $\bar{\mathbb{P}}$ is defined according to the maximum delay variation rate,

$$\bar p_{\bar i\bar j}=\begin{cases}1/\bar s & \begin{cases}\bar j\in(\bar d_{min}+1,\ \bar d_{max}+1), & d_{min}=0,\\ \bar j\in(\bar d_{min},\ \bar d_{max}), & d_{min}\ne 0,\end{cases}\\ 0, & \text{otherwise},\end{cases} \tag{6}$$

In order to regulate the MJLSD (1) under transition probabilities subject to polytopic uncertainties, we adopt the following mode-dependent state-feedback control law,

$$u_k=\mathcal{K}_{\bar\theta_k,\theta_k,k}z_k,\ \forall k\ge 0, \tag{7}$$

with augmented state z_k, initial condition z_0, namely

$$\begin{aligned} z_k &:= [x_k\ x_{k-1}\ \ldots\ x_{k-d_{max}}]^T,\\ z_0 &:= [\varphi_0(0)\ \varphi_0(-1)\ \ldots\ \varphi_0(-d_{max})]^T, \end{aligned} \tag{8}$$

and the feedback-gain $\mathcal{K}_{\bar\theta_k,\theta_k,k}=[K_{1,\theta_k,k}\ K_{2,\theta_k,k}\ \ldots\ K_{\bar s,\theta_k,k}]\in\mathbb{R}^{m\times n_d}$, where $n_d=d_{max}n+n$. Observe that the control law (7) takes into account mode-dependent and delay-independent state-feedback control. Therefore, as the proposed controller dismisses the real-time knowledge of time-delay d_k, we will transform the MJLSD (1) into a correspondent delay-independent system, such that the proposed control law guarantees closed-loop stability. In the following section we describe this procedure.

2.2 AUGMENTED MODEL DESCRIPTION

Based on the augmented system approach (Hetel et al., 2008) and the time-delay exhibiting Markovian behavior (Qiu et al., 2015), we can rewrite the System (1) as an augmented delay-free MJLS in terms of the augmented state (8) and governed by two mutually independent Markov chains, such that

$$z_{k+1}=F_{\bar\theta_k,\theta_k,k}z_k+G_{\bar\theta_k,\theta_k,k}u_k, \tag{9}$$

for each $k=0,1,\ldots,N$, where the augmented matrices $F_{\bar\theta_k,\theta_k,k}\in\mathbb{R}^{n_d\times n_d}$ and $G_{\bar\theta_k,\theta_k,k}\in\mathbb{R}^{n_d\times m}$, $\forall\theta_k\in\mathcal{S}$, $\bar\theta_k\in\bar{\mathcal{S}}$ and $\ell=1,\ldots,\nu$ are defined by

$$\begin{aligned} F_{\bar\theta_k,\theta_k,k} &:= \left[\begin{array}{cccc|c} A_{\theta_k,k} & \overbrace{0\ \cdots\ 0}^{\bar\theta_k+d_{min}-2} & A_{d,\theta_k,k} & \cdots & 0\\ \hline & & I_{d_{max}n} & & 0_{d_{max}n\times n}\end{array}\right],\\ G_{\bar\theta_k,\theta_k,k} &:= \begin{bmatrix} B_{\theta_k,k}\\ 0_{d_{max}n\times m}\end{bmatrix}, \end{aligned} \tag{10}$$

where $\bar\theta_k+d_{min}-2$ represents the number of null matrices $0_{n\times n}$ between $A_{\theta_k,k}$ and $A_{d,\theta_k,k}$. If $\bar\theta_k+d_{min}-2=1$, then the first column of $F_{\bar\theta_k,\theta_k,k}$ is composed of the element $A_{\theta_k,k}+A_{d,\theta_k,k}$. In Qiu et al. (2015), in order to regulate the MJLS with Markov time-delay, two Markov processes with uncertain jump variables are proposed.

2.3 ROBUST CONTROL PROBLEM

Our main goal is to design the optimal state-feedback control sequence $\mathcal{U}^* = \{u_k^*\}_{k=0}^N$ to guarantee the closed-loop stability of the MJLSD (1) with transition probabilities subject to polytopic uncertainties. To fulfill this objective, we apply the lifting method which allows us to develop a robust recursive regulator for MJLSD (1) in terms of the equivalent Markovian system (9) governed by two independent Markov chains.

Thus, we consider the following min-max constrained optimization problem to minimize the expected value of quadratic cost function J_N under the maximum influence of the polytopic uncertainty $\delta\mathbb{P}$, i.e., for $k = 0, 1, \ldots, N$,

$$\begin{aligned} &\min_{u_k,\, z_{k+1}} \max_{\delta\mathbb{P}} \mathbb{E}\Big\{ J_N \,\Big|\, \mathcal{O}_0 \Big\} \\ &\text{subject to } z_{k+1} = F_{\bar{\theta}_k,\theta_k,k} z_k + G_{\bar{\theta}_k,\theta_k,k} u_k, \end{aligned} \tag{11}$$

with

$$J_N = z_N^T \mathcal{P}_{\bar{\theta}_N,\theta_N,N}\, z_N + \sum_{k=0}^{N-1} (z_k^T \mathcal{Q}_{\bar{\theta}_k,\theta_k,k}\, z_k + u_k^T \mathcal{R}_{\bar{\theta}_k,\theta_k,k}\, u_k), \tag{12}$$

where $\mathcal{O}_0 := \{\bar{\theta}_0, \theta_0, z_0, x_0\}$, and $\mathcal{P}_{\bar{\theta}_N,\theta_N,N} \succ 0$, $\mathcal{Q}_{\bar{\theta}_k,\theta_k,k} \succ 0$ and $\mathcal{R}_{\bar{\theta}_k,\theta_k,k} \succ 0$ are the weighting matrices. Notice that the solution of problem (11)-(12) yields the Polytopic Robust Regulator, which differs from that found in Bueno et al. (2022), since here we are dealing with a Markovian system with two independent Markov chains and random state delay as well.

3. REGULARIZED LEAST-SQUARES WITH POLYTOPIC UNCERTAINTIES

In this section, we present an approach that sits at the foundation of the regulation algorithm proposed in this paper. It consists of solving an adapted version of the regularized least-squares problem in order to accommodate uncertainties with polytopic structure, as is presented in the following lemma.

Lemma 1. Consider the regularized least-squares problem with polytopic uncertainty

$$\min_{x} \max_{\delta\mathcal{A},\delta b} \Big(\mathcal{F}(x) = \|x\|_{\mathscr{Q}}^2 + \|(\mathcal{A}_0 + \delta\mathcal{A})\, x - (b_0 + \delta b)\|_{\mathscr{W}}^2 \Big), \tag{13}$$

where $x \in \mathbb{R}^a$ is an unknown vector, $\mathcal{A}_0 \in \mathbb{R}^{a\times b}$ and $b_0 \in \mathbb{R}^a$ are known parametric matrices, $\mathscr{Q} \succ 0$ and $\mathscr{W} \succ 0$ are weighting matrices. The parametric uncertainties $\{\delta\mathcal{A}, \delta b\}$ are bounded by the convex polyhedron

$$[\delta\mathcal{A}\ \ \delta b] = \mathscr{H}\boldsymbol{\Phi}\left[\bar{\mathcal{A}}\ \ \bar{b}\right], \quad \alpha \in \Lambda_\nu, \tag{14}$$

in which $\mathscr{H} \in \mathbb{R}^{a\times p\nu}$ is a known matrix, $\boldsymbol{\Phi} = \mathbf{diag}(\alpha_1, \ldots, \alpha_\nu) \otimes I_p$ is a coefficient matrix, such that $\alpha \in \Lambda_\nu$, and $\mathcal{A}^{(\ell)} \in \mathbb{R}^{p\times b}$, $b^{(\ell)} \in \mathbb{R}^p$ are the known vertices of the polytope, with $\ell = 1, \ldots, \nu$, such that $\bar{\mathcal{A}} = [\bar{\mathcal{A}}^{(1)} \ldots \bar{\mathcal{A}}^{(\nu)}]^T$, and $\bar{b} = [\bar{b}^{(1)} \ldots \bar{b}^{(\nu)}]^T$.

Then, the solution x^* and minimum value $\mathcal{F}(x^*)$ to problem (13)-(14) are given by

$$\begin{bmatrix} x^* \\ \mathcal{F}(x^*) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b_0 \\ 0 & \bar{b} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} \mathscr{Q}^{-1} & 0 & 0 & I \\ 0 & \mathscr{W}(\hat{\lambda})^{-1} & 0 & \mathcal{A}_0 \\ 0 & 0 & \hat{\lambda}^{-1} I & \bar{\mathcal{A}} \\ I & \mathcal{A}_0^T & \bar{\mathcal{A}}^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ b_0 \\ \bar{b} \\ 0 \end{bmatrix}, \tag{15}$$

with a non-negative scalar parameter $\hat{\lambda}$ obtained by the following optimization problem

$$\hat{\lambda} = \arg \min_{\lambda > \|\mathscr{H}^T \mathscr{W} \mathscr{H}\|} \Upsilon(\lambda), \tag{16}$$

where $\Upsilon(\lambda) := \|x(\lambda)\|_{\mathscr{Q}}^2 + \|\mathcal{A}_0 x(\lambda) - b_0\|_{\mathscr{W}(\lambda)}^2 + \lambda\|\bar{\mathcal{A}}x(\lambda) - \bar{b}\|$,

$$\begin{aligned} x(\lambda) &:= (\mathscr{Q}(\lambda) + \mathcal{A}_0^T \mathscr{W}(\lambda)\mathcal{A}_0)^T(\mathcal{A}_0^T \mathscr{W}(\lambda) b_0 + \lambda\bar{\mathcal{A}}^T \bar{b}), \\ \mathscr{Q}(\lambda) &:= \mathscr{Q} + \lambda\bar{\mathcal{A}}^T \mathcal{A}, \\ \mathscr{W}(\lambda) &:= (\mathscr{W}^{-1} - \lambda^{-1}\mathscr{H}\mathscr{H}^T)^{-1}. \end{aligned}$$

Proof. It follows from solution obtained in Cerri and Terra (2017), which is equivalently rewritten to address polytopic uncertainties (see Bueno et al. (2022) for further details). According to (Nikoukhah et al., 1992, Lemma 2.1), we have the positiveness of the central block matrix in (15).

Remark 1. The solution to the regularized least-squares problem with polytopic uncertainty, outlined in Lemma 1, depends on the solution of an auxiliary optimization problem (16) in order to obtain the optimal $\hat{\lambda}$ parameter. However, as discussed in Sayed (2001), we can adopt the practical approximation $\hat{\lambda} = \beta\|\mathscr{H}^T \mathscr{W} \mathscr{H}\|$, for some $\beta > 1$, to avoid the explicit minimization of $\Upsilon(\lambda)$.

4. POLYTOPIC ROBUST REGULATOR

The first step to solve the optimization problem (11)-(12) is to transform it into N one-step min-max optimization problems through the application of the dynamic programming. Thus, we yield

$$\begin{aligned} &\min_{u_k, z_{k+1}} \max_{\delta\mathbb{P}} \mathbb{E}\Big\{ J_k \Big| \mathcal{O}_k \Big\} \\ &\text{subject to the MJLS (9)} \end{aligned} \tag{17}$$

with the expected value of one-step quadratic cost function

$$\mathbb{E}\Big\{ J_k \Big| \mathcal{O}_k \Big\} = z_{k+1}^T \Psi_{\bar{\theta}_k,\theta_k,k+1} z_{k+1} + z_k^T \mathcal{Q}_{\bar{\theta}_k,\theta_k,k} z_k + u_k^T \mathcal{R}_{\bar{\theta}_k,\theta_k,k} u_k,$$

in which $\mathcal{O}_k := \{\bar{\theta}_k, \theta_k, z_k, x_k\}$, and $\Psi_{\bar{\theta}_k,\theta_k,k+1}$ is the operator defined by

$$\begin{aligned} \Psi_{\bar{\theta}_k,\theta_k,k+1} &= \mathbb{E}\{\mathcal{P}_{\bar{\theta}_k,\theta_k,k+1} | \mathcal{O}_k\} \\ &= \sum_{\bar{j}=1}^{\bar{s}} \sum_{j=1}^{s} \bar{p}_{\bar{i}\bar{j}}\, (p_{ij}^{(0)} + \delta p_{ij})\mathcal{P}_{\bar{j},j,k+1} \\ &= \Psi_{\bar{\theta}_k,\theta_k,k+1}^{(0)} + \delta\Psi_{\bar{\theta}_k,\theta_k,k+1}, \end{aligned}$$

where

$$\Psi_{\bar{\theta}_k,\theta_k,k+1}^{(0)} := \sum_{\bar{j}=1}^{\bar{s}} \sum_{j=1}^{s} \bar{p}_{\bar{i}\bar{j}}\, p_{ij}^{(0)} \mathcal{P}_{\bar{j},j,k+1}$$

$$\text{and } \delta\Psi_{\bar{\theta}_k,\theta_k,k+1} := \sum_{\bar{j}=1}^{\bar{s}} \sum_{j=1}^{s} \bar{p}_{\bar{i}\bar{j}}\, \delta p_{ij} \mathcal{P}_{\bar{j},j,k+1},$$

for all $\bar{j} \in \bar{\mathcal{S}}$, $j \in \mathcal{S}$, and $k = N-1, \ldots 0$. Notice that for all $\bar{\theta}_k = \bar{i} \in \bar{\mathcal{S}}$ and $\theta_k = i \in \mathcal{S}$, the operator $\delta\Psi_{\bar{i},i,k+1}$ can be rewritten by

$$\delta\Psi_{\bar{i},i,k+1} = \sum_{\bar{j}=1}^{\bar{s}} \bar{p}_{\bar{i}\bar{j}} \left(\sum_{j=1}^{s} \left(\sum_{\ell=1}^{\nu} \alpha_{\ell,k} p_{ij}^{(\ell)} \right) \mathcal{P}_{\bar{j},j,k+1} \right),$$

$$\delta\Psi_{\bar{i},i,k+1} = \sum_{\bar{j}=1}^{\bar{s}} \bar{p}_{\bar{i}\bar{j}} \Big((\alpha_{1,k} p_{i1}^{(1)} + \ldots + \alpha_{1,k} p_{i1}^{(\nu)}) \mathcal{P}_{\bar{j},1,k+1} + \ldots + (\alpha_{1,k} p_{is}^{(1)} + \ldots + \alpha_{1,k} p_{is}^{(\nu)}) \mathcal{P}_{\bar{j},s,k+1} \Big),$$

and expanding it and rewriting it with respect to each vertex, we have

$$\begin{aligned}\delta\Psi_{\bar{i},i,k+1} = \sqrt{\bar{p}_{\bar{i}1}} \Big\{ &\Big(\sqrt{\alpha_{1,k} p_{i1}^{(1)}}\Big) \mathcal{P}_{1,1,k+1}(\bullet) + \ldots + \\ &\Big(\sqrt{\alpha_{1,k} p_{is}^{(1)}}\Big) \mathcal{P}_{1,s,k+1}(\bullet) + \ldots + \Big(\sqrt{\alpha_{\nu,k} p_{i1}^{(\nu)}}\Big) \times \\ &\mathcal{P}_{1,1,k+1}(\bullet) + \ldots + \Big(\sqrt{\alpha_{\nu,k} p_{is}^{(\nu)}}\Big) \mathcal{P}_{1,s,k+1}(\bullet) \Big\} \sqrt{\bar{p}_{\bar{i}1}} \\ &+ \ldots + \sqrt{\bar{p}_{\bar{i}\bar{s}}} \Big\{ \Big(\sqrt{\alpha_{1,k} p_{i1}^{(1)}}\Big) \mathcal{P}_{\bar{s},1,k+1}(\bullet) + \ldots + \\ &\Big(\sqrt{\alpha_{1,k} p_{is}^{(1)}}\Big) \mathcal{P}_{\bar{s},s,k+1}(\bullet) + \ldots + \Big(\sqrt{\alpha_{\nu,k} p_{i1}^{(\nu)}}\Big) \times \\ &\mathcal{P}_{\bar{s},1,k+1}(\bullet) + \ldots + \Big(\sqrt{\alpha_{\nu,k} p_{is}^{(\nu)}}\Big) \mathcal{P}_{\bar{s},s,k+1}(\bullet) \Big\} \sqrt{\bar{p}_{\bar{i}\bar{s}}}.\end{aligned}$$

Therefore, $\delta\Psi_{\bar{i},i,k+1} = \delta\mathbf{p}_{\bar{\mathbf{i}},\mathbf{i}}^{\mathbf{T}} \mathscr{P}_{k+1} \delta\mathbf{p}_{\bar{\mathbf{i}},\mathbf{i}}$, where

$$\mathscr{P}_{k+1} := I_\nu \otimes \mathbf{diag}(\mathcal{P}_{1,1,k+1}, \ldots, \mathcal{P}_{1,s,k+1}, \ldots, \mathcal{P}_{\bar{s},1,k+1}, \ldots, \mathcal{P}_{\bar{s},s,k+1}),$$

$$\delta\mathbf{p}_{\bar{\mathbf{i}},\mathbf{i}} = \begin{bmatrix} \alpha_{1,k} I_{\bar{s}sn_d} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{\nu,k} I_{\bar{s}sn_d} \end{bmatrix}^{\circ 1/2} \left(\bar{p}_{\bar{i}}^{\circ 1/2} \otimes \begin{bmatrix} p_i^{(1)} \\ \vdots \\ p_i^{(\nu)} \end{bmatrix}^{\circ 1/2} \right),$$

with $\bar{p}_{\bar{i}} = [\bar{p}_{\bar{i}1} \cdots \bar{p}_{\bar{i}\bar{s}}]^T$, $p_i^{(l)} = \left[p_{i1}^{(l)} I_{n_d} \cdots p_{is}^{(l)} I_{n_d} \right]^T$, and $\alpha_k \in \Lambda_\nu$.

We now proceed to transform the one-step constrained optimization problem (17) into an unconstrained problem. By applying the penalty function method (see Luenberger and Ye (2008)), we include the constraint in the cost function as quadratic terms multiplied by a penalty parameter $\mu > 0$, which penalizes violations of this constraint. Thus, for a fixed $\mu > 0$, we rewrite problem (17) as the unconstrained problem

$$\min_{u_k, z_{k+1}} \max_{\delta\mathbb{P}} \mathcal{J}_k \tag{18}$$

with the new quadratic cost function given by

$$\begin{aligned}\mathcal{J}_k = &\begin{bmatrix} z_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} \Psi_{\bar{i},i,k+1}^{(0)} & 0 \\ 0 & \mathcal{R}_{\bar{i},i,k} \end{bmatrix} \begin{bmatrix} z_{k+1} \\ u_k \end{bmatrix} + \\ &\left(\begin{bmatrix} \delta\mathbf{p}_{\bar{\mathbf{i}},\mathbf{i}} & 0 \\ 0 & 0 \\ I_{n_d} & -G_{\bar{i},i,k} \end{bmatrix} \begin{bmatrix} z_{k+1} \\ u_k \end{bmatrix} - \begin{bmatrix} 0 \\ -I_{n_d} \\ F_{\bar{i},i,k} \end{bmatrix} z_k \right)^T \times \\ &\begin{bmatrix} \mathscr{P}_{k+1} & 0 & 0 \\ 0 & \mathcal{Q}_{\bar{i},i,k} & 0 \\ 0 & 0 & \mu I_{n_d} \end{bmatrix} (\bullet).\end{aligned} \tag{19}$$

Problem (18)-(19) is a special case of a regularized least-squares problem with polytopic uncertainty. Thereby, we apply the result in Lemma 1 to obtain the solution for this problem, as Theorem 1 states.

Theorem 1. Consider the robust regulation problem defined by (18)-(19), with given initial conditions z_0, $\mathcal{P}_{\bar{i},i,N} \succ 0$, $\mathcal{Q}_{\bar{i},i,k} \succ 0$, $\mathcal{R}_{\bar{i},i,k} \succ 0$, and fixed parameters $\mu > 0$ and $\beta > 1$. Then, for each $k = 0, 1, \ldots, N$, its solution recursively provides the future state z_{k+1}^*, the control input u_k^* and the minimized quadratic cost function $\mathcal{J}_k^*$ according to

$$\begin{bmatrix} z_{k+1}^* \\ u_k^* \\ \mathcal{J}_k^* \end{bmatrix} = \begin{bmatrix} I_{n_d} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & z_k \end{bmatrix}^T \begin{bmatrix} \mathcal{L}_{\bar{i},i,k} \\ \mathcal{K}_{\bar{i},i,k} \\ \mathcal{P}_{\bar{i},i,k} \end{bmatrix} z_k, \tag{20}$$

where, for $k = N, \ldots, 1, 0$, the closed-loop system matrix $\mathcal{L}_{\bar{i},i,k}$, the feedback-gain $\mathcal{K}_{\bar{i},i,k}$ and the solution of the Riccati equation $\mathcal{P}_{\bar{i},i,k}$ are recursively obtained from (21).

Proof. It follows directly from the mappings between problems (18) and (13):

$$\begin{aligned}&\mathscr{Q} := \begin{bmatrix} \Psi_{\bar{i},i,k+1} & 0 \\ 0 & \mathcal{R}_{\bar{i},i,k} \end{bmatrix}, \ \mathscr{W} := \begin{bmatrix} \mathscr{P}_{k+1} & 0 & 0 \\ 0 & \mathcal{Q}_{\bar{i},i,k} & 0 \\ 0 & 0 & \mu I_{n_d} \end{bmatrix}, \\ &x := \begin{bmatrix} z_{k+1} \\ u_k \end{bmatrix}, \ \mathcal{A}_0 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{n_d} & -G_{\bar{i},i,k} \end{bmatrix}, \ \delta\mathcal{A} := \begin{bmatrix} \delta\mathbf{p}_{\bar{\mathbf{i}},\mathbf{i}} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ &b_0 := \begin{bmatrix} 0 \\ -I_{n_d} \\ F_{\bar{i},i,k} \end{bmatrix} z_k, \ \delta b := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\end{aligned} \tag{22}$$

and, between the polytopic uncertainty models (19) and (14):

$$\begin{aligned}&\mathscr{H} := \begin{bmatrix} I_{\bar{s}s\nu n_d} \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{A}^{(\ell)} := \left[\bar{p}_{\bar{i}} \otimes p_i^{(\ell)} \ \ 0 \right]^{\circ 1/2}, \ b^{(\ell)} := 0, \\ &\ell = 1, \ldots, \nu, \ \bar{i} \in \bar{\theta}, \ i \in \theta.\end{aligned} \tag{23}$$

The symmetric matrix arrangement (21) is then obtained based on Lemma 1 with the identifications (22) and (23). Moreover, to compute the $\hat{\lambda}$, we consider the practical approximation discussed in Remark 1.

Remark 2. The penalty parameter μ can be understood as robustness measure of the polytopic robust regulator. It should be remarked that the solution to the PRR in Theorem 1 is guaranteed for any $\mu > 0$, which is adjusted beforehand and fixed during simulation. Moreover, according to Luenberger and Ye (2008), when $\mu \to \infty$, the solution to (18) approaches the optimal solution to the original problem (11) and we say that they are equivalent. Otherwise, the solution is suboptimal.

The next result shows that the symmetric matrix arrangement (21) in Theorem 1 also reduces to the form of well-known coupled Riccati equations, which facilitates the analysis of convergence and stability conditions for the proposed polytopic robust regulation problem.

Theorem 2. Consider the robust regulation problem (18)-(19) under the same conditions of Theorem 1. Then, the recursive algebraic solution is given by

$$\begin{aligned}\mathcal{P}_{\bar{i},i,k} = \mathcal{Q}_{\bar{i},i,k} + \mathcal{F}_{\bar{i},i,k}^T \Big(&\hat{\Psi}_{\bar{i},i,k+1} - \hat{\Psi}_{\bar{i},i,k+1} \mathcal{G}_{\bar{i},i,k} \times \\ &(\mathcal{R}_{\bar{i},i,k} + \mathcal{G}_{\bar{i},i,k}^T \hat{\Psi}_{\bar{i},i,k+1} \mathcal{G}_{\bar{i},i,k})^{-1} \mathcal{G}_{\bar{i},i,k}^T \hat{\Psi}_{\bar{i},i,k+1} \Big) \mathcal{F}_{\bar{i},i,k},\end{aligned} \tag{24}$$

where $\hat{\Psi}_{\bar{i},i,k+1} = (\mu^{-1} I_{n_d} + (\Psi^{(0)}_{\bar{i},i,k+1})^{-1}\hat{E}_p)^{-1}$,

$$\hat{E}_p = I_{n_d} - \overline{\mathbb{P}}^T_{\bar{i},i}(\lambda^{-1} I_{\bar{s}s\nu n_d} + \overline{\mathbb{P}}_{\bar{i},i}(\Psi^{(0)}_{\bar{i},i,k+1})^{-1}\overline{\mathbb{P}}^T_{\bar{i},i})^{-1}\times \overline{\mathbb{P}}_{\bar{i},i}(\Psi^{(0)}_{\bar{i},i,k+1})^{-1}.$$

Proof. It follows from solving the system of simultaneous equations presented in (20)-(21).

4.1 CONVERGENCE AND STABILITY

The conditions for convergence and stability of the proposed polytopic robust regulator are analyzed when the MJLS (9) is time-invariant. We also assume that the probabilities matrices $p_{\bar{i}\bar{j}}$ and p_{ij}, and the polytope vertices $p_i^{(\ell)}$, with $\bar{i} \in \mathcal{S}$ and $i \in \mathcal{S}$, are constant. Nonetheless, we still consider that the uncertainty matrix $\delta\mathbb{P}$ is time-varying, since it depends on the polytope coefficients $\alpha_k \in \Lambda_\nu$. First, let us define $F = (F_{1,1}, \ldots, F_{1,s}, \ldots, F_{\bar{s},1}, \ldots, F_{\bar{s},s}) \in \mathbb{H}^{n_d \times n_d}$, $G = (G_{1,1}, \ldots, G_{1,s}, \ldots, G_{\bar{s},1}, \ldots, G_{\bar{s},s}) \in \mathbb{H}^{n_d \times m}$, $\mathcal{P}_k = (\mathcal{P}_{1,1,k}, \ldots, \mathcal{P}_{1,s,k}, \ldots, \mathcal{P}_{\bar{s},1,k}, \ldots, \mathcal{P}_{\bar{s},s,k}) \in \mathbb{H}^{n_d}_+$, $\mathcal{Q} = (\mathcal{Q}_{1,1}, \ldots, \mathcal{Q}_{1,s}, \ldots, \mathcal{Q}_{\bar{s},1}, \ldots, \mathcal{Q}_{\bar{s},s}) \in \mathbb{H}^{n_d}_+$, and $\mathcal{R} = (\mathcal{R}_{1,1}, \ldots, \mathcal{R}_{1,s}, \ldots, \mathcal{R}_{\bar{s},1}, \ldots, \mathcal{R}_{\bar{s},s}) \in \mathbb{H}^m_+$.

Theorem 3. Consider sequences $F \in \mathbb{H}^{n_d \times n_d}$, $G \in \mathbb{H}^{n_d \times m}$ and $\mathcal{Q} \in \mathbb{H}^{n_d}_+$ known a priori. Assume that $(F_{\bar{i},i}, G_{\bar{i},i})$ is stabilizable and $(\mathcal{Q}_{\bar{i},i}, F_{\bar{i},i})$ is detectable. Then, for any initial condition $\mathcal{P}_{N,N} \in \mathbb{H}^{n_d}_+$, and fixed $\mu > 0$ and $\beta > 1$, $\mathcal{P}_k \in \mathbb{H}^{n_d}_+$ converges to the unique stabilizing solution $\mathcal{P} = (\mathcal{P}_{1,1,k}, \ldots, \mathcal{P}_{1,s,k}, \ldots, \mathcal{P}_{\bar{s},1,k}, \ldots, \mathcal{P}_{\bar{s},s,k}) \in \mathbb{H}^{n_d}_+$ of the algebraic Riccati equation (24), such that the closed-loop matrix $\mathcal{L}_{\bar{i},i}$, with $\bar{i} \in \bar{\mathcal{S}}$ and $i \in \mathcal{S}$, of the MJLS (9) is stable.

Proof. Notice that (24) conforms with the class of coupled algebraic Riccati equations. That being said, through direct matrix mappings and according to Costa et al. (2005), stabilizability of $(F_{\bar{i},i}, G_{\bar{i},i})$ and detectability of $(\mathcal{Q}_{\bar{i},i}, F_{\bar{i},i})$ guarantee the convergence of $\mathcal{P}_{\bar{i},i,k}$ in (24) to the unique stabilizing solution $\mathcal{P} \in \mathbb{H}^{n_d}_+$ that stabilizes the closed-loop matrix $\mathcal{L}_{\bar{i},i}$ in (21).

5. NUMERICAL EXAMPLE

The effectiveness of the proposed Polytopic Robust Regulator (PRR) is evaluated by a system model studied in Li et al. (2021). We further compare our result with alternative existing robust controllers.

Consider a discrete-time MJLSD as described in (1) with three operating modes (proposed by Li et al. (2021))

$$A_{1,k} = \begin{bmatrix} 0.2 & 0.5 \\ 0 & 0.6 \end{bmatrix},\ A_{d,1,k} = \begin{bmatrix} 0.2 & 0.5 \\ 0.1 & 0 \end{bmatrix},\ B_{1,k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$A_{2,k} = \begin{bmatrix} 0.6 & 0.3 \\ 0.1 & 0 \end{bmatrix},\ A_{d,2,k} = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.5 \end{bmatrix},\ B_{2,k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$A_{3,k} = \begin{bmatrix} 0.1 & 0.2 \\ 1 & 0.4 \end{bmatrix},\ A_{d,3,k} = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.3 \end{bmatrix},\ B_{3,k} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

with initial conditions $x_0 = [0.5\ 0.5]^T$ and $\varphi_0(k) = x_0$, for all $k \leq 0$, $\pi_i = 1/3 \cdot \mathbf{1}_s^T$, and the uncertain transition probability matrix $\mathbb{P}$ with two polytope vertices,

$$\mathbb{P}_0 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix},\ \mathbb{P}^{(1)} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbb{P}^{(2)} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Two cases of time-delay are considered. In the first, we assume that $d_{max} = 2$, whereas in the second, $d_{max} = 10$, with $d_{min} = 1$ and $\triangle d = 1$, for both cases.

For the first time-delay situation, the known transition probability matrix $\bar{\mathbb{P}}$, as described in (6), is given by

$$\bar{\pi}_{\bar{i}} = 0.5 \cdot \mathbf{1}_{\bar{s}}^T,\ \bar{\mathbb{P}} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

and for the second one, it can be easily obtained following the guidelines defined in (6).

We evaluate the proposed PRR (Theorem 1) by comparing its performance with the LMI-type controllers developed in Dzung and Hien (2017, Thm. 2) and Li et al. (2021, Thm. 2). We apply Theorem 1 with the initialization data

$$\mathcal{P}_{\bar{i},i,N} = \mathcal{Q}_{\bar{i},i,k} = I_{n_d},\ \mathcal{R}_{\bar{i},i,k} = I_m,\ \forall \bar{i} = 1, \ldots, \bar{s},$$
$$i = \{1, 2, 3\},\ \mu = 10^{15},\ \beta = 1.5.$$

For comparison purposes, an amount of 1000 Monte Carlo realizations were performed, considering a time horizon $N = 40$. The system operating mode θ_k and the time-delay d_k are selected based on their probabilities $\mathbb{P}_0 + \delta\mathbb{P}$ and $\bar{\mathbb{P}}$, respectively. Moreover, at each time step, the polytope coefficients $\alpha_k \in \Lambda_\nu$, defined in (4), are randomly selected. Both control approaches are applied to MJLSD (1) with transition probabilities subject to polytopic uncertainties.

The averaged norms of regulated states are depicted in Figures 1 and 2. For the two time-delay situations, the proposed PRR outperforms the other robust controller approaches in terms of a faster convergence. Additionally, in the second, more severe, delay case, the controller in Li

$$\begin{bmatrix} \mathcal{L}_{\bar{i},i,k} \\ \mathcal{K}_{\bar{i},i,k} \\ \mathcal{P}_{\bar{i},i,k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{n_d} \\ 0 & 0 & F_{\bar{i},i,k} \\ 0 & 0 & 0 \\ I_{n_d} & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix}^T \begin{bmatrix} (\Psi^{(0)}_{\bar{i},i,k+1})^{-1} & 0 & 0 & 0 & 0 & I_{n_d} & 0 \\ 0 & \mathcal{R}^{-1}_{\bar{i},i,k} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & \mathcal{Q}_{\bar{i},i,k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu^{-1} I_{n_d} & 0 & I_{n_d} & -G_{\bar{i},i,k} \\ 0 & 0 & 0 & 0 & \Gamma & \overline{\mathbb{P}}^{1/2}_{\bar{i},i} & 0 \\ I_{n_d} & 0 & 0 & I_{n_d} & (\overline{\mathbb{P}}^{1/2}_{\bar{i},i})^T & 0 & 0 \\ 0 & I_m & 0 & -G^T_{\bar{i},i,k} & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_{n_d} \\ F_{\bar{i},i,k} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (21)$$

where $\Gamma := \hat{\lambda}^{-1} I_{\bar{s}s\nu n_d}$, $\hat{\lambda} := \beta\|\mathscr{P}_{k+1}\|$, for some $\beta > 1$, and

$$\overline{\mathbb{P}}_{\bar{i},i} := \left[\bar{p}^{1/2}_{\bar{i},1}\ \ldots\ \bar{p}^{1/2}_{\bar{i},\bar{s}}\right]^T \otimes \left[(p_i^{(1)})^T\ \ldots\ (p_i^{(\nu)})^T\right]^T \text{ with } p_i^{(\ell)} := [p^{(\ell)}_{i1} I_{n_d} \ldots p^{(\ell)}_{is} I_{n_d}]^T.$$

et al. (2021, Thm. 2) was unable to regulate the MJLSD, presenting an unfeasible solution. For this reason, it is not shown in Figure 2. This emphasizes how the time-delay can cause instability and degrade its performance.

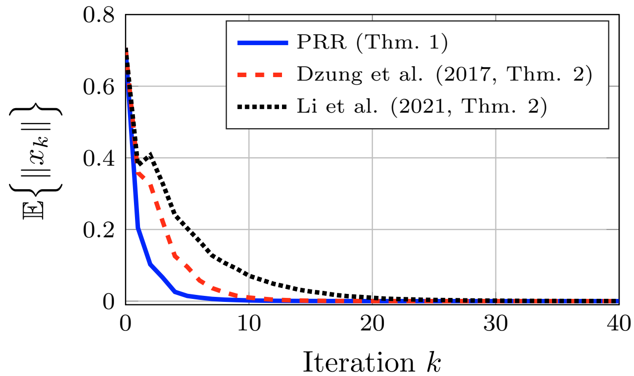

Fig. 1. Mean norms of states of the robust controllers when $d_{max} = 2$.

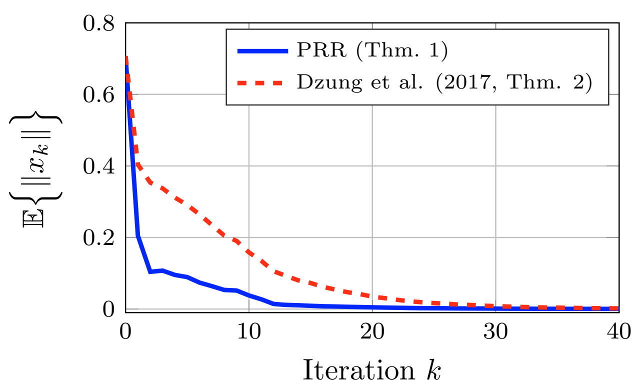

Fig. 2. Mean norms of states of the robust controllers when $d_{max} = 10$.

6. CONCLUSION

We proposed a recursive robust regulator for discrete-time MJLS with random state delays and polytopic uncertain probabilities. By applying the augmented system approach, we yield a delay-free MJLS governed by two independent Markov chain. Based on the resulting augmented MJLS, and combined it with a penalty parameter, we designed a robust recursive control law in terms of coupled Riccati equations, which is well suited for online applications due its recursiveness. A numerical example demonstrated that the proposed PRR outperforms other robust approaches. Furthermore, it showed faster convergence in a situation where one of the considered LMI-type approaches returned an infeasible solution. In future works, we will extend this result to address MJLSD with polytopic uncertainties affecting also the model parameters.

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