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Schrödinger and Dirac operators with the Aharonov–Bohm and magnetic-solenoid fields

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Abstract
We construct all self-adjoint Schrödinger and Dirac operators (Hamiltonians) with both the pure Aharonov–Bohm (AB) field and the so-called magnetic-solenoid field (a collinear superposition of the AB field and a constant magnetic field). We perform a spectral analysis for these operators, which includes finding spectra and spectral decompositions, or inversion formulae. In constructing the Hamiltonians and performing their spectral analysis, we follow, respectively, the von Neumann theory of self-adjoint extensions of symmetric operators and the Krein method of guiding functionals.

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1. Introduction

The Aharonov–Bohm (AB) effect [1] plays an important role in quantum theory refining the status of electromagnetic potentials in this theory. First, this effect was discussed in relation to a study of the interaction between a non-relativistic charged particle and an infinitely long and infinitesimally thin magnetic field of a solenoid (further AB field) which yields a magnetic flux Φ (a similar effect has been discussed earlier by Ehrenberg and Siday [2]). It was discovered that particle wave functions vanish at the solenoid line. In spite of the fact that the magnetic field vanishes out of the solenoid, the phase shift in the wave functions is proportional to the corresponding magnetic flux [3]. A nontrivial particle scattering by the solenoid is interpreted as a possibility for quantum particles to 'feel' potentials of the corresponding electromagnetic field. Indeed, potentials of the AB field do not vanish out of the solenoid. For the first time, a construction of self-adjoint (s.a. in what follows) Schrödinger operators with the AB field was given in [4]. The need for s.a. extensions of the Dirac Hamiltonian with the AB field in $2+1$ dimensions was recognized in [5–7]. s.a. extensions of the Dirac Hamiltonian with the AB field in $3+1$ dimensions were found in [8]; see also [9, 10]. The physically motivated boundary conditions for particle scattering by the AB field and a Coulomb center were studied in [11, 12]. A splitting of Landau levels in a superposition of a parallel uniform magnetic field and the AB field (further magnetic-solenoid field (MSF)) gives an example of the AB effect for bound states. First exact solutions of the Schrödinger equation with the MSF (non-relativistic case) were studied in [13]. Exact solutions of the relativistic wave equations (Klein–Gorgon and Dirac) with the MSF were obtained in [14–16] and were used later to study the AB effect in cyclotron and synchrotron radiations; see [15–17]. Later on, the problem of self-adjointness of the Dirac Hamiltonian with the MSF was studied in [18, 19].

In this work, we construct systematically all the s.a. Schrödinger and Dirac operators with both the pure AB field and the MSF. Then, we perform a spectral analysis for these Hamiltonians, which includes finding spectra and spectral decompositions, or inversion formulae. In constructing the Hamiltonians and performing their spectral analysis, we follow, respectively, the theory of s.a. extensions of symmetric differential operators [20, 21, 25] and the Krein method of guiding functionals [20, 21]. Examples of similar consideration are given in [22], where a nonrelativistic particle in the Calogero and Krazter potential fields is considered, and in [23], where a Dirac particle in the Coulomb field of arbitrary charge is considered. However, due to the peculiarities of the three-dimensional (3D) problem under

consideration, we use a necessary generalization of the approach [23].

Recall that the AB field of an infinitely thin solenoid (with constant flux Φ) along the axis $z = x^3$ can be described by electromagnetic potentials $A^{\mu}_{\rm AB}(x)$, $\mu = 0, 1, 2, 3$, $x = (x^0, \mathbf{r})$, $\mathbf{r} = (x^k,\ k = 1, 2, 3)$, $x^0 = ct$,

$$A^{\mu}_{\rm AB} = (0, \mathbf{A}_{\rm AB}), \quad \mathbf{A}_{\rm AB} = (A^k_{\rm AB},\ k = 1, 2, 3), \quad A^3_{\rm AB} = 0,$$
$$A^1_{\rm AB} = -\frac{\Phi \sin\varphi}{2\pi\rho}, \quad A^2_{\rm AB} = \frac{\Phi\cos\varphi}{2\pi\rho},$$

where ρ, φ are cylindrical coordinates, $x^1 = \rho\cos\varphi$ and $x^2 = \rho\sin\varphi$. The magnetic field of an AB solenoid has the form $\mathbf{B}_{\rm AB} = (0, 0, B_{\rm AB})$. It is easy to see that outside the z-axis the magnetic field $\mathbf{B}_{\rm AB} = \operatorname{rot}\mathbf{A}_{\rm AB}$ is equal to zero. Nevertheless, for any surface Σ with a boundary L being any contour (even an infinitely small one) around the z-axis, the circulation of the vector potential along L does not vanish and reads $\oint_L \mathbf{A}_{\rm AB}\,\mathrm{d}\mathbf{l} = \Phi$. If one interprets this circulation as the flux of the magnetic field $\mathbf{B}_{\rm AB}$ through the surface Σ,

$$\int_{\Sigma} \mathbf{B}_{\rm AB}\,\mathrm{d}\boldsymbol{\sigma} = \oint_L \mathbf{A}_{\rm AB}\,\mathrm{d}\mathbf{l} = \Phi,$$

then we obtain an expression for the magnetic field,

$$B_{\rm AB} = \Phi\delta(x^1)\delta(x^2),$$

where the term 'infinitely thin solenoid' comes from.

One can see that $\mathbf{A}_{\rm AB} = -\operatorname{rot}\Psi$, $\Psi = (0, 0, \frac{\Phi}{2\pi}\ln\rho)$, such that $\operatorname{div}\mathbf{A}_{\rm AB} = 0$, and again

$$\mathbf{B}_{\rm AB} = \operatorname{rot}\mathbf{A}_{\rm AB} = (0, 0, B_{\rm AB}),$$
$$B_{\rm AB} = \frac{\Phi}{2\pi}\Delta\ln\rho = \Phi\delta(x^1)\delta(x^2).$$

In cylindrical coordinates, we have

$$\frac{e}{c\hbar}A^1_{\rm AB} = -\phi\rho^{-1}\sin\varphi, \quad \frac{e}{c\hbar}A^2_{\rm AB} = \phi\rho^{-1}\cos\varphi, \quad \phi = \Phi/\Phi_0,$$

where Φ_0 is a fundamental unit of magnetic flux,

$$\Phi_0 = 2\pi c\hbar/e = 4135\times 10^{-7}\,\mathrm{G\,cm}^2$$

(recall that $e > 0$ is the absolute value of the electron charge).

The MSF is defined as a superposition of a constant uniform magnetic field of strength B directed along the axis z and the AB field with flux Φ in the same direction. The MSF is given by electromagnetic potentials of the form $A^{\mu} = (0, \mathbf{A})$, $\mathbf{A} = \left(A^k,\ k = 1, 2, 3\right)$,

$$A^1 = A^1_{\rm AB} - \frac{Bx^2}{2}, \quad A^2 = A^2_{\rm AB} + \frac{Bx^1}{2}, \quad A^3 = 0. \tag{1}$$

The potentials (1) define the magnetic field $\mathbf{B}$ of the form

$$\mathbf{B} = (0, 0, B + B_{\rm AB}).$$

In cylindrical coordinates, the potentials of the MSF have the form

$$\frac{e}{c\hbar}A^1 = -\tilde{\phi}\rho^{-1}\sin\varphi, \quad \frac{e}{c\hbar}A^2 = \tilde{\phi}\rho^{-1}\cos\varphi, \quad A^3 = 0,$$
$$\tilde{\phi} = \phi + \frac{\epsilon_B\gamma\rho^2}{2}, \quad \gamma = \frac{e|B|}{c\hbar} > 0, \quad \epsilon_B = \operatorname{sgn} B. \tag{2}$$

For further consideration, it is convenient to introduce the following representation:

$$\phi = \epsilon_B(\phi_0 + \mu), \quad \phi_0 = [\epsilon_B\phi] \in \mathbb{Z}, \quad \mu = \epsilon_B\phi - \phi_0,$$
$$0 \leqslant \mu < 1. \tag{3}$$

The quantity μ is called the mantissa of the magnetic flux and, in fact, determines all the physical effects in the AB field; see, e.g., [16].

2. s.a. Schrödinger Hamiltonians

In this section, we consider 2D and 3D nonrelativistic motions of a particle of mass m_e and charge $q = \epsilon_q e$, $\epsilon_q = \operatorname{sgn} q = \pm 1$ (positron or electron) in the MSF. The canonical formulation of the problem is the following. The starting point is the 'formal Schrödinger Hamiltonian' $\check{H}$ with the MSF that is, respectively, a 2D or 3D s.a. differential operation well-known from physics textbooks. In three dimensions, it is given by

$$\check{H} = \frac{1}{2m_e}\left(\check{\mathbf{p}} - \frac{q}{c}\mathbf{A}\right)^2, \quad \check{\mathbf{p}} = -\mathrm{i}\hbar\nabla, \ \nabla = \left(\partial_x,\ \partial_y,\ \partial_z\right). \tag{4}$$

It is convenient to represent $\check{H}$ as a sum of two terms, $\check{H}^{\perp}$ and $\check{H}^{\parallel}$,

$$\check{H} = \check{H}^{\perp} + \check{H}^{\parallel},$$

where the 2D s.a. differential operation $\check{H}^{\perp}$, the 'formal 2D Schrödinger Hamiltonian' with the MSF,

$$\check{H}^{\perp} = M^{-1}\check{\mathcal{H}}^{\perp}, \quad \check{\mathcal{H}}^{\perp} = \left(-\mathrm{i}\nabla^{\perp} - \tfrac{q}{c\hbar}\mathbf{A}^{\perp}\right)^2,$$
$$M = 2m_e\hbar^{-2}, \quad \nabla^{\perp} = \left(\partial_x,\ \partial_y\right), \quad \mathbf{A}^{\perp} = \left(A^1, A^2\right), \tag{5}$$

where A^1 and A^2 are given by (2), corresponds to a 2D motion in the xy-plane perpendicular to the z-axis, while the 1D differential operation $\check{H}^{\parallel}$,

$$\check{H}^{\parallel} = \check{\mathcal{H}} = \frac{\check{p}_z^2}{2m_e}, \quad \check{p}_z = -\mathrm{i}\hbar\partial_z,$$

corresponds to a 1D free motion along the z-axis.

The problem to be solved is to construct s.a. nonrelativistic 2D and 3D Hamiltonians $\hat{H}^{\perp}$ and $\hat{H}$ associated with the respective s.a. differential operations $\check{H}^{\perp}$ and $\check{H}$ and to perform a spectral analysis for these operators.

We begin with the 2D problem. We successively consider the case of the pure AB field, with $B = 0$, and then the case of the MSF, with $B \neq 0$. In the following subsection, we generalize the obtained results to three dimensions.

2.1. The 2D case

2.1.1. Reduction to the radial problem. In the case of two dimensions, the space of the particle quantum states is the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^2)$ of square-integrable functions $\psi(\boldsymbol{\rho})$, $\boldsymbol{\rho} = (x, y)$, with the scalar product

$$(\psi_1, \psi_2) = \int \overline{\psi_1(\boldsymbol{\rho})}\psi_2(\boldsymbol{\rho})\,\mathrm{d}\boldsymbol{\rho}, \quad \mathrm{d}\boldsymbol{\rho} = \mathrm{d}x\,\mathrm{d}y = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi.$$

A quantum Hamiltonian should be defined as an s.a. operator in this Hilbert space. It is more convenient to deal with s.a. operators associated with the s.a. differential operation $\check{\mathcal{H}}^{\perp} = M\check{H}^{\perp}$ defined in (5).

The construction is essentially based on the requirement for rotation symmetry, which certainly holds in a classical description of the system. This requirement is formulated as the requirement of the invariance of an s.a. Hamiltonian under rotations around the solenoid line, the z-axis. As in classical mechanics, the rotation symmetry allows separating the polar coordinates ρ and φ and reducing the 2D problem to a 1D radial problem.

The group of rotations $SO(2)$ in $\mathbb{R}^2$ naturally acts in the Hilbert space $\mathfrak{H}$ by unitary operators: if $S \in SO(2)$, then the corresponding operator $\hat{U}_S$ is defined by the relation $(\hat{U}_S\psi)(\boldsymbol{\rho}) = \psi(S^{-1}\boldsymbol{\rho})$, $\psi \in \mathfrak{H}$.

The Hilbert space $\mathfrak{H}$ is a direct orthogonal sum of subspaces $\mathfrak{H}_m$, which are the eigenspaces of the representation $\hat{U}_S$,

$$\mathfrak{H} = \sum_{m\in\mathbb{Z}}^{\oplus} \mathfrak{H}_m, \quad \hat{U}_S\mathfrak{H}_m = \mathrm{e}^{-\mathrm{i}m\theta}\mathfrak{H}_m,$$

where θ is the rotation angle corresponding to S.

It should be noted that $\mathfrak{H}_m$ consists of eigenfunctions $\psi_m(\boldsymbol{\rho})$ for the angular momentum operator $\hat{L}_z = -\mathrm{i}\hbar\partial/\partial\varphi$,

$$\hat{L}_z\psi_m(\boldsymbol{\rho}) = \hbar m\psi_m(\boldsymbol{\rho}), \quad \psi_m(\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi\rho}}\mathrm{e}^{\mathrm{i}m\varphi} f_m(\rho), \quad \forall\psi_m \in \mathfrak{H}_m.$$

It is convenient to change the indexing, $m \to l$, $\mathfrak{H}_m \to \mathfrak{H}_l$, $\psi_m(\boldsymbol{\rho}) \to \psi_l(\boldsymbol{\rho})$ as follows $m = \epsilon(\phi_0 - l)$, such that

$$\hat{L}_z\psi_l(\boldsymbol{\rho}) = \hbar\epsilon(\phi_0 - l)\psi_l(\boldsymbol{\rho}), \quad \forall\psi_l \in \mathfrak{H}_l.$$

We define a rotationally invariant initial symmetric operator $\hat{\mathcal{H}}^{\perp}$ associated with $\check{\mathcal{H}}^{\perp}$ as follows:

$$\hat{\mathcal{H}}^{\perp} : \begin{cases} D_{\mathcal{H}^{\perp}} = \{\psi(\boldsymbol{\rho}) : \quad \psi \in \mathcal{D}(\mathbb{R}^2\backslash\{0\})\}, \\ \hat{\mathcal{H}}^{\perp}\psi = \check{\mathcal{H}}^{\perp}\psi, \quad \forall\psi \in D_{\mathcal{H}^{\perp}}, \end{cases}$$

where $\mathcal{D}(\mathbb{R}^2\backslash\{0\})$ is the space of smooth and compactly supported functions vanishing in the neighborhood of the point $\boldsymbol{\rho} = 0$. The domain $D_{\mathcal{H}^{\perp}}$ is dense in $\mathfrak{H}$ and the symmetricity of $\hat{\mathcal{H}}^{\perp}$ is obvious.

In polar coordinates ρ and φ, the operation $\check{\mathcal{H}}^{\perp}$ becomes

$$\check{\mathcal{H}}^{\perp} = -\partial_\rho^2 - \rho^{-1}\partial_\rho + \rho^{-2}(\mathrm{i}\partial_\varphi + \epsilon_q\tilde{\phi})^2, \tag{6}$$

where $\tilde{\phi}$ is given by (2).

For every l, the relation

$$(S_l f)(\boldsymbol{\rho}) = \psi_l(\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi\rho}}\mathrm{e}^{\mathrm{i}\epsilon(\phi_0 - l)\varphi} f_l(\rho), \tag{7}$$

where $f = f(\rho) \in L^2(\mathbb{R}_+)$ and $f_l(\rho) = f(\rho)$, determines a unitary operator $S_l : L^2(\mathbb{R}_+) \longmapsto \mathfrak{H}_l$, where $L^2(\mathbb{R}_+)$ is the Hilbert space of square-integrable functions on the semi-axis $\mathbb{R}_+$ with scalar product

$$(f, g) = \int_{\mathbb{R}_+} \overline{f(\rho)}g(\rho)\mathrm{d}\rho.$$

For every l, we define a linear operator V_l from $\mathfrak{H}$ to $L^2(\mathbb{R}_+)$ by setting

$$(V_l\psi)(\rho) = \sqrt{\frac{\rho}{2\pi}}\int_0^{2\pi} \psi(\rho, \varphi)\,\mathrm{e}^{-\mathrm{i}\epsilon(\phi_0 - l)\varphi}\,\mathrm{d}\varphi. \tag{8}$$

If $\psi \in \mathfrak{H} = \sum_{l\in\mathbb{Z}} \psi_l$, $\psi_l \in \mathfrak{H}_l$, then we have $\psi_l = S_l V_l\psi$ for all l. In other words, $V_l = S_l^{-1}P_l$, where P_l is the orthogonal projector onto the subspace $\mathfrak{H}_l$. However, we prefer to work with V_l rather than P_l because the latter cannot be reasonably defined in the 3D case, where the Hilbert state space should be decomposed into a direct integral instead of a direct sum.

Clearly, $V_l\psi \in \mathcal{D}(\mathbb{R}_+)$ for any $\psi \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\})$, and it follows from (6) and (8) that

$$V_l\hat{\mathcal{H}}^{\perp}\psi = \hat{h}(l)V_l\psi, \quad \psi \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\}), \tag{9}$$

where the initial symmetric operators $\hat{h}(l)$ are defined on $D_{h(l)} = \mathcal{D}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$, where they act as

$$\check{h}(l) = -\partial_\rho^2 + \rho^{-2}\left[\left(l + \mu + \gamma\rho^2/2\right)^2 - 1/4\right]. \tag{10}$$

In view of (9), for any $\psi \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\})$, the $\mathfrak{H}_l$-component $(\hat{\mathcal{H}}^{\perp}\psi)_l$ of $\hat{\mathcal{H}}^{\perp}\psi$ can be written as

$$(\hat{\mathcal{H}}^{\perp}\psi)_l = S_l V_l\hat{\mathcal{H}}^{\perp}\psi = S_l\hat{h}(l)S_l^{-1}S_l V_l\psi = S_l\hat{h}(l)S_l^{-1}\psi_l. \tag{11}$$

Suppose we have an (not necessarily closed) operator $\hat{f}_l$ in $\mathfrak{H}_l$ for each l. We define the operator

$$\hat{f} = \sum_{l\in\mathbb{Z}}^{\oplus} \hat{f}_l \tag{12}$$

in $\mathfrak{H}$ by setting

$$\hat{f}\psi = \sum_{l\in\mathbb{Z}} \hat{f}_l\psi_l, \quad \psi = \sum_{l\in\mathbb{Z}} \psi_l.$$

The domain D_f of $\hat{f}$ consists of all $\psi = \sum_{l\in\mathbb{Z}} \psi_l \in \mathfrak{H}$ such that $\psi_l \in D_{f_l}$ for all l and the series $\sum_{l\in\mathbb{Z}} \hat{f}_l\psi_l$ converges in $\mathfrak{H}$. The operator $\hat{f}$ is closed (s.a.) iff all $\hat{f}_l$ are closed (respectively, s.a.). For every l, we have $D_{f_l} = D_f \cap \mathfrak{H}_l$.

We say that a closed operator $\hat{f}$ in $\mathfrak{H}$ is rotationally invariant if it can be represented in the form (12) for some family of operators $\hat{f}_l$ in $\mathfrak{H}_l$.

By (11), the direct sum of the operators $S_l\hat{h}(l)S_l^{-1}$ is an extension of $\hat{\mathcal{H}}^{\perp}$:

$$\hat{\mathcal{H}}^{\perp} \subseteq \sum_{l\in\mathbb{Z}}^{\oplus} S_l\hat{h}(l)S_l^{-1}. \tag{13}$$

Let $\hat{h}_{\mathfrak{e}}(l)$ be s.a. extensions of the symmetric operators $\hat{h}(l)$. Then the operators

$$\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}(l) = S_l\hat{h}_{\mathfrak{e}}(l)S_l^{-1} \tag{14}$$

are s.a. extensions of $S_l\hat{h}(l)S_l^{-1}$, and it follows from (13) that the orthogonal direct sum

$$\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp} = \sum_{l\in\mathbb{Z}}^{\oplus} \hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}(l) \tag{15}$$

represents rotationally invariant s.a. extensions of the initial operator $\hat{\mathcal{H}}^{\perp}$.

Conversely, let $\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}$ be a rotationally invariant s.a. extension of $\hat{\mathcal{H}}^{\perp}$. Then it has the form (15), where $\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}(l)$ are s.a. operators in $\mathfrak{H}_l$. Let us set $\hat{h}_{\mathfrak{e}}(l) = S_l^{-1}\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}(l)S_l$. For all l, $\hat{h}_{\mathfrak{e}}(l)$ are s.a. operators in $L^2(\mathbb{R}_+)$. If $f \in \mathcal{D}(\mathbb{R}_+)$, then $S_l f \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\}) \cap \mathfrak{H}_l$, and (13) and (15) imply that

$$\begin{aligned} S_l\hat{h}(l)f &= S_l\hat{h}(l)S_l^{-1}S_l f = \hat{\mathcal{H}}^{\perp}S_l f = \hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}S_l f = \hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}(l)S_l f \\ &= S_l\hat{h}_{\mathfrak{e}}(l)f. \end{aligned}$$

Hence, $\hat{h}(l)f = \hat{h}_{\mathfrak{e}}(l)f$, i.e. $\hat{h}_{\mathfrak{e}}(l)$ is an s.a. extension of $\hat{h}(l)$. We thus conclude that $\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}$ can be represented in the form (15), where $\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}(l)$ are given by (14) and $\hat{h}_{\mathfrak{e}}(l)$ are s.a. extensions of $\hat{h}(l)$.

The problem of constructing a rotationally invariant s.a. Hamiltonian $\hat{\mathcal{H}}_{\mathfrak{e}}^{\perp}$ is thus reduced to constructing s.a. radial Hamiltonians $\hat{h}_{\mathfrak{e}}(l)$.

We first consider the case of a pure AB field where $B = 0$. In such a case, we set $\epsilon_B = 1$ and $\epsilon = \epsilon_q$.

2.1.2. s.a. radial Hamiltonians with the AB field. In this case, we have $\gamma = 0$, and s.a. radial differential operations $\check{h}(l)$ (10) become

$$\check{h}(l) = -\partial_\rho^2 + \alpha\rho^{-2}, \quad \alpha = \varkappa_l^2 - 1/4, \quad \varkappa_l = |l+\mu|, \quad l \in \mathbb{Z}.$$

It is easy to see that this differential operation and the corresponding initial symmetric operator $\hat{h}(l)$ are actually identical to the respective operation and operator encountered in studying the Calogero problem; see [22]. We can therefore directly carry over the previously obtained results to s.a. extensions of $\hat{h}(l)$.

First region: $\alpha \geqslant 3/4$. In this region, we have $(l+\mu)^2 \geqslant 1$, which is equivalent to

$$l \geqslant 1-\mu \quad \text{or} \quad l \leqslant -1-\mu.$$

Because $l \in \mathbb{Z}$ and $0 \leqslant \mu < 1$, we have to distinguish the cases of $\mu = 0$ and $\mu > 0$:

$$\mu = 0 : l \leqslant -1 \quad \text{or} \quad l \geqslant 1, \quad \text{i.e.} \quad l \neq 0,$$

$$\mu > 0 : l \leqslant -2 \quad \text{or} \quad l \geqslant 1, \quad \text{i.e.} \quad l \neq 0, -1.$$

For such l, the initial symmetric operator $\hat{h}(l)$ has zero deficiency indices, is essentially s.a. and its unique s.a. extension is $\hat{h}_{\mathfrak{e}}(l) = \hat{h}_{(1)}(l) = \hat{h}^+(l)$ with the domain[3]

$$\begin{aligned} D^*_{\hat{h}(l)}(\mathbb{R}_+) = \{\psi_* : \psi_*, \psi_*' \text{ are a.c. in } \mathbb{R}_+,\ \psi_*, \\ \check{h}(l)\psi_* \in L^2(\mathbb{R}_+)\}. \end{aligned}$$

[3] Here and in what follows, we use a.c. for absolutely continuous.

The spectrum of $\hat{h}_{(1)}(l)$ is simple and continuous and coincides with the positive semiaxis, spec $\hat{h}_{(1)}(l) = \mathbb{R}_+$.

The generalized eigenfunctions $U_{\mathcal{E}}$,

$$U_{\mathcal{E}}(\rho) = (\rho/2)^{1/2} J_{\varkappa_l}(\sqrt{\mathcal{E}}\rho), \quad \hat{h}_{(1)}(l)U_{\mathcal{E}} = \mathcal{E}U_{\mathcal{E}}, \quad \mathcal{E} \in \mathbb{R}_+,$$

of $\hat{h}_{(1)}(l)$ form a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

Second region: $-1/4 < \alpha < 3/4$. In this region, we have $0 < (l+\mu)^2 < 1$, which is equivalent to

$$-\mu < l < 1-\mu \quad \text{or} \quad -1-\mu < l < -\mu. \tag{16}$$

If $\mu = 0$, inequalities (16) have no solutions for $l \in \mathbb{Z}$. If $\mu > 0$, these inequalities have two solutions $l = l_a$, where, for brevity, we introduce the notation

$$l_a = a, \quad a = 0, -1.$$

So, in the second region, we remain with the case of $\mu > 0$.

For each $l = l_a (a = 0, -1)$ there exists a one-parameter $U(1)$-family of s.a. Hamiltonians $\hat{h}_{\lambda_a}(l_a)$ parameterized by the real parameter $\lambda_a \in \mathbb{S}(-\pi/2, \pi/2)$, where $\mathbb{S}(a,b) = [a,b]$, $a \sim b$. These Hamiltonians are specified by the asymptotic s.a. boundary conditions at the origin,

$$\begin{aligned} \psi_{\lambda_a}(\rho) = C\left[(\kappa_0\rho)^{1/2+\varkappa_a}\cos\lambda_a + (\kappa_0\rho)^{1/2-\varkappa_a}\sin\lambda_a\right] \\ + O(\rho^{3/2}), \end{aligned} \tag{17}$$

$$D_{h_{\lambda_a}(l_a)} = \{\psi \in D^*_{\hat{h}(l_a)}(\mathbb{R}_+),\ \psi \text{ satisfy (17)}\}, \tag{18}$$

where $\varkappa_a \equiv \varkappa_{l_a} = |\mu + a|$, $0 < \varkappa_a < 1$ and C is an arbitrary constant, whereas k_0 is a constant of dimension of inverse length.

For $\lambda_a \neq (-\pi/2, 0)$, the spectrum of each of $\hat{h}_{\lambda_a}(l_a)$ is simple and continuous and spec $\hat{h}_{\lambda_a}(l_a) = \mathbb{R}_+$.

The generalized eigenfunctions $U_{\mathcal{E}}$,

$$\begin{aligned} U_{\mathcal{E}}(\rho) &= \sqrt{\frac{\rho}{2Q_a}}\left[J_{\varkappa_a}\left(\sqrt{\mathcal{E}}\rho\right) + \tilde{\lambda}_a\left(\sqrt{\mathcal{E}}/2\kappa_0\right)^{2\varkappa_a} J_{-\varkappa_a}\left(\sqrt{\mathcal{E}}\rho\right)\right], \\ Q_a &= 1 + 2\tilde{\lambda}_a(\mathcal{E}/4)^{\varkappa_a}\cos(\pi\varkappa_a) + (\tilde{\lambda}_a)^2(\mathcal{E}/4)^{2\varkappa_a} > 0, \\ \tilde{\lambda}_a &= \Gamma(1-\varkappa_a)\Gamma^{-1}(1+\varkappa_a)\tan\lambda_a, \\ \check{h}_{\lambda_a}(l_a)U_{\mathcal{E}} &= \mathcal{E}U_{\mathcal{E}}, \quad \mathcal{E} \in \mathbb{R}_+, \end{aligned} \tag{19}$$

of the Hamiltonian $\hat{h}_{\lambda_a}(l_a)$ form a complete orthonormalized set in the Hilbert space $L^2(\mathbb{R}_+)$.

For $\lambda_a \in (-\pi/2, 0)$, the spectrum of each of $\hat{h}_{\lambda_a}(l_a)$ is simple, but in addition to the continuous part of the spectrum, there exists one negative level $\mathcal{E}^{(-)}_{\lambda_a} = -4k_0^2|\tilde{\lambda}_a|^{-\varkappa_a^{-1}}$, such that spec $\hat{h}_{\lambda_a}(l_a) = \mathbb{R}_+ \cup \{\mathcal{E}^{(-)}_{\lambda_a}\}$.

In this case, the generalized eigenfunctions $U_{\mathcal{E}}$ of the continuous spectrum, $\mathcal{E} \geqslant 0$, are given by the same (19), while the eigenfunction $U^{(-)}$ corresponding to the discrete level $\mathcal{E}^{(-)}_{\lambda_a}$ is

$$U^{(-)}(\rho) = \sqrt{\frac{2\rho|\mathcal{E}^{(-)}_{\lambda_a}|\sin(\pi\varkappa_a)}{\pi\varkappa_a}}\, K_{\varkappa_a}\left(\sqrt{|\mathcal{E}^{(-)}_{\lambda_a}|}\rho\right),$$

and they together form a complete orthonormalized system in each Hilbert space $L^2(\mathbb{R}_+)$.

Third region: $\alpha = -1/4$. In this region, we have $l + \mu = 0$. If $\mu = 0$, this equation has a unique solution $l = l_0 = 0$, while if $\mu > 0$, there are no solutions, and we remain with only the case of $\mu = 0$.

For $l = l_0$, there exists a one-parameter $U(1)$-family of s.a. Hamiltonians $\hat{h}_{\mathfrak{e}}(l_0) = \hat{h}_\lambda(l_0)$ parameterized by the real parameter $\lambda \in \mathbb{S}(-\pi/2, \pi/2)$. These Hamiltonians are specified by the asymptotic s.a. boundary conditions at the origin

$$\psi_\lambda(\rho) = c\left[\rho^{1/2}\ln(\kappa_0\rho)\cos\lambda + \rho^{1/2}\sin\lambda\right] + O(\rho^{3/2}\ln\rho), \tag{20}$$

(the constants C and k_0 are of the same meaning as in (17)), and their domains are given by

$$D_{h_\lambda(l_0)} = \{\psi \in D^*_{\check{h}(l_0)}(\mathbb{R}_+),\ \psi \text{ satisfy (20)}\}. \tag{21}$$

The spectrum of $\hat{h}_\lambda(l_0)$ is simple. For $|\lambda| = \pi/2$, the spectrum is continuous and nonnegative, spec $\hat{h}_{\pm\pi/2}(l_0) = \mathbb{R}_+$. For $|\lambda| < \pi/2$, in addition to the continuous part of the spectrum, $\mathcal{E} \geqslant 0$, there exists one negative level $\mathcal{E}^{(-)}_\lambda = -4\kappa_0^2\exp[2(\tan\lambda - \mathbf{C})]$, where $\mathbf{C}$ is the Euler constant, such that

$$\operatorname{spec}\hat{h}_\lambda(l_0) = \left\{\mathcal{E}^{(-)}_\lambda\right\} \cup \mathbb{R}_+, \quad |\lambda| < \pi/2.$$

The generalized and normalized eigenfunctions $U_\mathcal{E}$ of the continuous spectrum are

$$U_\mathcal{E}(\rho) = \sqrt{\frac{\rho}{2(\tilde{\lambda}^2 + \pi^2/4)}}\left[\tilde{\lambda}J_0\left(\sqrt{\mathcal{E}}\rho\right) + \frac{\pi}{2}N_0\left(\sqrt{\mathcal{E}}\rho\right)\right],$$
$$\tilde{\lambda} = \tan\lambda - \mathbf{C} - \ln\left(\sqrt{\mathcal{E}}/2\kappa_0\right), \quad \hat{h}_\lambda(l_0)U_\mathcal{E} = \mathcal{E}U_\mathcal{E},$$
$$\mathcal{E} \in \mathbb{R}_+, \quad |\lambda| \leqslant \pi/2,$$

while the normalized eigenfunction $U^{(-)}$ corresponding to the discrete level is

$$U^{(-)}(\rho) = \sqrt{2\rho|\mathcal{E}^{(-)}_\lambda|}K_0\left(\sqrt{|\mathcal{E}^{(-)}_\lambda|}\rho\right),$$
$$\hat{h}_\lambda(l_0)U^{(-)} = \mathcal{E}^{(-)}_\lambda U^{(-)}, \quad |\lambda| < \pi/2,$$

and they together form a complete orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+)$.

Complete spectrum and inversion formulae. In the previous subsubsections, we have constructed all s.a. radial Hamiltonians associated with the s.a. differential operations $\check{h}(l)$ as s.a. extensions of the symmetric operators $\hat{h}(l)$ for any $l \in \mathbb{Z}$ and for any ϕ_0 and μ. We assemble our previous results into two groups.

For $\mu = 0$, we have the following s.a. radial Hamiltonians:

$$\hat{h}_1(l), \quad l \neq l_0, \quad D_{h_1(l)} = D^*_{\check{h}(l)}(\mathbb{R}_+),$$
$$\hat{h}_\lambda(l_0), \quad \lambda \in \mathbb{S}(-\pi/2, \pi/2), \quad D_{h_\lambda(l_0)} \text{ is given by (29).}$$

For $\mu > 0$, they are

$$\hat{h}_1(l), \quad l \neq l_a = a = 0, -1, \quad D_{h_1(l)} = D^*_{\check{h}(l)}(\mathbb{R}_+),$$
$$\hat{h}_{\lambda_a}(l_a), \quad \lambda_a \in \mathbb{S}(-\pi/2, \pi/2), \quad D_{h_{\lambda_a}(l_a)} \text{ is given by (25).}$$

Each set of possible s.a. radial Hamiltonians $\hat{h}_{\mathfrak{e}}(l)$ generates s.a. Hamiltonians in accordance with the relations (14) and (15). As a final result, we have a family of s.a. rotationally invariant 2D Schrödinger operators $\hat{H}^\perp_{\mathfrak{e}} = M^{-1}\hat{\mathcal{H}}^\perp_{\mathfrak{e}}$ associated with the s.a. differential operation $\check{H}^\perp$ (5) with $B = 0$.

When presenting the spectrum and inversion formulae for $\hat{H}^\perp_{\mathfrak{e}}$, we also consider the case of $\mu = 0$ and the case of $\mu > 0$ separately. We let E denote the spectrum points of $\hat{H}^\perp_{\mathfrak{e}}$ and let Ψ_E denote the corresponding (generalized) eigenfunctions. The spectrum points of the operators $\hat{h}_{\mathfrak{e}}(l)$ and $\hat{H}^\perp_{\mathfrak{e}}$ are evidently related by $\mathcal{E} = ME$. In addition, when writing formulae for eigenfunctions Ψ_E of the operator $\hat{H}^\perp_{\mathfrak{e}}$ in terms of eigenfunctions $U_\mathcal{E}$ of the operators $\hat{h}_{\mathfrak{e}}(l)$, we have to introduce the factor $(2\pi\rho)^{-1/2}\mathrm{e}^{\mathrm{i}\epsilon_q(\phi_0 - l)\varphi}$ in accordance with equation (7) with $\epsilon = \epsilon_q$ (because $\epsilon_B = 1$), to make the substitutions $\mathcal{E} = ME$ and $\mathcal{E}^{(-)}_{\lambda_a} = ME^{(-)}_{\lambda_a}$, $\mathcal{E}^{(-)}_\lambda = ME^{(-)}_\lambda$ for the respective points of the continuous spectrum and discrete spectrum and, in addition, to multiply eigenfunctions of the continuous spectrum of the operators $\hat{h}_{\mathfrak{e}}(l)$ by the factor $\sqrt{M}$ because of the change of the spectral measure $\mathrm{d}\mathcal{E}$ to the corresponding spectral measure[4] $\mathrm{d}E$.

For $\mu = 0$, there is a family of s.a. 2D Schrödinger operators $\hat{H}^\perp_{\mathfrak{e}} = \hat{H}^\perp_\lambda$ parameterized by a real parameter $\lambda \in \mathbb{S}(-\pi/2, \pi/2)$,

$$\hat{H}^\perp_\lambda = \sum_{l\in\mathbb{Z}, l\neq l_0}^{\oplus} \hat{H}^\perp(l) \oplus \hat{H}^\perp_\lambda(l_0),$$
$$\hat{H}^\perp(l) = M^{-1}S_l\hat{h}_{(1)}(l)S_l^{-1}, \quad l \neq l_0,$$
$$\hat{H}^\perp_\lambda(l_0) = M^{-1}S_{l_0}\hat{h}_\lambda(l_0)S_{l_0}^{-1}.$$

The spectrum of $\hat{H}^\perp_\lambda$ is given by

$$\operatorname{spec}\hat{H}^\perp_\lambda = \mathbb{R}_+ \cup \left\{\begin{matrix} E^{(-)}_\lambda = -4M^{-1}\kappa_0^2\exp[2(\tan\lambda - \mathbf{C})], & |\lambda| < \pi/2 \\ \varnothing, \quad \lambda = \pm\pi/2 \end{matrix}\right\}.$$

The complete system of orthonormalized (generalized) eigenfunctions of $\hat{H}^\perp_\lambda$ consists of the generalized eigenfunctions $\Psi_{l,E}(\boldsymbol{\rho})$ of the continuous spectrum,

$$\Psi_{l,E}(\boldsymbol{\rho}) = (M/4\pi)^{1/2}\,\mathrm{e}^{\mathrm{i}\epsilon_q(\phi_0 - l)\varphi}J_{\varkappa_l}\left(\sqrt{ME}\rho\right), \quad l \neq l_0,$$
$$E \geqslant 0,$$
$$\Psi^\lambda_{l_0,E}(\rho) = \sqrt{\frac{M}{4\pi(\tilde{\lambda}^2 + \pi^2/4)}}\mathrm{e}^{\mathrm{i}\epsilon_q\phi_0\varphi}\left[\tilde{\lambda}J_0\left(\sqrt{ME}\rho\right) + \frac{\pi}{2}N_0\left(\sqrt{ME}\rho\right)\right],$$
$$\tilde{\lambda} = \tan\lambda - \mathbf{C} - \ln\left(\sqrt{ME}/2\kappa_0\right),$$

[4] From a physical standpoint, the latter is related to the change of the 'normalization of the eigenfunctions of the continuous spectrum to δ function' from $\delta(\mathcal{E} - \mathcal{E}')$ to $\delta(E - E')$.

and (in the case of $|\lambda| < \pi/2$) the eigenfunction $\Psi^{\lambda}_{l_0}(\boldsymbol{\rho})$ corresponding to the discrete level $E^{(-)}_{\lambda}$,

$$\Psi^{\lambda}_{l_0}(\boldsymbol{\rho}) = M\sqrt{|E^{(-)}_{\lambda}|/\pi}\,\mathrm{e}^{\mathrm{i}\epsilon_q\phi_0\varphi}K_0\left(\sqrt{M|E^{(-)}_{\lambda}|}\rho\right),$$

such that

$$\check{H}^{\perp}\Psi_{l,E}(\boldsymbol{\rho}) = E\Psi_{l,E}(\boldsymbol{\rho}),$$
$$\check{H}^{\perp}\Psi^{\lambda}_{l_0,E}(\boldsymbol{\rho}) = E\Psi^{\lambda}_{l_0,E}(\boldsymbol{\rho}), \quad E \geqslant 0,$$
$$\hat{H}^{\perp}_{\lambda}\Psi^{\lambda}_{l_0}(\boldsymbol{\rho}) = E^{(-)}_{\lambda}\Psi^{\lambda}_{l_0}(\boldsymbol{\rho}), \quad |\lambda| < \pi/2.$$

The corresponding inversion formulae have the form

$$\Psi(\boldsymbol{\rho}) = \sum_{l\in\mathbb{Z}, l\neq l_0}\int_0^{\infty}\Phi_l(E)\Psi_{l,E}(\boldsymbol{\rho})\mathrm{d}E + \int_0^{\infty}\Phi_{l_0}(E)\Psi^{\lambda}_{l_0,E}(\boldsymbol{\rho})\mathrm{d}E + \Phi_{l_0}\Psi^{\lambda}_{l_0}(\boldsymbol{\rho}),$$
$$\Phi_l(E) = \int \mathrm{d}\boldsymbol{\rho}\overline{\Psi_{l,E}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}),$$
$$\Phi_{l_0}(E) = \int \mathrm{d}\boldsymbol{\rho}\overline{\Psi^{\lambda}_{l_0,E}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}),$$
$$\Phi_{l_0} = \int \mathrm{d}\boldsymbol{\rho}\overline{\Psi^{\lambda}_{l_0}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}), \quad \forall\Psi \in L^2(\mathbb{R}^2);$$
$$\int \mathrm{d}\boldsymbol{\rho}\,|\Psi(\boldsymbol{\rho})|^2 = \sum_{l\in\mathbb{Z}}\int_0^{\infty}|\Phi_l(E)|^2\,\mathrm{d}E + \left|\Phi_{l_0}\right|^2,$$

and the terms with Φ_{l_0} and $\Psi^{\lambda}_{l_0}(\boldsymbol{\rho})$ are absent in the case of $|\lambda| = \pi/2$.

For $\mu > 0$, there is a family of s.a. 2D Schrödinger operators $\hat{H}^{\perp}_{\mathfrak{e}} = \hat{H}^{\perp}_{\{\lambda_a\}}$, $a = 0, -1$, parameterized by two real parameters $\lambda_a \in \mathbb{S}\,(-\pi/2, \pi/2)$,

$$\hat{H}^{\perp}_{\{\lambda_a\}} = \sum_{l\in\mathbb{Z}, l\neq l_a}^{\oplus}\hat{H}^{\perp}(l) \oplus \sum_a^{\oplus}\hat{H}^{\perp}_{\lambda_a}(l_a),$$
$$\hat{H}^{\perp}(l) = M^{-1}S_l\hat{h}_{(1)}(l)S_l^{-1}, \quad \neq l_a,$$
$$\hat{H}^{\perp}_{\lambda_a}(l_a) = M^{-1}S_{l_a}\hat{h}_{\lambda_a}(l_a)S_{l_a}^{-1}.$$

The spectrum of $\hat{H}^{\perp}_{\{\lambda_a\}}$ is given by

$$\operatorname{spec}\hat{H}^{\perp}_{\{\lambda_a\}} = \mathbb{R}_+ \cup \left\{\begin{matrix} E^{(-)}_{\lambda_a} = -4M^{-1}k_0^2|\tilde{\lambda}_a|^{-\varkappa_a^{-1}}, & \lambda_a \in (-\pi/2, 0) \\ \varnothing, \quad \lambda_a \notin (-\pi/2, 0) \end{matrix}\right\},$$

where $\varkappa_a = |\mu + a|$, $\tilde{\lambda}_a = \Gamma(1-\varkappa_a)\Gamma^{-1}(1+\varkappa_a)\tan\lambda_a$.

A complete system of orthonormalized (generalized) eigenfunctions of $\hat{H}^{\perp}_{\{\lambda_a\}}$ consists of the generalized eigenfunctions $\Psi_{l,E}(\boldsymbol{\rho})$, $l \neq l_a$, and $\Psi^{\lambda_a}_{l_a,E}(\boldsymbol{\rho})$ of the continuous spectrum,

$$\Psi_{l,E}(\boldsymbol{\rho}) = (M/4\pi)^{1/2}\,\mathrm{e}^{\epsilon_q(\phi_0-l)\varphi}J_{\varkappa_l}\left(\sqrt{ME}\rho\right),$$
$$\varkappa_l = |l+\mu|, \quad l \neq l_a,$$
$$\Psi^{\lambda_a}_{l_a,E}(\boldsymbol{\rho}) = \sqrt{\frac{1}{4\pi Q_a}}\mathrm{e}^{\mathrm{i}\epsilon_q(\phi_0-l_a)\varphi}\Big[J_{\varkappa_a}\left(\sqrt{ME}\rho\right) + \tilde{\lambda}_a\left(\sqrt{ME}/2\kappa_0\right)^{2\varkappa_a}J_{-\varkappa_a}\left(\sqrt{ME}\rho\right)\Big],$$
$$Q_a = 1 + 2\tilde{\lambda}_a\,(ME/4)^{\varkappa_a}\cos(\pi\varkappa_a) + (\tilde{\lambda}_a)^2\,(ME/4)^{2\varkappa_a}, \quad E \geqslant 0,$$

and (in the case of $\lambda_a \in (-\pi/2, 0)$) the eigenfunctions $\Psi^{\lambda_a}_{l_a}(\boldsymbol{\rho})$ corresponding to the discrete levels $E^{(-)}_{\lambda_a}$,

$$\Psi^{\lambda_a}_{l_a}(\boldsymbol{\rho}) = \sqrt{\frac{M^2|E^{(-)}_{\lambda_a}|\sin(\pi\varkappa_a)}{\pi^2\varkappa_a}}\mathrm{e}^{\mathrm{i}\epsilon_q(\phi_0-l_a)\varphi}K_{\varkappa_a}\left(\sqrt{|ME^{(-)}_{\lambda_a}|}\rho\right),$$

such that

$$\check{H}^{\perp}\Psi_{l,E}(\boldsymbol{\rho}) = E\Psi_{l,E}(\boldsymbol{\rho}), \quad l \neq l_a,$$
$$\check{H}^{\perp}\Psi^{\lambda_a}_{l_a,E}(\boldsymbol{\rho}) = E\Psi^{\lambda_a}_{l_a,E}(\boldsymbol{\rho}), \quad E \geqslant 0,$$
$$\hat{H}^{\perp}_{\{\lambda_a\}}\Psi^{\lambda_b}_{l_b}(\boldsymbol{\rho}) = E^{(-)}_{\lambda_b}\Psi^{\lambda_b}_{l_b}(\boldsymbol{\rho}), \quad b = 0, -1.$$

The corresponding inversion formulae have the form

$$\Psi(\boldsymbol{\rho}) = \sum_{l\in\mathbb{Z},\ l\neq l_a}\int_0^{\infty}\Phi_l(E)\Psi_{l,E}(\boldsymbol{\rho})\mathrm{d}E + \sum_a\left[\int_0^{\infty}\Phi_{l_a}(E)\Psi^{\lambda_a}_{l_a,E}(\boldsymbol{\rho})\mathrm{d}E + \Phi_{l_a}\Psi^{\lambda_a}_{l_a}(\boldsymbol{\rho})\right],$$
$$\forall\Psi \in L^2(\mathbb{R}^2),$$
$$\Phi_l(E) = \int \mathrm{d}\boldsymbol{\rho}\overline{\Psi_{l,E}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}), \quad l \neq l_a,$$
$$\Phi_{l_a}(E) = \int \mathrm{d}\boldsymbol{\rho}\overline{\Psi^{\lambda_a}_{l_a,E}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}),$$
$$\Phi_{l_a} = \int \mathrm{d}\boldsymbol{\rho}\overline{\Psi^{\lambda_a}_{l_a}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}),$$
$$\int \mathrm{d}\boldsymbol{\rho}\,|\Psi(\boldsymbol{\rho})|^2 = \sum_{l\in\mathbb{Z}}\int_0^{\infty}|\Phi_l(E)|^2\,\mathrm{d}E + \sum_a\left|\Phi_{l_a}\right|^2,$$

and the terms with Φ_{l_a} and $\Psi^{\lambda_a}_{l_a}(\boldsymbol{\rho})$ are absent in the case of $\lambda_a \notin (-\pi/2, 0)$.

We now consider the case of the MSF where $B \neq 0$.

2.1.3. s.a. radial Hamiltonians with MSF. In this case, the radial differential operation $\check{h}(l)$ is given by (10) with $\gamma = \frac{e|B|}{c\hbar} \notin 0$, or

$$\check{h}(l) = -\partial_\rho^2 + g_1\rho^{-2} + g_2\rho^2 + \mathcal{E}^{(0)}_l,$$
$$g_1 = \varkappa_l^2 - 1/4, \quad \varkappa_l = |l+\mu|, \quad g_2 = \gamma^2/4,$$
$$\mathcal{E}^{(0)}_l = \gamma(l+\mu).$$

Up to the constant term $\mathcal{E}^{(0)}_l$, this s.a. differential operation is identical to the 1D Schrödinger operation $-d_x^2 + g_1x^{-2} + g_2x^2$. Its singular part was studied by us recently in [22]. We note that as in the case of the pure AB field, the division to different regions of g_1 is actually determined by the same term $g_1\rho^{-2}$ singular at the origin and independent of the value of B.

The first region: $g_1 \geqslant 3/4$. In this region, we have $(l+\mu)^2 \geqslant 1$, such that

$$\mu = 0 : l \leqslant -1 \quad \text{or} \quad l \geqslant 1 \quad \text{i.e. } l \neq l_0,$$
$$\mu > 0 : l \leqslant -2 \quad \text{or} \quad l \geqslant 1 \quad \text{i.e. } l \neq l_a.$$

For such l, the initial symmetric operator $\hat{h}(l)$ has zero-deficiency indices. It is essentially s.a. and its unique s.a. extension is $\hat{h}_1(l) = \hat{h}^+(l)$ with the domain $D^*_{\hat{h}(l)}(\mathbb{R}_+)$. The spectrum of $\hat{h}_1(l)$ is simple, discrete and given by

$$\operatorname{spec} \hat{h}_1(l) = \left\{\mathcal{E}_{l,m} = \gamma\,(1+|l+\mu|+(l+\mu)+2m)\,,\ m\in\mathbb{Z}_+\right\}. \tag{22}$$

The eigenfunctions $U^{(1)}_{l,m}$,

$$\begin{aligned} U^{(1)}_{l,m}(\rho) &= Q_{l,m}\,(\gamma/2)^{1/4+\varkappa_l/2}\,\rho^{1/2+\varkappa_l}\mathrm{e}^{-\gamma\rho^2/4}\Phi(-m,1+\varkappa_l;\gamma\rho^2/2),\\ Q_{l,m} &= \left(\frac{\sqrt{2\gamma}\,\Gamma(1+\varkappa_l+m)}{m!\,\Gamma^2(1+\varkappa_l)}\right)^{1/2}, \end{aligned} \tag{23}$$

of the Hamiltonian $\hat{h}_1(l)$ form a complete orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+)$.

The second region: $-1/4 < g_1 < 3/4$. In this region, we have $0 < (l+\mu)^2 < 1$, or equivalently (16). We know that if $\mu = 0$, these inequalities have no solutions for $l\in\mathbb{Z}$, while if $\mu > 0$ there are the two solutions, $l = l_a = a$, $a = 0, -1$. Therefore, we again remain with the case of $\mu > 0$.

For each $l = l_a$, there exists a one-parameter $U(1)$-family of s.a. radial Hamiltonians $\hat{h}_{\lambda_a}(l_a)$ parameterized by a real parameter $\lambda_a \in \mathbb{S}\,(-\pi/2,\pi/2)$. These Hamiltonians are specified by the asymptotic s.a. boundary conditions at $\rho\to 0$,

$$\begin{aligned} \psi_{\lambda_a}(\rho) = C\left[\left(\sqrt{\gamma/2}\rho\right)^{1/2+\varkappa_a}\sin\lambda_a + \left(\sqrt{\gamma/2}\rho\right)^{1/2-\varkappa_a}\cos\lambda_a\right]\\ + O(\rho^{3/2}), \end{aligned} \tag{24}$$

where $\kappa_a = |\mu+a|$, $0<\kappa_a<1$ and C is an arbitrary constant[5], and their domains are given by

$$D_{h_{\lambda_a}(l_a)} = \{\psi : \psi\in D^*_{\hat{h}(l_a)}(\mathbb{R}_+),\quad \psi \text{ satisfy (24)}\}. \tag{25}$$

The spectrum of $\hat{h}_{\lambda_a}(l_a)$ is simple, discrete and is bounded from below,

$$\operatorname{spec}\hat{h}_{\lambda_a}(l_a) = \left\{\mathcal{E}_{a,m} = \tau_{a,m} + \mathcal{E}^{(0)}_{l_a},\ m\in\mathbb{Z}_+\right\},$$

where $\tau_{a,m}$ are solutions of the equation $\omega_{\lambda_a}(\tau_{a,m}) = 0$,

$$\begin{aligned} &\omega_{\lambda_a}(W) = \omega_+(W)\sin\lambda_a + \omega_-(W)\cos\lambda_a,\\ &\omega_\pm(W) = \Gamma(1\pm\varkappa_a)/\,\Gamma(1/2\pm\varkappa_a/2 - W/2\gamma). \end{aligned} \tag{26}$$

The eigenfunctions $U^{(2)}_{\lambda_a,m}$,

$$\begin{aligned} U^{(2)}_{\lambda_a,m}(\rho) &= Q_{a,m}\left[u_+(\rho;\tau_{a,m})\sin\lambda_a + u_-(\rho;\tau_{a,m})\cos\lambda_a\right],\\ Q_{a,m} &= \left(\frac{\tilde{\omega}_{\lambda_a}(\tau_{a,m})}{\sqrt{2\gamma}\varkappa_a\omega'_{\lambda_a}(\tau_{a,m})}\right)^{1/2},\\ \tilde{\omega}_{\lambda_a}(W) &= \omega_+(W)\cos\lambda_a - \omega_-(W)\sin\lambda_a,\\ u_\pm(\rho;W) &= (\gamma/2)^{1/4\pm\varkappa_a/2}\,\rho^{1/2\pm\varkappa_a}\,\mathrm{e}^{-\gamma\rho^2/4}\Phi(1/2\pm\varkappa_a/2\\ &\quad - W/2\gamma, 1\pm\varkappa_a;\gamma\rho^2/2), \end{aligned} \tag{27}$$

[5] In comparison with (17), we fix the dimensional parameter k_0 by $k_0 = \sqrt{\gamma/2}$.

of the Hamiltonian $\hat{h}_{\lambda_a}(l_a)$ form a complete orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+)$.

For $\lambda_a = \pm\pi/2$ and $\lambda_a = 0$, one can easily obtain explicit expressions for the spectrum and eigenfunctions. For $\lambda_a = \pm\pi/2$, they are given by the respective formulae (22) and (23) with the substitutions $l\to l_a$ and $\varkappa_l\to\varkappa_a$. For $\lambda_a = 0$, these formulae are modified by the additional substitution $\varkappa_a\to-\varkappa_a$. In addition, one can see that in each interval $(\tau^{(\pm\pi/2)}_{a,m}, \tau^{(\pm\pi/2)}_{a,m+1})$, $m\in\{-1\}\cup\mathbb{Z}_+$, where $\tau^{(\pm\pi/2)}_{a,m}$, $m\in\mathbb{Z}_+$, are solutions of the equation $\omega_{\pm\pi/2}(\tau_{a,m}) = 0$ and we set formally $\tau^{(\pm\pi/2)}_{a,-1} = -\infty$, there is one solution $\tau_{a,m}$ of the equation $\omega_{\lambda_a}(\tau_{a,m}) = 0$ for a fixed $\lambda_a\in(-\pi/2,\pi/2)$; the solution $\tau_{a,m}$ increases monotonically from $\tau^{(\pm\pi/2)}_{a,m}+0$ to $\tau^{(\pm\pi/2)}_{a,m+1}-0$ when λ_a changes from $-\pi/2+0$ to $\pi/2-0$.

The third region: $g_1 = -1/4$. In this region, we have $l+\mu = 0$. Thus, we remain with only the case of $\mu = 0$ with $l = l_0 = 0$.

For $l = l_0$, there exists a one-parameter $U(1)$-family of s.a. radial Hamiltonians $\hat{h}_\lambda(l_0)$, parameterized by the real parameter $\lambda\in\mathbb{S}\,(-\pi/2,\pi/2)$. These Hamiltonians are specified by the asymptotic s.a. boundary conditions at $\rho\to 0$,

$$\psi_\lambda(\rho) = C\left[\rho^{1/2}\ln\left(\sqrt{\gamma/2}\rho\right)\cos\lambda + \rho^{1/2}\sin\lambda\right] + O(\rho^{3/2}\ln\rho), \tag{28}$$

where C is an arbitrary constant and their domains are given by

$$D_{h_\lambda(l_0)} = \{\psi : \psi\in D^*_{\hat{h}(l_0)}(\mathbb{R}_+),\ \psi \text{ satisfy (28)}\}. \tag{29}$$

The spectrum of $\hat{h}_\lambda(l_0)$ is simple, discrete and is bounded from below, and

$$\operatorname{spec}\hat{h}_\lambda(l_0) = \left\{\mathcal{E}_m,\ m\in\begin{cases}\mathbb{Z}_+, & \lambda = \pm\pi/2\\ \{-1\}\cup\mathbb{Z}_+, & |\lambda|<\pi/2\end{cases}\right\},$$

where $\mathcal{E}_m$ are solutions of the equation $\omega_\lambda(\mathcal{E}_m) = 0$,

$$\omega_\lambda(W) = \cos\lambda[\psi(\alpha_0) - 2\psi(1)] - \sin\lambda,\quad \alpha_0 = 1/2 - W/2\gamma. \tag{30}$$

The limit $\lambda\to\pm\pi/2$ in this equation and its solutions are described by the equation $\psi^{-1}(\alpha_0) = 0$ or $1/2 - \mathcal{E}^{(\pm\pi/2)}_m/2\gamma = -m$, $m\in\mathbb{Z}_+$, and by the solution $\mathcal{E}^{(\pm\pi/2)}_m = \gamma(1+2m)$.

A qualitative description of the spectrum is given above. One can see that in each interval $(\mathcal{E}^{(\pm\pi/2)}_m, \mathcal{E}^{(\pm\pi/2)}_{m+1})$, $m\in\{-1\}\cup\mathbb{Z}_+$, there is one solution $\mathcal{E}_m$ (for a fixed $\lambda\in(-\pi/2,\pi/2)$) of equation (30) (we set formally $E^{(\pm\pi/2)}_{-1} = -\infty$); the solution $\mathcal{E}_m$ increases monotonically from $\mathcal{E}^{(\pm\pi/2)}_m + 0$ to $\mathcal{E}^{(\pm\pi/2)}_{m+1} - 0$ when λ changes from $\pi/2-0$ to $-\pi/2+0$.

The eigenfunctions $U^{(3)}_{\lambda,m}$,

$$\begin{aligned} U^{(3)}_{\lambda,m} &= Q_{\lambda,m}\left[u_1(\rho;\mathcal{E}_m)\sin\lambda + u_3(\rho;\mathcal{E}_m)\cos\lambda\right],\\ u_1(\rho;W) &= (\gamma/2)^{1/4}\,\rho^{1/2}\mathrm{e}^{-\gamma\rho^2/4}\Phi(\alpha_0,1;\gamma\rho^2/2),\\ u_3(\rho;W) &= u_1(\rho;W)\ln\left(\sqrt{\gamma/2}\rho\right) + (\gamma/2)^{1/4}\rho^{1/2}\mathrm{e}^{-\gamma\rho^2/4}\\ &\quad\times\partial_\mu\Phi(1/2+\mu-W/2\gamma, 1+2\mu;\gamma\rho^2/2)\big|_{\mu=0}, \end{aligned}$$

$$Q_{\lambda,m} = \left[-\frac{\tilde{\omega}_\lambda(\mathcal{E}_m)}{\sqrt{2\gamma}\,\omega'_\lambda(\mathcal{E}_m)}\right]^{1/2},$$
$$\tilde{\omega}_\lambda(W) = \sin\lambda[\psi(\alpha_0) - 2\psi(1)] + \cos\lambda, \tag{31}$$

of the Hamiltonians $\hat{h}_\lambda(l_0)$ form a complete orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+)$.

We note that the spectrum and eigenfunctions in the case of $\lambda = \pm\pi/2$ can be obtained from the respective formulae for the first region in the formal limit $l \to 0$.

Complete spectrum and inversion formulae. In previous subsubsections, we constructed all the s.a. radial Hamiltonians associated with the s.a. differential operation $\check{h}(l)$ as s.a. extensions of the symmetric operator $\hat{h}(l)$ for any $l \in \mathbb{Z}$ and for any ϕ_0, μ and B. We assemble our previous results into two groups.

For $\mu = 0$, we have

$$\hat{h}_1(l), \quad l \neq l_0 = 0, \quad D_{h_1(l)} = D^*_{\check{h}(l)}(\mathbb{R}_+),$$
$$\hat{h}_\lambda(l_0), \quad \lambda \in \mathbb{S}(-\pi/2, \pi/2),$$

and the domain $D_{h_\lambda(l_0)}$ is given by equation (29).

For $\mu > 0$, we have

$$\hat{h}_1(l), \quad l \neq l_a = a = 0, -1, \quad D_{h_1(l)} = D^*_{\check{h}(l)}(\mathbb{R}_+),$$
$$\hat{h}_{\lambda_a}(l_a), \quad \lambda_a \in \mathbb{S}(-\pi/2, \pi/2),$$

and the domain $D_{h_{\lambda_a}(l_a)}$ is given by equation (25).

As a result, each set of possible s.a. radial Hamiltonians $\hat{h}_{\mathfrak{e}}(l)$ generates s.a. rotationally invariant Schrödinger operators $\hat{H}^\perp_{\mathfrak{e}} = M^{-1}\hat{\mathcal{H}}^\perp_{\mathfrak{e}}$ in accordance with relations (14) and (15). As in the case of the pure AB field where $B = 0$, we let E denote the spectrum points of $\hat{H}^\perp_{\mathfrak{e}}$.

It is convenient to change the indexing l, m of the spectrum points and eigenfunctions to l, n, as follows:

$$n = n(l,m) = \begin{cases} m, & l \leqslant -1, \\ m + l, & l \geqslant 0, \end{cases} \quad m \in \mathbb{Z}_+, \ l \in \mathbb{Z},$$
$$m = m(n,l) = \begin{cases} n, & l \leqslant -1, \\ n - l, & 0 \leqslant l \leqslant n, \end{cases} \quad n \in \mathbb{Z}_+, \ l \in \mathbb{Z}, \tag{32}$$

and then interchange their position, such that, finally, the indices l, m are replaced by indices n, l.

When writing formulae for eigenfunctions $\Psi_{n,l}$ of an operator $\hat{H}^\perp_{\mathfrak{e}}$ in terms of eigenfunctions $U_{l,m}$ of the operators $\hat{h}_{\mathfrak{e}}(l)$, we have to introduce the factor $(2\pi\rho)^{-1/2}\,\mathrm{e}^{\mathrm{i}\epsilon(\phi_0 - l)\varphi}$ in accordance with equation (7) and make the substitution $\mathcal{E}_{l,m} = ME_{n,l}$ for the corresponding spectrum points.

The final result is the following. There is a family of s.a. 2D Schrödinger operators $\hat{H}^\perp_{\mathfrak{e}}$ parameterized by real parameters λ_*, such that $\hat{H}^\perp_{\mathfrak{e}} = \hat{H}^\perp_{\lambda_*}$,

$$\hat{H}^\perp_{\lambda_*} = \sum_{l\in\mathbb{Z}, l\neq l_*}^{\oplus} \hat{H}^\perp(l) \oplus \sum_{l_*}^{\oplus} \hat{H}^\perp_{\lambda_*}(l_*),$$
$$\hat{H}^\perp(l) = M^{-1} S_l \hat{h}_1(l) S_l^{-1}, \ l \neq l_*,$$
$$\hat{H}^\perp_{\lambda_*}(l_*) = M^{-1} S_{l_*} \hat{h}_{\lambda_*}(l_*) S_{l_*}^{-1},$$
$$l_* = \begin{cases} l_0, & \mu = 0 \\ l_a, & \mu > 0 \end{cases},$$
$$\lambda_* = \begin{cases} \lambda \in \mathbb{S}(-\pi/2, \pi/2), & \mu = 0, \\ \lambda_a \in \mathbb{S}(-\pi/2, \pi/2), & \mu > 0. \end{cases} \tag{33}$$

The spectrum of $\hat{H}^\perp_{\lambda_*}$ is given by

$$\operatorname{spec} \hat{H}^\perp_{\lambda_*} = \{\cup_{l\in\mathbb{Z}, l\neq l_*}(E_{n,l}, n \in \mathbb{Z}_+)\} \cup \{\cup_{l=l_*}(E_n^{(\lambda_*)}, n \in \mathbb{Z}_+)\},$$
$$E_{n,l} = \gamma M^{-1}[1 + 2n + 2\theta(l)\mu], \quad l \leqslant n, \quad l \neq l_*,$$
$$\theta(l) = \begin{cases} 1, & l \geqslant 0, \\ 0, & l < 0, \end{cases} \tag{34}$$

$$E_n^{(\lambda)}: \begin{cases} \omega_\lambda(ME_n^{(\lambda)}) = 0, & |\lambda| < \pi/2, \\ E_n^{(\pm\pi/2)} = \gamma M^{-1}(1 + 2n), \end{cases} \quad \mu = 0,$$
$$\begin{cases} E_n^{(\lambda_a)} = M^{-1}\left[\tau_{a,n} + \gamma(a + \mu)\right], & \omega_{\lambda_a}(\tau_{a,n}) = 0, \\ E_n^{(\pm\pi/2)} = \gamma M^{-1}[1 + 2n + 2\theta(a)\mu], \end{cases}$$
$$n \in \mathbb{Z}_+, \ \mu > 0, \tag{35}$$

where $\omega_\lambda(W)$ and $\omega_{\lambda_a}(W)$ are given by respective equations (30) and (26).

A complete set of orthonormalized eigenfunctions of $\hat{H}^\perp_{\lambda_*}$ consists of the functions $\Psi_{n,l}(\boldsymbol{\rho})$, $l \neq l_*$, and $\Psi^{\lambda_*}_{n,l_*}(\boldsymbol{\rho})$,

$$\Psi_{n,l}(\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi\rho}}\mathrm{e}^{\mathrm{i}\epsilon(\phi_0 - l)\varphi} U^{(1)}_{l,m(n,l)}(\rho), \tag{36}$$

where $U^{(1)}_{l,m}(\rho)$ are given by equations (23) and (we note that $m(n, l_*) = n$)

$$\Psi^{\lambda}_{n,l_0}(\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi\rho}}\mathrm{e}^{\mathrm{i}\epsilon\phi_0\varphi} U^{(3)}_{\lambda,n}(\rho), \quad \mu = 0,$$
$$\Psi^{\lambda_a}_{n,l_a}(\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi\rho}}\mathrm{e}^{\mathrm{i}\epsilon(\phi_0 - l_a)\varphi} U^{(2)}_{\lambda_a,n}(\rho), \quad \mu > 0,$$

where $U^{(3)}_{\lambda,n}(\rho)$ and $U^{(2)}_{\lambda_a,n}(\rho)$ are given by respective equations (31) and (27), such that

$$\hat{H}^\perp_{\lambda_*}\Psi_{n,l}(\boldsymbol{\rho}) = E_{n,l}\Psi_{n,l}(\boldsymbol{\rho}), \quad l \neq l_*,$$
$$\hat{H}^\perp_{\lambda_*}(\boldsymbol{\rho})\Psi^{\lambda_*}_{n,l_*} = E_n^{(\lambda_*)}\Psi^{\lambda_*}_{n,l_*}(\boldsymbol{\rho}).$$

We note that for the case of $\lambda = \pm\pi/2$, $l = l_0 = 0$, and for the case of $\lambda_a = \pm\pi/2$, $l = l_a = a = 0, -1$, the energy eigenvalues $E_n^{(\lambda)}$ and $E_n^{(\lambda_a)}$ and the corresponding eigenfunctions Ψ^{λ}_n and $\Psi^{\lambda_a}_n$ are given by respective equations (34) and (36) extended to all values of l.

The corresponding inversion formulae have the form

$$\Psi(\boldsymbol{\rho}) = \sum_{l\in\mathbb{Z},\ l\neq l_*}\sum_{n\in\mathbb{Z}_+}\Phi_{n,l}\Psi_{n,l}(\boldsymbol{\rho}) + \sum_{l_*,n\in\mathbb{Z}_+}\Phi_{n,l_*}\Psi^{\lambda_*}_{n,l_*}(\boldsymbol{\rho}),$$
$$\Phi_{n,l} = \int \mathrm{d}\boldsymbol{\rho}\,\overline{\Psi_{n,l}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}), \quad l \neq l_*,$$
$$\Phi_{n,l_*} = \int \mathrm{d}\boldsymbol{\rho}\,\overline{\Psi^{\lambda_*}_{n,l_*}(\boldsymbol{\rho})}\Psi(\boldsymbol{\rho}),$$
$$\int \mathrm{d}\boldsymbol{\rho}\,|\Psi(\boldsymbol{\rho})|^2 = \sum_{l\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_+}\left|\Phi_{n,l}\right|^2, \quad \forall\Psi \in L^2(\mathbb{R}^2).$$

2.2. The 3D case

In three dimensions, we start with the differential operation $\check{H}$ (4). The initial symmetric operator $\hat{H}$ associated with $\check{H}$ is defined on the domain $D_H = \mathcal{D}(\mathbb{R}^3\backslash\mathbb{R}_z) \in \mathfrak{H} = L^2(\mathbb{R}^3)$, where $\mathcal{D}(\mathbb{R}^3\backslash\mathbb{R}_z)$ is the space of smooth and compactly supported functions vanishing in the neighborhood of the z-axis. The domain D_H is dense in $\mathfrak{H}$, and the symmetricity of $\hat{H}$ is obvious. An s.a. Schrödinger operator must be defined as an s.a. extension of $\hat{H}$.

There is an evident space symmetry in the classical description of the system, the symmetry with respect to rotations around the z-axis and translations along this axis, which is manifested as the invariance of the classical Hamiltonian under these space transformations. The key point in constructing a quantum description of the system is the requirement of the invariance of the Schrödinger operator under the same transformations. Namely, let $\mathbb{G}$ be the group of the above space transformations S: $\mathbf{r} \mapsto S\mathbf{r}$. This group is unitarily represented in $\mathfrak{H}$: if $S \in \mathbb{G}$, then the corresponding operator U_S is defined by

$$(U_S\psi)(\mathbf{r}) = \psi(S^{-1}\mathbf{r}), \quad \forall\psi \in \mathfrak{H}.$$

The operator $\hat{H}$ evidently commutes[6] with U_S for any S.

We search only for s.a. extensions $\hat{H}_{\mathfrak{e}}$ of $\hat{H}$ that also commute with U_S for any S. This condition is the explicit form of the invariance, or symmetry, of a Schrödinger operator under the space transformations. As in classical mechanics, this symmetry allows separating the cylindrical coordinates ρ, φ and z and reducing the 3D problem to a 1D radial problem. Let $L^2(\mathbb{R}\times\mathbb{R}_+)$ denote the space of square-integrable functions with respect to the Lebesgue measure $\mathrm{d}p_z\mathrm{d}\rho$ on $\mathbb{R}\times\mathbb{R}_+$, and let $V : \sum_{l\in\mathbb{Z}}^{\oplus} L^2(\mathbb{R}\times\mathbb{R}_+) \longmapsto \mathfrak{H}$ be the unitary operator defined by the relationship

$$(Vf)(\rho,\varphi,z) = \frac{1}{2\pi\sqrt{\rho}}\int_{\mathbb{R}_z}\mathrm{d}p_z\sum_{l\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}(\epsilon(\phi_0-l)\varphi+p_z z)} f(l,p_z,\rho).$$

Similarly to the preceding subsection, it is natural to expect that any s.a. Schrödinger operator $\hat{H}_{\mathfrak{e}}$ can be represented in the form

$$\hat{H}_{\mathfrak{e}} = V\int_{\mathbb{R}_z}\mathrm{d}p_z\sum_{l\in\mathbb{Z}}\hat{h}_{\mathfrak{e}}(l,p_z)V^{-1},$$

where $\hat{h}_{\mathfrak{e}}(l,p_z)$ for fixed l and p_z is an s.a. extension of the symmetric operator $\hat{h}(l,p_z) = \hat{h}(l) + p_z^2/2m_e$ in $L^2(\mathbb{R}_+)$ and the operator $\hat{h}(l)$ in $L^2(\mathbb{R}_+)$ is defined on the domain $D_{h(l)} = \mathcal{D}(\mathbb{R}_+)$, where it acts as

$$\check{h}(l) = -\partial_\rho^2 + \rho^{-2}\left[\left(l+\mu+\gamma\rho^2/2\right)^2 - 1/4\right].$$

[6] We remind the reader of the notion of commutativity in this case (where one of the operators, U_S, is bounded and defined everywhere): we say that the operators $\hat{H}$ and U_S commute if $U_S\hat{H} \subseteq \hat{H}U_S$; that is, if $\psi \in D_H$, then also $U_S\psi \in D_H$ and $U_S\hat{H}\psi = \hat{H}U_S\psi$.

The correct expression for $\hat{H}_{\mathfrak{e}}$ can be written in terms of a suitable direct integral,

$$\hat{H}_{\mathfrak{e}} = V\int_{\mathbb{R}_z}^{\oplus}\mathrm{d}p_z\sum_{l\in\mathbb{Z}}^{\oplus}\hat{h}_{\mathfrak{e}}(l,p_z)V^{-1}.$$

Its rigorous justification will be discussed in a publication of A Smirnov.

The inversion formulae in three dimensions are obtained by the following modifications to the 2D inversion formulae:

(1) $\sum_{l\in\mathbb{Z}}\int\mathrm{d}E \to \int\mathrm{d}p_z\sum_{l\in\mathbb{Z}}\int\mathrm{d}E^{\perp}$, where $E^{\perp}$ are spectrum points of 2D s.a. Schrödinger operators $\hat{H}_{\mathfrak{e}}^{\perp}$, whereas the eigenvalues (spectrum points) E of the 3D s.a. Schrödinger operators $\hat{H}_{\mathfrak{e}}$ are $E = E^{\perp} + p_z^2/2m$, $p_z \in \mathbb{R}$.
(2) The contribution of discrete spectrum points of the 2D s.a. Schrödinger operator $\hat{H}_{\mathfrak{e}}^{\perp}$ has to be multiplied by $\int dp_z$.
(3) Eigenfunctions of 2D s.a. Schrödinger operators $\hat{H}_{\mathfrak{e}}^{\perp}$ have to be multiplied by $(2\pi\hbar)^{-1/2}\,\mathrm{e}^{\mathrm{i}p_z z/\hbar}$ in order to obtain eigenfunctions of the 3D s.a. Schrödinger operators $\hat{H}_{\mathfrak{e}}$.
(4) The extension parameters λ_a and λ have to be replaced by the functions $\lambda_a(p_z)$ and $\lambda(p_z)$.

2.2.1. s.a. Schrödinger operators with the AB field. For the case of $\mu = 0$, there is a family of s.a. 3D Schrödinger operators parameterized by a real-valued function $\lambda(p_z) \in \mathbb{S}(-\pi/2,\pi/2)$, $p_z \in \mathbb{R}$.

The spectrum of $\hat{H}_{\lambda(p_z)}$ is given by

$$\operatorname{spec}\hat{H}_{\{\lambda(p_z)\}} = \mathbb{R}_+ \cup \left\{\begin{array}{l} p_z^2/2m_e - 4M^{-1}\kappa_0^2\exp\left[2(\tan\lambda(p_z)-\mathbf{C})\right], \\ \quad |\lambda(p_z)| < \pi/2 \\ \varnothing, \quad \lambda(p_z) = \pm\pi/2 \end{array}\right\}.$$

A complete system of orthonormalized generalized eigenfunctions of $\hat{H}_{\lambda(p_z)}$ consists of functions $\Psi_{l,p_z,E^{\perp}}(\mathbf{r})$, $l \neq l_0$, and $\Psi^{\lambda(p_z)}_{l_0,p_z,E^{\perp}}(\mathbf{r})$,

$$\Psi_{l,p_z,E^{\perp}}(\mathbf{r}) = \left(8\pi^2\hbar/M\right)^{-1/2}\mathrm{e}^{\mathrm{i}p_z z/\hbar+\mathrm{i}\epsilon_q(\phi_0-l)\varphi}J_{\varkappa_l}(\sqrt{ME^{\perp}}\rho),$$

$$\Psi^{\lambda(p_z)}_{l_0,p_z,E^{\perp}}(\mathbf{r}) = (8\pi^2\hbar(\tilde{\lambda}^2+\pi^2/4)/M)^{-1/2}\mathrm{e}^{\mathrm{i}p_z z/\hbar+\mathrm{i}\epsilon_q\phi_0\varphi} \times\left[\tilde{\lambda}J_0\left(\sqrt{ME^{\perp}}\rho\right)+\frac{\pi}{2}N_0(\sqrt{ME^{\perp}}\rho)\right],$$

$$\tilde{\lambda} = \tan\lambda(p_z) - \mathbf{C} - \ln\left(\sqrt{ME^{\perp}}/2\kappa_0\right),$$

and functions $\Psi^{\lambda(p_z)}_{l_0,p_z}(\mathbf{r})$,

$$\Psi^{\lambda(p_z)}_{l_0,p_z}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\hbar}}\mathrm{e}^{\mathrm{i}p_z z/\hbar+\mathrm{i}\epsilon_q\phi_0\varphi} \times\left\{\begin{array}{l} \sqrt{2M^2|E^{\perp(-)}_{\lambda(p_z)}|}K_0\left(\sqrt{M|E^{\perp(-)}_{\lambda(p_z)}|}\rho\right), \\ \quad |\lambda(p_z)| < \pi/2 \\ 0, \quad \lambda(p_z) = \pm\pi/2 \end{array}\right.,$$

$$E^{\perp(-)}_{\lambda(p_z)} = -4M^{-1}\kappa_0^2\exp 2(\tan\lambda(p_z)-\mathbf{C}),$$

such that

$$\check{H}\Psi_{l,p_z,E^\perp}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^\perp\right)\Psi_{l,p_z,E^\perp}(\mathbf{r}), \quad E^\perp \geqslant 0,$$
$$\check{H}\Psi^{\lambda(p_z)}_{l_0,p_z,E^\perp}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^\perp\right)\Psi^{\lambda(p_z)}_{l_0,p_z,E^\perp}(\mathbf{r}), \quad E^\perp \geqslant 0,$$
$$\check{H}\Psi^{\lambda(p_z)}_{l_0,p_z}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^{\perp(-)}_{\lambda(p_z)}\right)\Psi^{\lambda(p_z)}_{l_0,p_z}(\mathbf{r}).$$

The corresponding inversion formulae have the form

$$\Psi(\mathbf{r}) = \int \mathrm{d}p_z \left[\sum_{l\in\mathbb{Z},\ l\neq 0}\int_0^\infty \Phi_{l,p_z}(E^\perp)\Psi_{l,p_z,E^\perp}(\mathbf{r})\mathrm{d}E^\perp + \int_0^\infty \Phi_{l_0,p_z}(E^\perp)\Psi^{\lambda(p_z)}_{l_0,p_z,E^\perp}(\mathbf{r})\mathrm{d}E^\perp + \Phi_{l_0,p_z}\Psi^{\lambda(p_z)}_{l_0,p_z}(\mathbf{r})\right], \quad \forall\Psi\in L^2(\mathbb{R}^3),$$
$$\Phi_{l,p_z}(E^\perp) = \int \overline{\Psi_{l,p_z,E^\perp}(\mathbf{r})}\Psi(\mathbf{r})\mathrm{d}\mathbf{r}, \quad l\neq l_0,$$
$$\Phi_{l,p_z}(E^\perp) = \int \overline{\Psi^{\lambda(p_z)}_{l_0,p_z,E^\perp}(\mathbf{r})}\Psi(\mathbf{r})\mathrm{d}\mathbf{r},$$
$$\Phi_{l_0,p_z} = \int \overline{\Psi^{\lambda(p_z)}_{l_0,p_z}(\mathbf{r})}\Psi(\mathbf{r})\,\mathrm{d}\mathbf{r},$$
$$\int |\Psi(\mathbf{r})|^2\,\mathrm{d}\mathbf{r} = \int \mathrm{d}p_z\left[\sum_{l\in\mathbb{Z}}\int_0^\infty \left|\Phi_{l,p_z}(E^\perp)\right|^2\mathrm{d}E^\perp + \left|\Phi_{l_0,p_z}\right|^2\right].$$

For the case $\mu > 0$, there is a family of s.a. 3D Hamiltonians $\hat{H}_{\{\lambda_a(p_z)\}}$ parameterized by two real-valued functions $\lambda_a(p_z)\in\mathbb{S}(-\pi/2,\pi/2)$, $a=0,-1$, $p_z\in\mathbb{R}$.

The spectrum of $\hat{H}_{\{\lambda_a(p_z)\}}$ is given by

$$\operatorname{spec}\hat{H}_{\{\lambda_a(p_z)\}} = \left\{\begin{array}{c} p_z^2/2m_\mathrm{e} - 4M^{-1}k_0^2|\tilde{\lambda}_a|^{-\varkappa_a^{-1}}, \\ \lambda_a(p_z)\in(-\pi/2,0) \\ \varnothing,\ \lambda_a(p_z)\notin(-\pi/2,0) \end{array}\right\}\cup\mathbb{R}_+,$$
$$\varkappa_a = |\mu + a|,\ \tilde{\lambda}_a = \Gamma(1-\varkappa_a)\Gamma^{-1}(1+\varkappa_a)\tan\lambda_a(p_z),$$
$$\varkappa_a = |\mu + a|.$$

A complete orthonormalized system in $L^2(\mathbb{R}^3)$ consists of both generalized eigenfunctions $\Psi_{l,p_z,E^\perp}(\mathbf{r})$, $l\neq l_a$, and $\Psi^{\lambda_a(p_z)}_{l_a,p_z,E^\perp}(\mathbf{r})$,

$$\Psi_{l,p_z,E^\perp}(\mathbf{r}) = \left(8\pi^2\hbar/M\right)^{-1/2}\mathrm{e}^{\mathrm{i}p_z z/\hbar + \mathrm{i}\epsilon_q(\phi_0 - l)\varphi}J_{\varkappa_l}(\sqrt{ME^\perp}\rho),$$
$$\Psi^{\lambda_a(p_z)}_{l_a,p_z,E^\perp}(\mathbf{r}) = \left(8\pi^2\hbar Q_a\right)^{-1/2}\mathrm{e}^{\mathrm{i}p_z z/\hbar + \mathrm{i}\epsilon_q(\phi_0 - l_a)\varphi} \times\left[J_{\varkappa_a}\left(\sqrt{ME^\perp}\rho\right) + \left(\sqrt{ME^\perp}/2\kappa_0\right)^{2\varkappa_a} \times\tilde{\lambda}_a\,J_{-\varkappa_a}\left(\sqrt{ME^\perp}\rho\right)\right],$$
$$Q_a = 1 + 2\left(ME^\perp/4\right)^{\varkappa_a}\tilde{\lambda}_a\cos(\pi\varkappa_a) + (ME/4)^{2\varkappa_a}\tilde{\lambda}_a^2,$$

and eigenfunctions $\Psi^{\lambda_a(p_z)}_{l_a,p_z}(\mathbf{r})$,

$$\Psi^{\lambda_a(p_z)}_{l_a,p_z}(\mathbf{r}) = \left(2\pi^2\hbar\right)^{-1}\mathrm{e}^{\mathrm{i}p_z z/\hbar + \mathrm{i}(l_a + \epsilon_q\phi_0)\varphi} \times\begin{cases}\sqrt{\frac{M^2|E^{\perp(-)}_{\lambda_a(p_z)}|\sin(\pi\varkappa_a)}{2\pi\varkappa_a}}K_{\varkappa_a}\left(\sqrt{M|E^{\perp(-)}_{\lambda_a(p_z)}|}\rho\right), \\ \quad \lambda_a(p_z)\in(-\pi/2,0), \\ 0,\ \lambda_a(p_z)\notin(-\pi/2,0),\end{cases}$$
$$E^{\perp(-)}_{\lambda_a(p_z)} = -4M^{-1}\kappa_0^2\exp 2(\tan\lambda_a(p_z) - \mathbf{C}),$$

such that

$$\check{H}\Psi_{l,p_z,E^\perp}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^\perp\right)\Psi_{l,p_z,E^\perp}(\mathbf{r}), \quad E^\perp \geqslant 0,$$
$$\check{H}\Psi^{\lambda_a(p_z)}_{l_a,p_z,E^\perp}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^\perp\right)\Psi^{\lambda_a(p_z)}_{l_a,p_z,E^\perp}(\mathbf{r}), \quad E^\perp \geqslant 0,$$
$$\check{H}\Psi^{\lambda_a(p_z)}_{l_a,p_z}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^{\perp(-)}_{\lambda_a(p_z)}\right)\Psi^{\lambda_a(p_z)}_{l_a,p_z}(\mathbf{r}).$$

The corresponding inversion formulae have the form

$$\Psi(\mathbf{r}) = \int \mathrm{d}p_z\left[\sum_{l\in\mathbb{Z},\ l\neq l_a}\int_0^\infty \Phi_{l,p_z}(E^\perp)\Psi_{l,p_z,E^\perp}(\mathbf{r})\mathrm{d}E^\perp + \sum_a\int_0^\infty \Phi_{l_a,p_z}(E^\perp)\Psi^{\lambda_a(p_z)}_{l_a,p_z,E^\perp}(\mathbf{r})\mathrm{d}E^\perp + \Phi_{l_a,p_z}\Psi^{\lambda_a(p_z)}_{l_a,p_z}(\mathbf{r})\right], \quad \forall\Psi\in L^2(\mathbb{R}^3),$$
$$\Phi_{l,p_z}(E^\perp) = \int \mathrm{d}\mathbf{r}\overline{\Psi_{l,p_z,E^\perp}(\mathbf{r})}\Psi(\mathbf{r}), \quad E^\perp\geqslant 0, \quad l\neq l_a,$$
$$\Phi_{l_a,p_z}(E^\perp) = \int \mathrm{d}\mathbf{r}\overline{\Psi^{\lambda_a(p_z)}_{l_a,p_z,E^\perp}(\mathbf{r})}\Psi(\mathbf{r}), \quad E^\perp\geqslant 0,$$
$$\Phi_{l_a,p_z} = \int \mathrm{d}\mathbf{r}\overline{\Psi^{\lambda_a(p_z)}_{l_a,p_z}(\mathbf{r})}\Psi(\mathbf{r}),$$
$$\int \mathrm{d}\mathbf{r}\,|\Psi(\mathbf{r})|^2 = \int \mathrm{d}p_z\left[\sum_{l\in\mathbb{Z}}\int_0^\infty \left|\Phi_{l,p_z}(E^\perp)\right|^2\mathrm{d}E^\perp + \sum_a\left|\Phi_{l_a,p_z}\right|^2\right].$$

2.2.2. s.a. Schrödinger operators with MSF. There is a family of s.a. 3D Schrödinger operators $\hat{H}_{\lambda_*(p_z)}$ parameterized by real-valued functions $\lambda_*(p_z)\in\mathbb{S}(-\pi/2,\pi/2)$, $p_z\in\mathbb{R}$), where λ_* are defined by equation (33).

The spectrum of $\hat{H}_{\lambda_*(p_z)}$ is given by

$$\operatorname{spec}\hat{H}_{\lambda_*(p_z)} = \left\{p_z^2/2m_\mathrm{e} + E_n^{\perp\lambda_*(p_z)},\ n\in\mathbb{Z}_+\right\}\cup[\gamma M^{-1},\infty),$$

where $E_n^{\perp\lambda_*(p_z)}$ are defined by equations (34) and (35) with the substitution $\lambda_*\to\lambda_*(p_z)$.

A complete system of generalized orthonormalized eigenfunctions of $\hat{H}_{\lambda_*(p_z)}$ consists of functions $\Psi_{p_z,l,n}(\mathbf{r})$, $l\neq l_*$, and $\Psi^{\lambda_*(p_z)}_{p_z,l_*,,n}(\mathbf{r})$, $n\in\mathbb{Z}_+$,

$$\Psi_{p_z,l,n}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\hbar\rho}}\mathrm{e}^{\mathrm{i}p_z z/\hbar + \epsilon(\phi_0 - l)\varphi}U^{(1)}_{l,m(n,l)}(\rho), \quad l\neq l_*, \tag{37}$$

where l_* are defined by equation (33), $m(n,l)$ is given by (32) and $U^{(1)}_{l,m}(\rho)$ are given by equations (23),

$$\Psi^{\lambda(p_z)}_{p_z,l_0,n}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\hbar\rho}}\mathrm{e}^{\mathrm{i}p_z z/\hbar+\mathrm{i}\epsilon\phi_0\varphi}U^{(3)}_{\lambda(p_z),n}(\rho), \quad \mu=0,$$

$$\Psi^{\lambda_a(p_z)}_{p_z,l_a,n}(\boldsymbol{\rho}) = \frac{1}{2\pi\sqrt{\hbar\rho}}\mathrm{e}^{\mathrm{i}p_z z/\hbar+\mathrm{i}\epsilon(\phi_0-l_a)\varphi}U^{(2)}_{\lambda_a(p_z),n}(\rho), \quad \mu>0,$$

where $U^{(3)}_{\lambda(p_z),n}(\rho)$ and $U^{(2)}_{\lambda_a(p_z),n}(\rho)$ are given by the respective equations (31) and (27) with the substitution $\lambda_* \to \lambda_*(p_z)$, such that

$$\check{H}\Psi_{p_z,l,n}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^{\perp}_{n,l}\right)\Psi_{p_z,l,n}(\mathbf{r}), \quad l\neq l_*,$$

$$\check{H}\Psi^{\lambda_*(p_z)}_{p_z,l_*,,n}(\mathbf{r}) = \left(p_z^2/2m_\mathrm{e} + E^{\perp\lambda_*(p_z)}_{n}\right)\Psi^{\lambda_*(p_z)}_{p_z,l_*,,n}(\mathbf{r}), \tag{38}$$

where

$$E^{\perp}_{n,l} = \gamma M^{-1}[1+2n+2\theta(l)\mu], \quad l\leqslant n, \quad l\neq l_*, \ n\in\mathbb{Z}_+. \tag{39}$$

We note that for $\lambda(p_z)=\lambda_a(p_z)=\pm\pi/2$, the energy eigenvalues and corresponding eigenfunctions $\Psi_{p_z,l,n}(\mathbf{r})$ are given by equations (38), (39) and (37) extended to all values of l.

The corresponding inversion formulae have the form

$$\Psi(\mathbf{r}) = \int \mathrm{d}p_z \sum_{n\in\mathbb{Z}_+}\left[\sum_{l\in\mathbb{Z},\ l\neq l_*}\Phi_{p_z,l,n}\Psi_{p_z l,m(n,l)}(\mathbf{r}) + \sum_{l=l_*}\Phi_{p_z,l_*,n}\Psi^{\lambda_*(p_z)}_{p_z,l_*,n}(\mathbf{r})\right],$$

$$\Phi_{p_z,l,n} = \int \mathrm{d}\mathbf{r}\overline{\Psi_{p_z,l,n}(\mathbf{r})}\Psi(\mathbf{r}), \quad l\neq l_*,$$

$$\Phi_{p_z,l_*,n} = \int d\mathbf{r}\overline{\Psi^{\lambda(p_z)}_{p_z,l_*,n}(\mathbf{r})}\Psi(\mathbf{r}),$$

$$\int \mathrm{d}\mathbf{r}\,|\Psi(\mathbf{r})|^2 = \int \mathrm{d}p_z \sum_{l\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_+}\left|\Phi_{p_z,l,n}\right|^2, \quad \forall\Psi\in L^2(\mathbb{R}^3).$$

3. S.a. Dirac Hamiltonians with MSF

3.1. Generalities

In this section, we set $c=\hbar=1$. Written in the form of the Schrödinger equation, the Dirac equation with the MSF reads

$$\mathrm{i}\frac{\partial\Psi(x)}{\partial t} = \check{H}\Psi(x), \quad x=\left(x^0,\mathbf{r}\right),$$
$$\mathbf{r}=\left(x^k,\ k=1,2,3\right), \quad x^0=t,$$

where $\Psi(x)=\{\psi_\alpha(x),\ \alpha=1,\ldots,4\}$ is a four-spinor and $\check{H}$ is the s.a. Dirac differential operation, the 'formal Dirac Hamiltonian',

$$\check{H} = \boldsymbol{\alpha}\left(\check{\mathbf{p}}-\epsilon_q e\mathbf{A}\right)+m_\mathrm{e}\beta,$$

where the vector potential $\mathbf{A}$ is given by (2), $\boldsymbol{\alpha}=\left(\gamma^0\gamma^k,\ k=1,2,3\right)$, $\beta=\gamma^0$ and γ^μ, $\mu=0,1,2,3$, are Dirac γ matrices.

The space of quantum states for a particle is the Hilbert space $\mathfrak{H}=\boldsymbol{L}^2\left(\mathbb{R}^3\right)$ of square-integrable bispinors $\Psi(\mathbf{r})$ with the scalar product

$$(\Psi_1,\Psi_2) = \int \mathrm{d}\mathbf{r}\Psi_1^+(\mathbf{r})\Psi_2(\mathbf{r}), \ \mathrm{d}\mathbf{r}=\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3=\rho\mathrm{d}\,\rho\mathrm{d}\varphi\,\mathrm{d}z,$$

where ρ,φ and z are the cylindric coordinates. The Hilbert space $\mathfrak{H}$ can be presented as

$$\mathfrak{H} = \sum_{\alpha=1}^{4}{}^{\oplus}\,\mathfrak{H}_\alpha, \quad \mathfrak{H}_\alpha = L^2(\mathbb{R}^3).$$

Our first aim is to construct all s.a. Dirac operators (Dirac Hamiltonians) associated with the s.a. differential operation $\check{H}$ using the general approach presented in [25]. In particular, the construction is based on the known spatial symmetry in the problem[7] , which allows separating the cylindric coordinates ρ,φ and z.

It is convenient to choose the following representation for γ matrices:

$$\gamma^0=\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}, \quad \gamma^1=\begin{pmatrix}\mathrm{i}\sigma^2 & 0\\ 0 & -\mathrm{i}\sigma^2\end{pmatrix},$$
$$\gamma^2=\begin{pmatrix}-\mathrm{i}\sigma^1 & 0\\ 0 & \mathrm{i}\sigma^1\end{pmatrix}, \quad \gamma^3=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},$$
$$\gamma^5=-\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3=-\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \quad \Sigma^3=\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}.$$

Written in cylindric coordinates, the differential operation $\check{H}$ then becomes

$$\check{H} = \mathrm{diag}\left(Y+m_\mathrm{e}\sigma^3,\ Y-m_\mathrm{e}\sigma^3\right)+\check{p}_z\ \mathrm{antidiag}(\sigma^3,\sigma^3),$$

where

$$Y = Q[\sigma^3\partial_\rho+\rho^{-1}(\mathrm{i}\partial_\varphi+\epsilon_q\tilde{\phi})],$$
$$Q=\sigma^1\sin\varphi-\sigma^2\cos\varphi, \quad Q^2=1,$$

and

$$\epsilon_q\tilde{\phi} = \epsilon(\phi_0+\mu+\gamma\rho^2/2), \quad \phi_0=[\epsilon_B\phi]=\epsilon_B\phi-\mu,$$
$$0\leqslant\mu<1, \quad \gamma=e|B|>0.$$

This operation commutes with the s.a. differential operations

$$\check{p}_z = -\mathrm{i}\partial_z,\ \check{S}_z=\gamma^5\left(\gamma^3-m_\mathrm{e}^{-1}\check{p}_z\right),$$
$$\check{J}_z = -\mathrm{i}\partial_\varphi+\tfrac{1}{2}\Sigma^3=\mathrm{diag}\left(\check{\jmath}_z,\check{\jmath}_z\right), \quad \check{\jmath}_z=-\mathrm{i}\partial_\varphi+\sigma^3/2,$$

where $\Sigma^3=\mathrm{diag}(\sigma^3,\sigma^3)$.

We pass to the p_z representation for bispinors, $\Psi(\mathbf{r})\to\tilde{\Psi}(p_z,\boldsymbol{\rho})$,

$$\Psi(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\int \mathrm{e}^{\mathrm{i}p_z z}\tilde{\Psi}(p_z,\boldsymbol{\rho})\mathrm{d}p_z,$$
$$\tilde{\Psi}(p_z,\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi}}\int \mathrm{e}^{-\mathrm{i}p_z z}\Psi(\mathbf{x})\mathrm{d}z.$$

[7] By the spatial symmetry, we mean the invariance under rotations around the solenoid axis and under the translations along this axis.

In this representation, the operation $\check{J}_z$ is the same, while $\check{H}$ and $\check{S}_z$, respectively, become

$$\begin{aligned}\check{H} \to \check{H}\,(p_z) &= \check{H} = \mathrm{diag}\left(Y + m_\mathrm{e}\sigma^3, Y - m_\mathrm{e}\sigma^3\right)\\ &\quad + p_z\ \mathrm{antidiag}\left(\sigma^3, \sigma^3\right),\\ \check{S}_z \to \check{S}_z\,(p_z) &= m_\mathrm{e}^{-1} p_z\ \mathrm{antidiag}\ (I, I) + \mathrm{diag}(I, -I).\end{aligned}$$

We decompose the bispinor $\tilde{\Psi}(p_z, \boldsymbol{\rho})$ for a fixed p_z into two orthogonal components that are the eigenvectors for the spin matrix $\hat{S}_z(p_z)$:

The space of four-spinors $\tilde{\Psi}(p_z, \boldsymbol{\rho})$ with a fixed p_z is the direct orthogonal sum of two eigenspaces of $\hat{S}_z(p_z)$,

$$\tilde{\Psi}(p_z, \boldsymbol{\rho}) = \tilde{\Psi}_1(p_z, \boldsymbol{\rho}) + \tilde{\Psi}_{-1}(p_z, \boldsymbol{\rho}),$$

where

$$\begin{aligned}\tilde{\Psi}_1(p_z, \boldsymbol{\rho}) &= \left(\frac{M + m_\mathrm{e}}{2M}\right)^{1/2} \begin{pmatrix} \chi_1 \\ p_z\,(M + m_\mathrm{e})^{-1}\,\chi_1 \end{pmatrix}\\ &= \chi_1(p_z, \boldsymbol{\rho}) \otimes e_1(p_z),\\ \tilde{\Psi}_{-1}(p_z, \boldsymbol{\rho}) &= \left(\frac{M + m_\mathrm{e}}{2M}\right)^{1/2} \begin{pmatrix} -p_z\,(M + m_\mathrm{e})^{-1}\,\chi_{-1} \\ \chi_{-1} \end{pmatrix}\\ &= \chi_{-1}(p_z, \boldsymbol{\rho}) \otimes e_{-1}(p_z),\\ e_1(p_z) &= \left(\frac{M + m_\mathrm{e}}{2M}\right)^{1/2} \begin{pmatrix} 1 \\ p_z\,(M + m_\mathrm{e})^{-1} \end{pmatrix},\\ e_{-1}(p_z) &= -\,\mathrm{i}\sigma^2 e_1(p_z),\end{aligned} \tag{40}$$

and $e_s(p_z), s = \pm 1$, are two orthonormalized bispinors, $e_r^+(p_z) e_s^+(p_z) = \delta_{rs}$, and $\chi_s(p_z, \boldsymbol{\rho})$ are some doublets. The space of bispinors $\tilde{\Psi}(p_z, \boldsymbol{\rho})$ with a fixed p_z is the direct orthogonal sum of two eigenspaces of $\hat{S}_z(p_z)$,

$$\hat{S}_z\,(p_z)\,\tilde{\Psi}_s(p_z, \boldsymbol{\rho}) = s\frac{M}{m_\mathrm{e}}\tilde{\Psi}_s(p_z, \boldsymbol{\rho}),$$

$$M = \sqrt{m_\mathrm{e}^2 + p_z^2}, \quad s = \pm 1.$$

We thus obtain a one-to-one correspondence between bispinors $\Psi(\mathbf{r})$ and pairs of doublets $\chi_s(p_z, \boldsymbol{\rho})$,

$$\Psi(\mathbf{r}) \Longleftrightarrow \tilde{\Psi}_s(p_z, \boldsymbol{\rho}) \Longleftrightarrow \chi_s(p_z, \boldsymbol{\rho}),$$

such that $\|\,\Psi\,\|^2 = \sum_s \|\,\chi_s\,\|^2 = \sum_s \int \mathrm{d}p_z\,\mathrm{d}\boldsymbol{\rho}\,\chi_s^+(p_z, \boldsymbol{\rho})\chi_s(p_z, \boldsymbol{\rho})$.

The differential operations $\check{H}$ and $\check{J}_z$ induce the differential operations $\check{\mathbf{h}}$ and $\check{j}_z$ in the space of doublets $\chi_s(p_z, \boldsymbol{\rho})$:

$$\check{H}\,(p_z)\,\tilde{\Psi}_s = \check{\mathbf{h}}\,(s, p_z)\,\chi_s \otimes e_s, \quad \check{J}_z\,(p_z)\,\tilde{\Psi}_s = \check{j}_z \chi_s \otimes e_s,$$

$$\check{\mathbf{h}}\,(s, p_z) = Q\left[\sigma^3\partial_\rho + \rho^{-1}\left(\mathrm{i}\partial_\varphi + \epsilon_q\tilde{\phi}\right)\right] + sM\sigma^3.$$

The s.a. operator $\hat{j}_z$ associated with $\check{j}_z$ has a discrete spectrum, its eigenvalues are all half-integers labeled here by integers l as $\epsilon(\phi_0 - l + 1/2)$,

$$\hat{j}_z\xi_l(\varphi) = [\epsilon(\phi_0 - l + 1/2)]\xi_l(\varphi), \quad l \in \mathbb{Z}.$$

It is convenient to represent vectors $\xi_l(\varphi) \equiv \xi_l(p_z, \rho, \varphi)$ of the corresponding eigenspaces, as

$$\begin{aligned}\xi_l(\varphi) &= (2\pi)^{-1/2}\,\mathrm{e}^{\mathrm{i}[\epsilon(\phi_0 - l + 1/2) - \sigma^3/2]\varphi}\vartheta_l\\ &= S_l(\varphi)\frac{1}{\sqrt{2\pi\rho}}F(l, p_z, \rho),\\ S_l(\varphi) &= \mathrm{e}^{\mathrm{i}\epsilon(\phi_0 - l + 1/2)\varphi}\ \mathrm{antidiag}\left(i\,\mathrm{e}^{\mathrm{i}\varphi/2}, -\mathrm{e}^{-\mathrm{i}\varphi/2}\right),\\ S_l^+(\varphi)\,S_l(\varphi) &= I,\end{aligned} \tag{41}$$

where $\vartheta_l = \vartheta_l(p_z, \rho)$ and $F(l, p_z, \rho)$ are arbitrary doublets independent of φ.

The space of doublets $\chi_s(p_z, \boldsymbol{\rho})$ is a direct orthogonal sum of the eigenspaces of the operator $\hat{j}_z$, which means that the doublets allow the representations

$$\chi_s(p_z, \boldsymbol{\rho}) = \sum_{l\in\mathbb{Z}} \frac{1}{\sqrt{2\pi\rho}} S_l(\varphi) F(s, l, p_z, \rho),$$

and the factor $1/\sqrt{2\pi\rho}$ is introduced for further convenience.

The operation $\check{\mathbf{h}}\,(s, p_z)$ induces an operation $\check{h}(s, l)$ ('radial Hamiltonian' depending on the parameter p_z as well) in the space of doublets F,

$$\begin{aligned}\check{\mathbf{h}}\,(s, p_z)\,\chi_s &= \sum_{l\in\mathbb{Z}} \frac{1}{\sqrt{2\pi\rho}} S_l(\varphi)\check{h}(s, l) F(s, l, p_z; \rho),\\ \check{h}(s, l) &= \mathrm{i}\sigma^2\partial_\rho + \epsilon(\gamma\rho/2 + \rho^{-1}\varkappa_l)\sigma^1 - sM\sigma^3,\end{aligned} \tag{42}$$

where $\kappa_l = l + \mu - 1/2$.

In the Hilbert space $\mathbb{L}^2\,(\mathbb{R}_+) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ of doublets $F(\rho)$ (with p_z fixed), we define the initial symmetric radial Hamiltonian $\hat{h}(s, l)$ associated with the s.a. differential operation $\check{h}(s, l)$ and acting on the domain $D_{h(s,l)}$,

$$D_{h(s,l)} = \mathfrak{D}(\mathbb{R}_+) = \mathcal{D}(\mathbb{R}_+) \oplus \mathcal{D}(\mathbb{R}_+). \tag{43}$$

3.2. Solutions of radial equations

I. We first consider the homogeneous equation

$$[\check{h}(s, l) - W]F(\rho) = 0 \tag{44}$$

and some of its useful solutions.

We let f and g denote the respective upper and lower components of doublets F, $F = (f/g)$. Then equation (44) is equivalent to the set of radial equations for the doublet components

$$\begin{aligned}f' - \epsilon(\gamma\rho/2 + \rho^{-1}\varkappa_l)f + (W - sM)g &= 0,\\ g' + \epsilon(\gamma\rho/2 + \rho^{-1}\varkappa_l)g - (W + sM)f &= 0,\end{aligned} \tag{45}$$

where the prime denotes derivatives with respect to ρ.

We let $\check{h}_+ = \check{h}_+(s, l)$ and $\check{h}_- = \check{h}_-(s, l)$ denote the differential operation $\check{h}$ with $\epsilon = 1$ and $\epsilon = -1$, respectively. We then have

$$\begin{aligned}\check{h}_+(s, l) &= \mathrm{i}\sigma^2\partial_\rho + \left(\gamma\rho/2 + \rho^{-1}\varkappa_l\right)\sigma^1 - sM\sigma^3,\\ \check{h}_-(s, l) &= \mathrm{i}\sigma^2\partial_\rho - \left(\gamma\rho/2 + \rho^{-1}\varkappa_l\right)\sigma^1 - sM\sigma^3\\ &= \mathrm{i}\sigma^2\left[\mathrm{i}\sigma^2\partial_\rho + (\gamma\rho/2 + \rho^{-1}\varkappa)\sigma^1 + sM\sigma^3\right]\left(\mathrm{i}\sigma^2\right)^+\\ &= \mathrm{i}\sigma^2\check{h}_+(-s, l)\left(\mathrm{i}\sigma^2\right)^+.\end{aligned}$$

It follows that solutions $F_- = F_-(s, l, E_-(s); \rho)$ of the equation $[\check{h}_- - E_-(s)]F_- = 0$ are bijectively related to solutions $F_+ = F_+(s, l, , E_+(s); \rho)$ of the equation $[\check{h}_+ - E_+(s)]F_+ = 0$ as follows:

$$F_-(s, l, E_-(s); \rho) = \mathrm{i}\sigma^2 F_+(-s, l, E_+(-s); \rho),$$
$$E_-(s) = E_+(-s).$$

That is why we consider below the case of $\epsilon = \mathrm{sgn}\,(qB) = 1$ only and omit the subscript '+'.

The set (45) can be reduced to second-order differential equations for both f and g. For example, we have the following set of equations equivalent to (45):

$$f'' - \left[(\gamma\rho/2)^2 + \frac{\varkappa_l(\varkappa_l - 1)}{\rho^2} - w + \gamma\left(\varkappa_l + \frac{1}{2}\right)\right] f = 0,$$
$$g = (W - sM)^{-1}\left[-f' + (\gamma\rho/2 + \rho^{-1}\varkappa_l) f\right], \quad w = W^2 - M^2. \tag{46}$$

By the substitution

$$f(\rho) = z^{a/2}\mathrm{e}^{-z/2} p(z), \quad z = \gamma\rho^2/2, \quad a = 1/2 \pm (\varkappa_l - 1/2),$$

we reduce the first equation (46) to the equation for $p(z)$ that is the equation for confluent hypergeometric functions,

$$z\partial_z^2 p + (\beta - z)\partial_z p - \alpha p = 0, \quad \beta = a + 1/2,$$
$$\alpha = a/2 + \varkappa_l/2 + 1/2 - w/2\gamma. \tag{47}$$

Known solutions of equation (47) allow solutions of equations (44) to be obtained.

In what follows, we use the following solutions $F_1(\rho; s, W)$, $F_2(\rho; s, W)$ and $F_3(\rho; s, W)$ of equation (44):

$$F_1 = \rho^{1/2-l-\mu}\,\mathrm{e}^{-z/2} \times \begin{pmatrix} -(2\beta_1)^{-1}(W - sM)\rho\Phi(\alpha_1 + 1, \beta_1 + 1; z) \\ \Phi(\alpha_1, \beta_1; z) \end{pmatrix},$$
$$F_2 = \rho^{l+\mu-1/2}\,\mathrm{e}^{-z/2}\begin{pmatrix} \Phi(\alpha_2, \beta_2; z) \\ (2\beta_2)^{-1}(W + sM)\rho\Phi(\alpha_2, \beta_2 + 1; z) \end{pmatrix},$$
$$F_3 = \rho^{1/2-l-\mu}\,\mathrm{e}^{-z/2}\begin{pmatrix} 2^{-1}(W - sM)\rho\Psi(\alpha_1 + 1, \beta_1 + 1; z) \\ \Psi(\alpha_1, \beta_1; z) \end{pmatrix}, \tag{48}$$

where

$$\beta_1 = 1 - l - \mu, \quad \alpha_1 = -w/2\gamma, \quad \beta_2 = l + \mu,$$
$$\alpha_2 = l + \mu - w/2\gamma, \quad \omega_1 = \omega_1(s, W) = \frac{2\,(\gamma/2)^{\beta_2}\,\Gamma(\beta_1)}{(W + sM)\Gamma(\alpha_1)},$$
$$\omega_2 = \omega_2(W) = \frac{\Gamma(\beta_2)}{\Gamma(\alpha_2)}.$$

All the solutions F_1, F_2, and F_3 are real-entire in W, and $F_3 = \omega_2 F_1 - \omega_1 F_2$.

The solutions (48) have the following asymptotic behavior at the origin and at infinity.

As $\rho \to 0$, we have

$$F_1 = \rho^{1/2-l-\mu}\left(-(2\beta_1)^{-1}(W - sM)\rho/1\right)\tilde{O}(\rho^2),$$
$$F_2 = \rho^{l+\mu-1/2}\left(1/(2\beta_2)^{-1}(W + sM)\rho 1\right)\tilde{O}(\rho^2),$$
$$f_3 = \frac{(W - sM)\Gamma(\beta_1)}{2\,(\gamma/2)^{\beta_1}\,\Gamma(\alpha_1 + 1)}\rho^{l+\mu-1/2} \times \begin{cases} \tilde{O}(\rho^2), & l \leqslant -1, \\ \tilde{O}(\rho^{2-2\mu}), & l = 0, \quad \mu > 0, \\ \tilde{O}(\rho^2 \ln\rho), & l = 0, \quad \mu = 0, \end{cases}$$
$$g_3 = \frac{\Gamma(\beta_2)}{\Gamma(\alpha_2)}\rho^{1/2-l-\mu}\begin{cases} \tilde{O}(\rho^2), & l \geqslant 1, \\ \tilde{O}(\rho^{2\mu}), & l = 0, \quad \mu > 0, \end{cases} \tag{49}$$

where $F_3 = (f_3/g_3)$.

As $\rho \to \infty$, we have

$$F_1 = \frac{(\gamma/2)^{\alpha_1-\beta_1}\,\Gamma(\beta_1)}{\Gamma(\alpha_1)}\rho^{-\varkappa_l+2\alpha_1-2\beta_1}\,\mathrm{e}^{z/2} \times (\gamma\rho(W + sM)^{-1}/1)\tilde{O}(\rho^{-2}),$$
$$F_2 = \frac{(\gamma/2)^{\alpha_1}\,\Gamma(\beta_2)}{\Gamma(\alpha_2)}\rho^{\varkappa_l+2\alpha_1}\,\mathrm{e}^{z/2} \times \left(1/(\gamma\rho)^{-1}(W + sM)\right)\tilde{O}(\rho^{-2}),$$
$$F_3 = (\gamma/2)^{-\alpha_1}\,\rho^{\varkappa_l-2\alpha_1}\,\mathrm{e}^{-z/2} \left((\gamma\rho)^{-1}(W - sM)/1\right)\tilde{O}(\rho^{-2}).$$

We define the Wronskian $\mathrm{Wr}(F, \tilde{F})$ of the two doublets $F = (f/g)$ and $\tilde{F} = (\tilde{f}/\tilde{g})$ by

$$\mathrm{Wr}(F, \tilde{F}) = f\tilde{g} - g\tilde{f} = \mathrm{i}F\sigma^2\tilde{F}.$$

If $(\check{h} - W)F = (\check{h} - W)\tilde{F} = 0$, then $\mathrm{Wr}(F, \tilde{F}) = C = \mathrm{const}$. Solutions F and $\tilde{F}$ are linearly independent iff $C \neq 0$. It is easy to see that $\mathrm{Wr}(F_1, F_2) = -1$.

If $\mathrm{Im}\,W > 0$, the solutions F_1, F_2, and F_4 are pairwise linearly independent,

$$\mathrm{Wr}(F_1, F_3) = \omega_1(W), \quad \mathrm{Wr}(F_2, F_3) = \omega_2(W).$$

Taking the asymptotics of the linearly independent solutions F_1 and F_3 into account, we find that there are no square integrable solutions of equation (44) with $\mathrm{Im}\,W \neq 0$ and $|l| \geqslant 1$ or $l = 0$, $\mu = 0$. This implies that in these cases, the deficiency indices of $\hat{h}(s, l)$ are zero. In the case of $l = 0$, $\mu > 0$, the solution F_3 is square integrable, which implies that the deficiency indices of $\hat{h}(s, 0)$ are equal to (1, 1).

For any l and μ, the asymptotic behavior of any solution F of equation (44) at the origin, as $\rho \to 0$, is not more singular than $\rho^{-|\kappa_l|}$, $F(\rho) = O(\rho^{-|\kappa_l|})$.

II. We now consider the inhomogeneous equation

$$(\check{h}(s, l) - W)F(\rho) = \Psi(\rho), \quad \forall\Psi \in \mathbb{L}^2(\mathbb{R}_+).$$

Its general solution allows the representations

$$F(\rho) = c_1 F_d(\rho; W) + c_2 F_3(\rho; W) + \omega_d^{-1} \times \left[F_d(\rho; W)\int_\rho^\infty F_3(r; W)\Psi(r)\mathrm{d}r + F_3(\rho; W) \times \int_0^\rho F_d(r; W)\Psi(r)\mathrm{d}r\right],$$
$$\omega_d = \mathrm{Wr}(F_d, F_3), \quad d = \begin{cases} d = 1, & l \leqslant 0, \\ d = 2, & l \geqslant 1. \end{cases} \tag{50}$$

A simple estimate of the integral terms in the right-hand side (rhs) of (50) using the Cauchy–Bunyakovskii inequality shows that they are bounded as $\rho \to \infty$. It follows that $F \in \mathbb{L}^2(\mathbb{R}_+)$ implies $c_1 = 0$.

An evaluation shows that as $\rho \to 0$, the integral terms are of the order of $O(\rho^{1/2})$ (up to the factor $\ln \rho$ for $|\kappa_l| = 1/2$). In this case, $F \in \mathbb{L}^2(\mathbb{R}_+)$ implies $c_2 = 0$, and we find

$$F(\rho) = \omega_d^{-1}\left[F_d(\rho; W)\int_\rho^\infty F_3(r; W)\Psi(r)\mathrm{d}r + F_3(\rho; W) \times \int_0^\rho F_d(r; W)\Psi(r)\mathrm{d}r\right]. \tag{51}$$

For $|\kappa_l| \leqslant 1/2$, the doublet $F_3(\rho; W)$ is square-integrable, and a solution $F(\rho) \in \mathbb{L}^2(\mathbb{R}_+)$ allows the representation

$$\begin{aligned} F(\rho) &= b\omega_1^{-1}F_1(\rho; W) + c_2F_3(\rho; W) \\ &\quad + \omega_1^{-1}\left[F_3(\rho; W)\int_0^\rho F_1(r; W)\Psi(r)\mathrm{d}r \right. \\ &\quad \left. - F_1(\rho; W)\int_0^\rho F_3(r; W)\Psi(r)\mathrm{d}r\right], \\ F(\rho) &= b\omega_1^{-1}F_1(\rho; W) + c_2F_3(\rho; W) + O(\rho^{1/2}), \quad \rho \to 0, \end{aligned} \tag{52}$$

where

$$b = \int_0^\infty F_3(r; W)\Psi(r)\mathrm{d}r.$$

We use representations (50)–(52) to determine the Green functions for s.a. radial Hamiltonians.

3.3. s.a. radial Hamiltonians

3.3.1. Generalities. We proceed to construct s.a. radial Hamiltonians $\hat{h}_\mathfrak{e}(s, l)$ in the Hilbert space $\mathbb{L}^2(\mathbb{R}_+)$ as s.a. extensions of the initial symmetric radial operators $\hat{h}(s, l)$ (43) and analyze the corresponding spectral problems.

The action of all of the following operators associated with the differential operations $\check{h}(s, l)$ is given by $\check{h}(s, l)$; therefore we cite only their domains.

We begin with the adjoint $\hat{h}^+(s, l)$ of $\hat{h}(s, l)$. Its domain D_{h^+} is the natural domain for $\check{h}(s, l)$,

$$\begin{aligned} D_{h^+} &= D^*_{\check{h}(s,l)}(\mathbb{R}_+) \\ &= \left\{F_*(\rho) : F_* \text{ a.c. in } \mathbb{R}_+,\ F_*, \check{h}(s, l)F_* \in \mathbb{L}^2(\mathbb{R}_+)\right\}. \end{aligned}$$

The quadratic asymmetry form $\Delta_{h^+}(F_*)$ of $\hat{h}^+(s, l)$ is expressed in terms of the local quadratic form

$$[F_*, F_*](\rho) = \overline{g(\rho)}f(\rho) - \overline{f(\rho)}g(\rho), \ F_* = (f/g)$$

as follows:

$$\Delta_{h^+}(F_*) = (F_*, \hat{h}^+F_*) - (\hat{h}^+F_*, F_*) = -[F_*, F_*](\rho)|_0^\infty.$$

We can prove that $\lim_{\rho\to\infty}F_*(\rho) = 0$ for any $F_* \in D^*_{\check{h}}(\mathbb{R}_+)$. Indeed, because F_* and $\check{h}(s, l)F_*$ are square-integrable at infinity, the combination

$$F_*' - (\gamma\rho/2)\sigma^3F_* = -\mathrm{i}\sigma^2[\check{h}(s, l)F_* - (\varkappa_l/\rho)\sigma^1F_* + sM\sigma^3F_*]$$

is also square-integrable at infinity. This implies that f and $f' - (\gamma\rho/2)f$, together with g and $g' + (\gamma\rho/2)g$, are square-integrable at infinity. We consider the identity

$$\begin{aligned} |f(\rho)|^2 &= \int_a^\rho [\overline{\partial f(r)}f(r) + \overline{f(r)}\partial f(r)]\mathrm{d}r \\ &\quad + \gamma\int_a^\rho r|f(r)|^2\mathrm{d}r + |f(a)|^2, \quad \partial = \partial_\rho - \gamma\rho/2. \end{aligned}$$

The rhs of this identity has a limit (finite or infinite) as $\rho \to \infty$. Therefore, $|f(\rho)|$ also has a limit as $\rho \to \infty$. This limit has to be zero because $f(\rho)$ is square-integrable at infinity. In the same way, we can verify that $g(\rho) \to 0$ as $\rho \to \infty$.

To analyze the behavior of F_* at the origin, we consider the relationship

$$\Psi = \check{h}(s, l)F_*, \ \Psi, F_* \in \mathbb{L}^2(\mathbb{R}_+), \tag{53}$$

or

$$\begin{aligned} f' - \left(\gamma\rho/2 + \rho^{-1}\varkappa_l\right)f = -\chi_2, \ g' + \left(\gamma\rho/2 + \rho^{-1}\varkappa_l\right)g = \chi_1, \\ \chi = (\chi_1/\chi_2) = \Psi + sM\sigma^3F_* \in \mathbb{L}^2(\mathbb{R}_+), \end{aligned}$$

as an equation for F_* at a given χ. The general solution of these equations allows the representation

$$\begin{aligned} f(\rho) &= \rho^{\varkappa_l}\,\mathrm{e}^{\gamma\rho^2/4}\left[c_1 + \int_\rho^\infty r^{-\varkappa_l}\,\mathrm{e}^{-\gamma r^2/4}\chi_2(r)\mathrm{d}r\right], \\ g(\rho) &= \rho^{-\varkappa_l}\,\mathrm{e}^{-\gamma\rho^2/4}\left[c_2 + \int_{\rho_0}^\rho r^{\varkappa_l}\,\mathrm{e}^{\gamma r^2/4}\chi_1(r)\mathrm{d}r\right]. \end{aligned} \tag{54}$$

It turns out that the asymptotic behavior of the functions f and g at the origin crucially depends on the value of l. Therefore, our exposition is naturally divided into subsections related to the corresponding regions. We distinguish three regions of l.

3.3.2. The first region: $\kappa_l \leqslant -1/2$. In this region, we have

$$l \leqslant \begin{cases} -1, & \mu > 0, \\ 0, & \mu = 0. \end{cases}$$

The representation (54) allows the estimation of an asymptotic behavior of doublets $F_* \in D^*_{\check{h}(s,l)}(\mathbb{R}_+)$ at the origin for the first region:

$$\begin{aligned} f(\rho) &= \rho^{-|\varkappa_l|}\,\mathrm{e}^{\gamma\rho^2/4}\left[\tilde{c}_1 - \int_0^\rho r^{|\varkappa_l|}\,\mathrm{e}^{-\gamma r^2/4}\chi_2(r)\mathrm{d}r\right] \\ &= \tilde{c}_1\rho^{-|\varkappa_l|} + O(\rho^{1/2}), \\ \tilde{c}_1 &= c_1 + \int_0^\infty r^{|\varkappa_l|}\,\mathrm{e}^{-\gamma r^2/4}\chi_2(r)\,\mathrm{d}r. \end{aligned}$$

The condition $f \in L^2(\mathbb{R}_+)$ implies $\tilde{c}_1 = 0$, and therefore, $f(\rho) = O(\rho^{1/2})$ as $\rho \to 0$. As to $g(\rho)$, we find that

$$g(\rho) = \begin{cases} O(\rho^{1/2}), & \varkappa_l < -1/2, \\ O(\rho^{1/2}\ln\rho), & \varkappa_l = -1/2 \ \ (l = 0, \mu = 0), \end{cases} \quad \rho \to 0.$$

We thus find that $F_*(\rho) \to 0$ as $\rho \to 0$, which implies that $\Delta_{h^+}(F_*) = 0$, $\forall F_* \in D^*_{\check{h}(s,l)}(\mathbb{R}_+)$. This means that the

deficiency indices of each of the symmetric operators $\hat{h}(s,l)$ in the first region are zero. Therefore, there exists only one s.a. extension $\hat{h}_{\mathfrak{e}}(s,l,p_z)=\hat{h}_{(1)}(s,l)=\hat{h}^{+}(s,l)$ of $\hat{h}(s,l)$, i.e. a unique s.a. radial Hamiltonian with given s and l, its domain is the natural domain, $D_{\hat{h}_{(1)}(s,l)}=D^{*}_{\hat{h}(s,l)}(\mathbb{R}_{+})$.

The representation (51) with $d=1$ implies that the Green function for the s.a. Hamiltonian $\hat{h}_{(1)}(s,l)$ is given by

$$G(\rho,\rho';W)=\frac{1}{\omega_1(W)}\begin{cases}F_3(\rho;W)\otimes F_1(\rho';W), & \rho>\rho',\\ F_1(\rho;W)\otimes F_3(\rho';W), & \rho<\rho'.\end{cases} \tag{55}$$

Unfortunately, we cannot use representation (48) for F_3 as a sum of two terms directly for all values of μ because both are singular at $\mu=0$ (although the sum is not). To cover the total range of μ, we use another representation for F_3.

We let $F_{dl}(\rho;W)$ denote the functions $F_d(\rho;W)$, $d=1,2,3$, with a fixed l and represent F_{3l} as

$$\begin{aligned}F_{3l}&=\omega_1[A_{1l}F_{1l}+F_{4l}],\ A_{1l}=A_{1l}(W)\\&=\Omega_1(W)-\Gamma(\beta_2)P_{1l}(W),\\F_{4l}&=F_{4l}(\rho;W)=\Gamma(\beta_2)P_{1l}(W)F_{1l}(\rho;W)-F_{2l}(\rho;W),\\\Omega_1(W)&=\frac{\omega_2(W)}{\omega_1(W)},\\P_{1l}(W)&=\frac{(W+sM)(\gamma/2)^{|l|}\Gamma(\alpha_1)}{2|l|!\Gamma(\alpha_1-|l|)}.\end{aligned}$$

Using the relation (see [24])

$$\lim_{\beta\to-n}\frac{1}{\Gamma(\beta)}\Phi(\alpha,\beta;x)=\frac{x^{n+1}\Gamma(\alpha+n+1)}{(n+1)!\Gamma(\alpha)}\times\Phi(\alpha+n+1,n+2;x), \tag{56}$$

we can verify that

$$\Gamma^{-1}(\beta_2)F_{2l}(\rho;W)\big|_{\mu\to0}=P_{1l}(W)F_{1l}(\rho;W)|_{\mu=0}.$$

Taking the latter relation into account, it is easy to see that in the first region, A_{1l} and F_{4l} are finite for $\mu\geqslant0$, as well as ω_1 and F_{1l}, and also that $P_{1l}(E)$ and $F_{4l}(\rho;E)$ are real.

The Green function is then represented as

$$\begin{aligned}G(\rho,\rho';W)&=A_{1l}(W)F_{1l}(\rho;W)\otimes F_{1l}(\rho';W)\\&+\begin{cases}F_{4l}(\rho;W)\otimes F_{1l}(\rho';W), & \rho>\rho',\\ F_{1l}(\rho;W)\otimes F_{4l}(\rho';W), & \rho<\rho'\end{cases}\end{aligned} \tag{57}$$

for all $\mu\geqslant0$.

We choose the guiding functional $\Phi_1(F;W)$ for the s.a. operator $\hat{h}_{(1)}(s,l)$ in the form

$$\Phi_1(F;W)=\int_0^{\infty}F_1(\rho;W)F(\rho),$$
$$F(\rho)\in\mathbb{D}=D_r(\mathbb{R}_{+})\cap D_{\hat{h}_{(1)}(s,l)}.$$

It is easy to prove that the guiding functional is simple. It follows that the spectrum of $\hat{h}_{(1)}(s,l)$ is simple.

Using representation (57) for the Green function, we find that the derivative $\sigma'(E)=[\pi F_1^2(\rho;W)]^{-1}\mathrm{Im}\,G(\rho,\rho;E+\mathrm{i}0)$ of the spectral function is given by

$$\sigma'(E)=\pi^{-1}\mathrm{Im}\,A_{1l}(E+\mathrm{i}0). \tag{58}$$

It is easy to prove that $\mathrm{Im}\,A_{1l}(E+\mathrm{i}0)$ is continuous in μ for $\mu\geqslant0$, such that it is sufficient to find $\sigma'(E)$ only for the case of $\mu>0$, where equation (58) is more simple,

$$\sigma'(E)=\frac{(W+sM)(\gamma/2)^{-\beta_2}\Gamma(\beta_2)}{2\pi\Gamma(\beta_1)\Gamma(\alpha_2)}\bigg|_{W=E}\mathrm{Im}\,\Gamma(\alpha_1)|_{W=E+\mathrm{i}0}. \tag{59}$$

It is easy to see that $\sigma'(E)$ may differ from zero only at the points E_k defined by the relation $\alpha_1=-k$ ($\Gamma(\alpha_1)=\infty$), or $M^2-E_k^2=-2\gamma k$, which yields

$$E_k=\pm M_k,\quad M_k=\sqrt{M^2+2\gamma k},\quad M_0=M,\quad k\in\mathbb{Z}_{+}.$$

The presence of the factor $(E+sM)$ in the rhs of (59) implies that the points $E=-sM=-sM_0$ do not belong to the spectrum of $\hat{h}_{(1)}(s,l)$. In what follows it is convenient to change the numeration of the spectrum points. We introduce an index $\mathfrak{n}(s)$:

$$\mathfrak{n}(s)\in\mathcal{Z}(s)=\{n_{\zeta}(s)\},\quad \zeta=\pm,$$
$$n_{+}(s)\in\begin{cases}\mathbb{Z}_{+}, & s=1,\\ \mathbb{N}, & s=-1,\end{cases}\qquad n_{-}(s)\in\begin{cases}-\mathbb{N}, & s=1,\\ \mathbb{Z}_{-}, & s=-1.\end{cases} \tag{60}$$

Then we can write

$$E_k=\pm M_k\Longrightarrow E_{\mathfrak{n}(s)}=\zeta M_{|\mathfrak{n}(s)|},\quad \mathfrak{n}(s)\in\mathcal{Z}(s).$$

Finally, we obtain

$$\sigma'(E)=\sum_{\mathfrak{n}(s)\in\mathcal{Z}(s)}Q^2_{\mathfrak{n}(s)}\delta(E-E_{\mathfrak{n}(s)}),$$
$$Q_{\mathfrak{n}(s)}=\sqrt{\frac{(\gamma/2)^{\beta_1}\Gamma(\beta_1+|\mathfrak{n}(s)|)\left(1+sME_k^{-1}\right)}{|\mathfrak{n}(s)|!\Gamma^2(\beta_1)}},$$
$$\beta_1=1+|l|-\mu.$$

Thus, the simple spectrum of $\hat{h}_{(1)}(s,l)$ is given by $\mathrm{spec}\,\hat{h}_{(1)}(s,l)=\{E_{\mathfrak{n}(s)},\ \mathfrak{n}(s)\in\mathcal{Z}(s)\}$. The eigenvectors

$$\overset{\mathrm{I}}{U}_{\mathfrak{n}(s)}=\overset{\mathrm{I}}{U}_{\mathfrak{n}(s)}(s,l,p_z;\rho)=Q_{\mathfrak{n}(s)}F_1(\rho;E_{\mathfrak{n}(s)}),\ \mathfrak{n}(s)\in\mathcal{Z}(s), \tag{61}$$

of $\hat{h}_{(1)}(s,l)$ form a complete orthonormalized system in the space $\mathbb{L}^2(\mathbb{R}_{+})$ of doublets $F(\rho)$.

3.3.3. The second region: $\kappa_l\geqslant1/2$. In this region, we have $l\geqslant1$.

The representation (54) yields the following estimates for an asymptotic behavior of doublets $F_*\in D^*_{\hat{h}(s,l)}(\mathbb{R}_{+})$ at the origin for the second region:

$$\begin{cases}f(\rho)=\begin{cases}O(\rho^{1/2}), & \varkappa_l>1/2,\\ O(\rho^{1/2}\ln\rho), & \varkappa_l=1/2,\end{cases}\\ g(\rho)=O(\rho^{1/2}),\end{cases}\qquad \rho\to0.$$

It follows that $F_*(\rho)\to0$ as $\rho\to0$, which implies that $\Delta_{h^{+}}(F_*)=0$, $\forall F_*\in D^*_{\hat{h}(s,l)}(\mathbb{R}_{+})$. This means that the deficiency indices of each of the symmetric operators $\hat{h}(s,l)$

in the second region are also zero. Therefore, there exists only one s.a. extension $\hat{h}_{\mathfrak{e}}(s,l,p_z)=\hat{h}_{(2)}(s,l)=\hat{h}^{+}(s,l)$ of $\hat{h}(s,l)$, i.e. a unique s.a. radial Hamiltonian with given s and l, its domain is the natural domain, $D_{\hat{h}_{(2)}(s,l)}=D^{*}_{\check{h}(s,l)}(\mathbb{R}_+)$.

The representation (51) with $d=2$ implies that the Green function for the s.a. Hamiltonian $\hat{h}_{(2)}(s,l)$ is given by

$$G(\rho,\rho';W)=\omega_2^{-1}(W)\begin{cases}F_{3l}(\rho;W)\otimes F_{2l}(\rho';W), & \rho>\rho',\\ F_{2l}(\rho;W)\otimes F_{3l}(\rho';W), & \rho<\rho'.\end{cases}$$

Again, the representation (48) for F_3 as a sum of two terms is not applicable directly for $\mu=0$. We therefore use the following representation for F_3:

$$\begin{aligned}&F_{3l}=\omega_2\left(F_{5l}-A_{2l}F_{2l}\right),\\ &A_{2l}=A_{2l}(W)=\Omega_2(W)+\Gamma(\beta_1)P_{2l}(W),\\ &F_{5l}=F_{5l}(\rho;W)=F_{1l}(\rho;W)+\Gamma(\beta_1)P_{2l}(W)F_{2l}(\rho;W),\\ &\Omega_2(W)=\frac{\omega_1(W)}{\omega_2(W)},\qquad P_{2l}(W)=\frac{(W-sM)(\gamma/2)^{l-1}\Gamma(\alpha_1+l)}{2(l-1)!\,\Gamma(\alpha_1+1)}.\end{aligned}$$

Using relation (56), we can verify that

$$\Gamma^{-1}(\beta_1)F_{1l}(\rho;W)\big|_{\mu\to 0}=-\,P_{2l}(W)F_{2l}(\rho;W)|_{\mu=0}.$$

Taking the latter relation into account, it is easy to see that A_{2l} and F_{5l} are finite for $\mu\geqslant 0$, as well as ω_2 and F_{2l}, in the second region, and $P_{2l}(E)$ and $F_{5l}(\rho;E)$ are real.

The Green function is then represented as

$$\begin{aligned}G(\rho,\rho';W)=&-A_{2l}(W)F_{2l}(\rho;W)\otimes F_{2l}(\rho';W)\\ &+\begin{cases}F_{5l}(\rho;W)\otimes F_{2l}(\rho';W), & \rho>\rho',\\ F_{2l}(\rho;W)\otimes F_{5l}(\rho';W), & \rho<\rho'.\end{cases}\end{aligned}\tag{62}$$

for all $\mu\geqslant 0$.

We choose the guiding functional $\Phi_2(F;W)$ for the s.a. operator $\hat{h}_{(2)}(s,l)$ in the form

$$\begin{aligned}&\Phi_2(F;W)=\int_0^\infty F_2(\rho;W)F(\rho),\\ &F(\rho)\in\mathbb{D}=D_r(\mathbb{R}_+)\cap D_{\hat{h}_{(2)}(s,l)}.\end{aligned}$$

It is easy to prove that the guiding functional is simple. It follows that the spectrum of $\hat{h}_{(2)}(s,l)$ is simple.

Using representation (62) for the Green function, we find that the derivative $\sigma'(E)$ of the spectral function is given by

$$\sigma'(E)=-\pi^{-1}\,\mathrm{Im}\,A_{2l}(E+\mathrm{i}0).\tag{63}$$

It is easy to prove that $\mathrm{Im}\,A_{2l}(E+\mathrm{i}0)$ is continuous in μ for $\mu\geqslant 0$, such that it is sufficient to find $\sigma'(E)$ only for the case of $\mu>0$ where equation (63) is more simple,

$$\sigma'(E)=\frac{(W-sM)\,(\gamma/2)^{\beta_2}\,\Gamma(\beta_1)}{\pi\gamma\Gamma(\beta_2)\Gamma(1+\alpha_1)}\bigg|_{W=E}\mathrm{Im}\,\Gamma(\alpha_2)|_{W=E+\mathrm{i}0}.$$

It is easy to see that $\sigma'(E)$ may differ from zero only at the points E_k defined by the relation $\alpha_2=-k$ ($\Gamma(\alpha_2)=\infty$) or

$$M^2-E_k^2+2\gamma(l+\mu)=-2\gamma k,\quad k\in\mathbb{Z}_+,$$

which yields

$$E_k=\pm\sqrt{M^2+2\gamma(k+l+\mu)}=\pm M_{k+l+\mu},\quad k\in\mathbb{Z}_+.$$

All the points E_k are the spectrum points.

It is convenient to change indexing k for $\mathfrak{n}(s)$,

$$\begin{aligned}E_k\Longrightarrow E_{\mathfrak{n}(s)}=\sigma M_{|\mathfrak{n}(s)|+\mu},\ \{\mathfrak{n}(s)\in\mathcal{Z}(s),\ |\mathfrak{n}(s)|\geqslant l\}\\ \times(n_\sigma(s)=\sigma(k+l),\ k\in\mathbb{Z}_+).\end{aligned}$$

Thus, we finally obtain

$$\sigma'(E)=\sum_{\mathfrak{n}\in\mathcal{Z},|\mathfrak{n}|\geqslant l}Q^2_{\mathfrak{n}(s)}\delta(E-E_{\mathfrak{n}}),$$

$$Q_{\mathfrak{n}(s)}=\sqrt{\frac{(\gamma/2)^{l+\mu}\,\Gamma(|\mathfrak{n}(s)|+\mu)(1-sME^{-1}_{\mathfrak{n}(s)})}{(|\mathfrak{n}(s)|-l)!\Gamma^2(l+\mu)}}.$$

The simple spectrum of $\hat{h}_{(2)}(s,l)$ is given by

$$\mathrm{spec}\,\hat{h}_{(2)}(s,l)=\left\{E_{\mathfrak{n}(s)},\ \mathfrak{n}(s)\in\mathcal{Z},\ |\mathfrak{n}(s)|\geqslant l\right\}.$$

The eigenvectors

$$\overset{\mathrm{II}}{U}_{\mathfrak{n}(s)}=\overset{\mathrm{II}}{U}_{\mathfrak{n}(s)}(s,l,p_z;\rho)=Q_{\mathfrak{n}(s)}F_2(\rho;E_{\mathfrak{n}(s)}),\quad \mathfrak{n}(s)\in\mathcal{Z}(s),\tag{64}$$

of the Hamiltonian $\hat{h}_{(2)}(s,l)$ form a complete orthonormalized set in the space $\mathbb{L}^2(\mathbb{R}_+)$ of doublets $F(\rho)$.

3.3.4. The third region: $|\varkappa_l|<1/2$. In this region, we have $l=l_0=0$, and $\varkappa_l$ is reduced to $\varkappa_0=\mu-1/2$, $\mu>0$.

Representation (54) yields the following asymptotic behavior of doublets $F_*\in D^{*}_{\check{h}(s,l_0)}(\mathbb{R}_+)$ as $\rho\to 0$:

$$F_*(\rho)=\begin{cases}f(\rho)=c_1(m_{\mathrm{e}}\rho)^{\varkappa_0}\\ g(\rho)=c_2(m_{\mathrm{e}}\rho)^{-\varkappa_0}\end{cases}+O(\rho^{1/2}).$$

It follows that $\Delta_{h^+}(F_*)=\overline{c_2}c_1-\overline{c_1}c_2$. Such a representation for the quadratic form $\Delta_{h^+}(F_*)$ implies that the deficiency indices of the initial symmetric operator $\hat{h}(s,l_0)$ are $m_\pm=1$. The condition $\Delta_{h^+}(F_*)=0$ yields asymptotic boundary conditions as $\rho\to 0$,

$$F(\rho)=c\begin{pmatrix}(m_{\mathrm{e}}\rho)^{\varkappa_0}\cos\lambda\\ (m_{\mathrm{e}}\rho)^{-\varkappa_0}\sin\lambda\end{pmatrix}+O(\rho^{1/2}),\tag{65}$$

with a fixed $\lambda\in\mathbb{S}(-\pi/2,\pi/2)$ (note that λ depend on s and p_z, $\lambda=\lambda(s,p_z)$) define a maximum subspace in $D^{*}_{\check{h}(s,l_0)}(\mathbb{R}_+)$ where $\Delta_{h^+}=0$. This subspace is the domain of an s.a. operator that is an s.a. extension of $\hat{h}(s,l_0)$.

We thus find that there exists a one-parameter $U(1)$ family of s.a. radial Hamiltonians $\hat{h}_\lambda(s,l_0)$ parameterized by the real parameter $\lambda\in\mathbb{S}(-\pi/2,\pi/2)$. These Hamiltonians are specified by the domains

$$D_{h_\lambda(s,l_0)}=\left\{F(\rho):F(\rho)\in D^{*}_{\check{h}_\lambda(s,l_0)}(\mathbb{R}_+),\ F\text{ satisfy (65)}\right\}.\tag{66}$$

According to representation (52), which certainly holds for the doublets F belonging to $D_{h_\lambda(s,l_0)}$, and (49), the

asymptotic behavior of F as $\rho \to 0$ reads

$$F = \begin{pmatrix} -c_2\omega_1\rho^{\varkappa_0} \\ (b\omega_1^{-1} + c_2\omega_2)\rho^{-\varkappa_0} \end{pmatrix} + O(\rho^{1/2}).$$

On the other hand, F satisfies boundary conditions (65), whence it follows that there must be

$$c_2 = -\frac{b\cos\lambda}{\omega_1\omega_{(\lambda)}}, \quad \omega_{(\lambda)} = \omega_2\cos\lambda + m_e^{-2\varkappa_0}\omega_1\sin\lambda. \tag{67}$$

Then representation (52) for F with c_2 given by (67) implies that the Green function for the s.a. Hamiltonian $\hat{h}_\lambda(s, l_0)$ is given by

$$G(\rho, \rho'; W) = \Omega^{-1}(W)F_{(\lambda)}(\rho; W) \otimes F_{(\lambda)}(\rho'; W) + \begin{cases} \tilde{F}_{(\lambda)}(\rho; W) \otimes F_{(\lambda)}(\rho'; W), & \rho > \rho', \\ F_{(\lambda)}(\rho; W) \otimes \tilde{F}_{(\lambda)}(\rho'; W), & \rho < \rho', \end{cases} \tag{68}$$

where

$$F_{(\lambda)}(\rho; W) = m_e^{-\varkappa_0}F_1(\rho; W)\sin\lambda + m_e^{\varkappa_0}F_2(\rho; W)\cos\lambda,$$
$$\tilde{F}_{(\lambda)}(\rho; W) = m_e^{-\varkappa_0}F_1(\rho; W)\cos\lambda - m_e^{\varkappa_0}F_2(\rho; W)\sin\lambda,$$
$$\Omega(W) = \frac{\omega_{(\lambda)}(W)}{\tilde{\omega}_{(\lambda)}(W)}, \quad m_e^{\varkappa_0}F_3 = \tilde{\omega}_{(\lambda)}F_{(\lambda)} + \omega_{(\lambda)}\tilde{F}_{(\lambda)},$$
$$\tilde{\omega}_{(\lambda)}(W) = \omega_2\sin\lambda - m_e^{-2\varkappa_0}\omega_1\cos\lambda.$$

We note that the doublets $F_{(\lambda)}(\rho; W)$ and $\tilde{F}_{(\lambda)}(\rho; W)$ are real-entire in W, and the doublet $F_{(\lambda)}(\rho; W)$ satisfies asymptotic s.a. boundary conditions (65).

We choose the guiding functional $\Phi_\lambda(F; W)$ for the s.a. operator $\hat{h}_\lambda(s, l_0)$ in the form

$$\Phi_{(\lambda)}(F; W) = \int_0^\infty F_{(\lambda)}(\rho; W)F(\rho),$$
$$F(\rho) \in \mathbb{D} = D_r(\mathbb{R}_+) \cap D_{\hat{h}_\lambda(s,l_0)}.$$

It is easy to prove that the guiding functional is simple. It follows that the spectrum of $\hat{h}_\lambda(s, l)$ is simple.

Using the representation (68) for the Green function, we find that the derivative $\sigma'(E)$ of the spectral function is given by

$$\sigma'(E) = \pi^{-1}\operatorname{Im}\Omega^{-1}(E + \mathrm{i}0).$$

Because $\Omega(E)$ is real, $\sigma'(E)$ differs from zero only at the zero points E_k of the function $\Omega(E)$, $\Omega(E_k) = 0$, and we find that

$$\sigma'(E) = \sum_k Q_k^2\delta(E - E_k), \quad Q_k = \left[-\Omega'(E)\right]^{-1/2},$$
$$\Omega'(E_k) < 0.$$

Thus, the simple spectrum of $\hat{h}_\lambda(s, l_0)$ is given by spec $\hat{h}_\lambda(s, l_0) = \{E_k,\ k \in \mathbb{Z}\}$. The eigenvectors

$$\overset{\text{III}}{U}_k = \overset{\text{III}}{U}_k(\lambda, s, p_z; \rho) = Q_kF_{(\lambda)}(\rho; E_k),\ k \in \mathbb{Z}, \tag{69}$$

of $\hat{h}_\lambda(s, l_0)$ form a complete orthonormalized system in the space $\mathbb{L}^2(\mathbb{R}_+)$.

For $\lambda = 0$ and $\lambda = \pm\pi/2$, we can evaluate the spectrum explicitly.

I. First, we consider the case $\lambda = \pi/2$. In this case, we have

$$F_{(\pi/2)}(\rho; W) = m_e^{-\varkappa_0}F_1(\rho; W),$$
$$\Omega(W) = m_e^{-2\varkappa_0}\omega_1(W)\omega_2^{-1}(W),$$

and

$$\sigma'(E) = \left.\frac{m_e^{2\varkappa_0}\Gamma(\beta_2)(W + sM)}{2\pi\,(\gamma/2)^{\beta_2}\,\Gamma(\beta_1)\Gamma(\alpha_2)}\right|_{W=E} \operatorname{Im}\Gamma(\alpha_1)|_{W=E+\mathrm{i}0}. \tag{70}$$

As in the first region, $\sigma'(E)$ differs from zero only at the points (for which we will use the notation $\mathcal{E}_k$) defined by the relationship $\alpha_1 = -k$ ($\Gamma(\alpha_1) = \infty$), or by

$$\frac{M^2 - \mathcal{E}_k^2}{2\gamma} = -k, \quad E_k = \pm M_k, \quad k \in \mathbb{Z}_+.$$

The presence of the factor $(E + sM)$ in the rhs of (70) implies that the points $E = -sM = -sM_0$ do not belong to the spectrum of $\hat{h}_{\pi/2}(s, l_0)$. Thus,

$$\mathcal{E}_k = (\operatorname{sgn} k)M_{|k|},\ |k| \geqslant 1, \quad \mathcal{E}_0 = sM; \quad k \in \mathbb{Z}.$$

Using (60), we change the indexing of the spectrum points,

$$\mathcal{E}_k \Longrightarrow \mathcal{E}_{\mathfrak{n}(s)} = \zeta M_{|\mathfrak{n}(s)|}, \quad \mathfrak{n} = \mathfrak{n}(s) \in \mathcal{Z}(s).$$

Then, we finally obtain

$$\sigma'(E) = \sum_{\mathfrak{n}(s)\in\mathcal{Z}(s)} m_e^{2\varkappa_0}Q^2_{\pi/2|\mathfrak{n}(s)}\delta(E - \mathcal{E}_{\mathfrak{n}(s)}),$$
$$Q_{\pi/2|\mathfrak{n}(s)} = \sqrt{\frac{\Gamma(|\mathfrak{n}(s)| + 1 - \mu)(1 + sM\mathcal{E}^{-1}_{\mathfrak{n}(s)})}{(\gamma/2)^{\beta_2}\,|\mathfrak{n}|!\Gamma^2(1 - \mu)}}.$$

Thus, the simple spectrum of $\hat{h}_{\pi/2}(s, l_0)$ is given by spec $\hat{h}_{\pi/2}(s, l_0) = \{\mathcal{E}_{\mathfrak{n}(s)},\ \mathfrak{n}(s) \in \mathcal{Z}(s)\}$. The eigenvectors

$$\overset{\text{III}}{U}_{\pi/2|\mathfrak{n}(s)} = \overset{\text{III}}{U}_{\pi/2|\mathfrak{n}(s)}(\pi/2, s, l_0, p_z; \rho)$$
$$= Q_{\pi/2|\mathfrak{n}(s)}F_1(\rho; \mathcal{E}_{\mathfrak{n}(s)}),\ \mathfrak{n}(s) \in \mathcal{Z}(s),$$

of $\hat{h}_{\pi/2}(s, l_0)$ form a complete orthonormalized system in the space $\mathbb{L}^2(\mathbb{R}_+)$ of doublets $F(\rho)$.

We note that the spectrum, spectral function and eigenfunctions of $\hat{h}_{\pi/2}(s, l_0)$ can be obtained from the respective expressions from the first region, $\varkappa_l \leqslant -1/2$, by the substitution $l = 0$. We also note that for $\mu < 1/2$, the function $F_{(\pi/2)}(\rho; W) = m_e^{-\varkappa_0}F_1(\rho; W)$ has minimal singularity in the family of functions $F_{(\lambda)}(\rho; W)$; in fact, it is nonsingular; for $\mu > 1/2$, the function $F_{(0)}(\rho; W) = m_e^{\varkappa_0}F_2(\rho; W)$ has a minimal singularity in the family of $F_{(\lambda)}(\rho; W)$; in fact, $F_{(0)}(\rho; W)$ is not singular at all; for $\mu = 1/2$, all functions of the family $F_{(\lambda)}(\rho; W)$ have the same type of asymptotics: $F_{(\lambda)}(\rho; W) = O(1)$ as $\rho \to 0$.

We obtain the same results for the spectrum and complete orthonormalized set of the eigenvectors for the case $\lambda = -\pi/2$.

II. In the same manner, for the case $\lambda = 0$, we obtain $F_{(0)}(\rho; W) = m_e^{\varkappa_0} F_2(\rho; W)$; the simple spectrum of $\hat{h}_0(s, l_0)$ is given by

$$\operatorname{spec} \hat{h}_0(s, l_0) = \{E_{\mathfrak{n}}(0),\ \mathfrak{n} \in \mathcal{Z}\}, \quad \mathcal{Z} = \left\{n_\zeta \in \zeta\mathbb{Z}_+,\ \zeta = \pm\right\},$$

and

$$\sigma'(E) = \sum_{\mathfrak{n}\in\mathcal{Z}} m_e^{-2\varkappa_0} Q_{0|\mathfrak{n}}^2 \delta(E - E_{\mathfrak{n}}(0)),$$

$$Q_{0|\mathfrak{n}} = \sqrt{\frac{(\gamma/2)^\mu\, \Gamma(|\mathfrak{n}| + \mu)(1 - sME_{\mathfrak{n}}^{-1}(0))}{|\mathfrak{n}|!\Gamma^2(\mu)}},$$

where $E_{0|\mathfrak{n}}$ are solutions of the equation

$$\alpha_2 = \mu - (E_{\mathfrak{n}}^2(0) - M^2)/2\gamma = -|\mathfrak{n}|, \quad E_{\mathfrak{n}}(0) = \zeta M_{|\mathfrak{n}|+\mu},$$

and $n_+ = 0$ and $n_- = 0$ are considered different elements of $\mathcal{Z}$.

The eigenvectors $\overset{\text{III}}{U}_{0|\mathfrak{n}} = \overset{\text{III}}{U}_{0|\mathfrak{n}}(0, s, l_0, p_z; \rho) = Q_{0|\mathfrak{n}} F_2(\rho; E_{\mathfrak{n}}(0))$, $\mathfrak{n} \in \mathcal{Z}$, of the Hamiltonian $\hat{h}_0(s, l_0)$ form a complete orthonormalized system in the space $\mathbb{L}^2(\mathbb{R}_+)$ of doublets $F(\rho)$.

We note that the spectrum, spectral function and eigenfunctions of $\hat{h}_0(s, l_0)$ can be obtained from the respective expressions for the second region, $\varkappa_l \geqslant 1/2$, by the substitution $l = 0$. We also recall that for $\mu > 1/2$, the function $F_{(0)}(\rho; W) = m_e^{\varkappa_0} F_2(\rho; W)$ has minimal singularity at the origin in the family of functions $F_{(\lambda)}(\rho; W)$; in fact, $F_{(0)}(\rho; W)$ is completely nonsingular.

III. Now, we consider the general case $|\lambda| < \pi/2$. In this case we can equivalently write

$$\begin{aligned}
\sigma'(E) &= -\left(\pi \cos^2\lambda\right)^{-1} \operatorname{Im} \omega^{-1}(E + \mathrm{i}0) \\
&= \sum_{k\in\mathbb{Z}} Q_k^2 \delta(E - E_k(\lambda)), \\
\omega(W) &= t(W) + \tan\lambda,\ \omega'(E_k(\lambda)) > 0, \\
Q_k &= \left[\sqrt{\omega'(E_k(\lambda))}\cos\lambda\right]^{-1}, \\
t(W) &= \kappa \frac{(W + sM)\Gamma(-w/2\gamma)}{m_e \Gamma(\mu - w/2\gamma)}, \\
\kappa &= \frac{(2m_e^2/\gamma)^\mu \Gamma(\mu)}{2\Gamma(1-\mu)} > 0, \\
t(E_k(\lambda)) &= -\tan\lambda,\ t'(E_k(\lambda)) > 0, \\
\partial_\lambda E_k(\lambda) &= -\left[t'(E_k(\lambda))\cos^2\lambda\right]^{-1} < 0.
\end{aligned}$$

The function

$$t(E) = \kappa m_e^{-1} \Gamma^{-1}(\mu - w/2\gamma)(E + sM)\Gamma(-w/2\gamma)$$

has the properties $t(\mathcal{E}_{\mathfrak{n}(s)} \pm 0) = \mp\infty$; $t(E_{\mathfrak{n}}(0)) = 0$. Thus, we obtain:

(a) $s = 1$.
In each interval $(\mathcal{E}_{n_- -1}, \mathcal{E}_{n_-})$, $n_- \leqslant -1$, for a fixed $\lambda \in (-\pi/2, \pi/2)$, there exists an eigenvalue $E_{n_-}(\lambda)$ which increases monotonically from $\mathcal{E}_{n_- -1} + 0$ (passing $E_{n_-}(0)$) to $\mathcal{E}_{n_-} - 0$ as λ changes from $\pi/2 - 0$ (passing 0) to $-\pi/2 + 0$; in the interval $(\mathcal{E}_{-1}, \mathcal{E}_{n_+=0})$, for a fixed $\lambda \in (-\pi/2, \pi/2)$, there exists an eigenvalue $E_{n_-=0}(\lambda)$, which increases monotonically from $\mathcal{E}_{-1} + 0$ (passing $E_{n_-=0}(0)$) to $\mathcal{E}_{n_+=0} - 0$ as λ changes from $\pi/2 - 0$ (passing 0) to $-\pi/2 + 0$; in each interval $(\mathcal{E}_{n_+}, \mathcal{E}_{n_+ +1})$, $n_+ \geqslant 0$, for a fixed $\lambda \in (-\pi/2, \pi/2)$, there exists an eigenvalue $E_{n_+}(\lambda)$ which increases monotonically from $\mathcal{E}_{n_+} + 0$ (passing $E_{n_+}(0)$) to $\mathcal{E}_{n_+ +1} - 0$ as λ changes from $\pi/2 - 0$ (passing 0) to $-\pi/2 + 0$.

(b) $s = -1$.
In each interval $(\mathcal{E}_{n_- -1}, \mathcal{E}_{n_-})$, $n_- \leqslant 0$, for a fixed $\lambda \in (-\pi/2, \pi/2)$, there exists an eigenvalue $E_{n_-}(\lambda)$ which increases monotonically from $\mathcal{E}_{n_- -1} + 0$ (passing $E_{n_-}(0)$) to $\mathcal{E}_{n_-} - 0$ as λ changes from $\pi/2 - 0$ (passing 0) to $-\pi/2 + 0$; in the interval $(\mathcal{E}_{n_-=0}, \mathcal{E}_{n_+=1})$, for a fixed $\lambda \in (-\pi/2, \pi/2)$, there exists an eigenvalue $E_{n_+=0}(\lambda)$ which increases monotonically from $\mathcal{E}_{n_-=0} + 0$ (passing $E_{n_-=0}(0)$) to $\mathcal{E}_{n_+=1} - 0$ as λ changes from $\pi/2 - 0$ (passing 0) to $-\pi/2 + 0$; in each interval $(\mathcal{E}_{n_+}, \mathcal{E}_{n_+ +1})$, $n_+ \geqslant 1$, for a fixed $\lambda \in (-\pi/2, \pi/2)$, there exists an eigenvalue E_{n_+} which increases monotonically from $\mathcal{E}_{n_+} + 0$ (passing $E_{n_+}(0)$) to $\mathcal{E}_{n_+ +1} - 0$ as λ changes from $\pi/2 - 0$ (passing 0) to $-\pi/2 + 0$.

4. Summary

We have constructed all s.a. radial Hamiltonians $\hat{h}_{\mathfrak{e}}(s, l, p_z)$ as s.a. extensions of the symmetric operators $\hat{h}(s, l, p_z)$ for any s, l and p_z and for any values of ϕ_0, μ and γ. The complete s.a. Dirac operators $\hat{H}_{\mathfrak{e}}$ associated with the Dirac differential operation $\check{H}$ are constructed from the sets of $\hat{h}_{\mathfrak{e}}(s, l, p_z)$ by means of a procedure of 'a direct summation over s and l and a direct integration over p_z'. Each set of possible s.a. radial Hamiltonians $\hat{h}_{\mathfrak{e}}(s, l, p_z)$ generates a translationary-rotationally-invariant s.a. Hamiltonian $\hat{H}_{\mathfrak{e}}$. Namely, let $\mathbb{G}$ be the group of the above space transformations S: $\mathbf{r} \mapsto S\mathbf{r}$. This group is unitarily represented in $\mathfrak{H}$: if $S \in \mathbb{G}$, then the corresponding operator U_S is defined by

$$(U_S\, \psi)(\mathbf{r}) = \mathrm{e}^{-\mathrm{i}\theta\Sigma^3/2} \psi(S^{-1}\mathbf{r}), \quad \forall \psi \in \mathfrak{H},$$

where θ is the rotation angle of the vector $\boldsymbol{\rho}$ around the z-axis. The operator $\hat{H}$ evidently commutes with U_S for any S. We consider only such s.a. extensions $\hat{H}_{\mathfrak{e}}$ of $\hat{H}$ that also commute with U_S for any S. This condition is the explicit form of the invariance, or symmetry, of a quantum Hamiltonian under the space transformations. As in classical mechanics, this symmetry allows the seperation of the cylindrical coordinates ρ, φ and z and the reduction of the 3D problem to a 1D radial problem. Let V be a unitary operator defined by the relationship

$$\begin{aligned}
&(Vf)(\rho, \varphi, z) \\
&= \frac{1}{2\pi\sqrt{\rho}} \int_{\mathbb{R}_z} \mathrm{d}p_z \sum_{l\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}p_z z} \left[S_l(\varphi) F(s, l, p_z, \rho)\right] \otimes e_s(p_z),
\end{aligned}$$

where $S_l(\varphi)$ and $e_s(p_z)$ are given by, respectively, (41) and (40). It is natural to expect that any s.a. Hamiltonian $\hat{H}_{\mathfrak{e}}$ can be represented in the form

$$\hat{H}_{\mathfrak{e}} = V \int_{\mathbb{R}_z} \mathrm{d}p_z \sum_{s=\pm1} \sum_{l\in\mathbb{Z}} \hat{h}_{\mathfrak{e}}(s, l, p_z) V^{-1},$$

where $\hat{h}_{\mathfrak{e}}(s,l,p_z)$ for fixed s, l and p_z is an s.a. extension of the symmetric operator $\hat{h}(s,l,p_z)$ associated with the differential operation $\check{h}(s,l,p_z)$ given by equation (43). The operator $\hat{h}(s,l,p_z)$ is defined on the domain $D_{h(s,l,p_z)} = \mathcal{D}(\mathbb{R}_+) \subset \mathbb{L}^2(\mathbb{R}_+, \mathrm{d}\rho)$ in the Hilbert space $\mathbb{L}^2(\mathbb{R}_+, \mathrm{d}\rho)$ of functions $F(\rho,l,p_z)$ with the scalar product

$$(F_1(s,l,p_z), F_2(s,l,p_z)) = \int_{\mathbb{R}_+} \overline{F_1(s,l,p_z,\rho)} F_2(s,l,p_z,\rho)\mathrm{d}\rho.$$

An exact expression for $\hat{H}_{\mathfrak{e}}$ is

$$\hat{H}_{\mathfrak{e}} = V \int_{\mathbb{R}_z}^{\oplus} \mathrm{d}p_z \sum_{s=\pm 1}^{\oplus} \sum_{l\in\mathbb{Z}}^{\oplus} \hat{h}_{\mathfrak{e}}(s,l,p_z) V^{-1}.$$

Its rigorous justification will be discussed in a work by A Smirnov.

The inversion formulae in Hilbert space $\mathfrak{H}$ are correspondingly obtained from the known radial inversion formulae by a procedure of summation over s, l and integration over p_z. It should be noted that here we must consider the extension parameter λ as a function of s and p_z, $\lambda = \lambda(s,p_z)$. In what follows, $\int \mathrm{d}p_z$ means $\int_{-\infty}^{\infty} \mathrm{d}p_z$. Thus, we can summarize as follows:

For $\mu = 0$, there is a unique s.a. Dirac operator $\hat{H}_{\mathfrak{e}}$. Its spectrum is simple and given by

$$\operatorname{spec} \hat{H}_{\mathfrak{e}} = (-\infty, -m_{\mathrm{e}}] \cup [m_{\mathrm{e}}, \infty).$$

The generalized eigenfunctions $\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r})$ of $\hat{H}_{\mathfrak{e}}$,

$$\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\rho}} \mathrm{e}^{\mathrm{i}p_z z} S_l(\varphi) F_{\mathfrak{n}(s)}(s,l,p_z;\rho) \otimes e_s(p_z),$$

$$F_{\mathfrak{n}(s)}(s,l,p_z;\rho) = \begin{cases} \overset{\mathrm{I}}{U}_{\mathfrak{n}(s)}(s,l,p_z;\rho), & l \leqslant 0, \\ \overset{\mathrm{II}}{U}_{\mathfrak{n}(s)}(s,l,p_z;\rho), & 1 \leqslant l \leqslant |\mathfrak{n}(s)|, \end{cases}$$

$$\check{H}\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}) = E_{s,p_z,\mathfrak{n}(s),l}\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}),$$

$$E_{s,p_z,\mathfrak{n}(s),l} = \zeta\sqrt{m_{\mathrm{e}}^2 + p_z^2 + 2\gamma\,|\mathfrak{n}(s)|}, \quad \mathfrak{n}(s) \in \mathcal{Z}(s), \quad l \leqslant |\mathfrak{n}(s)|,$$

where $\mathcal{Z}(s)$ is defined by equation (60), and the doublets $\overset{\mathrm{I}}{U}_{\mathfrak{n}(s)}(s,l,p_z;\rho)$ and $\overset{\mathrm{II}}{U}_{\mathfrak{n}(s)}(s,l,p_z;\rho)$ are given by the respective equations (61) and (64), form a complete orthonormalized system in the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^3)$ of the Dirac bispinors. The latter means that the following inversion formulae exist:

$$\Psi(\mathbf{r}) = \int \mathrm{d}p_z \sum_{s=\pm 1} \sum_{\mathfrak{n}(s)\in\mathcal{Z}(s)} \sum_{l\leqslant|\mathfrak{n}(s)|} \Phi_{s,p_z,\mathfrak{n}(s),l}\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}),$$

$$\Phi_{s,p_z,\mathfrak{n}(s),l} = \int \overline{\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r})}\Psi(\mathbf{r})\mathrm{d}\mathbf{r},$$

$$\int |\Psi(\mathbf{r})|^2 \mathrm{d}\mathbf{r} = \int \mathrm{d}p_z \sum_{s=\pm 1} \sum_{\mathfrak{n}(s)\in\mathcal{Z}(s)} \sum_{l\leqslant|\mathfrak{n}(s)|} |\Phi_{s,l,p_z,n}|^2, \quad \forall\Psi \in \boldsymbol{L}^2(\mathbb{R}^3).$$

We note that for $\lambda = 0$ and $\pm\pi/2$, the spectrum at $l = 0$ can be found explicitly, see the third region.

For $\mu > 0$, there is a family of s.a. Dirac operators $\hat{H}_{\{\lambda(s,p_z)\}}$ parameterized by two real-valued functions $\lambda(s,p_z)$, $\lambda \in \mathbb{S}(-\pi/2, \pi/2)$, $s = \pm 1$. Their spectra are degenerated and continuous.

A complete set of generalized eigenfunctions of $\hat{H}_{\{\lambda(s,p_z)\}}$ consists of $\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r})$ and $\Psi^{\lambda(s,p_z)}_{s,p_z,k,l_0}(\mathbf{r})$. These bispinors have the form

$$\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\rho}} \mathrm{e}^{\mathrm{i}p_z z} S_l(\varphi) F_{\mathfrak{n}(s)}(s,l,p_z;\rho) \otimes e_s(p_z),$$

$$F_n(s,l,p_z;\rho) = \begin{cases} \overset{\mathrm{I}}{U}_n(s,l,p_z;\rho), & l \leqslant -1, \\ \overset{\mathrm{II}}{U}_n(s,l,p_z;\rho), & 1 \leqslant l \leqslant |\mathfrak{n}(s)|, \end{cases}$$

$$\mathfrak{n}(s) \in \mathcal{Z}(s), \quad l \leqslant |\mathfrak{n}(s)|, \quad l \neq 0,$$

and

$$\Psi^{\lambda(s,p_z)}_{s,p_z,k,l_0}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\rho}} \mathrm{e}^{\mathrm{i}p_z z} S_{l_0}(\varphi) \overset{\mathrm{III}}{U}_k(\lambda(s,p_z),s,p_z;\rho) \otimes e_s(p_z), \quad k \in \mathbb{Z},$$

where $\overset{\mathrm{III}}{U}_n(\lambda(s,p_z),s,p_z;\rho)$ are given by (69) with the substitution $\lambda(s) \to \lambda(s,p_z)$, such that

$$\check{H}\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}) = E_{s,p_z,\mathfrak{n}(s),l}\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}), \quad l \leqslant |\mathfrak{n}(s)|, \quad l \neq 0,$$

$$E_{s,p_z,\mathfrak{n}(s),l} = \sigma\sqrt{m_{\mathrm{e}}^2 + p_z^2 + 2\gamma[|\mathfrak{n}(s)| + \theta(l)]},$$

$$\theta(l) = \begin{cases} 0, & l \leqslant 0 \\ 1, & l \geqslant 1 \end{cases},$$

$$\check{H}\Psi^{\lambda(s,p_z)}_{s,p_z,k,l_0}(\mathbf{r}) = E^{\lambda(s,p_z)}_{s,p_z,k,l_0}\Psi^{\lambda(s,p_z)}_{s,p_z,k,l_0}(\mathbf{r}),$$

$$E^{\lambda(s,p_z)}_{s,p_z,k,l_0} : \Omega\left(\lambda, E^{\lambda}_{s,p_z,k,l_0}\right) = 0,$$

$$\Omega(\lambda, W) = \frac{\cos\lambda + a(W)\sin\lambda}{\sin\lambda - a(W)\cos\lambda},$$

$$a(W) = \frac{2m_{\mathrm{e}}^{2\mu-1}(\gamma/2)^{1-\mu}\Gamma(\mu)\Gamma(1-\mu-w/2\gamma)}{(W+sM)\Gamma(1-\mu)\Gamma(-w/2\gamma)}.$$

In the case under consideration, the corresponding inversion formulae have the form

$$\Psi(\mathbf{r}) = \int \mathrm{d}p_z \sum_{s=\pm 1} \left[\sum_{\mathfrak{n}(s)\in\mathcal{Z}(s)} \sum_{l\leqslant|\mathfrak{n}(s)|, l\neq 0} \Phi_{s,p_z,\mathfrak{n}(s),l}\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r}) + \sum_k \Phi_{s,p_z,k,l_0}\Psi^{\lambda(s,p_z)}_{s,p_z,k,l_0}(\mathbf{r}) \right], \quad \forall\Psi \in \boldsymbol{L}^2(\mathbb{R}^3),$$

$$\Phi_{s,p_z,\mathfrak{n}(s),l} = \int \overline{\Psi_{s,p_z,\mathfrak{n}(s),l}(\mathbf{r})}\Psi(\mathbf{r})\mathrm{d}\mathbf{r}, \quad l \neq 0,$$

$$\Phi_{s,p_z,k,l_0} = \int \overline{\Psi^{\lambda(s,p_z)}_{s,p_z,k,l_0}(\mathbf{r})}\Psi(\mathbf{r})\mathrm{d}\mathbf{r},$$

$$\int |\Psi(\mathbf{r})|^2 \mathrm{d}\mathbf{r} = \int \mathrm{d}p_z \sum_{s=\pm 1} \left[\sum_{\mathfrak{n}(s)\in\mathcal{Z}(s)} \sum_{l\leqslant|\mathfrak{n}(s)|, l\neq 0} \left|\Phi_{s,p_z,\mathfrak{n}(s),l}\right|^2 + \sum_k \left|\Phi_{s,p_z,k,l_0}\right|^2 \right].$$

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