ISSN 0016-2663, *Functional Analysis and Its Applications*, 2025, *Vol.* 59, *No.* 2, *pp.* 107–113. ©*Pleiades Publishing, Ltd.*, 2025.
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A Note on Lattice Knots

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Abstract. The aim of this note is to share the observation that the set of elementary operations of Turing on lattice knots can be reduced to just one type of simple local switches.

Key words: knots, cubiculated knots.

DOI: 10.1134/S1234567825020016

1. Introduction

The subject of this note is the study of knots in $\mathbb{R}^3$ that can be composed out of unit intervals that are parallel to elements of the standard basis. It can be easily shown that each tame knot can be represented in this form (such representation is often refered to as a *cubulation* [1]). We will see that any isotopy of knots carries over to a simple combinatorially defined equivalence relation on lattice knots. Namely, the following holds.

Theorem 1. *Any two cubulated knots are isotopic if and only if one can be obtained from the other by a sequence of the folowing operations* (*see Fig.* 1) *and their inverses as soon as each step does not create self-intersections.*

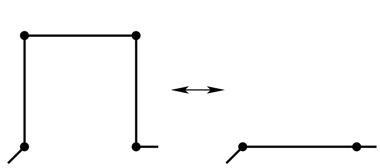

Fig. 1. Elementary switches.

It is rather curious that the first-ever survey of the subject of topology, written by Dehn and Heegaard for Klein's Encyklopädie der mathematischen Wissenschaften [2], was partially devoted to lattice knots. Lacking the modern language of topological spaces, they work with cubulated knots and combinatorial equivalence from the outset.

Also, Alan Turing, in his famous article "Solvable and Unsolvable Problems" [7], addresses the problem of algorithmic distinction of cubulated knots. It should also be noted that lattice knots is a subject of interest of biologists as lattice models of polymers (see, e.g., [3], [8]).

Grid diagrams, which play the key role in a combinatorial description of knot Floer homology [6], give rise to lattice knots. On the other hand, cubulation is the first step in the construction of a grid diagram of a given knot. At last, we would like to remark that the higher dimensional analogs of cubulated knots also generated some literature (see, e.g., [4]).

A version of the main theorem seems to be a folklore knowledge, mentioned as such, e.g., by Turing in [7] and Przytycki. To the best of the authors' knowledge, the first paper where its proof was written up is [5]. However, all the aforementioned authors require more operations as elementary. In particular, they include doubling (in the sense of our definition below) as a basic operation. The main novelty of our paper is the establishment of the fact that doubling follows from the rest of the operations appearing in [7] and [5].

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2020 *Mathematics Subject Classification*: 57K10.

Consider the set $\mathcal{N}$ obtained as the union of all translations by integer vectors of the union of the three coordinate lines:

$$\mathbb{Z}^3 + \{(x, y, z)\colon x = y = 0 \text{ or } x = z = 0 \text{ or } z = y = 0\} \subset \mathbb{R}^3.$$

Definition 1. An embedding $f\colon S^1 \to \mathbb{R}^3$ such that $f(S^1) \subset \mathcal{N}$ is called a *lattice knot.*

Remark 1. Lattice knots are also referred to as *cubic knots* in the literature [5]. It is easy to show that any tame knot is isotopic to a lattice knot.

In the present paper, we will represent a lattice knot by an element of the commutator subgroup of the free group $\mathbb{F}_3$ in the following way. With each lattice knot we associate a reduced word in the alphabet

$$\{x, y, z, \bar{x}, \bar{y}, \bar{z}\},$$

which encodes linear embeddings of each unit-length segment of a knot.

Example 1. A lattice trefoil, depicted in Fig. 2, is represented in this notation by

$$T = xxz\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{z}\bar{z}xxxxzzz\bar{x}\bar{x}y\bar{z}\bar{z}xxx\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}yy.$$

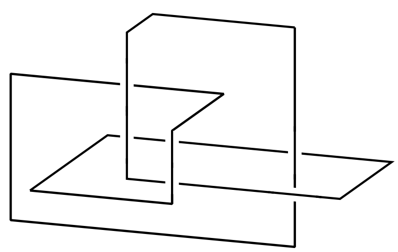

Fig. 2. Lattice trefoil knot T.

Definition 2. The abelianization map $\mathrm{Ab}\colon \mathbb{F}_3 \to \mathbb{Z}^3$ is defined by the formula

$$\mathrm{Ab}\colon w \mapsto \big(|w|_x - |w|_{\bar{x}},\ |w|_y - |w|_{\bar{y}}, |w|_z - |w|_{\bar{z}}\big),$$

where $|w|_s$ is the number of occurrences of the symbol s in the reduced word w.

Definition 3. A *subword* of a word w is a set of consecutive symbols of w. We will denote it by $w[n..m]$, where n is the index of the first symbol and m is the index of the last. The numbering starts at 1, and $w[n..m]$ is the empty word if $m < n$ by convention.

Now, we are going to recast the definition of a lattice knot in terms of words.

Definition 4. A *lattice knot* is a reduced word w in the alphabet $\{x, y, z, \bar{x}, \bar{y}, \bar{z}\}$ such that
(1) $\mathrm{Ab}(w) = 0$;
(2) no proper subword w' of w has the property $\mathrm{Ab}(w') = 0$.

Remark 2. These two conditions amount to the closeness and the absence of self-intersections of a knot, respectively.

The following definition introduces the main operation of the paper.

Definition 5. The operation which takes a subword v in w and substitutes it with $\bar{x}vx$ or $xv\bar{x}$ (analogously, for y and z), once the result (after a reduction, if necessary) is again a knot, will be called an *elementary switch.* We will use the notation $w[v \to \bar{x}vx]$, $w[v \to \bar{y}vy]$, $w[v \to \bar{z}vz]$, etc.

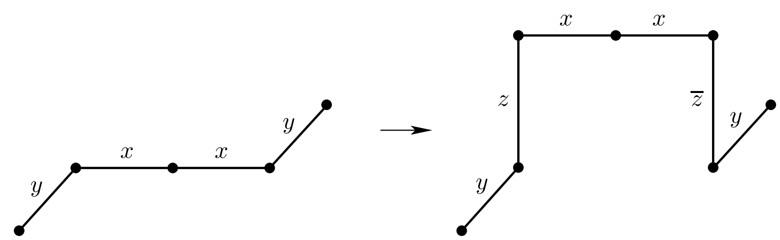

Fig. 3. Switch $xx \to zxx\bar{z}$.

A typical elementary switchm corresponding to the substitution $xx \to zxx\bar{z}$, is depicted in Fig. 3.

Geometrically, there can be two types of behavior, depending on whether the result is a reduced word or not. This is illustrated by Fig. 1.

Definition 6. The operation which takes a word and substitutes each x with xx is called the *doubling in the x-direction*. The operation of doubling in the directions y and z are defined likewise.

Remark 3. Doubling can be thought of as the same knot positioned in a refined lattice, see Fig. 4.

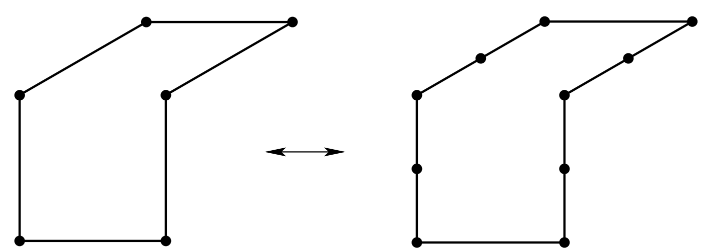

Fig. 4. Doubling.

2. Doubling

The purpose of this section is to prove that doubling in the z-direction can be achieved by repeated application of elementary switches.

It is done in stages depicted in the figure above for the case of the trefoil knot of Example 1. The idea is very simple. First, we lift the uppermost layer of the knot, which is always possible since there are no branches of the knot above it. Then, we can lift the next layer up because there are no horizontal segments between it and the already lifted layer to cause an intersection and, eventually, we will be able to lift all layers. The rest of the paper will be devoted to formalizing this argument.

Definition 7. The set of all entries $w[m]$ of a word w with the property $|w[1..(m-1)]|_z = |[1..m]|_z = n$ is called the *n-th layer of w*. In other words, it is the set of all entries $w[m] \neq z$ such that $|w[1..m]|_z = n$. Generally, a layer is some number of subwords since it need not be connected.

The n-th layer is simply the set of segments that intersect with the plane parallel to the xy-plane and with the z-coordinate equal to n. Fig. 6 shows an example of a knot with 2 layers, the 1-layer is displayed in red. Note that it is not connected.

Example 2. In Example 1, the zeroth layer is highlighted in bold:

$$\boldsymbol{xx}z\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{z}\bar{z}xxxxzzz\bar{x}\bar{x}y\bar{z}\bar{z}\boldsymbol{xxx\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{x}yy}.$$

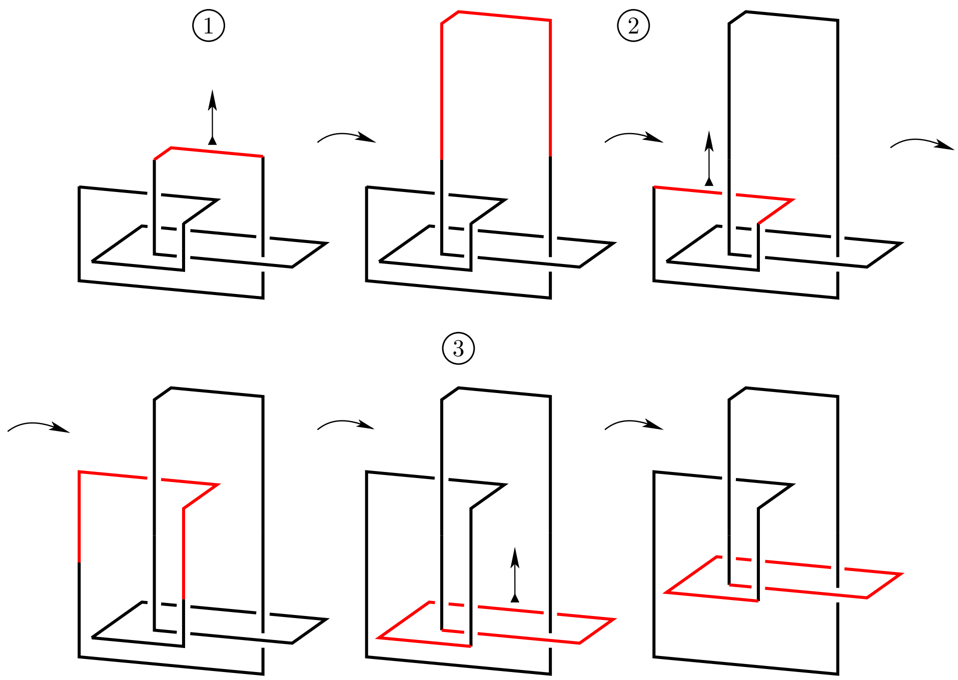

Fig. 5. Stages of doubling.

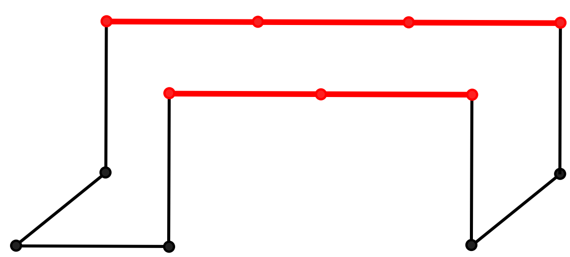

Fig. 6. Example of a layer.

The (-1)-st level is

$$T = xxz\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{z}\bar{z}\boldsymbol{xxxx}zzz\bar{x}\bar{x}y\bar{z}\bar{z}xxx\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{x}yy,$$

and the second layer is

$$xxz\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{z}\bar{z}xxxxzzz\boldsymbol{\bar{x}\bar{x}y}\bar{z}\bar{z}xxx\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{x}yy.$$

Definition 8. We say that the n-th layer *has k vacant layers above* if all the layers of order $n+1, \dots, n+k$ are empty.

Definition 9. The *elevation of the n-th layer by k* is the composition of the operations $w[v_i \to z^k v_i \bar{z}^k]$ (which clearly commute) for all subwords v_i that make up the n-th layer, followed by reduction.

Example 3. The first move of Fig. 5 is the elevation of the second layer by 3. More formally, it is represented as

$$\begin{aligned} &xxz\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{z}\bar{z}xxxxzzz\underline{\bar{x}\bar{x}y}\bar{z}\bar{z}xxx\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{x}yy \\ &\quad\mapsto xxz\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{z}\bar{z}xxxxzzz\boldsymbol{zzz}\underline{\bar{x}\bar{x}y}\boldsymbol{\bar{z}\bar{z}\bar{z}}\bar{z}\bar{z}xxx\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{x}yy. \end{aligned}$$

Remark 4. The first condition of Definition 4 is automatically satisfied after an elevation is applied because z and $\bar{z}$ are added in pairs, and reduction kills pairs.

The proof of the following lemma is rather obvious from the geometric point of view. It is essentially contained in Fig. 7.

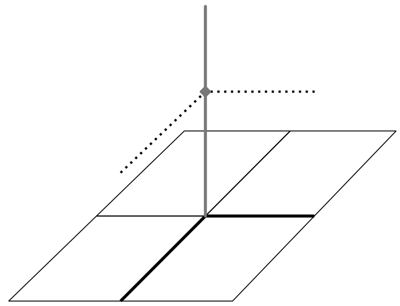

Fig. 7. Elevation.

If, after a lifting, the n-th layer intersects another branch of the knot, then this other branch consists of two vertical segments in the viscinity of the hypothetical intersection point. But then, an intersection already took place in the original knot.

For the sake of completeness, we include a formal proof.

Lemma 1. *If the n-th layer of a knot w has one vacant layer above, then the elevation w' of the layer by one is also a knot. Moreover, the n-th layer of w' becomes vacant, the $(n+1)$-st layer of w' coincides with the n-th layer of w, and all other layers remain the same.*

Proof. Suppose that after the elevation of the n-th layer, the new word, which we denote by w', does not satisfy the second condition of Definition 4. That is, there is a proper subword $w'[n_1'..n_2']$ such that $\mathrm{Ab}(w'[n_1'..n_2']) = 0$, or, equivalently, there are two different numbers n_1', n_2' such that

$$\mathrm{Ab}(w'[1..n_1']) = \mathrm{Ab}(w'[1..n_2']).$$

Let us put back all the pairs $\bar{z}z$ and $z\bar{z}$ that were cancelled out during reduction. Let us denote by N_1 and N_2 the new places of $w'(n_1')$ and $w'(n_2')$ in this new (non-reduced) word W.

Consider three complementary cases:

Case 1. Suppose N_1 and N_2 are such that $W[N_1..N_2]$ contains equal numbers of z's and $\bar{z}$'s added during the elevation. Then the intersection existed already in w.

Case 2. Now, consider the case when $W[N_1..N_2]$ contains more added z's than $\bar{z}$'s. Since z and $\bar{z}$ are added by the elevation in an alternating manner, there is exactly one more z added than the number of $\bar{z}$'s. Now,

$$|W[N_1..N_2]|_z = 0,$$

so

$$|w[n_1..n_2]|_{\bar{z}} = 1,$$

where $w[n_1..n_2]$ is the subword of w obtained by removing all added z's and $\bar{z}$'s from $W[N_1..N_2]$.

On the other hand,

$$|W[1..N_2]|_z = n + 1,$$

so

$$|W[1..N_1]|_z = n + 1.$$

That means that $W[N_1] = z$ or $\bar{z}$; otherwise, it belonged to the $(n+1)$-st layer of w, which is empty. If $W[N_1]$ were z that was added by the elevation, then $|W[(N_1+1)..N_2]|_z = 0$; thus, the statement gets reduced to Case 1. If $W[N_1]$ is z that was already in w, then $W[N_1+1]$ is necessarily z that was already in w: otherwise, the $(n+1)$-st layer in w would not be empty. If $W[N_1] = \bar{z}$ then $|w[(n_1+1)..n_2]|_z = 0$, and we, again, arrive to a contradiction.

Case 3. When there is one more added $\bar{z}$ than the numeber z's in $W[n_1..n_2]$, by an argument identical to Case 2, one proves that intersections also do not appear after the elevation. Alternatively, one may use the fact that a cyclic permutation of a word gives the same knot, and the complement of $w'[n_1'..n_2']$ satisfying the assumption of Case 3 satisfies the assumption of Case 2.

The last statement of the lemma follows from the observation that only the n-th and $(n+1)$-st layers are affected by the operation of elevation. Namely, the n-th layer dissapears and becomes the $(n+1)$-th layer in the new knot. □

Now, from Lemma 1 we can establish the doubling in any direction by induction on the number of layers.

Theorem 2. *Let w be a knot. Then the doubling of w in any direction and w are connected by a sequence of elementary switches.*

Proof. Let m be the maximal number such that the m-th layer is nonempty. Clearly, it possesses a vacant layer above. Hence, we can elevate it m times, ending up with a knot with $2m$ layers with all the layers between $(m-1)$ and $(2m)$ vacant. We can then elevate the $(m-1)$-st layer by $m-1$, and so on all the way through to the zeroth level. □

3. Isotopy of knots

A theorem of Hinojosa–Verjovsky–Marcotte, [5; Theorem 1], adapted to the terminology of this paper, states the following.

Theorem 3. *Two lattice knots w_1 and w_2 are isotopic if and only if one can be obtained from the other by a sequence of elementary switches, doubling, and their inverses.*

Our Theorem 2 ensures that doubling follows from elementary switches. Or, in the terminology of [5], that any (M1) move is a sequence of (M2) moves. Thus, combining two results, we arrive at the following result.

Main theorem. *A pair of lattice knots w_1 and w_2 are isotopic if and only if they can be connected by a sequence of elementary switches.*

Funding

A. Grishkov was supported by FAPESP (grant 2018/23690-6), CNPq (grants 10 307593/2023-1 and 406932/2023-9), and in accordance with the state task of the IM SB RAS, project FWNF-2022-003.

Conflict of Interest

The authors of this work declare that they have no conflicts of interest.

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Received 25 September 2024
In final form 14 October 2024
Accepted 17 October 2024

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