GLOBAL MILD SOLUTIONS FOR THE NONAUTONOMOUS 2D NAVIER-STOKES EQUATIONS WITH IMPULSE EFFECTS

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Global mild solutions for the nonautonomous 2D Navier-Stokes equations with impulse effects

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Abstract

The present paper deals with existence and uniqueness of global mild solutions for the 2D Navier-Stokes equations with impulses. Using the framework of nonautonomous dynamical systems, we extend previous results considering the 2D Navier-Stokes equations with impulse effects and allowing that the nonlinear terms are explicitly time-dependent. Additionally, we present sufficient conditions to obtain dissipativity (boundedness) for solutions starting in bounded sets.

1 Introduction

The Navier-Stokes equations (NSEs) represent a formulation of the Newton's laws of motion for a continuous distribution of matter in a fluid state, characterized by an inability to support shear stresses, see [13]. The NSEs allow to determine the velocity field and the pressure of fluids confined in regions of the space, and they are used to describe many different physics phenomena as weather, water flow in tubes, ocean currents and others. Moreover, these equations are useful in several fields of knowledge such as petroleum industry, plasma physics, meteorology, thermo-hydraulics, among others (see [29] for instance). Due to this fact, these equations have been attracted to the attention of several mathematicians since they play an important role for applications. See [1, 5, 8–10, 13–17, 19–30] and the references therein.

On the other hand, the theory of impulsive dynamical systems has been shown to be a powerful tool to model real-world problems in physics, technology, biology, among others. Because of this fact, the interest in the study of impulsive dynamical systems has increasing considerably. For recents trends on this subject we indicate the works [2–4, 11, 12, 18, 31, 32] and the references therein.

However, the study of Navier-Stokes equations with impulse effects is really scarce, there are few papers concerning with this topic in the literature. The most of them are concerned with impulsive control problems for stochastic Navier-Stokes equations (see [25–27], for instance). Therefore, motivated by this fact, in this paper, we investigate existence and uniqueness of mild solutions for the impulsive NSEs

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + q(t)(u \cdot \nabla)u - \nu \Delta u + \nabla p = \phi(t, u), & (t, x) \in \left((0, +\infty) \setminus \bigcup_{k=1}^{+\infty} \{t_k\} \right) \times \Omega, \\
\operatorname{div} u = 0, & (t, x) \in (0, +\infty) \times \Omega, \\
u = 0, & (t, x) \in (0, +\infty) \times \partial\Omega, \\
u(0, \cdot) = u_0 & x \in \Omega, \\
u(t_k^+, \cdot) - u(t_k^-, \cdot) = I_k(u(t_k, \cdot)), & x \in \Omega, \ k = 1, 2, \ldots,
\end{cases}
\tag{1.1}
$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$. Here $u = (u_1, u_2)$ denotes the velocity field of a fluid filling Ω, p is its scalar pressure and $\nu > 0$ is its viscosity. We will assume that q is a bounded function, ϕ is a nonlinearity which will be specified later, $\{t_k\}_{k \in \mathbb{N}} \subset (0, +\infty)$ is a sequence of impulse times such that $\lim\limits_{t \to +\infty} t_k = +\infty$, $u(t_k, \cdot) = u(t_k^+, \cdot) = \lim\limits_{\delta \to 0+} u(t_k + \delta, \cdot)$, $u(t_k^-, \cdot) = \lim\limits_{\delta \to 0+} u(t_k - \delta, \cdot)$ and I_k, $k \in \mathbb{N}$, are the impulse operators.

Besides to impulsive actions in the system (1.1), we also allow that the external force ϕ is not continuous and depends on the solution u.

We point out that the Navier-Stokes equations with impulses make sense physically and allow to describe more precisely the phenomena modeled by these equations, since u represents the velocity of the field of a fluid and moreover, the external force ϕ in this case does not need to be continuous. It is well known that the phenomena which occur in the environment have impulsive behavior and the functions which model them have several discontinuities. Therefore, with this impulsive model, we intend to give a more precisely description of the Navier-Stokes equations.

The system (1.1) without impulse conditions was studied in the classical monograph [6], where ϕ is a function of time $t \in \mathbb{R}$. More precisely, the author studies existence and uniqueness of global mild solutions for the non-impulsive equation

$$
\frac{\partial u}{\partial t} + q(t)(u \cdot \nabla)u - \nu \Delta u + \nabla p = \phi(t),
$$

subject to the conditions $\operatorname{div} u = 0$ and $u|_{\partial\Omega} = 0$, where Ω is a bounded smooth domain in $\mathbb{R}^2$.

Our goal here is to write a weaker formulation of the system (1.1) and then, we intend to investigate the existence and uniqueness of mild solutions. In order to do this, we start by considering some notations which can be found in [28] and [29], for instance. Let $\mathbb{L}^2(\Omega) =$

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$(L^2(\Omega))^2$ and $\mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^2$ endowed, respectively, with the inner products

$$(u,v) = \sum_{j=1}^{2} \int_{\Omega} u_j \cdot v_j \ dx, \quad u = (u_1, u_2), \ v = (v_1, v_2) \in \mathbb{L}^2(\Omega),$$

and

$$((u,v)) = \sum_{j=1}^{2} \int_{\Omega} \nabla u_j \cdot \nabla v_j dx, \quad u = (u_1, u_2), \ v = (v_1, v_2) \in \mathbb{H}_0^1(\Omega)$$

and norms $|\cdot| = (\cdot, \cdot)^{1/2}$ and $\|\cdot\| = ((\cdot, \cdot))^{1/2}$.

Now, we consider the following sets:

$$\mathcal{E} = \{v \in (C_0^\infty(\Omega))^2 : \ \nabla \cdot v = 0 \text{ in } \Omega\},$$

$$V = \text{closure of } \mathcal{E} \text{ in } \mathbb{H}_0^1(\Omega)$$

and

$$H = \text{closure of } \mathcal{E} \text{ in } \mathbb{L}^2(\Omega).$$

The space H is a Hilbert space with the scalar product $(\cdot, \cdot)$ induced by $\mathbb{L}^2(\Omega)$ and the space V is a Hilbert space with the scalar product $((u,v))$ induced by $\mathbb{H}_0^1(\Omega)$.

The space V is contained in H, it is dense in H and by the Poincare's Inequality, the inclusion $i : V \hookrightarrow H$ is continuous. Denote by V' and H' the dual spaces of V and H, respectively. The adjoint operator i^* is linear and continuous from H' to V', $i^*(H')$ is dense in V' and i^* is one to one since $i(V) = V$ is dense in H. Moreover, by the Riesz representation Theorem, we can identify H and H' and write

$$V \subset H \equiv H' \subset V',$$

where each space is dense in the following one and the injections are continuous.

As a consequence of the previous identifications, the scalar product in H, (f, u), of $f \in H$ and $u \in V$ is the same as the duality product between V' and V, $\langle f, u \rangle$, i.e.,

$$\langle f, u \rangle = (f, u), \quad \text{for all } f \in H \text{ and for all } u \in V.$$

Also, for each $u \in V$, the form

$$v \in V \mapsto \nu((u, v)) \in \mathbb{R}$$

is linear and continuous on V. Therefore, there exists an element of V' which we denote by Au such that

$$\langle Au, v \rangle = \nu((u, v)), \text{ for all } v \in V.$$

Notice that the mapping $u \mapsto Au$ is linear, continuous and it is an isomorphism from V to V'.

Based on it, we consider the following weak formulation of (1.1):

$$
\begin{cases}
\dfrac{d}{dt}(u,v) + \nu((u,v)) + b(t)(u,u,v) = \langle \phi(t,u), v \rangle, & v \in V,\ t > 0,\ t \neq t_k, \\
u(t_k) - u(t_k^-) = I_k(u(t_k^-)), & k \in \mathbb{N}, \\
u(0) = u_0 \in H,
\end{cases}
\tag{1.2}
$$

where $\phi(t,u) \in V'$ and $b(t) : V \times V \times V \to \mathbb{R}$ is given by

$$
b(t)(u,v,w) = q(t) \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.
$$

The weak formulation (1.2) is equivalent to the impulsive system

$$
\begin{cases}
u' + Au + B(t)(u,u) = \phi, & \text{in } V',\ t > 0,\ t \neq t_k, \\
u(t_k) - u(t_k^-) = I_k(u(t_k^-)), & k \in \mathbb{N}, \\
u(0) = u_0 \in H,
\end{cases}
\tag{1.3}
$$

where $u' = du/dt$, $A : V \to V'$ is the Stokes operator defined by

$$
\langle Au, v \rangle = \nu((u,v)), \text{ for all } u, v \in V,
$$

and $B(t) : V \times V \to V'$ is a bilinear operator defined by

$$
\langle B(t)(u,v), w \rangle = b(t)(u,v,w), \text{ for all } u, v, w \in V.
$$

In Section 2, we consider the following general impulsive system

$$
\begin{cases}
u' + Au + B(\sigma(\cdot,\omega))(u,u) = f(\cdot, \sigma(\cdot,\omega), u), & t > 0,\ t \in I,\ t \neq t_k,\ k \in \mathbb{N}, \\
u(t_k) - u(t_k^-) = I_k(u(t_k^-)), & k \in \mathbb{N}, \\
u(0) = u_0 \in H,
\end{cases}
\tag{1.4}
$$

where $f : I \times \mathcal{M} \times H \to H$ is a piecewise continuous function with respect to $t \in \mathbb{R}$, non-stationary and also depends on the solution u. All the conditions of system (1.4) will be specified later. We prove the existence and uniqueness of global mild solutions for the system (1.4) when $\mathcal{M}$ is compact, see Theorems 2.3 and 2.4. The case when $\mathcal{M}$ is not compact is considered in Theorem 2.5.

In Section 3, we prove existence and uniqueness of global mild solutions for the 2D NSEs with impulses (1.1) via system (1.4). We also give sufficient conditions to obtain dissipativity for the system (1.1). All the results from this paper hold for the non-impulsive case.

2 Preliminaries

Let $(\mathcal{M}, d)$ be a metric space and $(\mathcal{M}, \mathbb{R}, \sigma)$ be a dynamical system on $\mathcal{M}$, i.e., $\sigma : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ is a continuous mapping which satisfies the following properties:

i) $\sigma(0, \omega) = \omega, \quad \omega \in \mathcal{M}$;

ii) $\sigma(s, \sigma(t, \omega)) = \sigma(s + t, \omega), \quad t, s \in \mathbb{R}, \omega \in \mathcal{M}$.

Let H be a real or complex Hilbert space and $\mathcal{L}(H) = \{T : H \to H : T$ is linear and bounded$\}$ equipped with the operational norm. Let $A : D(A) \subset H \to H$ be a self-adjoint operator such that

$$\mathrm{Re} \langle Au, u \rangle \geqslant a|u|_H^2, \tag{2.1}$$

for all $u \in D(A)$ and $a > 0$. It follows by [5, Lemma 6.20] that $\mathbb{C} \setminus (-\infty, a] \subset \rho(A)$ (resolvent of A), and there exists a constant $M \geqslant 1$ such that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(H)} \leqslant \frac{M}{|\lambda - a|}, \quad \lambda \in \Sigma_{a,\varphi}, \, \lambda \neq a,$$

where $\Sigma_{a,\varphi} = \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| \leqslant \varphi\}, \varphi < \pi$. In particular, A is sectorial. It follows that $-A$ generates an analytic semigroup $\{e^{-At} : t \geqslant 0\} \subset \mathcal{L}(H)$ which satisfies

$$\|e^{-At}\|_{\mathcal{L}(H)} \leqslant Ke^{-at}, \tag{2.2}$$

for some constant $K > 0$, where $a > 0$ comes from (2.1).

Assuming that $0 \in \rho(A)$, we consider the scale of Hilbert spaces $X^\alpha = D(A^\alpha)$ of fractional power of the operator A endowed with the norm $\|\cdot\|_{X^\alpha} = \|A^\alpha \cdot\|_H$ ($X^0 = H$). If $\beta > \alpha \geqslant 0$, it is well known that X^β is a dense subspace of X^α with continuous inclusion and

$$\|e^{-At}\|_{\mathcal{L}(X^\alpha, X^\beta)} \leqslant C_{\alpha,\beta} \, t^{\alpha-\beta} e^{-at}. \tag{2.3}$$

Consider F a Hilbert space such that $H \subset F$ with inclusion dense and continuous. Denote by $\mathcal{L}(F, H) = \{T : F \to H : T$ is linear and bounded$\}$ equipped with the operational norm. We will assume that the semigroup $\{e^{-At} : t \geqslant 0\}$ satisfies:

i) $e^{-At} \in \mathcal{L}(F, H), \quad$ for all $t > 0$;

ii) There exists $0 \leqslant \alpha_1 < 1$, such that

$$\|e^{-At}\|_{\mathcal{L}(F,H)} \leqslant K_1 t^{-\alpha_1} e^{-at}, \quad \text{for all } t > 0. \tag{2.4}$$

We denote by $\mathcal{L}^2(H, F)$ the space of all continuous bilinear operators $\mathcal{B} : H \times H \to F$ equipped with the norm

$$\|\mathcal{B}\|_{\mathcal{L}^2(H,F)} = \sup\{|\mathcal{B}(u, v)|_F : |u|_H \leqslant 1, \, |v|_H \leqslant 1\}$$

and by $C(\mathcal{M}, \mathcal{L}^2(H, F))$ the space of all continuous mapping $B : \mathcal{M} \to \mathcal{L}^2(H, F)$.

Let $\{t_k\}_{k \in \mathbb{N}}$ be a strictly increasing sequence in $(0, +\infty)$ such that $\lim\limits_{t \to +\infty} t_k = +\infty$. Let $I \subset \mathbb{R}$ be an interval and $f : I \times \mathcal{M} \times H \to H$ and $I_k : H \to H$, $k \in \mathbb{N}$, be functions satisfying the following conditions:

(C1) For each fixed $t \in I$, $f(t, \cdot, \cdot)$ is continuous on $\mathcal{M} \times H$.

(C2) Let $\omega \in \mathcal{M}$ and $u \in H$. Then $\lim\limits_{s \to t} f(s, \omega, u) = f(t, \omega, u)$ for all $t \neq t_k$, $k \in \mathbb{N}$, the limit $\lim\limits_{s \to t_k-} f(s, \omega, u)$ exists and $\lim\limits_{s \to t_k+} f(s, \omega, u) = f(t_k, \omega, u)$, for all $k \in \mathbb{N}$.

(C3) There is a bounded function $M : \mathbb{R} \to \mathbb{R}_+$, such that for any interval $[a, b] \subset I$, we have

$$\int_a^b |\phi(s)| |f(s, \omega, u)|_H ds \leqslant \int_a^b M(s) |\phi(s)| ds$$

for all $\phi \in L^1[a, b]$, $\omega \in \mathcal{M}$ and $u \in H$.

(C4) There is a bounded function $L : \mathbb{R} \to \mathbb{R}_+$, such that for any interval $[a, b] \subset I$, we have

$$\int_a^b |\phi(s)| |f(s, \omega_1, u_1) - f(s, \omega_2, u_2)|_H \, ds \leqslant \int_a^b L(s) |\phi(s)| (d(\omega_1, \omega_2) + |u_1 - u_2|_H) ds$$

for all $\phi \in L^1[a, b]$, $\omega_1, \omega_2 \in \mathcal{M}$ and $u_1, u_2 \in H$.

(C5) There exists a constant $K_2 > 0$ such that

$$\sup_{k \in \mathbb{N}} \sup_{u \in H} |I_k(u)|_H \leqslant K_2.$$

(C6) There exists a constant $K_3 > 0$ such that

$$|I_k(u) - I_k(v)|_H \leqslant K_3 |u - v|_H$$

for all $u, v \in H$ and for all $k \in \mathbb{N}$.

Now, given $\omega \in \mathcal{M}$ and assuming all the conditions above, we consider the following impulsive system in the state space H:

$$\begin{cases} u' + Au + B(\sigma(\cdot, \omega))(u, u) = f(\cdot, \sigma(\cdot, \omega), u), & t > 0, \ t \in I, \ t \neq t_k, \ k \in \mathbb{N}, \\ u(t_k) - u(t_k^-) = I_k(u(t_k^-)), & k \in \mathbb{N}, \\ u(0) = u_0 \in H. \end{cases} \tag{2.5}$$

Remark 2.1. Since $\lim\limits_{k \to +\infty} t_k = +\infty$, it is clear that given a closed interval $[0, T]$, there exists at most a finite number of moments of impulses $t_1, t_2, \ldots, t_n \in [0, T]$ such that $0 < t_1 < t_2 < \ldots < t_n \leqslant T$. Thus, given $T > 0$ there is an integer $n_T > 0$ such that $t_{n_T} \leqslant T < t_{n_T+1}$.

Given $T > 0$, we consider the space $PC^+([0, T], H) = \{u : [0, T] \to H : u \text{ is continuous at } t \neq t_k, \text{ right-continuous at } t = t_k \text{ and the limit } \lim_{t \to t_k^-} u(t) \text{ exists for all } k = 1, \ldots, n_T\}$. It is well known that the space $PC^+([0, T], H)$ endowed with the norm $\|u\|_{PC^+} = \sup_{t \in [0,T]} |u(t)|_H$ is a Banach space.

In the sequel, we present the definition of a mild solution for the system (2.5).

Definition 2.2. Let $[0, T] \subset I$. We say that $u \in PC^+([0, T], H)$ is a *mild solution* of (2.5) if u satisfies the following integral equation:

$$u(t) = \begin{cases} e^{-At}u_0 + \int_0^t e^{-A(t-s)} g(s, \omega, u(s)) ds, & \text{if } 0 \leqslant t < t_1, \\[2mm] e^{-A(t-t_1)}[u(t_1^-) + I_1(u(t_1^-))] + \int_{t_1}^t e^{-A(t-s)} g(s, \omega, u(s)) ds, & \text{if } t_1 \leqslant t < t_2, \\[2mm] e^{-A(t-t_2)}[u(t_2^-) + I_2(u(t_2^-))] + \int_{t_2}^t e^{-A(t-s)} g(s, \omega, u(s)) ds, & \text{if } t_2 \leqslant t < t_3, \\[2mm] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\[2mm] e^{-A(t-t_k)}[u(t_k^-) + I_k(u(t_k^-))] + \int_{t_k}^t e^{-A(t-s)} g(s, \omega, u(s)) ds, & \text{if } t_k \leqslant t \leqslant T, \end{cases} \quad (2.6)$$

where $0 < t_1 < \ldots < t_k \leqslant T < t_{k+1}$ are the impulse times $(k = n_T)$ and $g(s, \omega, u(s)) = -B(\sigma(s, \omega))(u(s), u(s)) + f(s, \sigma(s, \omega), u(s))$, $s \in [0, T]$. System (2.6) can be rewritten in the following way

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} g(s, \omega, u(s)) ds + \sum_{0 < t_i < t} e^{-A(t-t_i)} I_i(u(t_i^-)).$$

Given $K \subset H$, we consider the following space of functions:

$PC_1^+([0, T] \times K \times \mathcal{M}, H) = \{\varphi : [0, T] \times K \times \mathcal{M} \to H : \text{ for all } (u, \omega) \in K \times \mathcal{M},$
$\quad \varphi(\cdot, u, \omega) \in PC^+([0, T], H) \text{ and for all } t \in [0, T], \varphi(t, \cdot, \cdot) : K \times \mathcal{M} \to H \text{ is continuous}\}.$

In our first result, namely Theorem 2.3, we assume that the space $\mathcal{M}$ is compact. In this case, $\left(C(\mathcal{M}, \mathcal{L}^2(H, F)), \|\cdot\|_\infty\right)$ is a Banach space, see [6], where

$$\|B\|_\infty = \sup_{\omega \in \mathcal{M}} \|B(\omega)\|_{\mathcal{L}^2(H,F)}.$$

Also, note that for all $u, v \in H$ and $\omega \in \mathcal{M}$, we have that

$$|B(\omega)(u, u) - B(\omega)(v, v)|_F \leqslant \|B\|_\infty (|u|_H + |v|_H)|u - v|_H, \quad (2.7)$$

and also

$$|B(\omega)(u, u)|_F \leqslant \|B\|_\infty |u|_H^2. \quad (2.8)$$

Theorem 2.3 ensures that the nonautonomous system (2.5) admits a unique mild solution in the sense of Definition 2.2.

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Theorem 2.3. *Let $u_0 \in H$, $r > 0$ and $\mathcal{M}$ be compact. Assume that (2.2), (2.4), (2.7) and conditions (C1) - (C6) hold. Then there exist positive numbers $\delta = \delta(u_0, r) > 0$, $T = T(u_0, r) > 0$ and a function $\varphi : [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M} \to H$ satisfying the following conditions:*

    i) $\varphi(0, u_0, \omega) = u_0$, for all $\omega \in \mathcal{M}$;

    ii) $|\varphi(t, u, \omega) - u_0|_H \leqslant r$ for all $(t, u, \omega) \in [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}$;

    iii) $\varphi \in PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)})$.

Moreover, the function $u : [0, T] \to H$ defined by $u(t) = \varphi(t, u_0, \omega)$ is the unique mild solution of system (2.5).

Proof. Let $\delta > 0$ and $T > 0$ be such that $[0, T] \subset I$. Given $\varphi \in PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, H)$, we define

$$S\varphi(t, u, \omega) = e^{-At}u + \int_0^t e^{-A(t-s)}g(s, \omega, \varphi(s))ds + \sum_{0 < t_i < t} e^{-A(t-t_i)}I_i(\varphi(t_i^-)),$$

where $\varphi(s) = \varphi(s, u, \omega)$ and $g(s, \omega, \varphi(s)) = -B(\sigma(s, \omega))(\varphi(s), \varphi(s)) + f(s, \sigma(s, \omega), \varphi(s))$, for all $s \in [0, T]$, $u \in \overline{B(u_0, \delta)}$ and $\omega \in \mathcal{M}$. Since functions in $PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)})$ are bounded, we can consider the distance

$$d_\infty(\varphi_1, \varphi_2) = \sup\{|\varphi_1(t, u, \omega) - \varphi_2(t, u, \omega)|_H : 0 \leqslant t \leqslant T, \ u \in \overline{B(u_0, \delta)}, \ \omega \in \mathcal{M}\},$$

for $\varphi_1, \varphi_2 \in PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)})$. It is not difficult to see that $(PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)}), d_\infty)$ is a complete metric space. For convenience, let us denote $\Gamma(\delta, T, r) = PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)})$ and $\Gamma(\delta, T) = PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, H)$.

    Assertion 1: $S \in C(\Gamma(\delta, T, r), \Gamma(\delta, T))$.

    In fact, at first note that $S\varphi \in \Gamma(\delta, T)$ for all $\varphi \in \Gamma(\delta, T, r)$.
    Now, let $\varphi_1, \varphi_2 \in \Gamma(\delta, T, r)$ and $(t, u, \omega) \in [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}$. By Condition (C4) there is a bounded function $L : \mathbb{R} \to \mathbb{R}_+$ such that

$$\int_0^t e^{-a(t-s)}|f(s, \sigma(s, \omega), \varphi_1(s)) - f(s, \sigma(s, \omega), \varphi_2(s))|_H \, ds \leqslant$$

$$\leqslant \int_0^t L(s)|\varphi_1(s) - \varphi_2(s)|_H \, ds \leqslant NT d_\infty(\varphi_1, \varphi_2), \tag{2.9}$$

where $N = \sup\limits_{s \in [0, T]} |L(s)|$.

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Then, using (2.2), (2.4), (2.7), (2.9) and Condition (C6), we have

$$|S\varphi_1(t,u,\omega) - S\varphi_2(t,u,\omega)|_H \leqslant$$

$$\leqslant \int_0^t \left| e^{-A(t-s)} \left[B(\sigma(s,\omega))(\varphi_1(s),\varphi_1(s)) - B(\sigma(s,\omega))(\varphi_2(s),\varphi_2(s)) \right] \right|_H ds +$$

$$+ \int_0^t \left| e^{-A(t-s)} \left[f(s,\sigma(s,\omega),\varphi_1(s)) - f(s,\sigma(s,\omega),\varphi_2(s)) \right] \right|_H ds +$$

$$+ \sum_{0 < t_i < t} |e^{-A(t-t_i)} [I_i(\varphi_1(t_i^-)) - I_i(\varphi_2(t_i^-))]|_H \leqslant$$

$$\leqslant 2\|B\|_\infty K_1(r + |u_0|_H) d_\infty(\varphi_1,\varphi_2) \int_0^t (t-s)^{-\alpha_1} e^{-a(t-s)} ds +$$

$$+ KNT d_\infty(\varphi_1,\varphi_2) + K K_3 d_\infty(\varphi_1,\varphi_2) \sum_{0 < t_i < t} e^{-a(t-t_i)} \leqslant$$

$$\leqslant \left(2\|B\|_\infty K_1(r + |u_0|_H) \frac{T^{-\alpha_1+1}}{-\alpha_1 + 1} + KNT + K K_3 n_T \right) d_\infty(\varphi_1,\varphi_2), \qquad (2.10)$$

where n_T is the number of impulses on the interval $[0,T]$. Hence, $S \in C(\Gamma(\delta,T,r), \Gamma(\delta,T))$.

Assertion 2: There are $\delta_1 = \delta_1(u_0,r) \in (0,\delta)$ and $T_1 = T_1(u_0,r) \in (0,T)$ such that $S : \Gamma(\delta_1,T_1,r) \to \Gamma(\delta_1,T_1,r)$.

In fact, let $\varphi \in \Gamma(\delta,T,r)$ and $(t,u,\omega) \in [0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}$. By (2.2) and Condition (C4), one can obtain a bounded function $L : \mathbb{R} \to \mathbb{R}_+$ such that

$$\left| \int_0^t e^{-A(t-s)} \left[f(s,\sigma(s,\omega),\varphi(s)) - f(s,\sigma(s,\omega),0) \right] ds \right|_H \leqslant$$

$$\int_0^t K e^{-a(t-s)} |f(s,\sigma(s,\omega),\varphi(s)) - f(s,\sigma(s,\omega),0)|_H ds \leqslant$$

$$\leqslant K \int_0^t L(s)|\varphi(s)|_H ds \leqslant KN(|u_0|_H + r)T, \qquad (2.11)$$

where $N = \sup_{s \in [0,T]} |L(s)|$.

Let $m(\delta,T) = \sup \left\{ |e^{-At}u - u_0|_H : t \in [0,T], \ u \in \overline{B(u_0,\delta)} \right\}$ and $M = \sup_{s \in [0,T]} |M(s)|$, where M is the function given by Condition (C3). Then, using (2.2), (2.4), (2.8), (2.11), Condition (C3) and Condition (C5), we obtain

$$|S\varphi(t,u,\omega) - u_0|_H \leqslant \left| e^{-At}u - u_0 \right|_H + \left| \int_0^t e^{-A(t-s)} B(\sigma(s,\omega))(\varphi(s),\varphi(s)) ds \right|_H +$$

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$$+\left|\int_0^t e^{-A(t-s)} f(s, \sigma(s,\omega), \varphi(s))ds\right|_H + \left|\sum_{0<t_i<t} e^{-A(t-t_i)} I_i(\varphi(t_i^-))\right|_H \leqslant$$

$$\leqslant m(\delta, T) + \int_0^t K_1 e^{-a(t-s)}(t-s)^{-\alpha_1}\|B\|_\infty |\varphi(s)|_H^2 ds +$$

$$+\left|\int_0^t e^{-A(t-s)}\left[f(s, \sigma(s,\omega), \varphi(s)) - f(s, \sigma(s,\omega), 0)\right]ds\right|_H + \left|\int_0^t e^{-A(t-s)} f(s, \sigma(s,\omega), 0)ds\right|_H +$$

$$+\sum_{0<t_i<t} K e^{-a(t-t_i)}|I_i(\varphi(t_i^-))|_H \leqslant$$

$$\leqslant m(\delta, T) + K_1\|B\|_\infty(|u_0|_H + r)^2 \frac{T^{-\alpha_1+1}}{1-\alpha_1} + KK_2 n_T +$$

$$+KNT(|u_0|_H + r) + \int_0^t K e^{-a(t-s)} M(s)ds \leqslant$$

$$\leqslant m(\delta, T) + K_1\|B\|_\infty(|u_0|_H + r)^2 \frac{T^{-\alpha_1+1}}{1-\alpha_1} + KN(|u_0|_H + r)T +$$

$$+KMT + KK_2 n_T := d_1(u_0, r, \delta, T).$$

Now, we note that $d_1(u_0, r, \delta, T) \to 0$ as $\delta \to 0$ and $T \to 0$. Thus, there are $\delta_1 = \delta_1(u_0, r) > 0, \delta_1 < \delta$, and $T_1 = T_1(u_0, r) > 0, T_1 < T$, such that $d_1(u_0, r, \delta', T') \leqslant r$ for all $\delta' \in (0, \delta_1]$ and $T' \in (0, T_1]$.

Assertion 3: There exist $T_0 = T_0(u_0, r) > 0$ and $\delta_0 = \delta_0(u_0, r) > 0$ such that $S : \Gamma(\delta_0, T_0, r) \to \Gamma(\delta_0, T_0, r)$ is a contraction.

In fact, take $T_2 > 0$ such that

$$2\|B\|_\infty K_1(r + |u_0|_H)\frac{T_2^{-\alpha_1+1}}{-\alpha_1+1} + KNT_2 + KK_3 n_{T_2} < 1.$$

It is enough to take $\delta_0 = \delta_1$ and $T_0 = \min\{T_1, T_2\}$ to conclude Assertion 3.

In conclusion, by the Banach fixed point Theorem, there exists a unique function $\varphi \in \Gamma(\delta_0, T_0, r)$ satisfying the system (2.6) on the interval $[0, T_0]$ and the result follows. $\square$

Theorem 2.4 gives sufficient conditions for the mild solution of system (2.5) to be prolongated on $\mathbb{R}_+$.

Theorem 2.4. *Suppose that $I = \mathbb{R}_+$ and the conditions of Theorem 2.3 hold. If the mild solution $\varphi(t, u_0, \omega)$ of system (2.5) is bounded, then it may be prolonged on $\mathbb{R}_+$.*

Proof. By Theorem 2.3, $\varphi(t, u_0, \omega)$ is the unique solution of system (2.5) passing through the point $u_0 \in H$ at time $t = 0$. This solution is defined on some maximal interval $[0, \alpha_{(u_0,\omega)})$. Let $\varphi(t, u_0, \omega)$ be bounded and suppose that $\alpha_{(u_0,\omega)} < \infty$. If $\alpha_{(u_0,\omega)} \neq t_k$ for all $k \in \mathbb{N}$, then defining $\varphi(\alpha_{(u_0,\omega)}, u_0, \omega) = \lim\limits_{t \to \alpha_{(u_0,\omega)}-} \varphi(t, u_0, \omega)$, it follows that $\varphi(t, u_0, \omega)$ may be extend on the interval $[0, \alpha_{(u_0,\omega)}]$ which is a contradiction. Now, suppose that $\alpha_{(u_0,\omega)} = t_k$ for some $k \in \mathbb{N}$. Since the limit $\lim\limits_{t \to t_k-} \varphi(t, u_0, \omega) = \varphi(t_k^-, u_0, \omega)$ exists and $I_k(\varphi(t_k^-, u_0, \omega)) \in H$, then we may use the proof of Theorem 2.3 and extend $\varphi(t, u_0, \omega)$ in some interval $[t_k, t_k + \epsilon)$, $\epsilon > 0$, with $\varphi(t_k, u_0, \omega) = \varphi(t_k^-, u_0, \omega) + I_k(\varphi(t_k^-, u_0, \omega))$ which is a contradiction. Hence, $\alpha_{(u_0,\omega)} = +\infty$. $\qquad\square$

Next, we deals with the case when $\mathcal{M}$ is not necessarily compact. Following the proof of Theorem 2.3, one can see that the compactness of $\mathcal{M}$ is used just when we work with the norm $\|\cdot\|_\infty$. So, if we suppose that $\mathcal{M}$ is a metric space such that $\|B\|_\infty = \sup\limits_{\omega \in \mathcal{M}} \|B(\omega)\|_{\mathcal{L}^2(H,F)} < \infty$, then we can drop the compactness of $\mathcal{M}$ in Theorem 2.3. In Theorem 2.4, we used the compactness of $\mathcal{M}$ just to assure the existence of the mild solution according to Theorem 2.3. Thus, we can state the next result which deals with existence and uniqueness of global mild solutions for the system (2.5) when $\mathcal{M}$ is not necessarily compact. The proof follows by the proofs of Theorems 2.3 and 2.4.

Theorem 2.5. *Let $u_0 \in H$ and $r > 0$. Assume that (2.2), (2.4), (2.7) and conditions (C1) - (C6) hold. Suppose that $\mathcal{M}$ is a metric space such that $\|B\|_\infty = \sup_{\omega \in \mathcal{M}} \|B(\omega)\|_{\mathcal{L}^2(H,F)} < \infty$. Then there exist positive numbers $\delta = \delta(u_0, r) > 0$, $T = T(u_0, r) > 0$ and a function $\varphi : [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M} \to H$ satisfying the following conditions:*

i) $\varphi(0, u_0, \omega) = u_0$ for all $\omega \in \mathcal{M}$;

ii) $|\varphi(t, u, \omega) - u_0|_H \leqslant r$ for all $(t, u, \omega) \in [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}$;

iii) $\varphi \in PC_1^+([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)})$.

Moreover, the function $u : [0, T] \to H$ defined by $u(t) = \varphi(t, u_0, \omega)$ is the unique mild solution of system (2.5). If $u(t)$ is bounded and $I = \mathbb{R}_+$, then it can be prolonged on $\mathbb{R}_+$.

Remark 2.6. Suppose that (2.2), (2.4), (2.7) and conditions (C1) - (C4) hold. By the proofs of the previous results we obtain the existence and uniqueness of global mild solutions for the following non-impulsive system:

$$\begin{cases} u' + Au + B(\sigma(t, \omega))(u, u) = f(t, \sigma(t, \omega), u), & t > 0, \\ u(0) = u_0 \in H. \end{cases}$$

3 The 2D Navier-Stokes equations with impulses

In this section, we present conditions to obtain the existence and uniqueness of global mild solutions for the following 2D Navier-Stokes equations with impulses

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + q(t)(u \cdot \nabla)u - \nu \Delta u + \nabla p = \phi(t, u), & (t, x) \in \left((0, +\infty) \setminus \displaystyle\bigcup_{k=1}^{+\infty} \{t_k\} \right) \times \Omega, \\
\operatorname{div} u = 0, & (t, x) \in (0, +\infty) \times \Omega, \\
u = 0, & (t, x) \in (0, +\infty) \times \partial\Omega, \\
u(0, \cdot) = u_0(\cdot) & x \in \Omega, \\
u(t_k, \cdot) - u(t_k^-, \cdot) = I_k(u(t_k^-, \cdot)), & x \in \Omega, \ k = 1, 2, \ldots,
\end{cases}
\tag{3.1}
$$

where Ω is an open and bounded set in $\mathbb{R}^2$ with $\partial\Omega \in C^2$, $u = (u_1, u_2)$ is the velocity field of a fluid, p is the scalar pressure, $\nu > 0$ is the kinematic viscosity of the fluid, $\phi = \phi(t, u) \in \mathbb{R}^2$ is the external body force, $q(t)$ is a bounded function, $\{t_k\}_{k\in\mathbb{N}} \subset (0, +\infty)$ is a sequence of impulses such that $\lim\limits_{t\to+\infty} t_k = +\infty$ and I_k is the impulse operator for each $k \in \mathbb{N}$.

Let

$$
\mathcal{E} = \{u \in (C_0^\infty(\Omega))^2 : \ \nabla \cdot u = 0 \text{ in } \Omega\},
$$

$$
V = \text{closure of } \mathcal{E} \text{ in } \mathbb{H}_0^1(\Omega)
$$

and

$$
H = \text{closure of } \mathcal{E} \text{ in } \mathbb{L}^2(\Omega),
$$

where $\mathbb{L}^2(\Omega) = (L^2(\Omega))^2$ and $\mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^2$ are endowed, respectively, with the inner products

$$
(u, v) = \sum_{j=1}^2 \int_\Omega u_j \cdot v_j \ dx, \quad u = (u_1, u_2), \ v = (v_1, v_2) \in \mathbb{L}^2(\Omega),
$$

and

$$
((u, v)) = \sum_{j=1}^2 \int_\Omega \nabla u_j \cdot \nabla v_j dx, \quad u = (u_1, u_2), \ v = (v_1, v_2) \in \mathbb{H}_0^1(\Omega),
$$

and norms $|\cdot| = (\cdot, \cdot)^{1/2}$ and $\|\cdot\| = ((\cdot, \cdot))^{1/2}$.

We assume the following general hypotheses throughout this section:

(H1) $\phi : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded function such that for each fixed $t \in \mathbb{R}_+$, $\phi(t, \cdot)$ is continuous on $\mathbb{R}^2$.

(H2) Let $x \in \mathbb{R}^2$. Then $\lim\limits_{s\to t} \phi(s, x) = \phi(t, x)$ for all $t \neq t_k$, $k \in \mathbb{N}$, the limit $\lim\limits_{s\to t_k-} \phi(s, x)$ exists and $\lim\limits_{s\to t_k+} \phi(s, x) = \phi(t_k, x)$, for all $k \in \mathbb{N}$.

(H3) There is $C > 0$ such that $|\phi(s,x) - \phi(s,y)| \leqslant C|x-y|$ for all $s \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}^2$.

(H4) There exists a constant $C_1 > 0$ such that

$$\sup_{k \in \mathbb{N}} \sup_{x \in \mathbb{R}^2} |I_k(x)| \leqslant C_1.$$

(H5) There exists a constant $C_2 > 0$ such that

$$|I_k(x) - I_k(y)| \leqslant C_2|x-y|$$

for all $x, y \in \mathbb{R}^2$ and for all $k \in \mathbb{N}$.

Now, denote by P the corresponding orthogonal projection $P : \mathbb{L}^2(\Omega) \to H$ and set the operators

$$A = -\nu P \Delta$$

and

$$\mathcal{B}(t)(u,v) = q(t)P((u \cdot \nabla)v).$$

It is well known that the Stokes operator A is positive self-adjoint with domain $D(A)$ dense in H, $0 \in \rho(A)$ and A^{-1} is compact. Also, there exists $\alpha > 0$ such that

$$\langle Au, u \rangle \geqslant \alpha |u|_H^2, \tag{3.2}$$

for all $u \in H$. We also have the following orthogonality property of the nonlinear term which is fundamental and expresses the conservation of energy by the inertial forces:

$$\langle \mathcal{B}(t)(u,v), v \rangle = 0 \tag{3.3}$$

for all $u, v \in H$ and for all $t \in \mathbb{R}_+$. For the above properties see, for instance, [6], [7] and [30].

We set the Hilbert spaces X^α, $\alpha \in (0,1]$, as the domain of the powers of A and we have

$$V = X^{\frac{1}{2}} \quad \text{and} \quad |u|_V = |\nabla u|.$$

Applying P in the equation

$$\frac{\partial u}{\partial t} + q(t)(u \cdot \nabla)u - \nu \Delta u + \nabla p = \phi(t,u),$$

we obtain the evolution equation

$$u' + Au + \mathcal{B}(t)(u,u) = \mathcal{F}(t,u), \tag{3.4}$$

where $\mathcal{F}(t,u) = P\phi(t,u)$ for all $t > 0$ and $u \in H$, $\langle Au, v \rangle = \nu((u,v))$ for all $u, v \in H$ and

$$\langle \mathcal{B}(t)(u,u), w \rangle = q(t) \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial u_j}{\partial x_i} w_j dx \quad \text{for all} \quad u, w \in H.$$

We also assume that:

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(A) $\mathcal{F} \in PC^+(\mathbb{R}_+ \times H, H)$;

(B) $\mathcal{B} \in C(\mathbb{R}_+, \mathcal{L}^2(H, F))$, where $F = D(A^{-\delta})$ for some $0 < \delta < 1$.

Denote Y by $C(\mathbb{R}_+, \mathcal{L}^2(H, F)) \times PC^+(\mathbb{R}_+ \times H, H)$ and let $(Y, \mathbb{R}_+, \sigma)$ be the semidynamical system of translations, that is, $\sigma(t, g) = g_t$ for all $g \in Y$ and $t \geqslant 0$. Now, set

$$\mathcal{M} := \mathcal{H}(\mathcal{B}, \mathcal{F}) = \overline{\{(\mathcal{B}_\tau, \mathcal{F}_\tau) : \tau \in \mathbb{R}_+\}},$$

where $\mathcal{B}_\tau(t) = \mathcal{B}(t + \tau)$ for all $t \in \mathbb{R}_+$ and $\mathcal{F}_\tau(t, u) = \mathcal{F}(t + \tau, u)$ for all $t \in \mathbb{R}_+$ and $u \in H$. If $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}) \in \mathcal{M}$ and $\tau \geqslant 0$ we consider $\widetilde{\mathcal{B}}_\tau(t) = \widetilde{\mathcal{B}}(t + \tau)$ and $\widetilde{\mathcal{F}}_\tau(t, u) = \widetilde{\mathcal{F}}(t + \tau, u)$ for all $(t, u) \in \mathbb{R}_+ \times H$.

According to [6], the equation

$$u' + Au + \widetilde{\mathcal{B}}(t)(u, u) = \widetilde{\mathcal{F}}(t, u), \tag{3.5}$$

where $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}) \in \mathcal{H}(\mathcal{B}, \mathcal{F})$, is called the $\mathcal{H}-$class along with the equation (3.4).

Define the mapping $B : \mathcal{M} \to L^2(H, F)$ by

$$B(\omega) = B(\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}) := \widetilde{\mathcal{B}}(0)$$

and the mapping $f : \mathbb{R}_+ \times \mathcal{M} \times H \to H$ by

$$f(t, \omega, u) = f(t, (\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}), u) := \widetilde{\mathcal{F}}(0, u).$$

Then equation (3.5) can be rewritten in the form

$$u' + Au + B(\sigma(t, \omega))(u, u) = f(t, \sigma(t, \omega), u). \tag{3.6}$$

From (3.3), we obtain

$$\langle B(\omega)(u, v), v \rangle = 0 \tag{3.7}$$

for all $u, v \in H$ and for all $\omega \in \mathcal{M}$.

With the above conditions, we cannot assure that $\mathcal{M}$ is compact. However, the space $\mathcal{M}$ is such that $\sup_{\omega \in \mathcal{M}} \|B(\omega)\|_{\mathcal{L}^2(H,F)} < \infty$ as shows the next result.

Lemma 3.1. $\sup_{\omega \in \mathcal{M}} \|B(\omega)\|_{\mathcal{L}^2(H,F)} < \infty$.

Proof. Since $q(t)$ is bounded, there is $L > 0$ such that $|q(t)| \leqslant L$ for all $t \geqslant 0$. Then

$$\sup_{\omega \in \mathcal{M}} \|B(\omega)\|_{\mathcal{L}^2(H,F)} = \sup_{\omega \in \{(\mathcal{B}_\tau, \mathcal{F}_\tau) : \tau \in \mathbb{R}_+\}} \|B(\omega)\|_{\mathcal{L}^2(H,F)} = \sup_{\tau \in \mathbb{R}_+} \|\mathcal{B}(\tau)\|_{\mathcal{L}^2(H,F)} =$$

$$= \sup_{\tau \in \mathbb{R}_+} \sup_{|u|_H \leqslant 1, |v|_H \leqslant 1} |\mathcal{B}(\tau)(u, v)|_F = \sup_{\tau \in \mathbb{R}_+} \sup_{|u|_H \leqslant 1, |v|_H \leqslant 1} |q(\tau)P((u \cdot \nabla)v)|_F \leqslant$$

$$\leqslant L \sup_{|u|_H \leqslant 1, |v|_H \leqslant 1} \|(u \cdot \nabla)v)\|_H \leqslant L \sup_{|u|_H \leqslant 1, |v|_H \leqslant 1} \|u\|_H \|\nabla v\|_H \leqslant$$

$$\leqslant L \sup_{|u|_H \leqslant 1, |v|_H \leqslant 1} \|u\|_H \|v\|_V = L \sup_{|u|_H \leqslant 1, |v|_V \leqslant 1} \|u\|_H \|v\|_V \leqslant L.$$

Therefore, we have the desired result. $\qquad \square$

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Since ϕ is bounded and we have Lemma 3.1, we may consider in $\mathcal{M}$ the metric $d_{\mathcal{M}}$ given by

$$d_{\mathcal{M}}(\omega_1, \omega_2) = d_{\mathcal{M}}((\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{F}}_1), (\widetilde{\mathcal{B}}_2, \widetilde{\mathcal{F}}_2)) = \|\widetilde{\mathcal{B}}_1 - \widetilde{\mathcal{B}}_2\|_\infty + \|\widetilde{\mathcal{F}}_1 - \widetilde{\mathcal{F}}_2\|_{PC_\infty^+},$$

where $\|\widetilde{\mathcal{F}}\|_{PC_\infty^+} = \sup\limits_{(t,u)\in[0,+\infty)\times H} |\widetilde{\mathcal{F}}(t,u)|_H$ and $\|\widetilde{\mathcal{B}}\|_\infty = \sup\limits_{\omega\in\mathcal{M}} \|\widetilde{\mathcal{B}}(\omega)\|_{\mathcal{L}^2(H,F)}$.

Let us consider the following impulsive system associated to (3.6):

$$\begin{cases} u' + Au + B(\sigma(t,\omega))(u,u) = f(t, \sigma(t,\omega), u), & u \in H,\ t > 0,\ t \neq t_k, \\ u(t_k) - u(t_k^-) = I_k(u(t_k^-)), & k \in \mathbb{N}, \\ u(0) = u_0, \end{cases} \qquad (3.8)$$

which is a weak formulation of (3.1). In the next lines, we show that f satisfies the conditions (C1), (C2), (C3) and (C4) presented in Section 2. This will help us to show that system (3.8) admits a unique global mild solution, see Theorems 3.4 and 3.5 in the sequel.

Lemma 3.2. *The mapping $f : \mathbb{R}_+ \times \mathcal{M} \times H \to H$ satisfies the conditions (C1), (C2), (C3) and (C4).*

Proof. First, let us show that f satisfies condition (C1). Let $t \in \mathbb{R}_+$ be fixed. Take $(\omega_n, u_n), (\omega_0, u_0) \in \mathcal{M} \times H$, $n = 1, 2, \ldots$, such that

$$d_{\mathcal{M}}(\omega_n, \omega_0) \to 0 \quad \text{and} \quad |u_n - u_0|_H \to 0$$

as $n \to +\infty$. Note that $\omega_0 = (\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}})$ and $\omega_n = (\widetilde{\mathcal{B}}_n, \widetilde{\mathcal{F}}_n)$, $n = 1, 2, 3, \ldots$. Moreover,

$$\|\widetilde{\mathcal{F}}_n - \widetilde{\mathcal{F}}\|_{PC_\infty^+} \to 0 \quad \text{as} \quad n \to +\infty.$$

Now, since $(\widetilde{\mathcal{B}}_n, \widetilde{\mathcal{F}}_n), (\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}) \in \mathcal{M}$, $n = 1, 2, 3, \ldots$, then there are sequences $\{\tau_k^n\}_{k\in\mathbb{N}}$ and $\{s_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}_+$ such that

$$\widetilde{\mathcal{F}}_n(r, u) = \lim_{k\to+\infty} \mathcal{F}_{\tau_k^n}(r, u) \quad \text{and} \quad \widetilde{\mathcal{F}}(r, u) = \lim_{k\to+\infty} \mathcal{F}_{s_k}(r, u)$$

for each $(r, u) \in \mathbb{R}_+ \times H$ and $n \in \mathbb{N}$. Then,

$$|f(t, \omega_n, u_n) - f(t, \omega_0, u_0)|_H = |\widetilde{\mathcal{F}}_n(0, u_n) - \widetilde{\mathcal{F}}(0, u_0)|_H = \lim_{k\to+\infty} |\mathcal{F}(\tau_k^n, u_n) - \mathcal{F}(s_k, u_0)|_H$$

$$\leqslant \lim_{k\to+\infty} |\mathcal{F}(\tau_k^n, u_n) - \mathcal{F}(\tau_k^n, u_0)|_H + \lim_{k\to+\infty} |\mathcal{F}(\tau_k^n, u_0) - \mathcal{F}(s_k, u_0)|_H$$

$$= \lim_{k\to+\infty} |P\phi(\tau_k^n, u_n) - P\phi(\tau_k^n, u_0)|_H + |\widetilde{\mathcal{F}}_n(0, u_0) - \widetilde{\mathcal{F}}(0, u_0)|_H$$

$$\leqslant \|P\||C|u_n - u_0|_H + |\widetilde{\mathcal{F}}_n(0, u_0) - \widetilde{\mathcal{F}}(0, u_0)|_H,$$

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where the last inequality follows by Condition (H3). Hence, $|f(t, \omega_n, u_n) - f(t, \omega_0, u_0)|_H \to 0$ as $n \to +\infty$.

The Condition (H2) implies in Condition (C2).

In order to show that Condition (C3) holds, we define the function $M : \mathbb{R} \to \mathbb{R}_+$ by $M(t) = \sup\{|\phi(s, u)|_H : s \in \mathbb{R}_+, u \in H\}$ for all $t \in \mathbb{R}$, which is well defined since ϕ is bounded by Condition (H1). Note that M is constant. Given $\omega = (\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}) \in \mathcal{M}$, there is a sequence $\{r_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ such that

$$f(t, \omega, u) = \widetilde{\mathcal{F}}(0, u) = \lim_{n \to +\infty} \mathcal{F}_{r_n}(0, u) = \lim_{n \to +\infty} P\phi(r_n, u),$$

for all $t \geqslant 0$ and $u \in H$. Then, for $[a, b] \subset \mathbb{R}_+$, we have

$$\int_a^b |\psi(s)||f(s, \omega, u)|_H ds = \int_a^b |\psi(s)||\widetilde{\mathcal{F}}(0, u)|_H ds \leqslant \int_a^b M(s)|\psi(s)|ds,$$

for all $\psi \in L^1[a, b]$, $\omega \in \mathcal{M}$ and $u \in H$.

Finally, we need to verify the Condition (C4). Define $L : \mathbb{R} \to \mathbb{R}_+$ by $L(t) = C + 1$, $t \in \mathbb{R}$, where C is the constant of the Condition (H3). Then, given $[a, b] \subset \mathbb{R}_+$, we have

$$\int_a^b |\psi(s)||f(s, \omega_1, u_1) - f(s, \omega_2, u_2)|_H ds = \int_a^b |\psi(s)||\widetilde{\mathcal{F}}_1(0, u_1) - \widetilde{\mathcal{F}}_2(0, u_2)|_H ds \leqslant$$

$$\leqslant \int_a^b |\psi(s)| \left(|\widetilde{\mathcal{F}}_1(0, u_1) - \widetilde{\mathcal{F}}_1(0, u_2)|_H + |\widetilde{\mathcal{F}}_1(0, u_2) - \widetilde{\mathcal{F}}_2(0, u_2)|_H \right) ds \leqslant$$

$$\leqslant \int_a^b |\psi(s)| \left(C\|P\||u_1 - u_2|_H + \|\widetilde{\mathcal{F}}_1 - \widetilde{\mathcal{F}}_2\|_{PC_\infty^+} \right) ds \leqslant$$

$$\leqslant \int_a^b L(s)|\psi(s)|(d_\mathcal{M}(\omega_1, \omega_2) + |u_1 - u_2|_H)ds,$$

for all $\psi \in L^1[a, b]$, $\omega_1 = (\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{B}}_1), \omega_2 = (\widetilde{\mathcal{F}}_2, \widetilde{\mathcal{B}}_2) \in \mathcal{M}$ and $u_1, u_2 \in H$. $\qquad\square$

Lemma 3.3. $\sup\{|f(t, \omega, u)|_H : t \geqslant 0, \omega \in \mathcal{M}, u \in H\} < \infty$.

Proof. If $t \geqslant 0$, $u \in H$ and $\omega = (\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}) \in \mathcal{M}$, we have

$$|f(t, \omega, u)|_H = |\widetilde{\mathcal{F}}(0, u)|_H = \lim_{n \to +\infty} |\mathcal{F}_{\tau_n}(0, u)|_H = \lim_{n \to +\infty} |P\phi(\tau_n, u)|_H \leqslant \|P\|\eta \leqslant \eta,$$

where $\eta > 0$ is a bound of ϕ since it is bounded by Condition (H1). Hence, the result follows. $\qquad\square$

By Lemma 3.3, we may define $\|f\|_1 = \sup\{|f(t, \omega, u)|_H : t \geqslant 0, \omega \in \mathcal{M}, u \in H\}$.

From Theorem 2.5, we have the following straightforward result of existence and uniqueness of mild solutions when $\mathcal{M}$ is not necessarily compact.

Theorem 3.4. *Under conditions (H1)-(H5), (A) and (B), the system (3.8) admits a unique mild solution $u : [0, T] \to H$ defined in some interval $[0, T]$ satisfying $u(0) = u_0$.*

The mild solution of system (3.8) may be prolonged on $\mathbb{R}_+$, see Theorem 3.5.

Theorem 3.5. *Suppose that $\displaystyle\sum_{i=1}^{+\infty} |I_i(u)|_H = \Gamma < \infty$ for all $u \in H$. Then:*

a) *The solution $\varphi(t, u_0, \omega)$ of the impulsive system (3.8) is bounded and therefore, it may be prolonged on $\mathbb{R}_+$;*

b) *$|\varphi(t, u_0, \omega)|_H \leqslant 2C(|u_0|_H) + \Gamma$, for all $t \geqslant 0$, $\omega \in \mathcal{M}$ and $u_0 \in H$, where*

$$C(r) = \begin{cases} r & \text{if } r \geqslant \dfrac{\|f\|_1}{\alpha}, \\[4mm] \dfrac{\|f\|_1}{\alpha} & \text{if } r \leqslant \dfrac{\|f\|_1}{\alpha} \end{cases}$$

and α is given by (3.2).

Proof. a) Let $u_0 \in H$ and $\omega \in \mathcal{M}$. By Theorem 3.4, there exists a unique solution $\varphi(t, u_0, \omega)$ of equation (3.8) passing through u_0 at time $t = 0$ and defined on some interval $[0, T_{(u_0, \omega)})$. Let $\psi(t; 0, u_0, \omega)$ be the solution of the non-impulsive equation (3.6) such that $\psi(0) = u_0$. Note that

$$\varphi(t, u_0, \omega) = \psi(t; 0, u_0, \omega) \quad \text{for} \quad t \in [0, T_{(u_0, \omega)}) \cap [0, t_1).$$

Define $\eta(t) = |\psi(t; 0, u_0, \omega)|_H^2$ for $t \in [0, T_{(u_0, \omega)}) \cap [0, t_1)$ and denote $\psi(t; 0, u_0, \omega)$ by $\psi(t)$. Using (3.2) and (3.7), we obtain

$$\eta'(t) = 2\langle \psi'(t), \psi(t) \rangle = -2\langle A\psi(t), \psi(t) \rangle - 2\langle B(\sigma(t, w))(\psi(t), \psi(t)), \psi(t) \rangle +$$

$$+ 2\langle f(t, \sigma(t, \omega), \psi(t)), \psi(t) \rangle \leqslant -2\alpha |\psi(t)|_H^2 + 2\|f\|_1 |\psi(t)|_H.$$

Thus,

$$\eta' \leqslant -2\alpha\eta + 2\|f\|_1 \eta^{1/2},$$

which implies that $\eta(t) \leqslant v(t)$ for all $t \in [0, T_{(u_0, \omega)}) \cap [0, t_1)$, where $v(t)$ is an upper solution of $v' = -2\alpha v + 2\|f\|_1 v^{1/2}$ such that $v(0) = \eta(0) = |u_0|_H^2$. Then

$$\eta(t) \leqslant \left[\left(|u|_H - \frac{\|f\|_1}{\alpha} \right) e^{-\alpha t} + \frac{\|f\|_1}{\alpha} \right]^2,$$

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that is,

$$|\varphi(t, u_0, \omega)|_H = |\psi(t; 0, u_0, \omega)|_H \leqslant \left(|u_0|_H - \frac{\|f\|_1}{\alpha}\right) e^{-\alpha t} + \frac{\|f\|_1}{\alpha}, \quad t \in [0, T_{(u_0, \omega)}) \cap [0, t_1).$$

Now, if $t \in [0, T_{(u_0, \omega)}) \cap [t_1, t_2)$, we use the previous argument and we obtain

$$
\begin{aligned}
|\varphi(t, u_0, \omega)|_H &\leqslant \left(|\psi(t_1; 0, u_0, \omega) + I_1(\psi(t_1; 0, u_0, \omega))|_H - \frac{\|f\|_1}{\alpha}\right) e^{-\alpha(t-t_1)} + \frac{\|f\|_1}{\alpha} \\
&\leqslant \left(|u_0|_H - \frac{\|f\|_1}{\alpha}\right) e^{-\alpha t} + |I_1(\psi(t_1; 0, u_0, \omega))|_H e^{-\alpha(t-t_1)} + \frac{\|f\|_1}{\alpha}.
\end{aligned}
$$

Continuing with this reasoning, if $t \in [0, T_{(u_0, \omega)}) \cap [t_k, t_{k+1})$, we get

$$|\varphi(t, u_0, \omega)|_H \leqslant \left(|u_0|_H - \frac{\|f\|_1}{\alpha}\right) e^{-\alpha t} + \sum_{i=1}^{k} |I_i(\psi(t_i; t_{i-1}, u_{i-1}^+, \omega))|_H e^{-\alpha(t-t_i)} + \frac{\|f\|_1}{\alpha}, \quad (3.9)$$

where we denote $t_0 = 0$, $u_0^+ = u_0$ and $u_i^+ = \psi(t_i; t_{i-1}, u_{i-1}^+, \omega) + I_i(\psi(t_i; t_{i-1}, u_{i-1}^+, \omega))$ for $i = 1, 2, \ldots$.

Since there exists a finite number of impulses on the interval $[0, T_{(u_0, \omega)})$ and $\sum_{i=1}^{+\infty} |I_i(u)|_H = \Gamma < \infty$ for all $u \in H$, then by (3.9) we get that $\varphi(t, u_0, \omega)$ is bounded. Consequently, by Theorem 2.5, it can be prolonged on $\mathbb{R}_+$.

b) Note that $|\varphi(t, u_0, \omega)|_H \leqslant |u_0|_H + \frac{\|f\|_1}{\alpha} + \Gamma$, for all $t \geqslant 0$, $u_0 \in H$ and $\omega \in \mathcal{M}$. Therefore, the result holds. $\qquad \square$

The system (3.8) is called *bounded dissipative* if there is a nonempty bounded set $B_0 \subset H$ such that for each bounded set $B \subset H$ there exists $T = T(B) > 0$ such that $\varphi(t, u_0, \omega) \in B_0$ for all $t \geqslant T$, $u_0 \in B$ and $\omega \in \mathcal{M}$. In this case, B_0 is called a *bounded attractor* for the system (3.8). In the next result, we obtain dissipativity for the system (3.8).

Theorem 3.6. *Suppose that* $\sum_{i=1}^{+\infty} |I_i(u)|_H = \Gamma < \infty$ *for all* $u \in H$. *Then the system (3.8) is bounded dissipative.*

Proof. By the proof of Theorem 3.5, we have

$$\lim_{t \to +\infty} \sup_{|u_0|_H \leqslant r, \, \omega \in \mathcal{M}} |\varphi(t, u_0, \omega)|_H \leqslant \frac{\|f\|_1}{\alpha} + \Gamma,$$

for all $r > 0$. Hence, the set $B_0 = \left\{u \in H : |u|_H \leqslant \frac{\|f\|_1}{\alpha} + \Gamma\right\}$ is a bounded attractor for the system (3.8). $\qquad \square$

In the last result, we present an estimative between two solutions with different initial data in the same fiber $\omega \in \mathcal{M}$.

Theorem 3.7. *Suppose that* $\sum_{i=1}^{+\infty} |I_i(u)|_H = \Gamma < \infty$ *for all* $u \in H$. *Let* $r_0 = 2\dfrac{\|f\|_1}{\alpha} + \Gamma$. *Then, for each* $k \in \mathbb{N}$, *we have*

$$|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|_H \leqslant (1 + C_2)^k e^{-(\alpha - 2\|B\|_\infty r_0 - C)t} |u_1 - u_2|_H,$$

for all $t \in [t_k, t_{k+1})$, $u_1, u_2 \in \overline{B(0, \alpha^{-1}\|f\|_1)}$ *and* $\omega \in \mathcal{M}$.

Proof. Let $t \in \mathbb{R}_+$, $\omega \in \mathcal{M}$, $u_1, u_2 \in \overline{B(0, \alpha^{-1}\|f\|_1)}$ and define $\eta(t) = \psi(t; 0, u_1, \omega) - \psi(t; 0, u_2, \omega)$, where $\psi_i := \psi(t; 0, u_i, \omega)$ is the solution of equation (3.6) without impulses defined on $\mathbb{R}_+$ and passing through u_i at time $t = 0$, $i = 1, 2$. By Theorem 3.5, we have

$$|\psi_i(t)|_H \leqslant 2\frac{\|f\|_1}{\alpha} + \Gamma = r_0, \quad \text{for all} \quad t \in [0, t_1] \quad \text{and} \quad i = 1, 2.$$

Given $\omega = (\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}}) \in \mathcal{M}$, there is a sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\widetilde{\mathcal{F}}(t, u) = \lim_{n \to +\infty} \mathcal{F}_{s_n}(t, u)$ for all $(t, u) \in \mathbb{R}_+ \times H$. By the definition of f, we have

$$|f(t, \sigma(t, \omega), \psi_1) - f(t, \sigma(t, \omega), \psi_2)|_H = |\widetilde{\mathcal{F}}_t(0, \psi_1) - \widetilde{\mathcal{F}}_t(0, \psi_2)|_H =$$

$$= \lim_{n \to +\infty} |\mathcal{F}_{s_n}(t, \psi_1) - \mathcal{F}_{s_n}(t, \psi_2)|_H = \lim_{n \to +\infty} |P\phi(s_n + t, \psi_1) - P\phi(s_n + t, \psi_2)|_H \leqslant \|P\|C|\eta(t)|_H.$$

Then, using (3.2) and the above estimative, we have

$$\frac{d}{dt}|\eta(t)|_H^2 = -2\langle A\eta(t), \eta(t)\rangle + 2\langle B(\sigma(t, \omega))(\psi_2, \psi_2) - B(\sigma(t, \omega))(\psi_1, \psi_1), \eta(t)\rangle +$$

$$+2\langle f(t, \sigma(t, \omega), \psi_1) - f(t, \sigma(t, \omega), \psi_2), \eta(t)\rangle \leqslant$$

$$\leqslant -2\alpha|\eta(t)|_H^2 + 4r_0\|B\|_\infty|\eta(t)|_H^2 + 2C\|P\||\eta(t)|_H^2$$

$$\leqslant -2\left(\alpha - 2\|B\|_\infty r_0 - C\right)|\eta(t)|_H^2 = -2\beta|\eta(t)|_H^2,$$

for all $t \in [0, t_1]$, where $\beta = \alpha - 2\|B\|_\infty r_0 - C$ and $\|P\| \leqslant 1$. Hence, $|\eta(t)|_H^2 \leqslant e^{-2\beta t}|\eta(0)|_H^2$, that is,

$$|\psi(t; 0, u_1, \omega) - \psi(t; 0, u_2, \omega)|_H \leqslant e^{-\beta t}|u_1 - u_2|_H \tag{3.10}$$

for all $t \in [0, t_1]$. Thus, if $0 \leqslant t < t_1$, we get

$$|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|_H = |\psi(t; 0, u_1, \omega) - \psi(t; 0, u_2, \omega)|_H \leqslant e^{-\beta t}|u_1 - u_2|_H.$$

Let $t_1 \leqslant t < t_2$ and $\eta_1(t) = \varphi(t, u_1, \omega) - \varphi(t, u_2, \omega) = \psi(t; t_1, u_1^+, \omega) - \psi(t; t_1, u_2^+, \omega)$, where $\psi(t; t_1, u_i^+, \omega)$ is the solution of (3.6) such that $\psi(t_1) = u_i^+$ and $u_i^+ = \psi(t_1; 0, u_i, \omega) + I_1(\psi(t_1; 0, u_i, \omega))$, $i = 1, 2$. Following the steps above to show (3.10), we obtain

$$|\eta_1(t)|_H \leqslant e^{-\beta(t - t_1)}|\eta_1(t_1)|_H, \quad \text{for all} \quad t \geqslant t_1.$$

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On the other hand, we have

$$
\begin{aligned}
|\eta_1(t_1)|_H &= |\psi(t_1; 0, u_1, \omega) + I_1(\psi(t_1; 0, u_1, \omega)) - \psi(t_1; 0, u_2, \omega) - I_1(\psi(t_1; 0, u_2, \omega))|_H \\
&\leqslant |\psi(t_1; 0, u_1, \omega) - \psi(t_1; 0, u_2, \omega)|_H + |I_1(\psi(t_1; 0, u_1, \omega)) - I_1(\psi(t_1; 0, u_2, \omega))|_H \\
&\leqslant (1 + C_2)|\psi(t_1; 0, u_1, \omega) - \psi(t_1; 0, u_2, \omega)|_H \\
&\leqslant (1 + C_2)e^{-\beta t_1}|u_1 - u_2|_H,
\end{aligned}
$$

where C_2 comes from Condition (H5). Consequently,

$$
|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|_H \leqslant (1 + C_2)e^{-\beta t}|u_1 - u_2|_H, \quad \text{for all} \quad t_1 \leqslant t < t_2.
$$

Continuing with this process, if $t_k \leqslant t < t_{k+1}$ we get

$$
|\varphi(t, u_1, \omega) - \varphi(t, u_2, \omega)|_H \leqslant (1 + C_2)^k e^{-\beta t}|u_1 - u_2|_H,
$$

for all $u_1, u_2 \in \overline{B(0, \alpha^{-1}\|f\|_1)}$ and $\omega \in \mathcal{M}$. $\qquad\square$

Remark 3.8. Theorems 3.4, 3.5, 3.6 and 3.7 also hold for the non-impulsive system

$$
\begin{cases}
u' + Au + B(\sigma(t, \omega))(u, u) = f(t, \sigma(t, \omega), u), & u \in H, \ t > 0, \\
u(0) = u_0,
\end{cases}
$$

with the obvious adaptations.

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