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Existence, regularization and upper-semicontinuity of uniform attractors for a nonautonomous semilinear evolution equation of second order

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Abstract

We investigate the forward dynamics of a nonautonomous semilinear wave-type evolution problem, which models propagation phenomena in nonlinear elastic rods and ion-acoustic waves. We establish global well-posedness and prove the existence of a family of uniform attractors under appropriate growth and dissipativity conditions. Additionally, we demonstrate upper-semicontinuity in a suitable space and derive regularity results in a more refined space. Finally, we characterize the uniform attractor through kernel sections for the problem under consideration.

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1 Introduction

In this paper, we investigate the forward dynamics of the following nonautonomous semilinear second-order evolution problem

$$\begin{cases} u_{tt} - \Delta u - \eta(t)\Delta u_t - \Delta u_{tt} = f(u) + g(x,t), & t > s,\ x \in \Omega, \\ u = 0, \quad t \geqslant s,\ x \in \partial\Omega, \\ u(s,x) = u_0(x), \quad u_t(s,x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded C^2 smooth domain in $\mathbb{R}^N$ with $N \geqslant 3$ and $\eta\colon \mathbb{R} \longrightarrow (0,\infty)$ is a uniformly continuous function satisfying

$$0 < a_1 \leqslant \eta(t) \leqslant a_2 < \infty, \quad t \in \mathbb{R}. \tag{1.2}$$

The nonlinear term $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz function that satisfies the following dissipativity condition:

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \tag{1.3}$$

where $\lambda_1 > 0$ denotes the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions on Ω. Additionally, f satisfies the polynomial growth condition:

$$|f(s_1) - f(s_2)| \leqslant c|s_1 - s_2|\big(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}\big), \quad s_1, s_2 \in \mathbb{R}, \tag{1.4}$$

for some constant $c > 0$ and exponent $1 < \rho < \frac{N+2}{N-2}$ and

$$f(s)s \leqslant \int_0^s f(r)\,dr := F(s), \quad s \in \mathbb{R}. \tag{1.5}$$

As a consequence of (1.4), the nonlinear term f also satisfies:

$$|f(s)| \leqslant c(1 + |s|^\rho), \quad s \in \mathbb{R}, \tag{1.6}$$

for some constant $c > 0$. Finally, the external source g is a differentiable function satisfying the conditions

$$g, g_t \in L^2_b(\mathbb{R}, L^2(\Omega)), \tag{1.7}$$

where $L^2_b(\mathbb{R}, L^2(\Omega))$ is a subspace of $L^2_{loc}(\mathbb{R}, L^2(\Omega))$, given by

$$L^2_{loc}(\mathbb{R}, L^2(\Omega)) = \left\{ h\colon \mathbb{R} \to L^2(\Omega) \;\middle|\; \int_{t_1}^{t_2} \|h(s)\|^2_{L^2(\Omega)}\,ds < \infty, \quad [t_1,t_2] \subset \mathbb{R} \right\},$$

with norm

$$\|h\|^2_{L^2_b(\mathbb{R},L^2(\Omega))} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|h(s)\|^2_{L^2(\Omega)}\,ds < \infty$$

and

$$W_b^{1,2}(\mathbb{R}, L^2(\Omega)) := \left\{ h \in W_{loc}^{1,2}(\mathbb{R}, L^2(\Omega)) \;\middle|\; h, h_t \in L^2_b(\mathbb{R}, L^2(\Omega)) \right\}$$

with norm

$$\|h\|^2_{W_b^{1,2}(\mathbb{R},L^2(\Omega))} = \|h\|^2_{L^2_b(\mathbb{R},L^2(\Omega))} + \|h_t\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}. \tag{1.8}$$

The model presented in (1.1) is inspired by its autonomous counterpart, widely studied in the context of asymptotic behavior (see [12, 19, 20, 26]) and references therein). It has important physical applications, such as wave propagation in nonlinear elastic rods and ion-acoustic waves, see [6, 18, 22, 30]. When the term Δu_{tt} is omitted, (1.1) reduces to the classical strongly damped wave equation [11]. For related nonautonomous models, see [4, 5, 7, 10, 11, 13].

From a historical point of view, in [12], problem (1.1) was considered in its autonomous version with $g = 0$ and $\eta(t) \equiv \mu$ (a constant function) and the authors were concerned about the well-posedness, existence and uniqueness of global solutions. Furthermore, the existence of a gradient-like global attractor for the problem was established using the semigroup approach. In [3], the authors significantly expanded upon the analysis introduced in [12] considering for that time a nonautonomous term (expressed by the time-dependent function $\eta(t)$, but yet with $g = 0$) and provided a complete survey about this problem in a *pullback* setting, ensuring the existence and robustness of a family of pullback exponential attractors, as well as establishing the existence of a pullback attractor whose sections possess uniformly bounded finite fractal dimension. Moreover, in [3], the authors also established the upper semicontinuity and regularity of the pullback attractor.

In this work, we revisit the nonautonomous problem, this time focusing on its *forward* dynamics, in the sense of uniform attractors. More precisely, we consider problem (1.1) with both time-dependent functions $g(t)$ and $\eta(t)$, and establish for the first time in the literature the existence, regularity, and upper semicontinuity of the uniform attractor associated with this problem. It is worth mentioning that the addition of a time-dependent function $g(t)$ satisfying assumption (1.7) (rather than a direct boundedness condition as used in previous nonautonomous second-order works) brings a considerable difficulty in establishing the existence of solutions and, consequently, the existence of its uniform attractor, since the construction of the required symbol space driving the system is strictly related to the nonautonomous terms of the problem. Still regarding the symbol space, in this work we prove the existence of a uniform attractor for (1.1) without necessarily requiring its symbol space to be a compact set (this compactness is a standard hypothesis, for instance, in the classical reference [14]). Moreover, we address also that, even with the uniform attractor not satisfying an invariance property, we were able to apply the bootstrapping technique in order to prove its regularization (for instance, in [3], this property was proved strongly based on the natural invariance of pullback attractors, building on the developments presented in [12]) and it may provide a framework for extending the same technique to a broader class of nonautonomous evolution equations whose asymptotic behavior is described by uniform attractors.

Finally, compared to [3], where the existence of the pullback attractor was obtained by a *smoothing property*, this work draws inspiration from [25]. In that paper, the authors studied the *forward* dynamics of a wave equation with nonlinear damping and developed a method (see [25, Theorem 4.2]) based on *contractive functions* to verify uniform asymptotic compactness, which is crucial for establishing the existence of uniform attractors. The method described in [25] was first inspired by results due to Chueshov and Lasiecka for autonomous systems as given in [16, Proposition 3.2] and [17, Proposition 2.10].

In addition, still related to [25], where a general (possibly nonlinear) damping term of the form $h(u_t)$ is considered, our case features a linear damping term given by $h(u_t) = \Delta u_t$, but with a nonautonomous coefficient $\eta(t)$, which adds further complexity to the model. Notably, the presence of the term Δu_{tt} in our equation is nonstandard in the classical literature, highlighting both the originality and the analytical challenges addressed in this work.

In general, compared with earlier results, we highlight the achievement of significant advances concerning the global well-posedness of problem (1.1) within this new *forward* nonautonomous framework. Furthermore, we contribute to a deeper understanding of its asymptotic behavior by providing a detailed forward analysis of problem (1.1), including, for the first time, the existence, regularity, and upper semicontinuity of the uniform attractor associated with this problem. These contributions extend the theory to nonautonomous cases and consolidate key qualitative properties of the corresponding dynamical system.

Back to the analysis of the evolution problem, under the previous assumptions, we consider system (1.1) in the Hilbert space $H_0^1(\Omega) \times H_0^1(\Omega)$ and according to the approaches outlined in [3] and [12], we will conduct a detailed analysis of problem (1.1) by introducing the change of variables (t, z), with $z = (I - \Delta)u$. This transformation leads us to the following system:

$$\begin{cases} z_{tt} + \eta(t)\Lambda z_t + \Lambda z = f^e(z) + g(x,t), \quad t > s, \ x \in \Omega, \\ z = 0, \quad t \geqslant s, \quad x \in \partial\Omega, \\ z(s) = z_0 \quad \text{and} \quad z_t(s) = w_0, \end{cases} \tag{1.9}$$

where $\Lambda = I - (I - \Delta)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, $f^e = f \circ (I - \Lambda)$ and $H^{-1}(\Omega)$ is the extrapolation space of $H^2(\Omega) \cap H_0^1(\Omega)$ generated by the realization of $-\Delta$ in $H^2(\Omega) \cap H_0^1(\Omega)$. The symbol $\mathcal{L}(X)$ denotes the space of all bounded linear operators from a Banach space X into itself.

The change of variable $w = z_t$ now leads us to the following first-order problem:

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} z \\ w \end{bmatrix} + \mathcal{Q}(t)\begin{bmatrix} z \\ w \end{bmatrix} = \mathcal{F}\left(t, \begin{bmatrix} z \\ w \end{bmatrix}\right), & t > s, \\ \begin{bmatrix} z(s) \\ w(s) \end{bmatrix} = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \end{cases} \tag{1.10}$$

where $\mathcal{Q}(t) \in \mathcal{L}(H^{-1}(\Omega) \times H^{-1}(\Omega))$ is given by

$$\mathcal{Q}(t) = \begin{bmatrix} 0 & -I \\ \Lambda & \eta(t)\Lambda \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I - (I - \Delta)^{-1} & \eta(t)(I - (I - \Delta)^{-1}) \end{bmatrix} \tag{1.11}$$

and $\mathcal{F}\colon \mathbb{R} \times H^{-1}(\Omega) \times H^{-1}(\Omega) \longrightarrow H^{-1}(\Omega) \times H^{-1}(\Omega)$ is given by

$$\mathcal{F}\left(t, \begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f^e(z) + g(x,t) \end{bmatrix}, \qquad \begin{bmatrix} z \\ w \end{bmatrix} \in H^{-1}(\Omega) \times H^{-1}(\Omega). \tag{1.12}$$

The article is organized as follows: In Sect. 2, we present the mathematical formulation of the problem and the main theoretical results ensuring the well-posedness of problem (1.1) in $H_0^1(\Omega) \times H_0^1(\Omega)$. Next, in Sect. 3, we construct the symbol space equipped with a suitable topology, which is explored in more detail in this section. Without assuming the compactness of the symbol space, we define a family of evolution processes associated with system (3.3) and establish the existence of the uniform attractor using the contractive functions technique (see Lemma A.9). Moreover, we establish further properties, including the upper semicontinuity of this family of uniform attractors and its regularity in $(H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$, which are discussed in Sects. 4 and 5, respectively. Finally, in Sect. 6, under more restrictive conditions, we assume the compactness of the symbol space with respect to the topology introduced in Sect. 3 to obtain a characterization of the uniform attractor via kernel sections (see Theorem 6.2). Additionally, we have included an Appendix A that presents some basic definitions and abstract results on the existence and uniqueness of solutions, as well as the theory of uniform attractors for systems of evolution processes. These concepts are essential for the comprehensive understanding of this work.

2 Local and global well-posedness results

Let us denote by Δ the closed extension in $H^{-1}(\Omega)$ of the Dirichlet Laplacian with domain $H^2(\Omega) \cap H_0^1(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$. By $\{X^\alpha : \alpha \in \mathbb{R}\}$ we mean the double-sided fractional power scale generated by $(X, \tilde{A})$, where $X = L^2(\Omega)$ and $\tilde{A} = I - \Delta$ (see [1] or [21]). In this case, we set $\Lambda = I - \tilde{A}^{-1}$. It is well-known that $X^{-\frac{1}{2}} = H^{-1}(\Omega)$, $X^{\frac{1}{2}} = H_0^1(\Omega)$ and $X^1 = H^2(\Omega) \cap H_0^1(\Omega)$ (see [23] or [27]).

In order to ensure the local and global well-posedness of solutions for problem (1.9) in $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$ (which is equivalent to solve (1.10) in $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$) we will apply the techniques developed in [3, 12, 29]. Consequently, we shall be able to obtain the same results for (1.1) in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$. For that, we will consider evolution processes $\{S_{-1/2}(t,s) : t \geqslant s\}$ in $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$ and $\{S(t,s) : t \geqslant s\}$ in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ associated respectively with the problems (1.9) and (1.1), which will be seen to be closely related to each other (see Theorems 2.14 and 2.16). To begin understanding the relationship between S and $S_{-1/2}$, let us first recall an essential result.

Lemma 2.1 [12, Lemma 2.3] *Let $s \geqslant 0$ and $r \geqslant -\frac{1}{2}$. Then the map given by*

$$\begin{aligned} \Phi_s : X^r \times X^r &\longrightarrow X^{r+s} \times X^{r+s} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto \begin{bmatrix} \tilde{A}^{-s} & 0 \\ 0 & \tilde{A}^{-s} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

is an isometric isomorphism. We denote $\Phi_s^{-1} = \Phi_{-s}$. In particular, Φ_1 is an isometric isomorphism from $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$ into $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$.

To the aim of this first part concerning the proof of the local well-posedness property, we shall remember some auxiliary results such as the continuity of the family of operators $\mathcal{Q}$ and the Lipschitz condition on the map $\mathcal{F}$. Before that, we recall some important continuous embeddings that will used throughout this work:

$$H_0^s(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow L^2(\Omega), \quad \text{if } \frac{1}{2} \geqslant \frac{1}{r} \geqslant \frac{1}{2} - \frac{s}{N} > 0, \tag{2.1}$$

see Theorem 1.1, Chapter 2, in [14], and

$$H_0^s(\Omega) \hookrightarrow X^{\frac{s}{2}} \hookrightarrow H^s(\Omega), \quad \text{for all } s \in \mathbb{R},$$

see Theorem 16.1 in [28]. Remember we are assuming that $N \geqslant 3$. By duality, we obtain

$$L^2(\Omega) \hookrightarrow L^{r'}(\Omega) \hookrightarrow X^{-\frac{s}{2}}, \quad \text{if } \frac{1}{r} + \frac{1}{r'} = 1 \text{ and } \frac{1}{2} \geqslant \frac{1}{r} \geqslant \frac{1}{2} - \frac{s}{N} > 0. \tag{2.2}$$

Lemma 2.2 [3, Lemma 3.2] *The map $\mathbb{R} \ni t \longmapsto \mathcal{Q}(t) \in \mathcal{L}(X^{-\frac{1}{2}} \times X^{-\frac{1}{2}})$ defined in* (1.11) *is continuous in the uniform operator topology.*

Lemma 2.3 [12, Lemma 2.4] *Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies condition* (1.4). *Then*

$$\begin{aligned} f^e : X^{-\frac{1}{2}} &\longrightarrow X^{-\frac{1}{2}} \\ \phi &\longmapsto f^e(\phi) : \Omega \subset \mathbb{R}^N \longrightarrow \mathbb{R} \\ &\qquad\qquad x \longmapsto f^e(\phi)(x) := f\big(\tilde{A}^{-1}\phi(x)\big) \end{aligned}$$

defines an operator from $X^{-\frac{1}{2}}$ into $X^{-\frac{1}{2}}$ which is Lipschitz continuous in bounded subsets of $X^{-\frac{1}{2}}$.

Lemma 2.4 *Assume that f satisfies condition (1.4) and g satisfies condition (1.7). Then the operator $\mathcal{F}\colon \mathbb{R}\times X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}\longrightarrow X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}$ defined in (1.12) is locally Lipschitz continuous in $X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}$ uniformly in t on bounded intervals.*

Proof It is an immediate consequence of Lemma 2.3 and the definition of $\mathcal{F}$. □

Lemma 2.5 *Under the same hypotheses of Lemma 2.4, the mapping $\mathcal{F}$ is locally bounded in $X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}$ uniformly in t on $\mathbb{R}$.*

Proof Let $D\in X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}$ be an arbitrary bounded set. If $t\in\mathbb{R}$ and $\begin{bmatrix}\phi\\ \psi\end{bmatrix}\in D$ then

$$\left\|\mathcal{F}\left(t,\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right)\right\|_{X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}}=\left\|f^e(\phi)+g(t)\right\|_{X^{-\frac{1}{2}}}\leqslant\left\|f^e(\phi)\right\|_{X^{-\frac{1}{2}}}+c_1\left\|g(t)\right\|_X,$$

for some $c_1>0$. On the other hand, by using (1.6) and the embeddings (2.1)–(2.2), we obtain

$$\begin{aligned}\left\|f^e(\phi)\right\|_{X^{-\frac{1}{2}}}&\leqslant c_2\left\|f^e(\phi)\right\|_{L^{\frac{2N}{N+2}}(\Omega)}\leqslant c_3\left(1+\left\|\tilde{A}^{-1}\phi\right\|^{\rho}_{L^{\frac{2N\rho}{N+2}}(\Omega)}\right)\\&\leqslant c_4\left(1+\left\|\tilde{A}^{-1}\phi\right\|^{\rho}_{X^{\frac{1}{2}}}\right)=c_4\left(1+\left\|\phi\right\|^{\rho}_{X^{-\frac{1}{2}}}\right).\end{aligned}$$

It follows by [24, Proposition 7.1] that

$$\begin{aligned}\|g(t)\|_{L^2(\Omega)}&\leqslant\sup_{r\in[t,t+1]}\|g(r)\|_{L^2(\Omega)}\leqslant C\|g\|_{W^{1,2}((t,t+1),L^2(\Omega))}\\&\leqslant C\left(\|g\|^2_{L^2((t,t+1),L^2(\Omega))}+\|g_t\|^2_{L^2((t,t+1),L^2(\Omega))}\right)^{\frac{1}{2}}\\&\leqslant C\left(\int_t^{t+1}\|g(r)\|^2_{L^2(\Omega)}\,dr+\int_t^{t+1}\|g_t(r)\|^2_{L^2(\Omega)}\,dr\right)^{\frac{1}{2}}\\&\leqslant C\left(\sup_{t\in\mathbb{R}}\int_t^{t+1}\|g(r)\|^2_{L^2(\Omega)}\,dr+\sup_{t\in\mathbb{R}}\int_t^{t+1}\|g_t(r)\|^2_{L^2(\Omega)}\,dr\right)^{\frac{1}{2}}\\&\leqslant C\left(\|g\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}+\|g_t\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}\right)^{\frac{1}{2}}\\&\leqslant C\|g\|_{W^{1,2}_b(\mathbb{R},L^2(\Omega))},\end{aligned}$$

where $C=2$ (see the proof of [24, Proposition 7.1] to notice that it is indeed independent of $t\in\mathbb{R}$).

Consequently, by (1.7), we conclude that there is a constant $M=M(D)>0$ such that

$$\left\|\mathcal{F}\left(t,\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right)\right\|_{X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}}\leqslant M,$$

and the result is proved. □

By Lemmas 2.2, 2.4, and 2.5, the conditions of Theorem A.20 are satisfied, ensuring the local well-posedness of solutions to problem (1.9).

Theorem 2.6 *Assume that f satisfies (1.4) and g satisfies (1.7). Then for each $z_0, w_0 \in X^{-\frac{1}{2}}$ and $s \in \mathbb{R}$, there exists a maximal time of existence $T^{max} = T^{max}(z_0, w_0) > s$ such that the problem (1.9) admits a unique solution*

$$z(\cdot) = z(\cdot, s, z_0) \in \mathcal{C}\Big([s, T^{max}), X^{-\frac{1}{2}}\Big) \cap \mathcal{C}^1\Big((s, T^{max}), X^{-\frac{1}{2}}\Big)$$

defined on the maximal interval of existence $[s, T^{max})$, where either $T^{max} = \infty$ or

$$\lim_{t \to (T^{max})^-} \Big(\|z(t, s, z_0)\|^2_{X^{-\frac{1}{2}}} + \|z_t(t, s, w_0)\|^2_{X^{-\frac{1}{2}}}\Big) = \infty.$$

Proof Since (1.9) is equivalent to system (1.10) (with $w = z_t$), the result is a consequence of Theorem A.20 applied to the first order problem (1.10). □

The local well-posedness of solutions to the non-autonomous second-order semi-linear evolution equation (1.1) follows from Lemma 2.1, combined with Theorem 2.6.

Theorem 2.7 (Local well-posedness) *Assume that f satisfies (1.4) and g satisfies (1.7). Then for each $u_0, v_0 \in X^{\frac{1}{2}}$ and $s \in \mathbb{R}$, there exists a maximal time of existence $\tau^{max}_{u_0,v_0} > s$ such that the problem (1.1) admits a unique solution*

$$u(\cdot) = u(\cdot, s, u_0) \in \mathcal{C}\Big([s, \tau^{max}_{u_0,v_0}), X^{\frac{1}{2}}\Big) \cap \mathcal{C}^1\Big((s, \tau^{max}_{u_0,v_0}), X^{\frac{1}{2}}\Big)$$

defined on the maximal interval of existence $[s, \tau^{max}_{u_0,v_0})$, where either $\tau^{max}_{u_0,v_0} = \infty$ or

$$\lim_{t \to (\tau^{max}_{u_0,v_0})^-} \Big(\|u(t, s, u_0)\|^2_{X^{\frac{1}{2}}} + \|u_t(t, s, v_0)\|^2_{X^{\frac{1}{2}}}\Big) = \infty.$$

The next step is to prove the global well-posedness of solutions to the problem (1.9), and consequently to the problem (1.1). In order to do that, we first recall some auxiliary results.

Lemma 2.8 [3, Lemma 3.7] *The inequality*

$$\frac{\lambda_1}{1+\lambda_1}\|\psi\|^2_{X^{-\frac{1}{2}}} \leqslant \|\psi\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}\psi\|^2_{X^{-\frac{1}{2}}}$$

holds for all $\psi \in X^{-\frac{1}{2}}$.

Remark 2.9 As a direct consequence of Lemma 2.8, we have the Poincaré inequality

$$\|\tilde{A}^{-\frac{1}{2}}\psi\|^2_{X^{-\frac{1}{2}}} \leqslant \frac{1}{1+\lambda_1}\|\psi\|^2_{X^{-\frac{1}{2}}}, \qquad \psi \in X^{-\frac{1}{2}}.$$

Lemma 2.10 [3, Lemma 3.8] *Assume that f satisfies conditions (1.3) and (1.4). Then the following properties hold:*

(i) *There exist $\nu_0 \in (0, \lambda_1)$ and $K_1 > 0$ such that*

$$\int_\Omega f(\tilde{A}^{-1}\psi)\tilde{A}^{-1}\psi\, dx \leqslant \frac{(\lambda_1 - \nu_0)}{1+\lambda_1}\|\psi\|^2_{X^{-\frac{1}{2}}} + K_1,$$

for all $\psi \in X^{-\frac{1}{2}}$.

(ii) *There exist $\nu_0 \in (0, \lambda_1)$ and $K_2 > 0$ such that*

$$\int_\Omega \int_0^{\tilde{A}^{-1}\psi} f(s)\, ds dx \leqslant \frac{(\lambda_1 - \nu_0)}{2(1+\lambda_1)}\|\psi\|^2_{X^{-\frac{1}{2}}} + K_2,$$

for all $\psi \in X^{-\frac{1}{2}}$.

Lemma 2.11 [12, Lemma 2.1] *The following equality holds:*

$$\left\langle \tilde{A}^{-\frac{1}{2}}\phi, \tilde{A}^{\frac{1}{2}}\psi \right\rangle_X = \int_\Omega \phi\psi\, dx, \qquad \phi \in L^{\frac{2N}{N+2}}(\Omega),\ \psi \in X^{\frac{1}{2}}.$$

Lemma 2.12 [15, Lemma 2.1] *For every $t, \tau, \beta \in \mathbb{R}$ with $t \geqslant \tau$, $\beta > 0$ and $g \in L^2_b(\mathbb{R}, L^2(\Omega))$, we have*

$$\sup_{t \geqslant \tau} \int_\tau^t e^{-\beta(t-s)}\|g(s)\|^2_{L^2(\Omega)}\, ds \leqslant \frac{1}{1-e^{-\beta}}\|g\|^2_{L^2_b(\mathbb{R}, L^2(\Omega))}.$$

In the following, Lemma 2.13 deals with a boundedness result for the solution of problem (1.9).

Lemma 2.13 *Assume that conditions (1.3), (1.4) and (1.7) hold and let $z_0, w_0 \in X^{-\frac{1}{2}}$ and $s \in \mathbb{R}$ be given. Then $z(\cdot, s, z_0)$ and $z_t(\cdot, s, w_0)$ satisfy for some constant $C = C(z_0, w_0) > 0$ the inequality*

$$\|z\|^2_{X^{-\frac{1}{2}}} + \|z_t\|^2_{X^{-\frac{1}{2}}} \leqslant C\Big(e^{(t-s)} + \frac{1}{1-e}\|g\|^2_{L^2_b(\mathbb{R}; L^2(\Omega))}\Big)$$

in $[s, T^{max})$, where $T^{max} = T^{max}(z_0, w_0) > s$ comes from Theorem 2.6.

Proof According to Theorem 2.6, there exists a solution $z(\cdot)=z(\cdot,s,z_0)$ of (1.9) in $X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}$ defined on some interval $[s,T^{max})$. Taking the inner product of (1.9) with z_t in $X^{-\frac{1}{2}}$, we obtain the following equality

$$\begin{aligned}&\langle z_{tt},z_t\rangle_{X^{-\frac{1}{2}}}+\eta(t)\big\langle(I-\tilde{A}^{-1})z_t,z_t\big\rangle_{X^{-\frac{1}{2}}}+\big\langle(I-\tilde{A}^{-1})z,z_t\big\rangle_{X^{-\frac{1}{2}}}\\&\quad=\big\langle f^e(z),z_t\big\rangle_{X^{-\frac{1}{2}}}+\big\langle g(t),z_t\big\rangle_{X^{-\frac{1}{2}}}\end{aligned}$$

for all $t\in[s,T^{max})$. Consequently,

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}}+\|z\|^2_{X^{-\frac{1}{2}}}-\ \|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right)\right]-\big\langle f^e(z),z_t\big\rangle X^{-\frac{1}{2}}\\&\quad=\ -\eta(t)\left(\|z_t\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z_t\|^2_{X^{-\frac{1}{2}}}\right)+\big\langle g(t),z_t\big\rangle_{X^{-\frac{1}{2}}}.\end{aligned}\tag{2.3}$$

Setting $F(t)=\displaystyle\int_0^t f(\tau)\,d\tau$, it follows by Lemma 2.11 that

$$\begin{aligned}\big\langle f^e(z),z_t\big\rangle_{X^{-\frac{1}{2}}}&=\big\langle\tilde{A}^{-\frac{1}{2}}f^e(z),\tilde{A}^{\frac{1}{2}}\tilde{A}^{-1}z_t\big\rangle_{L^2(\Omega)}=\int_\Omega f(\tilde{A}^{-1}z)\tilde{A}^{-1}z_t\,dx\\&=\int_\Omega\frac{d}{dt}\big(F(\tilde{A}^{-1}z)\big)\,dx=\frac{d}{dt}\left(\int_\Omega F(\tilde{A}^{-1}z)\,dx\right).\end{aligned}$$

Thus, Eq. (2.3) becomes

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}}+\|z\|^2_{X^{-\frac{1}{2}}}-\ \|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right)-\int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx\right]=\\&\qquad=\ -\eta(t)\left(\|z_t\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z_t\|^2_{X^{-\frac{1}{2}}}\right)+\big\langle g(t),z_t\big\rangle_{X^{-\frac{1}{2}}}.\end{aligned}\tag{2.4}$$

By Lemma 2.10, (ii), one can obtain $\nu_0\in(0,\lambda_1)$ and $K_2>0$ such that

$$-\int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx\geqslant-\frac{(\lambda_1-\nu_0)}{2(1+\lambda_1)}\|z\|^2_{X^{-\frac{1}{2}}}-K_2.\tag{2.5}$$

Condition (1.2) and Lemma 2.8 imply the following estimate

$$-\eta(t)\left(\|z_t\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z_t\|^2_{X^{-\frac{1}{2}}}\right)\leqslant-\frac{a_1\lambda_1}{1+\lambda_1}\|z_t\|^2_{X^{-\frac{1}{2}}}\leqslant 0,\tag{2.6}$$

and by Lemma 2.11, Young's and Poincaré's inequalities, we have

$$\begin{aligned}-\big\langle g(t),z_t\big\rangle_{X^{-\frac{1}{2}}}&=-\int_\Omega g(t)\tilde{A}^{-1}z_t\,dx\leqslant c\|g(t)\|^2_X+\frac{\nu_0}{4}\|\tilde{A}^{-\frac{1}{2}}z_t\|^2_{X^{-\frac{1}{2}}}\\&\leqslant c_1\|g(t)\|^2_X+\frac{\nu_0}{4(1+\lambda_1)}\|z_t\|^2_{X^{-\frac{1}{2}}},\end{aligned}\tag{2.7}$$

with $c_1 > 0$ and we used Remark 2.9.

Combining the estimates (2.4), (2.5), (2.6), and (2.7), we can write

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}} + \|z\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right) - \int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx\right] \leqslant \\
&\quad \leqslant c_1\|g(t)\|^2_X + \frac{\nu_0}{4(1+\lambda_1)}\|z_t\|^2_{X^{-\frac{1}{2}}} \\
&\quad \leqslant K_2 + c_1\|g(t)\|^2_X + \frac{\lambda_1}{4(1+\lambda_1)}\|z_t\|^2_{X^{-\frac{1}{2}}} - \int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx + \frac{\lambda_1}{2(1+\lambda_1)}\|z\|^2_{X^{-\frac{1}{2}}} \\
&\quad \overset{\text{Lemma } 2.8}{\leqslant} c_2 + c_2\|g(t)\|^2_X + \frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}} + \|z\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right) \\
&\quad - \int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx,
\end{aligned}
$$

where $c_2 \geqslant \max\{K_2, c_1\}$. By the Gronwall's inequality, we have

$$
\begin{aligned}
&\left[\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}} + \|z\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right) - \int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx\right] \\
&\quad \leqslant \tilde{c}e^{(t-s)} + \int_s^t e^{(t-r)}\big(c_2 + c_2\|g(r)\|^2_X\big)\,dr \\
&\quad \leqslant c\left(e^{(t-s)} + \int_s^t e^{(t-r)}\|g(r)\|^2_X\,dr\right), \qquad t \in \big[s, T^{max}\big),
\end{aligned}
\tag{2.8}
$$

for some constant $c = c(z_0, w_0) > 0$. But, since

$$
\begin{aligned}
&\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}} + \|z\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right) - \int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx \geqslant \\
&\overset{\text{Lemma } 2.8}{\geqslant} \frac{1}{2}\|z_t\|^2_{X^{-\frac{1}{2}}} + \frac{2\lambda_1}{4(1+\lambda_1)}\|z\|^2_{X^{-\frac{1}{2}}} - \int_\Omega F\big(\tilde{A}^{-1}z\big)\,dx \\
&\overset{(2.5)}{\geqslant} \frac{1}{2}\|z_t\|^2_{X^{-\frac{1}{2}}} + \frac{2\lambda_1}{4(1+\lambda_1)}\|z\|^2_{X^{-\frac{1}{2}}} - \frac{(\lambda_1-\nu_0)}{2(1+\lambda_1)}\|z\|^2_{X^{-\frac{1}{2}}} - K_2. \\
&\geqslant \frac{1}{2}\|z_t\|^2_{X^{-\frac{1}{2}}} + \frac{2\nu_0}{4(1+\lambda_1)}\|z\|^2_{X^{-\frac{1}{2}}} - K_2,
\end{aligned}
$$

it follows from (2.8) and Lemma 2.12 that

$$
\begin{aligned}
\|z\|^2_{X^{-\frac{1}{2}}} + \|z_t\|^2_{X^{-\frac{1}{2}}} &\leqslant C\left(e^{(t-s)} + \int_s^t e^{(t-r)}\|g(r)\|^2_X\,dr\right) \\
&\leqslant C\left(e^{(t-s)} + \sup_{t\geqslant s}\int_s^t e^{(t-r)}\|g(r)\|^2_X\,dr\right) \\
&\leqslant C\left(e^{(t-s)} + \frac{1}{1-e}\|g\|^2_{L^2_b(\mathbb{R};L^2(\Omega))}\right), \qquad t \in [s, T^{max}),
\end{aligned}
$$

for some constant $C = C(z_0, w_0) > 0$. □

As a consequence of Lemma 2.13, Theorem 2.6 and Theorem A.20, we can state the following global well-posedness result on solutions of the problem (1.9).

Theorem 2.14 *Assume that conditions* (1.3), (1.4) *and* (1.7) *hold and let* $z_0, w_0 \in X^{-\frac{1}{2}}$ *and* $s \in \mathbb{R}$ *be given. Then the solution* $z(\cdot) = z(\cdot, s, z_0)$ *of* (1.9) *exists globally in time. Moreover, the relation* $S_{-1/2}(t,s)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} z(t,s,z_0) \\ z_t(t,s,w_0) \end{bmatrix}$ *defines an evolution process in* $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$ *associated to the problem* (1.9) *which satisfies in* $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$ *the variation of constants formula*

$$S_{-1/2}(t,s)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = L_{-1/2}(t,s)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} + U_{-1/2}(t,s)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \tag{2.9}$$

where

$$L_{-1/2}(t,s) = I - \int_s^t \mathcal{Q}(\tau) L_{-1/2}(\tau,s)\, d\tau \tag{2.10}$$

and

$$U_{-1/2}(t,s)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = \int_s^t L_{-1/2}(t,\tau)\mathcal{F}\left(\tau, S_{-1/2}(\tau,s)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix}\right) d\tau. \tag{2.11}$$

Next, we state the version of Lemma 2.13 concerning the boundedness of the solution of problem (1.1), and also the global well-posedness result of solution of the problem (1.1). These results are consequences of Lemma 2.13, Theorem 2.14, and Lemma 2.1.

Lemma 2.15 *Assume that conditions* (1.3), (1.4) *and* (1.7) *hold and let* $u_0, v_0 \in X^{\frac{1}{2}}$ *and* $s \in \mathbb{R}$ *be given. Then* $u(\cdot, s, u_0)$ *and* $u_t(\cdot, s, v_0)$ *satisfy for some constant* $C = C(u_0, v_0) > 0$ *the inequality*

$$\|u\|^2_{X^{\frac{1}{2}}} + \|u_t\|^2_{X^{\frac{1}{2}}} \leqslant C\left(e^{(t-s)} + \frac{1}{1-e}\|g\|^2_{L^2_b(\mathbb{R};L^2(\Omega))}\right)$$

in $[s, \tau^{max}_{u_0,v_0})$*, where* $\tau^{max}_{u_0,v_0} > s$ *comes from Theorem* 2.7.

Theorem 2.16 (Global well-posedness) *Assume that conditions* (1.3), (1.4) *and* (1.7) *hold and let* $u_0, v_0 \in X^{\frac{1}{2}}$ *and* $s \in \mathbb{R}$ *be given. Then the solution* $u(\cdot) = u(\cdot, s, u_0)$ *of* (1.1) *exists globally in time. Moreover, the relation* $S(t,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u(t,s,u_0) \\ u_t(t,s,v_0) \end{bmatrix}$ *defines an evolution process in* $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ *associated with the problem* (1.1)*, which is given by* $S(t,s) = \Phi_1 S_{-1/2}(t,s)\Phi_{-1}$*, and satisfies in* $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ *the variation of constants formula*

$$S(t,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = L(t,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + U(t,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix},$$

where

$$L(t,s) = \Phi_1 L_{-1/2}(t,s)\Phi_{-1} \qquad \text{and} \qquad U(t,s) = \Phi_1 U_{-1/2}(t,s)\Phi_{-1},$$

and more specifically

$$U(t,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \int_s^t L(t,\tau)\Phi_1 \mathcal{F}\left(\tau, \Phi_{-1} S(\tau,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) d\tau.$$

3 Existence of the uniform attractor

3.1 Construction of the symbol space associated with the problem (1.1)

Let $y(t) = (u(t), u_t(t))$ and $Y = H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the finite energy norm

$$\|y\|_Y = \{\|u\|^2_{H_0^1(\Omega)} + \|u_t\|^2_{H_0^1(\Omega)}\}^{\frac{1}{2}}.$$

By considering $\sigma_0(t) = (\eta(t), g(x,t))$ and setting $A_{\sigma_0(t)}(u,v) = (v, \Delta u + \Delta v_t + f(u) + \eta(t)\Delta v + g(x,t))$, then the nonautonomous system (1.1) can be rewritten in the operator form

$$\begin{cases} \partial_t y = A_{\sigma_0(t)}(y), \\ y(s) = (u_0, v_0). \end{cases} \tag{3.1}$$

The function $\sigma_0(t) = (\eta(t), g(x,t))$ is known as the time symbol (or symbol) of Eq. (3.1). The reader may consult [14] for more details. Let

$$\eta_h(t) = \eta(t+h) \quad \text{and} \quad g_h(x,t) = g(x,t+h),$$

for all $t, h \in \mathbb{R}$ and $x \in \Omega$. Now, define

$$\Sigma_0 = \{(\eta_h, g_h) : h \in \mathbb{R}\} \subseteq \{\eta_h : h \in \mathbb{R}\} \times \{g_h : h \in \mathbb{R}\}.$$

Since $\eta \in C(\mathbb{R})$ is uniformly continuous and $\{\eta_h \in C([t_1,t_2], \mathbb{R}_+) : h \in \mathbb{R}\}$ is precompact in $C([t_1,t_2], \mathbb{R}_+)$ for all bounded subinterval $[t_1,t_2] \subset \mathbb{R}$ (Arzelá–Áscoli), it follows by [14, Proposition 2.1, Chapter V] that

$$\{\eta_h : h \in \mathbb{R}\} \text{ is precompact in } \Xi_1 = (\mathcal{C}(\mathbb{R}), d_{\Xi_1}),$$

with topology generated by the Fréchet metric $d_{\Xi_1}(\xi_1,\xi_2) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{d^n_{\Xi_1}(\xi_1,\xi_2)}{1+d^n_{\Xi_1}(\xi_1,\xi_2)}\right)$, where $d^n_{\Xi_1}(\xi_1,\xi_2) = \max_{t\in[-n,n]} |\xi_1(t) - \xi_2(t)|$, $n \in \mathbb{N}$. We recall that a sequence of functions $\{\xi_n\}_{n\in\mathbb{N}} \subset \Xi_1$ converges to a function $\xi \in \Xi_1$ (which will be denoted by $\xi_n \xrightarrow[n\to\infty]{\Xi_1} \xi$), if for any interval $[a,b] \subset \mathbb{R}$, $a < b$, there holds the convergence

$$\lim_{n\to\infty} \max_{t\in[a,b]} |\xi_n(t) - \xi(t)| = 0.$$

By condition (1.7), we have $g \in W_b^{1,2}(\mathbb{R}, L^2(\Omega))$. Consider the family

$$\{g_h : h \in \mathbb{R}\} \text{ in } \Xi_2,$$

where Ξ_2 represents the space $W_{loc}^{1,2}(\mathbb{R}, L^2(\Omega))$ endowed with the local 2-power mean convergence topology, that is, a sequence $\{g_n\}_{n\in\mathbb{N}} \subset \Xi_2$ converges to a function $g \in \Xi_2$ (which will be denoted by $g_n \xrightarrow[n\to\infty]{\Xi_2} g$), if for any interval $[a,b] \subset \mathbb{R}$, $a < b$, we have

$$\lim_{n\to\infty} \int_a^b \left(\|g_n(s) - g(s)\|^2_{L^2(\Omega)} + \|\partial_t g_n(s) - \partial_t g(s)\|^2_{L^2(\Omega)} \right) ds = 0.$$

We note that $\{g_h : h \in \mathbb{R}\}$ is not supposed to be necessarily precompact in Ξ_2. Let us denote the hull of the symbol σ_0, where $\sigma_0(t) = (\eta(t), g(x,t))$, by

$$\Sigma = \mathcal{H}(\sigma_0) = \overline{\Sigma_0}^{\Xi}.$$

where $\Xi := \Xi_1 \times \Xi_2$ is endowed with the product topology induced by Ξ_1 and Ξ_2.

Remark 3.1 According to [14, Chapter V, Sections 2–3], Σ is a complete metric space.

Proposition 3.2 *The following properties hold:*

(i) Σ *is bounded in* $\mathcal{C}_b(\mathbb{R}) \times W_b^{1,2}(\mathbb{R}, L^2(\Omega))$, *and for any* $\sigma \in \Sigma$, *we have*

$$\|\sigma\|_{\mathcal{C}_b(\mathbb{R})\times W_b^{1,2}(\mathbb{R},L^2(\Omega))} \leqslant a_2 + C\|g\|_{W_b^{1,2}(\mathbb{R},L^2(\Omega))},$$

where $C > 0$, $a_2 > 0$ *comes from* (1.2), $\|\cdot\|_{W_b^{1,2}(\mathbb{R},L^2(\Omega))}$ *is given by* (1.8) *and* $\mathcal{C}_b(\mathbb{R})$ *is endowed with the uniform norm* $\|\eta\|_\infty = \sup_{t\in\mathbb{R}} |\eta(t)|$. *In particular, for* $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, *we have* $a_1 \leqslant \sigma_1(t) \leqslant a_2$ *for all* $t \in \mathbb{R}$, *and* $\|\sigma_2\|_{W_b^{1,2}(\mathbb{R},L^2(\Omega))} \leqslant C\|g\|_{W_b^{1,2}(\mathbb{R},L^2(\Omega))}$.

(ii) *The translation group* $\{\theta_h : h \in \mathbb{R}\}$ *is continuous on* Ξ *for all* $h \in \mathbb{R}$.

(iii) *The translation group* $\{\theta_h : h \in \mathbb{R}\}$ *acting on* Σ *is invariant in* Σ, *that is,*

$$\theta_h\Sigma = \Sigma, \quad \textit{for all } h \in \mathbb{R}.$$

Proof (i) Let $\sigma = (\sigma_1, \sigma_2) \in \Sigma$. Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\max_{t\in[a,b]} |\eta(t+t_n) - \sigma_1(t)| \overset{n\to\infty}{\longrightarrow} 0$$

and

$$\lim_{n\to\infty}\int_a^b \left(\|g(s+t_n)-\sigma_2(s)\|^2_{L^2(\Omega)}+\|\partial_t g(s+t_n)-\partial_t\sigma_2(s)\|^2_{L^2(\Omega)}\right)ds=0, \tag{3.2}$$

whenever $[a,b]$ is a compact interval.

On one hand, it follows by condition (1.2) that $\sigma_1\in C_b(\mathbb{R})$ with $0<a_1\leqslant\sigma_1(t)\leqslant a_2$ for all $t\in\mathbb{R}$.

On the other hand, since

$$\|\sigma_2(s)\|^2_{L^2(\Omega)}\leqslant 2\left(\|g(s+t_n)-\sigma_2(s)\|^2_{L^2(\Omega)}+\|g(s+t_n)\|^2_{L^2(\Omega)}\right),$$
$$\|\partial_t\sigma_2(s)\|^2_{L^2(\Omega)}\leqslant 2\left(\|\partial_t g(s+t_n)-\partial_t\sigma_2(s)\|^2_{L^2(\Omega)}+\|\partial_t g(s+t_n)\|^2_{L^2(\Omega)}\right),$$

we derive for each $t\in\mathbb{R}$ that

$$\begin{aligned}\frac{1}{2}\int_t^{t+1}&\|\sigma_2(s)\|^2_{L^2(\Omega)}\,ds\\ &\leqslant\int_t^{t+1}\|g(s+t_n)-\sigma_2(s)\|^2_{L^2(\Omega)}\,ds+\int_{t+t_n}^{t+t_n+1}\|g(s)\|^2_{L^2(\Omega)}\,ds\\ &\leqslant\int_t^{t+1}\|g(s+t_n)-\sigma_2(s)\|^2_{L^2(\Omega)}\,ds+\sup_{r\in\mathbb{R}}\int_r^{r+1}\|g(s)\|^2_{L^2(\Omega)}\,ds\\ &=\int_t^{t+1}\|g(s+t_n)-\sigma_2(s)\|^2_{L^2(\Omega)}\,ds+\|g\|^2_{L^2_b(\mathbb{R},L^2(\Omega))},\end{aligned}$$

and by (3.2) it follows that $\|\sigma_2\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}\leqslant 2\|g\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}$. By an analogous argument, it follows that $\|\partial_t\sigma_2\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}\leqslant 2\|\partial_t g\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}$, hence

$$\|\sigma_2\|^2_{W^{1,2}_b(\mathbb{R},L^2(\Omega))}\leqslant 2\|g\|^2_{W^{1,2}_b(\mathbb{R},L^2(\Omega))}.$$

Consequently,

$$\|\sigma\|_{C_b(\mathbb{R})\times W^{1,2}_b(\mathbb{R},L^2(\Omega))}=\|\sigma_1\|_{C_b(\mathbb{R})}+\|\sigma_2\|_{W^{1,2}_b(\mathbb{R},L^2(\Omega))}\leqslant a_2+c\|g\|_{W^{1,2}_b(\mathbb{R},L^2(\Omega))}.$$

(ii) Let $h\in\mathbb{R}$ be fixed, and let $\{(\overline{\eta_n},\overline{g_n})\}_{n\in\mathbb{N}}$ be a sequence in Ξ such that $\overline{\eta_n}\xrightarrow[n\to\infty]{\Xi_1}\overline{\eta}$ and $\overline{g_n}\xrightarrow[n\to\infty]{\Xi_2}\overline{g}$. By [14, Proposition 2.3, Chapter V], we conclude that $\theta_h\overline{\eta_n}\xrightarrow[n\to\infty]{\Xi_1}\theta_h\overline{\eta}$.

Moreover, note that for any interval $[a, b] \subset \mathbb{R}$ with $a < b$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_a^b \left(\|\theta_h\overline{g_n}(s) - \theta_h\overline{g}(s)\|^2_{L^2(\Omega)} + \|\theta_h\partial_t\overline{g_n}(s) - \theta_h\partial_t\overline{g}(s)\|^2_{L^2(\Omega)}\right) ds \\
&\quad = \lim_{n\to\infty}\int_a^b \left(\|\overline{g_n}(s+h)-\overline{g}(s+h)\|^2_{L^2(\Omega)}+\|\partial_t\overline{g_n}(s+h)-\partial_t\overline{g}(s+h)\|^2_{L^2(\Omega)}\right) ds \\
&\quad = \lim_{n\to\infty}\int_{a+h}^{b+h} \left(\|\overline{g_n}(r) - \overline{g}(r)\|^2_{L^2(\Omega)} + \|\partial_t\overline{g_n}(r) - \partial_t\overline{g}(r)\|^2_{L^2(\Omega)}\right) dr \\
&\quad = 0,
\end{aligned}$$

where in last equality we used that $h \in \mathbb{R}$ is fixed and $\overline{g_n} \xrightarrow[n\to\infty]{\Xi_2} \overline{g}$. It follows that

$$\theta_h\overline{g_n} \xrightarrow[n\to\infty]{\Xi_2} \theta_h\overline{g},$$

and we complete the proof.
(iii) Let $h \in \mathbb{R}$ be fixed. If $\sigma \in \Sigma = \mathcal{H}(\sigma_0)$, then $\sigma(\cdot) = \lim_{n\to\infty}\sigma_0(\cdot + h_n)$ in Ξ, where $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$. By the continuity of $\theta_h : \Xi \to \Xi$, we have

$$\theta_h\sigma(\cdot) = \lim_{n\to\infty}\theta_h\sigma_0(\cdot + h_n) = \lim_{n\to\infty}\sigma_0(\cdot + h_n + h),$$

and, hence, $\theta_h\Sigma \subseteq \Sigma$.

Conversely, by the continuity of θ_{-h} as stated in item (ii), we obtain

$$\theta_{-h}\sigma(\cdot) = \lim_{n\to\infty}\theta_{-h}\sigma_0(\cdot + h_n) = \lim_{n\to\infty}\sigma_0(\cdot + h_n - h),$$

which implies that $\theta_{-h}\sigma(\cdot) \in \Sigma$. Therefore,

$$\sigma = \theta_h\theta_{-h}\sigma \in \theta_h\Sigma,$$

showing that $\Sigma \subseteq \theta_h\Sigma$. □

Remark 3.3 Assume that conditions (1.3), (1.4) and (1.7) hold. Let

$$A_{\sigma(t)}(u, v) = (v, \Delta u + \Delta v_t + f(u) + \overline{\eta}(t)\Delta v + \overline{g}(x, t)),$$

with $\sigma(t) = (\overline{\eta}(t), \overline{g}(x, t))$, $\sigma \in \Sigma$, with $\Sigma = \mathcal{H}(\sigma_0) = \overline{\Sigma_0}^{\Xi}$, and consider the system

$$\begin{cases} \partial_t(u, u_t) = A_{\sigma(t)}(u, u_t), \quad t > s,\ x \in \Omega, \\ u = 0,\ t \geqslant s,\ x \in \partial\Omega, \\ u(s, x) = u_0(x), \quad u_t(s, x) = v_0(x),\ x \in \Omega. \end{cases} \tag{3.3}$$

For $u \in X^{\frac{1}{2}}$, let $z = (I - \Delta)u \in X^{-\frac{1}{2}}$ and set $w(t) = z_t(t)$. Taking

$$B_{\sigma(t)}(z, w) = (w, -\overline{\eta}(t)\Lambda w - \Lambda z + f^e(z) + \overline{g}(x, t)),$$

with $\sigma(t) = (\overline{\eta}(t), \overline{g}(x, t))$, $\sigma \in \Sigma$, it follows that system (3.3) becomes

$$\begin{cases} \partial_t(z, z_t) = B_{\sigma(t)}(z, z_t), \quad t > s, \ x \in \Omega, \\ z = 0, \ t \geqslant s, \ x \in \partial\Omega, \\ z(s, x) = z_0(x), \quad z_t(s, x) = w_0(x), \ x \in \Omega, \end{cases} \tag{3.4}$$

where $\Lambda = I - (I - \Delta)^{-1} \in \mathcal{L}(X^{-\frac{1}{2}})$ and $f^e = f \circ (I - \Lambda)$. By the proofs of Theorem 2.6, Lemma 2.13 and Theorem 2.14, the system (3.4) generates a family of processes $\{S_{\sigma,-1/2}(t, s)\}_{\sigma\in\Sigma}$ in $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$ defined by

$$S_{\sigma,-1/2}(t, s)\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} z^\sigma(t,s,z_0) \\ z_t^\sigma(t,s,w_0) \end{bmatrix}$$

for all $t \geqslant s$, where $z^\sigma(\cdot) = z^\sigma(\cdot, s, z_0) \in \mathcal{C}\left([s, +\infty), X^{-\frac{1}{2}}\right) \cap \mathcal{C}^1\left((s, +\infty), X^{-\frac{1}{2}}\right)$ is a global solution of (3.4) satisfying

$$\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} \leqslant C\left(e^{(t-s)} + \frac{1}{1-e}\|g\|^2_{W_b^{1,2}(\mathbb{R};L^2(\Omega))}\right),$$

for all $t \geqslant s$. Moreover, the processes $\{S_{\sigma,-1/2}(t, s)\}_{\sigma\in\Sigma}$ satisfies (2.9), (2.10), and (2.11).

On the other hand, by the proofs of Theorem 2.7, Lemma 2.15 and Theorem 2.16, the system (3.3) generates a family of uniformly bounded processes $\{S_\sigma(t, s)\}_{\sigma\in\Sigma}$ in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ given by

$$S_\sigma(t, s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u^\sigma(t,s,u_0) \\ u_t^\sigma(t,s,v_0) \end{bmatrix}$$

for all $t \geqslant s$, where $u^\sigma(\cdot) = u^\sigma(\cdot, s, u_0) \in \mathcal{C}\left([s, +\infty), X^{\frac{1}{2}}\right) \cap \mathcal{C}^1\left((s, +\infty), X^{\frac{1}{2}}\right)$ is a global solution of (3.3) satisfying

$$\|u^\sigma\|^2_{X^{\frac{1}{2}}} + \|u_t^\sigma\|^2_{X^{\frac{1}{2}}} \leqslant C\left(e^{(t-s)} + \frac{1}{1-e}\|g\|^2_{W_b^{1,2}(\mathbb{R};L^2(\Omega))}\right),$$

for all $t \geqslant s$. Further,

$$S_\sigma(t, s) = \Phi_1 S_{-1/2,\sigma}(t, s)\Phi_{-1}$$

and

$$S_\sigma(t, s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = L_\sigma(t, s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + U_\sigma(t, s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix},$$

where

$$L_\sigma(t,s) = \Phi_1 L_{\sigma,-1/2}(t,s)\Phi_{-1} \qquad \text{and} \qquad U_\sigma(t,s) = \Phi_1 U_{\sigma,-1/2}(t,s)\Phi_{-1},$$

and more specifically

$$U_\sigma(t,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \int_s^t L_\sigma(t,\tau)\Phi_1\mathcal{F}_\sigma\left(\tau, \Phi_{-1}S_\sigma(\tau,s)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right)d\tau.$$

In what follows, we prove the existence of a uniformly bounded absorbing set for the system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$.

3.2 Existence of a uniformly absorbing set

Lemma 3.4 *Let $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ be the family of processes associated with system (3.3). Then there exists a bounded set $\mathcal{B} \subset X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ that uniformly (w.r.t. $\sigma \in \Sigma$) absorbs all bounded subsets of $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$, that is, for every bounded subset $D \subset X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ there exists an absorbing time $T_D \geqslant 0$ such that*

$$\bigcup_{\sigma\in\Sigma} S_\sigma(t,0)D \subset \mathcal{B}, \qquad \textit{for all} \quad t \geqslant T_D.$$

Proof Given $\sigma \in \Sigma$, $\sigma(t) = (\overline{\eta}(t), \overline{g}(x,t))$, and $z_0, w_0 \in X^{-\frac{1}{2}}$, let $S_{\sigma,-1/2}(t,0)\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} z^\sigma \\ z_t^\sigma \end{smallmatrix}\right]$ for all $t \geqslant 0$. Let $0 < b < \frac{\nu_0}{4(1+\lambda_1)}$ and define for any $t \geqslant 0$ the functionals

$$W_\sigma(t) = \frac{1}{2}\left(\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z^\sigma\|^2_{X^{-\frac{1}{2}}}\right) - \int_\Omega F(\tilde{A}^{-1}z^\sigma)\,dx$$

and

$$V_\sigma^b(t) = W_\sigma(t) + b\langle z^\sigma, z_t^\sigma\rangle_{X^{-\frac{1}{2}}},$$

where $F(t) = \displaystyle\int_0^t f(\tau)\,d\tau.$

Claim 1: There are constants $\tilde{c}, \tilde{\tilde{c}} > 0$ (which is independent of the choice of σ) such that

$$\begin{aligned}\frac{\nu_0}{2(1+\lambda_1)}\left(\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z_t^\sigma\|^2_{X^{-\frac{1}{2}}}\right) - \tilde{c} &\leqslant W_\sigma(t) \\ &\leqslant \tilde{\tilde{c}}\left(1 + \|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z^\sigma\|^{\rho+1}_{X^{-\frac{1}{2}}}\right),\end{aligned}$$

for all $t \geqslant 0$ and $\sigma \in \Sigma$.

Indeed, on one hand, by Lemma 2.10, (ii), one can obtain $\nu_0 \in (0, \lambda_1)$ and $\tilde{c} > 0$ such that

$$-\int_\Omega F\big(\tilde{A}^{-1}z^\sigma\big)\,dx \geqslant -\frac{(\lambda_1-\nu_0)}{2(1+\lambda_1)}\|z^\sigma\|^2_{X^{-\frac{1}{2}}} - \tilde{c},$$

and, consequently, by Lemma 2.8 we obtain

$$\begin{aligned}
W_\sigma(t) &\geqslant \frac{1}{2}\Big(\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z^\sigma_t\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z^\sigma\|^2_{X^{-\frac{1}{2}}}\Big) - \frac{(\lambda_1-\nu_0)}{2(1+\lambda_1)}\|z^\sigma\|^2_{X^{-\frac{1}{2}}} - \tilde{c}\\
&\geqslant \frac{\lambda_1}{2(1+\lambda)}\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \frac{1}{2}\|z^\sigma_t\|^2_{X^{-\frac{1}{2}}} - \frac{\lambda_1}{2(1+\lambda_1)}\|z^\sigma\|^2_{X^{-\frac{1}{2}}}\\
&+ \frac{\nu_0}{2(1+\lambda_1)}\|z^\sigma\|^2_{X^{-\frac{1}{2}}} - \tilde{c}\\
&= \frac{\nu_0}{2(1+\lambda_1)}\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \frac{1}{2}\|z^\sigma_t\|^2_{X^{-\frac{1}{2}}} - \tilde{c}\\
&\geqslant \frac{\nu_0}{2(1+\lambda_1)}\Big(\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z^\sigma_t\|^2_{X^{-\frac{1}{2}}}\Big) - \tilde{c},
\end{aligned} \tag{3.5}$$

for all $t \geqslant 0$ and $\sigma \in \Sigma$.

On the other hand, the estimate (1.6) implies

$$\begin{aligned}
\int_\Omega \big|F(\tilde{A}^{-1}z^\sigma)\big|\,dx &\leqslant c_1\Big(1 + \|\tilde{A}^{-1}z^\sigma\|^{\rho+1}_{L^{\rho+1}(\Omega)}\Big) \leqslant c_2\Big(1+\|\tilde{A}^{-1}z^\sigma\|^{\rho+1}_{X^{\frac{1}{2}}}\Big)\\
&= c_2\Big(1+\|z^\sigma\|^{\rho+1}_{X^{-\frac{1}{2}}}\Big)
\end{aligned}$$

where the constants $c_i > 0$ $(i = 1, 2)$ are independent of σ, resulting in

$$W_\sigma(t) \leqslant \tilde{\tilde{c}}\Big(1 + \|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z^\sigma_t\|^2_{X^{-\frac{1}{2}}} + \|z^\sigma\|^{\rho+1}_{X^{-\frac{1}{2}}}\Big), \qquad t \geqslant 0,\ \sigma \in \Sigma, \tag{3.6}$$

for some constant $\tilde{\tilde{c}} > 0$ (which is independent of the choice of σ).

The Claim 1 follows by (3.5) and (3.6).

Now, note that inequality (3.5) ensures that

$$\big|V^b_\sigma(t) - W_\sigma(t)\big| \leqslant b\Big(\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \|z^\sigma_t\|^2_{X^{-\frac{1}{2}}}\Big) \leqslant \frac{2(1+\lambda_1)b}{\nu_0}\big(W_\sigma(t)+\tilde{c}\big)$$

which implies

$$c_{1,b}W_\sigma(t) - c_{0,b} \leqslant V^b_\sigma(t) \leqslant c_{2,b}W_\sigma(t) + c_{0,b}, \qquad t \geqslant 0,\ \sigma \in \Sigma, \tag{3.7}$$

with

$$c_{0,b} = \frac{2\tilde{c}(1+\lambda_1)b}{\nu_0} > 0, \quad c_{1,b} = 1 - \frac{2(1+\lambda_1)b}{\nu_0} > 0, \quad \text{and}$$

$$c_{2,b} = 1 + \frac{2(1+\lambda_1)b}{\nu_0} > 0.$$

Derivating $V_\sigma^b(t)$ with respect to t, we obtain

$$\frac{d}{dt}V_\sigma^b(t) = \frac{d}{dt}W_\sigma(t) + \frac{d}{dt}\Big[b\langle z^\sigma, z_t^\sigma\rangle_{X^{-\frac{1}{2}}}\Big].$$

Claim 2: $\frac{d}{dt}W_\sigma(t) \leqslant -\Big(\frac{a_1\lambda_1}{(1+\lambda_1)} - \frac{\delta}{2}\Big)\|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} + \frac{c_3}{2\delta}\|\overline{g}(t)\|_X^2$, for some constant $c_3 > 0$ and for any $\delta > 0$ (they are independent of σ).

In fact, from (2.4) we have

$$\begin{aligned}\frac{d}{dt}W_\sigma(t) &= -\eta(t)\Big(\|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z_t^\sigma\|^2_{X^{-\frac{1}{2}}}\Big) + \big\langle\overline{g}(t), z_t^\sigma\big\rangle_{X^{-\frac{1}{2}}}\\ &\leqslant -\frac{a_1\lambda_1}{(1+\lambda_1)}\|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} + \frac{\delta}{2}\|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} + \frac{c_3}{2\delta}\|\overline{g}(t)\|^2_X\end{aligned}$$

for any $\delta > 0$.

Claim 3: $\frac{d}{dt}\Big[\langle z^\sigma, z_t^\sigma\rangle_{X^{-\frac{1}{2}}}\Big] \leqslant -W_\sigma(t) + \Big(-\frac{\lambda_1}{2(1+\lambda_1)} + \frac{a_2\delta_1}{2} + \frac{a_2\delta_2}{2(1+\lambda_1)} + \frac{\delta_3}{2}\Big)\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \Big(\frac{3}{2} + \frac{a_2}{2\delta_1} + \frac{a_2}{2\delta_2(1+\lambda_1)}\Big)\|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} + \frac{c_3}{2\delta_3}\|\overline{g}(t)\|^2_X$, for any choice of $\delta_1, \delta_2, \delta_3 > 0$.

Indeed, note that

$$\begin{aligned}\frac{d}{dt}\Big[\langle z^\sigma, z_t^\sigma\rangle_{X^{-\frac{1}{2}}}\Big] &= \|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} + \langle z^\sigma, z_{tt}^\sigma\rangle_{X^{-\frac{1}{2}}}\\ &= \|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} - \overline{\eta}(t)\langle z^\sigma, z_t^\sigma\rangle_{X^{-\frac{1}{2}}} + \overline{\eta}(t)\langle\tilde{A}^{-\frac{1}{2}}z^\sigma, \tilde{A}^{-\frac{1}{2}}z_t^\sigma\rangle_{X^{-\frac{1}{2}}}\\ &\quad - \Big(\|z^\sigma\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z^\sigma\|^2_{X^{-\frac{1}{2}}}\Big)\\ &\quad + \int_\Omega f\big(\tilde{A}^{-1}z^\sigma\big)\tilde{A}^{-1}z^\sigma\,dx + \big\langle\overline{g}(t), z^\sigma\big\rangle_{X^{-\frac{1}{2}}}\\ &= -W_\sigma(t) + \frac{3}{2}\|z_t^\sigma\|^2_{X^{-\frac{1}{2}}} - \frac{1}{2}\|z^\sigma\|^2_{X^{-\frac{1}{2}}} + \frac{1}{2}\|\tilde{A}^{-\frac{1}{2}}z^\sigma\|^2_{X^{-\frac{1}{2}}} - \overline{\eta}(t)\langle z^\sigma, z_t^\sigma\rangle_{X^{-\frac{1}{2}}}\\ &\quad + \overline{\eta}(t)\langle\tilde{A}^{-\frac{1}{2}}z^\sigma, \tilde{A}^{-\frac{1}{2}}z_t^\sigma\rangle_{X^{-\frac{1}{2}}} + \int_\Omega\Big(f(\tilde{A}^{-1}z^\sigma)\tilde{A}^{-1}z^\sigma - F(\tilde{A}^{-1}z^\sigma)\Big)\,dx\\ &\quad + \big\langle\overline{g}(t), z^\sigma\big\rangle_{X^{-\frac{1}{2}}}.\end{aligned}$$

An application of (1.5) yields

$$\int_\Omega\Big(f(\tilde{A}^{-1}z^\sigma)\tilde{A}^{-1}z^\sigma - F(\tilde{A}^{-1}z^\sigma)\Big)\,dx \leqslant 0,$$

and, consequently, by using Remark 2.9 and the Young's and Poincaré's inequalities,

$$\begin{aligned}
&\frac{d}{dt}\Big[\langle z^\sigma, z_t^\sigma\rangle_{X^{-\frac12}}\Big]\\
&\quad\leqslant -W_\sigma(t)+\frac32\|z_t^\sigma\|^2_{X^{-\frac12}}-\frac12\|z^\sigma\|^2_{X^{-\frac12}}+\frac12\|\tilde A^{-\frac12}z^\sigma\|^2_{X^{-\frac12}}\\
&\quad+a_2\Big(\frac{\delta_1}{2}\|z^\sigma\|^2_{X^{-\frac12}}+\frac{1}{2\delta_1}\|z_t^\sigma\|^2_{X^{-\frac12}}\Big)\\
&\quad\;+a_2\Big(\frac{\delta_2}{2}\|\tilde A^{-\frac12}z^\sigma\|^2_{X^{-\frac12}}+\frac{1}{2\delta_2}\|\tilde A^{-\frac12}z_t^\sigma\|^2_{X^{-\frac12}}\Big)+\frac{\delta}{2}\|z^\sigma\|^2_{X^{-\frac12}}+\frac{c_3}{2\delta}\|\overline{g}(t)\|^2_X\\
&\quad\leqslant -W_\sigma(t)+\Big(-\frac12+\frac{a_2\delta_1}{2}+\frac{1}{2(1+\lambda_1)}+\frac{a_2\delta_2}{2(1+\lambda_1)}+\frac{\delta_3}{2}\Big)\|z^\sigma\|^2_{X^{-\frac12}}\\
&\quad\;+\Big(\frac32+\frac{a_2}{2\delta_1}+\frac{a_2}{2\delta_2(1+\lambda_1)}\Big)\|z_t^\sigma\|^2_{X^{-\frac12}}+\frac{c_3}{2\delta_3}\|\overline{g}(t)\|^2_X,
\end{aligned}$$

for any choice of $\delta_1,\delta_2,\delta_3>0$.

Based on Claims 2 and 3, we conclude that

$$\begin{aligned}
\frac{d}{dt}V_\sigma^b(t)\leqslant -bW_\sigma(t)&-\Big(\frac{b\lambda_1}{2(1+\lambda_1)}-\frac{a_2\delta_1 b}{2}-\frac{a_2\delta_2 b}{2(1+\lambda_1)}-\frac{\delta_3 b}{2}\Big)\|z^\sigma\|^2_{X^{-\frac12}}\\
&-\Big(\frac{a_1\lambda_1}{1+\lambda_1}-\frac{\delta}{2}-\frac{3b}{2}-\frac{a_2 b}{2\delta_1}-\frac{a_2 b}{2\delta_2(1+\lambda_1)}\Big)\|z_t^\sigma\|^2_{X^{-\frac12}}\\
&+\frac{c_3(b+1)}{2\delta_3}\|\overline{g}(t)\|^2_X.
\end{aligned}$$

Choosing $\delta_1,\delta_2,\delta_3,b>0$ sufficiently small, one can find a constant $\beta=\beta(\delta_1,\delta_2,\delta_3,b)>0$ such that

$$\begin{aligned}
\frac{d}{dt}V_\sigma^b(t)&\leqslant -bW_\sigma(t)-\beta\Big(\|z^\sigma\|^2_{X^{-\frac12}}+\|z_t^\sigma\|^2_{X^{-\frac12}}\Big)+\frac{c_3(b+1)}{2\delta_3}\|\overline{g}(t)\|^2_X\\
&\leqslant -bW_\sigma(t)+\frac{c_3(b+1)}{2\delta_3}\|\overline{g}(t)\|^2_X\\
&\overset{(3.7)}{\leqslant} -c_bV_\sigma^b(t)+c_4\|\overline{g}(t)\|^2_X+\tilde c_b,
\end{aligned}$$

for all $t\geqslant 0$ and positive constants $c_4, c_b, \tilde c_b>0$ which are independent of σ. By Gronwall's inequality, Lemma 2.12 and Proposition 3.2, we obtain

$$\begin{aligned}
V_\sigma^b(t)&\leqslant V_\sigma^b(0)e^{-c_bt}+\int_0^t e^{-c_b(t-r)}\Big(c_4\|\overline{g}(r)\|^2_X+\tilde c_b\Big)\,dr\\
&\leqslant V_\sigma^b(0)e^{-c_bt}+c_4\sup_{t\geqslant0}\int_0^t e^{-c_b(t-r)}\|\overline{g}(r)\|^2_X\,dr+\tilde c_b\sup_{t\geqslant0}\Big(e^{-c_bt}\int_0^t e^{c_br}\,dr\Big)\\
&\leqslant V_\sigma^b(0)e^{-c_bt}+\frac{c_4}{1-e^{-c_b}}\|\overline{g}\|^2_{L_b^2(\mathbb{R},L^2(\Omega))}+\frac{\tilde c_b}{c_b}\sup_{t\geqslant0}(1-e^{-c_bt})
\end{aligned}$$

$$\leqslant V_\sigma^b(0)e^{-c_b t} + \frac{c_4}{1-e^{-c_b}}\|g\|^2_{W_b^{1,2}(\mathbb{R},L^2(\Omega))} + \frac{\tilde{c}_b}{c_b}$$
$$\leqslant V_\sigma^b(0)e^{-c_b t} + C_b,$$

with $C_b > 0$. Consequently, using (3.7), we have

$$W_\sigma(t) \leqslant cW_\sigma(0)e^{-c_b t} + c,$$

for some $c > 0$. Hence, due to (3.5), it results in

$$\|z^\sigma(t)\|^2_{X^{-\frac{1}{2}}} + \|z_t^\sigma(t)\|^2_{X^{-\frac{1}{2}}} \leqslant KW_\sigma(0)e^{-c_b t} + K, \quad \text{for all} \quad t \geqslant 0, \tag{3.8}$$

for some constant $K > 0$ which is independent of σ. In this way, let $D \subset X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$ be a bounded subset. If $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in D$ and $\left[\begin{smallmatrix} z^\sigma(0) \\ z_t^\sigma(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right]$, then it follows by (3.6) that

$$W_\sigma(0) \leqslant \tilde{\tilde{c}}\Big(1 + \|z_0\|^2_{X^{-\frac{1}{2}}} + \|w_0\|^2_{X^{-\frac{1}{2}}} + \|z_0\|^{\rho+1}_{X^{-\frac{1}{2}}}\Big) \leqslant R_D,$$

for some constant $R_D > 0$ which is independent of the choice of $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in D$ and $\sigma \in \Sigma$. Thus, according to (3.8), one can find $T_D > 0$ such that if $t \geqslant T_D$ then

$$\sup_{\sigma\in\Sigma} \left\| \begin{bmatrix} z^\sigma(t) \\ z_t^\sigma(t) \end{bmatrix} \right\|^2_{X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}} \leqslant 2K,$$

and the result follows by isometries. □

3.3 The uniform attractor

Lemma 3.5 [3, Lemma 4.3] *There exists $\gamma \in (0, 1)$ such that for any bounded subset $D \subset X^{\frac{1}{2}}$, there holds*

$$\left\| f(u_1) - f(u_2) \right\|_{L^{\frac{2N}{N+2}}(\Omega)} \leqslant c_1\|u_1 - u_2\|_{H^{1-\gamma}(\Omega)}, \quad \textit{for all} \quad u_1, u_2 \in D,$$

for some constant $c_1 = c_1(\gamma, D) > 0$.

Theorem 3.6 [8, Theorem II.5.16] (Aubin–Lions–Simon) *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. Assume that the embedding of B_1 in B_2 is continuous and the embedding of B_0 in B_1 is compact. Let $p, r \in [1, \infty]$. Given $T > 0$, consider the set*

$$E_{p,r} = \left\{ v \in L^p((0, T), B_0) : \frac{dv}{dt} \in L^r((0, T), B_2) \right\}.$$

(i) *If $p < \infty$, then the embedding of $E_{p,r}$ into $L^p((0, T), B_1)$ is compact.*
(ii) *If $p = \infty$ and $r > 1$, then the embedding of $E_{p,r}$ into $C^0((0, T), B_1)$ is compact.*

Lemma 3.7 *Let $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence and $g \in W_b^{1,2}(\mathbb{R}, L^2(\Omega))$. Assume that $\{u_n(t) : t \in \mathbb{R}\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, and for any $T_1 > 0$, $\{(u_n)_t(t) : t \in \mathbb{R}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty((0, T_1), L^2(\Omega))$. Then for any $T > 0$ and $C > 0$ there exist subsequences $\{u_{n_k}\}_{k\in\mathbb{N}}$ and $\{s_{n_k}\}_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T\int_s^T\int_\Omega e^{-C(T-t)}(g(x,t+s_{n_k})-g(x,t+s_{n_l}))(u_{n_k}-u_{n_l})_t \, dxdtds = 0.$$

Proof Since $\{u_n(t) : t \in \mathbb{R}\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and the embedding $H_0^1(\Omega) \hookrightarrow L^m(\Omega)$ is compact for $2 \leqslant m < \frac{2N}{N-2}$ (by Rellich–Kondrachov compactness theorem), we assume without loss of generality that

$$u_n(T) \to u_0 \quad \text{in} \quad L^m(\Omega), \quad 2 \leqslant m < \frac{2N}{N-2}.$$

Claim 1: The sequence $\{u_n\}_{n\in\mathbb{N}}$ is convergent, up to a subsequence, in $L^1((0, T), L^2(\Omega))$ and in $L^m((0, T), L^m(\Omega))$ with $2 \leqslant m < \frac{2N}{N-2}$.

In fact, consider the sets

$$\begin{aligned} E_{1,\infty} &= \{v \in L^1((0, T), H_0^1(\Omega)) : \ v_t \in L^\infty((0, T), L^2(\Omega))\}, \\ E_{m,\infty} &= \{v \in L^m((0, T), H_0^1(\Omega)) : \ v_t \in L^\infty((0, T), L^m(\Omega))\}. \end{aligned}$$

Since $\{u_n(t) : t \in \mathbb{R}\}_{n\in\mathbb{N}} \subset E_{1,\infty} \cap E_{m,\infty}$ and $H_0^1(\Omega) \hookrightarrow L^m(\Omega)$ is compact with $2 \leqslant m < \frac{2N}{N-2}$, then the conclusion of Claim 1 follows by Aubin-Lions-Simon Theorem (Theorem 3.6, (i)).

Claim 2: If $\zeta \leqslant 2$, then there exists $M = M(T) > 0$ such that

$$\left(\int_0^T \|g(x, s+s_i) - g(x, s+s_j)\|_{L^\zeta(\Omega)}^\zeta ds\right)^{\frac{1}{\zeta}} \leqslant M\|g\|_{L_b^2(\mathbb{R};L^2(\Omega))}$$

and

$$\left(\int_0^T \|g_t(x, s+s_i) - g_t(x, s+s_j)\|_{L^\zeta(\Omega)}^\zeta ds\right)^{\frac{1}{\zeta}} \leqslant M\|g_t\|_{L_b^2(\mathbb{R};L^2(\Omega))},$$

for all $i, j \in \mathbb{N}$.

In fact, define $g_{s_i}(x, s) = g(x, s+s_i)$ and $g_{s_j}(x, s) = g(x, s+s_j)$, for $t \in [0, T]$, $x \in \Omega$, and $i, j \in \mathbb{N}$. Denote by $\lfloor T \rfloor$ the greatest integer less than or equal to T. Since $\zeta \leqslant 2$ then there exists a constant $c > 0$ such that $\|g_{s_i} - g_{s_j}\|_{L^\zeta((0,T),L^\zeta(\Omega))}^2 \leqslant c\|g_{s_i} - g_{s_j}\|_{L^2((0,T),L^2(\Omega))}^2$. Consequently,

$$\|g_{s_i} - g_{s_j}\|_{L^\zeta((0,T),L^\zeta(\Omega))}^2$$

$$
\begin{aligned}
&\leqslant 2c\|g_{s_i}\|^2_{L^2((0,T),L^2(\Omega))} + 2c\|g_{s_j}\|^2_{L^2((0,T),L^2(\Omega))}\\
&\leqslant 2c\|g_{s_i}\|^2_{L^2((0,T),L^2(\Omega))} + 2c\|g_{s_j}\|^2_{L^2((0,T),L^2(\Omega))}\\
&= 2c\int_{s_i}^{s_i+T}\|g(x,s)\|^2_{L^2(\Omega)}ds + 2c\int_{s_j}^{s_j+T}\|g(x,s)\|^2_{L^2(\Omega)}ds\\
&\leqslant 2c\sum_{k=0}^{\lfloor T\rfloor}\int_{s_i+k}^{s_i+k+1}\|g(x,s)\|^2_{L^2(\Omega)}ds + 2c\sum_{k=0}^{\lfloor T\rfloor}\int_{s_j+k}^{s_j+k+1}\|g(x,s)\|^2_{L^2(\Omega)}ds\\
&\leqslant 4c(1+\lfloor T\rfloor)\|g\|^2_{L^2_b(\mathbb{R},L^2(\Omega))}.
\end{aligned}
$$

Analogously,

$$
\|(g_t)_{s_i} - (g_t)_{s_j}\|^2_{L^\zeta((0,T),L^\zeta(\Omega))} \leqslant 4c(1+\lfloor T\rfloor)\|g_t\|^2_{L^2_b(\mathbb{R},L^2(\Omega))},
$$

where $(g_t)_{s_i}(x,s) = g_t(x,s+s_i)$ and $(g_t)_{s_j}(x,s) = g_t(x,s+s_j)$. Taking $M = \sqrt{4c(1+\lfloor T\rfloor)}$, the claim is proved.

Claim 3: The following estimate holds

$$
\begin{aligned}
&\left|\int_0^T\int_s^T\int_\Omega e^{-C(T-\tau)}(g(x,\tau+s_i)-g(x,\tau+s_j))(u_i-u_j)_t(\tau)\,dxd\tau ds\right|\\
&\quad\leqslant MT\|g\|_{W^{1,2}_b(\mathbb{R},L^2(\Omega))}\|u_i(T)-u_j(T)\|_{L^m(\Omega)}\\
&\quad\quad + M(1+CT)\|g\|_{W^{1,2}_b(\mathbb{R},L^2(\Omega))}\|u_i-u_j\|_{L^m((0,T),L^m(\Omega))}\\
&\quad\quad + MT\|g_t\|_{W^{1,2}_b(\mathbb{R},L^2(\Omega))}\|u_i-u_j\|_{L^m((0,T),L^m(\Omega))},
\end{aligned}
$$

for all $i, j \in \mathbb{N}$, $C > 0$, and $2 \leqslant m < \frac{2N}{N-2}$.

In fact, let $w(t) = w_{i,j}(t) = u_i(t) - u_j(t)$ and

$$
G(x,t) = g_{i,j}(x,t) = g(x,t+s_i) - g(x,t+s_j),
$$

$i, j \in \mathbb{N}$, for $t \in [0,T]$ and $x \in \Omega$. Since

$$
\begin{aligned}
G(x,t)e^{-C(T-t)}w_t(t) = \frac{d}{dt}[G(x,t)e^{-C(T-t)}w(t)] - G_t(x,t)e^{-C(T-t)}w(t)\\
- G(x,t)Ce^{-C(T-t)}w(t),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left|\int_0^T\int_s^T\int_\Omega e^{-C(T-\tau)}G(x,\tau)w_t(\tau)\,dxd\tau ds\right| \leqslant\\
&\qquad\leqslant \int_0^T\int_\Omega \left|G(x,T)w(T)\right|dxds + \int_0^T\int_\Omega\left|G(x,s)w(s)\right|dxds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_s^T\int_\Omega \left|G_t(x,\tau)w(\tau)\right| dxd\tau ds\\
&+C\int_0^T\int_s^T\int_\Omega \left|G(x,\tau)w(\tau)\right| dxd\tau ds\\
&\leqslant T\int_\Omega \left|G(x,T)w(T)\right| dx+\int_0^T\int_\Omega \left|G(x,s)w(s)\right| dxds\\
&+T\int_0^T\int_\Omega \left|G_t(x,s)w(s)\right| dxds+CT\int_0^T\int_\Omega \left|G(x,s)w(s)\right| dxds\\
&\leqslant T\left(\int_\Omega \left|G(x,T)\right|^{m^*} dx\right)^{\frac{1}{m^*}}\left(\int_\Omega \left|w(T)\right|^{m} dx\right)^{\frac{1}{m}}\\
&+T\left(\int_0^T \left\|G_t(x,s)\right\|^{m^*}_{L^{m^*}(\Omega)}ds\right)^{\frac{1}{m^*}}\left(\int_0^T \|w(s)\|^{m}_{L^m(\Omega)}ds\right)^{\frac{1}{m}}\\
&+(1+CT)\left(\int_0^T \left\|G(x,s)\right\|^{m^*}_{L^{m^*}(\Omega)}ds\right)^{\frac{1}{m^*}}\left(\int_0^T \|w(s)\|^{m}_{L^m(\Omega)}ds\right)^{\frac{1}{m}}\\
&\leqslant M\|g\|_{W_b^{1,2}(\mathbb{R},L^2(\Omega))}\Big(T\|w(T)\|_{L^m(\Omega)}+(1+CT)\|w\|_{L^m((0,T),L^m(\Omega))}\Big)\\
&+MT\|g_t\|_{W_b^{1,2}(\mathbb{R},L^2(\Omega))}\|w\|_{L^m((0,T),L^m(\Omega))},
\end{aligned}
$$

and the assertion is proved.

By the Claim 1, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$
\begin{aligned}
&\|u_{n_k}(T)-u_{n_l}(T)\|_{L^m(\Omega)}\xrightarrow{k,l\to\infty}0,\\
&\|u_{n_k}-u_{n_l}\|_{L^m((0,T),L^m(\Omega))}\xrightarrow{k,l\to\infty}0,\quad 2\leqslant m<\frac{2N}{N-2}.
\end{aligned}
$$

Consequently,

$$
\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T\int_s^T\int_\Omega e^{-C(t-\tau)}(g(x,\tau+s_{n_k})-g(x,\tau+s_{n_l}))(u_{n_k}-u_{n_l})_t(\tau)dxd\tau ds=0,
$$

proving the result. □

Our next aim is to build a contractive function in order to apply the Lemma A.9.

Let Φ_{-1} be the isometry given in Lemma 2.1, $(z_0^i, w_0^i)\in\widetilde{\mathcal{B}}=\Phi_{-1}(\mathcal{B})\subset X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}$, where $\mathcal{B}$ is the absorbing set obtained in Lemma 3.4, and consider (z^i, z_t^i), $i=1,2$, solutions to the initial value problem

$$
\begin{cases}
z_{tt}^i+\eta_i(t)\Lambda z_t^i+\Lambda z^i=f^e(z^i)+g_i(x,t), & t>s,\ x\in\Omega,\\
z^i=0, \quad t\geqslant s, \quad x\in\partial\Omega,\\
z^i(s)=z_0^i \quad \text{and} \quad z_t^i(s)=w_0^i,
\end{cases}
$$

with $\sigma_i(x,t) = (\eta_i(t), g_i(t))$ in Σ. Then $z(t) = z^1(t) - z^2(t)$ is solution of system

$$\begin{cases} z_{tt} + \eta_1(t)\Lambda z_t^1 - \eta_2(t)\Lambda z_t^2 + \Lambda z = f^e(z^1) - f^e(z^2) \\ +g_1(t) - g_2(t), \quad t > s, x \in \Omega, \\ z = 0, \quad t \geqslant s, \quad x \in \partial\Omega, \\ z(s) = z_0^1 - z_0^2 \quad \text{and} \quad z_t(s) = w_0^1 - w_0^2, \end{cases} \tag{3.9}$$

or equivalently,

$$\begin{cases} z_{tt} + \eta_1(t)\Lambda z_t + (\eta_1(t) - \eta_2(t))\Lambda z_t^2 + \Lambda z = f^e(z^1) - f^e(z^2) \\ +g_1(t) - g_2(t), \quad t > s, \ x \in \Omega, \\ z = 0, \quad t \geqslant s, \quad x \in \partial\Omega, \\ z(s) = z_0^1 - z_0^2 \quad \text{and} \quad z_t(s) = w_0^1 - w_0^2. \end{cases}$$

Consider the functionals

$$E(t) = \frac{1}{2}\left(\|z(t)\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}} z(t)\|^2_{X^{-\frac{1}{2}}} + \|z_t(t)\|^2_{X^{-\frac{1}{2}}} \right)$$

and

$$E_b(t) = E(t) + b\langle z, z_t\rangle_{X^{-\frac{1}{2}}},$$

with $b > 0$.

Lemma 3.8 *There are constants $C_1, C_2 > 0$ such that*

$$C_1 E(t) \leqslant E_b(t) \leqslant C_2 E(t) \quad \textit{for all } t \geqslant 0,$$

for a sufficiently small $b > 0$.

Proof Note that

$$\begin{aligned} E_b(t) &\leqslant E(t) + \frac{b}{2}\|z\|^2_{X^{-\frac{1}{2}}} + \frac{b}{2}\|z_t\|^2_{X^{-\frac{1}{2}}} \\ &\overset{\text{Lemma 2.8}}{\leqslant} E(t) + \frac{b(1+\lambda_1)}{2\lambda_1}\left(\|z\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}} z\|^2_{X^{-\frac{1}{2}}} \right) + \frac{b}{2}\|z_t\|^2_{X^{-\frac{1}{2}}} \\ &\leqslant \frac{\lambda_1 + b(1+\lambda_1)}{\lambda_1} E(t), \end{aligned}$$

with $b > 0$.

On the other hand, choosing $0 < b < \frac{\lambda_1}{1+\lambda_1}$, applying Remark 2.9 and $\left|2b\langle z, z_t\rangle_{X^{-\frac{1}{2}}}\right| \leqslant b\|z\|^2_{X^{-\frac{1}{2}}} + b\|z_t\|^2_{X^{-\frac{1}{2}}}$, we obtain

$$E_b(t) \geqslant \left(\frac{1}{2} - \frac{b}{2} - \frac{1}{2(1+\lambda_1)} \right)\|z\|^2_{X^{-\frac{1}{2}}} + \frac{(1-b)}{2}\|z_t\|^2_{X^{-\frac{1}{2}}}$$

$$\geqslant \left(\frac{1}{2}-\frac{b}{2}-\frac{1}{2(1+\lambda_1)}\right)\left(\|z\|^2_{X^{-\frac{1}{2}}}+\|z_t\|^2_{X^{-\frac{1}{2}}}\right)$$
$$\geqslant \left(\frac{1}{2}-\frac{b}{2}-\frac{1}{2(1+\lambda_1)}\right)\left(\|z\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}+\|z_t\|^2_{X^{-\frac{1}{2}}}\right),$$

and denoting $C_1=\frac{1}{2}-\frac{b}{2}-\frac{1}{2(1+\lambda_1)}>0$, the proof is complete. □

Lemma 3.9 *There exist constants $C,\tilde{C}>0$ and some $\gamma\in(0,1)$ such that*

$$\frac{d}{dt}E_b(t)\leqslant -CE_b(t)+\tilde{C}\|z\|^2_{X^{\frac{-1-\gamma}{2}}}+b\langle g_1-g_2,z\rangle_{X^{-\frac{1}{2}}}+\langle g_1-g_2,z_t\rangle_{X^{-\frac{1}{2}}}$$
$$+\left\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z_t\right\rangle_{X^{-\frac{1}{2}}}+b\left\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z\right\rangle_{X^{-\frac{1}{2}}}$$

for sufficiently small $b>0$.

Proof Taking the inner product of (3.9) with z_t in $X^{-\frac{1}{2}}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}}+\|z\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right)\right)\\
&=-\eta_1(t)\left(\|z_t\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z_t\|^2_{X^{-\frac{1}{2}}}\right)+\left\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z_t\right\rangle_{X^{-\frac{1}{2}}}\\
&\quad+\left\langle f^e(z^1)-f^e(z^2),z_t\right\rangle_{X^{-\frac{1}{2}}}+\langle g_1-g_2,z_t\rangle_{X^{-\frac{1}{2}}}\\
&\leqslant -\frac{a_1\lambda_1}{1+\lambda_1}\|z_t\|^2_{X^{-\frac{1}{2}}}+\left\langle f^e(z^1)-f^e(z^2),z_t\right\rangle_{X^{-\frac{1}{2}}}\\
&\quad+\langle g_1-g_2,z_t\rangle_{X^{-\frac{1}{2}}}+\left\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z_t\right\rangle_{X^{-\frac{1}{2}}},
\end{aligned}\tag{3.10}$$

where we used Lemma 2.8. Also, using Lemma 2.8 and Remark 2.9, we have the following estimate

$$\begin{aligned}
&\frac{d}{dt}\left(b\langle z,z_t\rangle_{-\frac{1}{2}}\right)=b\|z_t\|^2_{X^{-\frac{1}{2}}}+b\langle z,z_{tt}\rangle_{X^{-\frac{1}{2}}}\\
&\leqslant b\|z_t\|^2_{X^{-\frac{1}{2}}}-b\eta_1(t)\langle z,z_t\rangle_{X^{-\frac{1}{2}}}+b\eta_1(t)\left\langle\tilde{A}^{-\frac{1}{2}}z,\tilde{A}^{-\frac{1}{2}}z_t\right\rangle_{X^{-\frac{1}{2}}}\\
&\quad-b\left(\|z\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right)\\
&\quad+b\left\langle f^e(z^1)-f^e(z^2),z\right\rangle_{X^{-\frac{1}{2}}}+b\langle g_1-g_2,z\rangle_{X^{-\frac{1}{2}}}+b\left\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z\right\rangle_{X^{-\frac{1}{2}}}\\
&\leqslant b\|z_t\|^2_{X^{-\frac{1}{2}}}+\frac{ba_2\delta_1}{2}\|z\|^2_{X^{-\frac{1}{2}}}+\frac{ba_2}{2\delta_1}\|z_t\|^2_{X^{-\frac{1}{2}}}+\frac{ba_2\delta_2}{2}\|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\\
&\quad+\frac{ba_2}{2\delta_2(1+\lambda_1)}\|z_t\|^2_{X^{-\frac{1}{2}}}-\frac{b\lambda_1}{1+\lambda_1}\|z\|^2_{X^{-\frac{1}{2}}}+b\left\langle f^e(z^1)-f^e(z^2),z\right\rangle_{X^{-\frac{1}{2}}}\\
&\quad+b\langle g_1-g_2,z\rangle_{X^{-\frac{1}{2}}}
\end{aligned}$$

$$+ b\Big\langle (\eta_2(t) - \eta_1(t))\Lambda z_t^2, z\Big\rangle_{X^{-\frac{1}{2}}},$$

for any choice of $\delta_1, \delta_2 > 0$.

According to Lemma 2.11, Lemma 3.5 and the continuous embedding $X^{\frac{1}{2}} \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, $N \geqslant 3$, we obtain

$$\begin{aligned}
\langle f^e(z^1) - f^e(z^2), z\rangle_{X^{-\frac{1}{2}}} &= \int_\Omega \big[f^e(z^1) - f^e(z^2)\big]\tilde{A}^{-1} z\, dx \\
&\leqslant \big\| f(\tilde{A}^{-1}z^1) - f(\tilde{A}^{-1}z^2)\big\|_{L^{\frac{2N}{N+2}}(\Omega)} \|\tilde{A}^{-1}z\|_{L^{\frac{2N}{N-2}}(\Omega)} \\
&\leqslant c_0 \big\|\tilde{A}^{-1}z^1 - \tilde{A}^{-1}z^2\big\|_{H^{1-\gamma}(\Omega)} \|\tilde{A}^{-1}z\|_{X^{\frac{1}{2}}} \\
&\leqslant c_0 \big\|z\big\|_{X^{\frac{-1-\gamma}{2}}} \|z\|_{X^{-\frac{1}{2}}} \\
&\leqslant c_0 \frac{\delta_3}{2} \big\|z\big\|^2_{X^{\frac{-1-\gamma}{2}}} + c_0 \frac{1}{2\delta_3} \|z\|^2_{X^{-\frac{1}{2}}},
\end{aligned}$$

and, analogously,

$$\begin{aligned}
\langle f^e(z^1) - f^e(z^2), z_t\rangle_{X^{-\frac{1}{2}}} &= \int_\Omega \big[f^e(z^1) - f^e(z^2)\big]\tilde{A}^{-1} z_t\, dx \\
&\leqslant c_0 \frac{\delta_4}{2} \big\|z\big\|^2_{X^{\frac{-1-\gamma}{2}}} + c_0 \frac{1}{2\delta_4} \|z_t\|^2_{X^{-\frac{1}{2}}},
\end{aligned}$$

for any choice of $\delta_3, \delta_4 > 0$, and some constant $c_0 > 0$.

Then, from last inequalities we deduce

$$\begin{aligned}
\frac{d}{dt} E_b(t) \leqslant &-\left(\frac{b\lambda_1}{1+\lambda_1} - \frac{ba_2\delta_1}{2} - \frac{bc_0\delta_3}{2} - \frac{ba_2\delta_2}{2(1+\lambda_1)}\right)\|z\|^2_{X^{-\frac{1}{2}}} \\
&- \left(\frac{a_1\lambda_1}{1+\lambda_1} - \frac{ba_2}{2\delta_2(1+\lambda_1)} - \frac{ba_2}{2\delta_1} - b - \frac{c_0\delta_4}{2}\right)\|z_t\|^2_{X^{-\frac{1}{2}}} \\
&+ \left(\frac{bc_0}{2\delta_3} + \frac{c_0}{2\delta_4}\right)\|z\|^2_{X^{\frac{-1-\gamma}{2}}} + b\langle g_1 - g_2, z\rangle_{X^{-\frac{1}{2}}} + \langle g_1 - g_2, z_t\rangle_{X^{-\frac{1}{2}}} \\
&+ \Big\langle (\eta_2(t) - \eta_1(t))\Lambda z_t^2, z_t\Big\rangle_{X^{-\frac{1}{2}}} + b\Big\langle (\eta_2(t) - \eta_1(t))\Lambda z_t^2, z\Big\rangle_{X^{-\frac{1}{2}}},
\end{aligned}$$

from where choosing sufficiently small $b, \delta_1, \delta_2, \delta_3, \delta_4 > 0$ and noting that the inequality $E(t) \leqslant \frac{1}{2}\Big(\|z\|^2_{X^{-\frac{1}{2}}} + \|z_t\|^2_{X^{-\frac{1}{2}}}\Big)$ holds, it follows that we can find constants $\tilde{\tilde{C}}, \tilde{C} > 0$, which depend only on $b, \delta_1, \delta_2, \delta_3, \delta_4$, satisfying

$$\begin{aligned}
\frac{d}{dt} E_b(t) \leqslant &-\tilde{\tilde{C}} E(t) + \tilde{C}\|z\|^2_{X^{\frac{-1-\gamma}{2}}} + b\langle g_1 - g_2, z\rangle_{X^{-\frac{1}{2}}} + \langle g_1 - g_2, z_t\rangle_{X^{-\frac{1}{2}}} \\
&+ \Big\langle (\eta_2(t) - \eta_1(t))\Lambda z_t^2, z_t\Big\rangle_{X^{-\frac{1}{2}}} + b\Big\langle (\eta_2(t) - \eta_1(t))\Lambda z_t^2, z\Big\rangle_{X^{-\frac{1}{2}}}
\end{aligned}$$

$$
\overset{\text{Lemma 3.8}}{\leqslant} -\frac{\tilde{\tilde{C}}}{C_2}E_b(t) + \tilde{C}\|z\|^2_{X^{\frac{-1-\gamma}{2}}} + b\langle g_1 - g_2, z\rangle_{X^{-\frac{1}{2}}} + \langle g_1 - g_2, z_t\rangle_{X^{-\frac{1}{2}}}
+ \Big\langle(\eta_2(t) - \eta_1(t))\Lambda z_t^2, z_t\Big\rangle_{X^{-\frac{1}{2}}} + b\Big\langle(\eta_2(t) - \eta_1(t))\Lambda z_t^2, z\Big\rangle_{X^{-\frac{1}{2}}},
$$

proving the result. □

Lemma 3.10 *Given any $\epsilon > 0$, there exists a sufficiently large time $T = T(\epsilon, \widetilde{\mathcal{B}}) > 0$ such that*

$$
\begin{aligned}
&|\|S_{\sigma_1,-\frac{1}{2}}(T,0)(z_0^1, w_0^1) - S_{\sigma_2,-\frac{1}{2}}(T,0)(z_0^2, w_0^2)\|_{X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}}\\
&\quad \leqslant \epsilon + \Psi_T\big((z_0^1, w_0^1), (z_0^2, w_0^2); \sigma_1, \sigma_2\big),
\end{aligned}
$$

where $\Psi_T = \left|\Psi_{T,\mathrm{aux}}\right|^{1/2}$, with

$$
\Psi_{T,\mathrm{aux}}\big((z_0^1, w_0^1), (z_0^2, w_0^2), \sigma_1, \sigma_2\big) = \frac{1}{T}\left(\Psi_{T,1,\mathrm{aux}} + \Psi_{T,2,\mathrm{aux}} + \Psi_{T,3,\mathrm{aux}}\right),
$$

and

$$
\begin{aligned}
&\Psi_{T,1,\mathrm{aux}}((z_0^1, w_0^1), (z_0^2, w_0^2), \sigma_1, \sigma_2) = \frac{\tilde{C}}{C_1}\int_0^T\int_s^T \|z(t)\|^2_{X^{\frac{-1-\gamma}{2}}}\,dt\,ds,\\
&\Psi_{T,2,\mathrm{aux}}((z_0^1, w_0^1), (z_0^2, w_0^2), \sigma_1, \sigma_2)\\
&= \frac{1}{C_1}\int_0^T\int_s^T be^{-C(T-t)}\langle g_1(t) - g_2(t), z(t)\rangle_{X^{-\frac{1}{2}}}\,dtds\\
&\quad + \frac{1}{C_1}\int_0^T\int_s^T e^{-C(T-t)}\langle g_1(t) - g_2(t), z_t(t)\rangle_{X^{-\frac{1}{2}}}\,dtds,
\end{aligned}
$$

$$
\begin{aligned}
&\Psi_{T,3,\mathrm{aux}}((z_0^1, w_0^1), (z_0^2, w_0^2), \sigma_1, \sigma_2)\\
&= \frac{1}{C_1}\int_0^T\int_s^T e^{-C(T-t)}\Big\langle(\eta_2(t) - \eta_1(t))\Lambda z_t^2(t), z_t(t)\Big\rangle_{X^{-\frac{1}{2}}}\,dtds\\
&\quad + \frac{1}{C_1}\int_0^T\int_s^T be^{-C(T-t)}\langle(\eta_2(t) - \eta_1(t))\Lambda z_t^2(t), z(t)\rangle_{X^{-\frac{1}{2}}}\,dtds.
\end{aligned}
$$

Here, $z = z^1 - z^2$ is the solution of Eq. (3.9), with initial data $(z_0^1, w_0^1), (z_0^2, w_0^2) \in \widetilde{\mathcal{B}}$, and parameters $\sigma_1, \sigma_2 \in \Sigma$. The constants $C, \tilde{C} > 0$ are independent of T.

Proof By Lemma 3.9, there exist constants $C, \tilde{C} > 0$ such that

$$
\begin{aligned}
\frac{d}{dt}E_b(t) &\leqslant -CE_b(t) + \tilde{C}\|z\|^2_{X^{\frac{-1-\gamma}{2}}}\\
&\quad + \langle(\eta_2(t) - \eta_1(t))\Lambda z_t^2, z_t\rangle_{X^{-\frac{1}{2}}} + b\langle(\eta_2(t) - \eta_1(t))\Lambda z_t^2, z\rangle_{X^{-\frac{1}{2}}}
\end{aligned}
$$

$$+ b\langle g_1 - g_2, z\rangle_{X^{-\frac{1}{2}}} + \langle g_1 - g_2, z_t\rangle_{X^{-\frac{1}{2}}}.$$

Consequently,

$$\begin{aligned}\frac{d}{dt}\left(E_b(t)e^{Ct}\right) \leqslant\ & e^{Ct}\tilde{C}\|z\|^2_{X^{\frac{-1-\gamma}{2}}} + e^{Ct}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2, z_t\rangle_{X^{-\frac{1}{2}}} \\ & + be^{Ct}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2, z\rangle_{X^{-\frac{1}{2}}} \\ & + be^{Ct}\langle g_1 - g_2, z\rangle_{X^{-\frac{1}{2}}} + e^{Ct}\langle g_1 - g_2, z_t\rangle_{X^{-\frac{1}{2}}},\end{aligned}$$

and integrating the previous inequality over $[s, T]$ (w.r.t. t), we obtain

$$\begin{aligned}E_b(T) \leqslant\ & E_b(s)e^{-C(T-s)} \\ & + \tilde{C}\int_s^T e^{-C(T-t)}\|z\|^2_{X^{\frac{-1-\gamma}{2}}}\, dt + \int_s^T be^{-C(T-t)}\langle g_1 - g_2, z\rangle_{X^{-\frac{1}{2}}}\, dt \\ & + \int_s^T e^{-C(T-t)}\langle g_1 - g_2, z_t\rangle_{X^{-\frac{1}{2}}} dt \\ & + \int_s^T e^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2, z_t\rangle_{X^{-\frac{1}{2}}}\, dt \\ & + \int_s^T be^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2, z\rangle_{X^{-\frac{1}{2}}}\, dt.\end{aligned}$$

Using Lemma 3.8 and the fact that $|E(s)| \leqslant \tilde{M}$ for some $\tilde{M} > 0$ (inspired by (3.8)), we conclude that

$$\begin{aligned}E(T) \leqslant\ & \frac{C_2}{C_1}\tilde{M}e^{-C(T-s)} \\ & + \frac{\tilde{C}}{C_1}\int_s^T e^{-C(T-t)}\|z\|^2_{X^{\frac{-1-\gamma}{2}}}\, dt + \frac{1}{C_1}\int_s^T be^{-C(T-t)}\langle g_1 - g_2, z\rangle_{X^{-\frac{1}{2}}}\, dt \\ & + \frac{1}{C_1}\int_s^T e^{-C(T-t)}\langle g_1 - g_2, z_t\rangle_{X^{-\frac{1}{2}}} dt \\ & + \frac{1}{C_1}\int_s^T e^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2, z_t\rangle_{X^{-\frac{1}{2}}}\, dt \\ & + \frac{1}{C_1}\int_s^T be^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2, z\rangle_{X^{-\frac{1}{2}}}\, dt.\end{aligned}$$

Integrating the above inequality over $[0, T]$ with respect to s, $0 \leqslant s \leqslant t \leqslant T$, we have

$$E(T)T \leqslant \frac{C_2\tilde{M}}{C_1}e^{-CT}\left(\frac{e^{CT}-1}{C}\right) + \frac{\tilde{C}}{C_1}\int_0^T\int_s^T \|z\|^2_{X^{\frac{-1-\gamma}{2}}}\, dtds$$

$$+\frac{1}{C_1}\int_0^T\int_s^T be^{-C(T-t)}\langle g_1-g_2,z\rangle_{X^{-\frac{1}{2}}}\,dtds$$
$$+\frac{1}{C_1}\int_0^T\int_s^T e^{-C(T-t)}\langle g_1-g_2,z_t\rangle_{X^{-\frac{1}{2}}}dtds$$
$$+\frac{1}{C_1}\int_0^T\int_s^T e^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z_t\rangle_{X^{-\frac{1}{2}}}\,dtds$$
$$+\frac{1}{C_1}\int_0^T\int_s^T be^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z\rangle_{X^{-\frac{1}{2}}}\,dtds.$$

Let $\epsilon>0$ and $\hat{M}>\frac{C_2\tilde{M}}{CC_1}$. Choose $T>0$ such that $\frac{\hat{M}}{T}<\epsilon^2$. Then

$$E(T)\leqslant\epsilon^2+\Psi_{T,aux}((z_0^1,w_0^1),(z_0^2,w_0^2);\sigma_1,\sigma_2),$$

where $\Psi_{T,\mathrm{aux}}((z_0^1,w_0^1),(z_0^2,w_0^2),\sigma_1,\sigma_2)=\frac{1}{T}(\Psi_{T,1,\mathrm{aux}}+\Psi_{T,2,\mathrm{aux}}+\Psi_{T,3,\mathrm{aux}})$, with

$$\Psi_{T,1,\mathrm{aux}}((z_0^1,w_0^1),(z_0^2,w_0^2),\sigma_1,\sigma_2)=\frac{\tilde{C}}{C_1}\int_0^T\int_s^T\|z\|^2_{X^{\frac{-1-\gamma}{2}}}dtds,$$

$$\Psi_{T,2,\mathrm{aux}}((z_0^1,w_0^1),(z_0^2,w_0^2),\sigma_1,\sigma_2)=\frac{1}{C_1}\int_0^T\int_s^T be^{-C(T-t)}\langle g_1-g_2,z\rangle_{X^{-\frac{1}{2}}}\,dtds$$
$$+\frac{1}{C_1}\int_0^T\int_s^T e^{-C(T-t)}\langle g_1-g_2,z_t\rangle_{X^{-\frac{1}{2}}}dtds,$$

$$\begin{aligned}&\Psi_{T,3,\mathrm{aux}}((z_0^1,w_0^1),(z_0^2,w_0^2),\sigma_1,\sigma_2)\\&=\frac{1}{C_1}\int_0^T\int_s^T e^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z_t\rangle_{X^{-\frac{1}{2}}}\,dtds\\&+\frac{1}{C_1}\int_0^T\int_s^T be^{-C(T-t)}\langle(\eta_2(t)-\eta_1(t))\Lambda z_t^2,z\rangle_{X^{-\frac{1}{2}}}\,dtds.\end{aligned}$$

Using Lemma 2.8, we conclude that

$$\begin{aligned}&\|S_{\sigma_1,-\frac{1}{2}}(T,0)(z_0^1,w_0^1)-S_{\sigma_2,-\frac{1}{2}}(T,0)(z_0^2,w_0^2)\|_{X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}}\\&\leqslant\epsilon+\Psi_T((z_0^1,w_0^1),(z_0^2,w_0^2);\sigma_1,\sigma_2),\end{aligned}$$

for all $(z_0^1,w_0^1),(z_0^2,w_0^2)\in\widetilde{\mathcal{B}}$ and $\sigma_1,\sigma_2\in\Sigma$ with $\Psi_T=|\Psi_{T,\mathrm{aux}}|^{\frac{1}{2}}$. □

Lemma 3.11 *The map $\Psi_{T,\mathrm{aux}}$, defined in Lemma 3.10, is a contractive function on $\widetilde{\mathcal{B}}\times\widetilde{\mathcal{B}}\times\Sigma\times\Sigma$. Consequently, Ψ_T is also a contractive function on $\widetilde{\mathcal{B}}\times\widetilde{\mathcal{B}}\times\Sigma\times\Sigma$.*

Proof Let $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$ and $\{(z_0^n, w_0^n)\}_{n\in\mathbb{N}}$ be a sequence of initial data in $\widetilde{\mathcal{B}} \subset X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$. Now, for each $n \in \mathbb{N}$, let $(z^n(t), z_t^n(t))$ be the corresponding solution of the system

$$\begin{cases} z_{tt}^n + \eta_n(t)\Lambda z_t^n + \Lambda z^n = f^e(z^n) + g_n(x,t), \quad t > s, \ x \in \Omega, \\ z^n = 0, \quad t \geqslant s, \quad x \in \partial\Omega, \\ z^n(s) = z_0^n \quad \text{and} \quad z_t^n(s) = w_0^n. \end{cases}$$

Since $\widetilde{\mathcal{B}}$ is bounded, it follows by Lemma 3.4 (see (3.8)) that $(z^n(t), z_t^n(t))$ is uniformly bounded in $X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}$.

Claim 1: There exists a subsequence such that

$$\lim_{k\to\infty}\lim_{l\to\infty} \Psi_{T,1,\mathrm{aux}}((z_0^{n_k}, w_0^{n_k}), (z_0^{n_l}, w_0^{n_l}), \sigma_{n_k}, \sigma_{n_l}) = 0.$$

In fact, since

$$\{z^n\}_{n\in\mathbb{N}} \subset \{w \in L^2((0,T), X^{-\frac{1}{2}}) : \ w_t \in L^\infty((0,T), X^{\frac{-1-\gamma}{2}})\}$$

and the embedding $X^{-\frac{1}{2}} \hookrightarrow X^{\frac{-1-\gamma}{2}}$ is compact, it follows from Theorem 3.6, item (i), that the sequence $\{z^n\}_{n\in\mathbb{N}}$ is convergent, up to a subsequence, in $L^2((0,T), X^{\frac{-1-\gamma}{2}})$. Consequently,

$$\begin{aligned} &|\Psi_{T,1,\mathrm{aux}}((z_0^{n_k}, w_0^{n_k}), (z_0^{n_l}, w_0^{n_l}), \sigma_{n_k}, \sigma_{n_l})| \\ &\quad\leqslant \frac{\tilde{C}}{C_1}\int_0^T\int_s^T \|z^{n_k}(t) - z^{n_l}(t)\|^2_{X^{\frac{-1-\gamma}{2}}}\,dtds \\ &\quad= \frac{\tilde{C}T}{C_1}\|z^{n_k}(t) - z^{n_l}(t)\|^2_{L^2((0,T),X^{\frac{-1-\gamma}{2}})} \overset{l,k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Claim 2: There exists a subsequence such that

$$\lim_{k\to\infty}\lim_{l\to\infty} \Psi_{T,2,\mathrm{aux}}((z_0^{n_k}, w_0^{n_k}), (z_0^{n_l}, w_0^{n_l}), \sigma_{n_k}, \sigma_{n_l}) = 0.$$

Indeed, at first, note that by Lemma 3.4 (see (3.8)), there exists $\mathcal{M} = \mathcal{M}(T) > 0$ such that

$$T^{\frac{3}{2}}(\lfloor T\rfloor + 1)^{\frac{1}{2}} \sup_{s\in[0,T]} \|u_t^{n_k}(s) - u_t^{n_l}(s)\|_{L^2(\Omega)} \leqslant \mathcal{M}$$

and

$$T^{\frac{3}{2}}(\lfloor T\rfloor + 1)^{\frac{1}{2}} \sup_{s\in[0,T]} \|u^{n_k}(s) - u^{n_l}(s)\|_{L^2(\Omega)} \leqslant \mathcal{M}.$$

On the other hand, for each $n \in \mathbb{N}$, there exists a sequence $\{r_p^n\}_{p\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\|g(x,t+r_p^n) - g_n(x,t)\|_{L_b^2(\mathbb{R},L^2(\Omega))} \overset{p\to\infty}{\longrightarrow} 0.$$

Thus, there exists $p_n > n$ such that $\|g(x,t+r_{p_n}^n) - g_n(x,t)\|_{L_b^2(\mathbb{R},L^2(\Omega))} < \dfrac{1}{n}$, for all $n \in \mathbb{N}$. Besides that, for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_0^T \int_s^T \int_\Omega |(g(x,t+r_{p_{n_k}}^{n_k}) - g_{n_k}(x,t))||\tilde{A}^{-1}(z^{n_k} - z^{n_l})_t|dxdtds \leqslant \\
&\quad \leqslant T \int_0^T \int_\Omega |(g(x,t+r_{p_{n_k}}^{n_k}) - g_{n_k}(x,t))||\tilde{A}^{-1}(z^{n_k} - z^{n_l})_t|dxdt \\
&\quad \leqslant T \left(\int_0^T \|g(x,t+r_{p_{n_k}}^{n_k}) - g_{n_k}(x,t)\|_{L^2(\Omega)}^2 dt\right)^{\frac{1}{2}} \\
&\qquad \left(\int_0^T \|\tilde{A}^{-1}(z^{n_k} - z^{n_l})_t\|_{L^2(\Omega)}^2 dt\right)^{\frac{1}{2}} \\
&\quad \leqslant T \left(\sum_{s=0}^{\lfloor T \rfloor} \int_s^{s+1} \|g(x,t+r_{p_{n_k}}^{n_k}) - g_{n_k}(x,t)\|_{L^2(\Omega)}^2 dt\right)^{\frac{1}{2}} \\
&\qquad \left(\int_0^T \|u_t^{n_k} - u_t^{n_l}\|_{L^2(\Omega)}^2 dt\right)^{\frac{1}{2}} \\
&\quad \leqslant T^{\frac{3}{2}}(\lfloor T \rfloor + 1)^{\frac{1}{2}} \|g(x,t+r_{p_{n_k}}^{n_k}) - g_{n_k}(x,t)\|_{L_b^2(\mathbb{R},L^2(\Omega))} \\
&\qquad \sup_{s\in[0,T]} \|u_t^{n_k}(s) - u_t^{n_l}(s)\|_{L^2(\Omega)} \leqslant \frac{\mathcal{M}}{n_k},
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\int_0^T \int_s^T \int_\Omega |(g(x,t+r_{p_{n_k}}^{n_k}) - g_{n_k}(x,t))||\tilde{A}^{-1}(z^{n_k} - z^{n_l})|dxdtds \leqslant \\
&\quad \leqslant T^{\frac{3}{2}}(\lfloor T \rfloor + 1)^{\frac{1}{2}} \|g(x,t+r_{p_{n_k}}^{n_k}) - g_{n_k}(x,t)\|_{L_b^2(\mathbb{R},L^2(\Omega))} \\
&\qquad \sup_{s\in[0,T]} \|u^{n_k}(s) - u^{n_l}(s)\|_{L^2(\Omega)} \\
&\quad \leqslant \frac{\mathcal{M}}{n_k}.
\end{aligned}$$

Consequently, using Lemma 3.7, we conclude that

$$\left| \int_0^T \int_s^T \int_\Omega e^{-C(T-t)}(g_{n_k}(x,t) - g_{n_l}(x,t))\tilde{A}^{-1}(z^{n_k} - z^{n_l})_t dxdtds \right| \leqslant$$

$$
\begin{aligned}
&\leqslant \int_0^T \int_s^T \int_\Omega |g(x,t+r^{n_k}_{p_{n_k}}) - g_{n_k}(x,t)||\tilde{A}^{-1}(z^{n_k}-z^{n_l})_t|dxdtds \\
&\quad + \left| \int_0^T \int_s^T \int_\Omega e^{-C(T-t)}(g(x,t+r^{n_k}_{p_{n_k}}) - g(x,t+r^{n_l}_{p_{n_l}}))(u^{n_k}-u^{n_l})_t dxdtds \right| \\
&\quad + \int_0^T \int_s^T \int_\Omega |g(x,t+r^{n_l}_{p_{n_l}}) - g_{n_l}(x,t)||\tilde{A}^{-1}(z^{n_k}-z^{n_l})_t|dxdtds \\
&\leqslant \frac{\mathcal{M}}{n_k} + \frac{\mathcal{M}}{n_l} + \left| \int_0^T \int_s^T \int_\Omega e^{-C(T-t)}(g(x,t+r^{n_k}_{p_{n_k}}) \right. \\
&\quad \left. -g(x,t+r^{n_l}_{p_{n_l}}))(u^{n_k}-u^{n_l})_t dxdt \right| \overset{l,k\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

and (similarly to Lemma 3.7 with u_t replaced by u)

$$
\begin{aligned}
&\left| \int_0^T \int_s^T \int_\Omega e^{-C(T-t)}(g_{n_k}(x,t) - g_{n_l}(x,t))\tilde{A}^{-1}(z^{n_k}-z^{n_l})dxdtds \right| \leqslant \\
&\quad \leqslant \frac{\mathcal{M}}{n_k} + \frac{\mathcal{M}}{n_l} + \left| \int_0^T \int_s^T \int_\Omega e^{-C(T-t)}(g(x,t+r^{n_k}_{p_{n_k}}) \right. \\
&\quad \left. -g(x,t+r^{n_l}_{p_{n_l}}))(u^{n_k}-u^{n_l})dxdt \right| \overset{l,k\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

In conclusion,

$$
\Psi_{T,2,\mathrm{aux}}((z_0^{n_k}, w_0^{n_k}), (z_0^{n_l}, w_0^{n_l}), \sigma_{n_k}, \sigma_{n_l}) \overset{l,k\to\infty}{\longrightarrow} 0.
$$

Claim 3: There exists a subsequence such that

$$
\lim_{k\to\infty} \lim_{l\to\infty} \Psi_{T,3,\mathrm{aux}}((z_0^{n_k}, w_0^{n_k}), (z_0^{n_l}, w_0^{n_l}), \sigma_{n_k}, \sigma_{n_l}) = 0.
$$

In fact, since $\{\eta_h : h \in \mathbb{R}\}$ is precompact in $\Xi_1 = (\mathcal{C}(\mathbb{R}), d_{\Xi_1})$, it follows that $\{\eta_n\}_{n\in\mathbb{N}}$, up to a subsequence, is a Cauchy sequence in Ξ_1. By Lemma 3.4 (see (3.8)), the sequences $\{z^n\}_{n\in\mathbb{N}}$ and $\{z_t^n\}_{n\in\mathbb{N}}$ are bounded in $X^{-\frac{1}{2}}$ and, consequently,

$$
\begin{aligned}
&\Psi_{T,3,\mathrm{aux}}((z_0^{n_k}, w_0^{n_k}), (z_0^{n_l}, w_0^{n_l}), \sigma_{n_k}, \sigma_{n_l}) \\
&\quad \leqslant \frac{1}{C_1} \int_0^T \int_s^T |\eta_{n_k}(t) - \eta_{n_l}(t)| \|\Lambda z_t^{n_k}\|_{X^{-\frac{1}{2}}} \|z_t^{n_k} - z_t^{n_l}\|_{X^{-\frac{1}{2}}} \, dtds \\
&\quad + \frac{b}{C_1} \int_0^T \int_s^T |\eta_{n_k}(t) - \eta_{n_l}(t)| \|\Lambda z_t^{n_k}\|_{X^{-\frac{1}{2}}} \|z^{n_k} - z^{n_l}\|_{X^{-\frac{1}{2}}} \, dtds \\
&\quad \leqslant C \int_0^T \int_0^T |\eta_{n_k}(t) - \eta_{n_l}(t)| \, dtds \\
&\quad \leqslant CT^2 \sup_{t\in[0,T]} |\eta_{n_k}(t) - \eta_{n_l}(t)| \overset{l,k\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Therefore, $\Psi_{T,\mathrm{aux}}$ is a contractive function on $\widetilde{\mathcal{B}} \times \widetilde{\mathcal{B}} \times \Sigma \times \Sigma$. As a direct consequence of the definition of a contractive function, it follows that

$$\Psi_T = \left|\Psi_{T,\mathrm{aux}}\right|^{1/2}$$

is also a contractive function on $\widetilde{\mathcal{B}} \times \widetilde{\mathcal{B}} \times \Sigma \times \Sigma$. □

As a consequence of Lemma A.9, Lemma 2.1 and Theorem A.11, we conclude the following result.

Theorem 3.12 *The family of processes $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ associated with system (3.3) admits a uniform attractor $\mathcal{A}_\Sigma$ (in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$) given by*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(\mathcal{B}) = \omega_{\tau,\Sigma}(\mathcal{B}) = \bigcup_{D\subset\mathscr{B}(X^{\frac{1}{2}}\times X^{\frac{1}{2}})} \omega_{\tau,\Sigma}(D), \qquad \textit{for all} \;\; \tau \in \mathbb{R},$$

where $\mathcal{B}$ is the absorbing set established in the Lemma 3.4.

4 Upper semicontinuity of the uniform attractor

Let $\epsilon \in [0,1]$. This section concerns the upper-semicontinuity of the uniform attractor $\mathcal{A}^\epsilon_\Sigma$ of the family of evolution processes $\{S^\epsilon_\sigma(t,s)\}_{\sigma\in\Sigma}$ associated with system

$$\begin{cases} u_{tt} - \Delta u - \eta_\epsilon(t)\Delta u_t - \Delta u_{tt} = f(u) + g(x,t), \;\; t > s, \;\; x \in \Omega, \\ u = 0, \;\; t \geqslant s, \;\; x \in \partial\Omega, \\ u(s,x) = u_0(x), \quad u_t(s,x) = v_0(x), \;\; x \in \Omega, \end{cases} \tag{4.1}$$

where $\eta_\epsilon : \mathbb{R} \longrightarrow (0,\infty)$ is uniformly continuous satisfying

$$0 < \eta_1 \leqslant \eta_\epsilon(t) \leqslant \eta_2 < \infty,$$

uniformly with respect to $\epsilon \in [0,1]$. Moreover, we assume that $\lim_{\epsilon\to 0^+} \|\eta_\epsilon - \eta_0\|_{L^\infty(\Omega)} = 0$.

Remark 4.1 The uniform attractor $\mathcal{A}^\epsilon_\Sigma$ with respect to the problem (4.1) is obtained by considering the hull $\Sigma_\epsilon = \mathcal{H}(\sigma^\epsilon_0) = \overline{\Sigma_0}^{\Xi}$ with symbol $\sigma^\epsilon_0(t) = (\eta_\epsilon(t), g(x,t))$ and $\Xi := \Xi_1 \times \Xi_2$, where $\Xi_1 = (\mathcal{C}(\mathbb{R}), d_{\Xi_1})$ and $\Xi_2 = W^{1,2}_{loc}(\mathbb{R}, L^2(\Omega))$ is endowed with the local 2-power mean convergence topology.

Remark 4.2 By the proof of Lemma 3.4, we may conclude that there exists a bounded set $\mathcal{B} \subset X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ such that for every bounded subset $D \subset X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ one can obtain an absorbing time $T_D \geqslant 0$ such that

$$\bigcup_{\epsilon\in[0,1]} \bigcup_{\sigma\in\Sigma} S^\epsilon_\sigma(t,0)D \subset \mathcal{B}, \qquad \text{for all} \quad t \geqslant T_D.$$

Lemma 4.3 *We have*

$$\operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}\left(S_\sigma^\epsilon(t,s)w_0, S_\sigma^0(t,s)w_0\right) \to 0 \quad as \quad \epsilon \to 0^+$$

in compacts subsets of $\mathbb{R}$ *uniformly for* $w_0 = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ *in bounded subsets of* $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$.

Proof The proof follows that of Lemma 4.13 in [3]. □

Theorem 4.4 *The family of uniform attractors* $\{\mathcal{A}_\Sigma^\epsilon\}_{\epsilon\in[0,1]}$ *is upper-semicontinuous at* $\epsilon_0 = 0$*, that is,*

$$\lim_{\epsilon\to 0}\left[\operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}\left(\mathcal{A}_\Sigma^\epsilon, \mathcal{A}_\Sigma^0\right)\right] = 0.$$

Proof Suppose to the contrary that there exist $\delta > 0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset [0,1]$ with $\lim_{n\to\infty}\epsilon_n = 0$ such that

$$\operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}(\mathcal{A}_\Sigma^{\epsilon_n}, \mathcal{A}_\Sigma^0) > \delta, \quad n \in \mathbb{N}.$$

Thus, one can obtain $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_\Sigma^{\epsilon_n}$ such that

$$\operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}(v_n, \mathcal{A}_\Sigma^0) > \delta, \quad n \in \mathbb{N}. \tag{4.2}$$

According to Remark 4.2, we can choose $t_0 > 0$ satisfying

$$\bigcup_{\epsilon\in[0,1]}\bigcup_{\sigma\in\Sigma} S_\sigma^\epsilon(t,0)\mathcal{B} \subset \mathcal{B}, \quad \text{for all } t \geqslant t_0.$$

Since $\mathcal{A}_\Sigma^0$ is the uniform attractor of $\{S_\sigma^0(t,0)\}_{\sigma\in\Sigma}$ and $\mathcal{B}$ is bounded, there exists $T_0 \geqslant t_0$ such that

$$\sup_{\sigma\in\Sigma}\operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}(S_\sigma^0(t,0)\mathcal{B}, \mathcal{A}_\Sigma^0) < \frac{\delta}{3}, \quad \text{for all} \quad t \geqslant T_0. \tag{4.3}$$

Moreover, using Theorem A.11, for each $k \in \mathbb{N}$, we have $v_k \in \mathcal{A}_\Sigma^\epsilon = \omega_{0,\Sigma}^\epsilon(\mathcal{B})$. Thus, there exist $\{x_n^k\}_{n\in\mathbb{N}} \subset \mathcal{B}$, $\{\sigma_n^k\}_{n\in\mathbb{N}} \subset \Sigma_\epsilon$ and $\{t_n^k\}_{n\in\mathbb{N}} \subset [0,\infty)$, with $\lim_{n\to\infty} t_n^k = \infty$, such that $\lim_{n\to\infty} S_{\sigma_n^k}^\epsilon(t_n^k,0)x_n^k = v_k$. Consequently, there exists an integer $n_k > k$ such that for $t_k = t_{n_k}^k$, $\sigma_k = \sigma_{n_k}^k$, and $x_k = x_{n_k}^k$, we have

$$\|S_{\sigma_k}^\epsilon(t_k,0)x_k - v_k\|_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}} < \frac{\delta}{3}, \quad k \in \mathbb{N}. \tag{4.4}$$

Fix $k_0 \in \mathbb{N}$ such that $t_{k_0} > T_0$. By Lemma 4.3, there exist $\epsilon_0 > 0$ such that

$$\sup_{x\in\mathcal{B}}\|S_{\sigma_{k_0}}^\epsilon(t_0,0)x - S_{\sigma_{k_0}}^0(t_0,0)x\|_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}} < \frac{\delta}{3}, \quad \text{for all } \epsilon < \epsilon_0. \tag{4.5}$$

Hence, using (4.3), (4.4), and (4.5), we conclude that

$$
\begin{aligned}
\operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}(v_{k_0}, \mathcal{A}_\Sigma^0) &\leqslant \operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}(v_{k_0}, S^{\epsilon}_{\sigma_{k_0}}(t_{k_0},0)x_{k_0}) \\
&\quad + \operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}(S^{\epsilon}_{\sigma_{k_0}}(t_{k_0},0)x_{k_0}, S^{0}_{\sigma_{k_0}}(t_{k_0},0)x_{n_0}) + \operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}(S^{0}_{\sigma_{k_0}}(t_{k_0},0)x_{k_0}, \mathcal{A}_\Sigma^0) \\
&< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta,
\end{aligned}
$$

which contradicts (4.2). Therefore, the family of uniform attractors $\{\mathcal{A}_\Sigma^\epsilon\}_{\epsilon\in[0,1]}$ is upper semicontinuous at $\epsilon_0 = 0$. □

5 Regularity of the uniform attractor

In this section, we provide a result regarding the regularity of that uniform attractor. More precisely, in Theorem 5.5, we shall prove that $\mathcal{A}_\Sigma$ is a bounded subset of $X^1 \times X^1$. Before that, we present some preliminary results.

Lemma 5.1 *Given $r \in [0,1]$, there are constants $K, \alpha > 0$ such that*

$$
\sup_{\sigma\in\Sigma} \left\| L_\sigma(t,0) \right\|_{\mathcal{L}(X^{\frac{1+r}{2}}\times X^{\frac{1+r}{2}})} \leqslant K e^{-\alpha t}, \qquad \textit{for all } \ t \geqslant 0.
$$

Proof It is a consequence of Lemma 4.1 and Lemma 4.7 in [3]. □

Remark 5.2 Suppose that for some given $r \in [0,1]$ and some given bounded set $D \subset X^{\frac{1+r}{2}} \times X^{\frac{1+r}{2}}$ the operators $\{L_\sigma(t,0)\}_{\sigma\in\Sigma}$ and $\{U_\sigma(t,0)\}_{\sigma\in\Sigma}$ are well-defined and satisfy (for some constant $C > 0$)

$$
\sup_{\sigma\in\Sigma} \left\| L_\sigma(t,0) \right\|_{\mathcal{L}(X^{\frac{1+r}{2}}\times X^{\frac{1+r}{2}})} \leqslant C, \qquad \text{for all } \ t \geqslant 0,
$$

and

$$
\sup_{\sigma\in\Sigma} \sup_{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\in D} \left\| U_\sigma(t,0) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{X^{\frac{1+r}{2}}\times X^{\frac{1+r}{2}}} \leqslant C, \qquad \text{for all } \ t \geqslant 0.
$$

Then it is immediate that

$$
\gamma_\Sigma(D,0) := \bigcup_{\sigma\in\Sigma} \gamma_\sigma(D,0) = \bigcup_{\sigma\in\Sigma} \bigcup_{t\geqslant 0} S_\sigma(t,0)D,
$$

is bounded in $X^{\frac{1+r}{2}} \times X^{\frac{1+r}{2}}$.

Lemma 5.3 [12, Lemma 3.1] *Let $\tilde{f}^e := f^e \circ \tilde{A}$. If f satisfies condition* (1.4) *then there exists a constant $c > 0$ such that*

$$
\left\| \tilde{f}^e(\phi) \right\|_{X^{\frac{\tilde{\sigma}-1}{2}}} \leqslant c\Big(1 + \|\phi\|^{\rho}_{X^{\frac{1}{2}}}\Big), \qquad \phi \in X^{\frac{1}{2}},
$$

where $\widetilde{\sigma} := \min\left\{1, \frac{N+2}{2} - \rho\frac{N-2}{2}\right\}$, *provided* $N \geqslant 3$.

Lemma 5.4 *Suppose that f satisfies conditions (1.3) and (1.4), and let us consider* $\widetilde{\sigma} = \min\left\{1, \frac{N+2}{2} - \rho\frac{N-2}{2}\right\}$. *Then*

$$\sup_{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in \mathcal{B}} \left\| U_\sigma(t,0)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}}} \leqslant C, \quad \text{for all} \quad t \geqslant 0 \quad \text{and} \quad \sigma \in \Sigma,$$

for some constant $C = C(\mathcal{B}) > 0$, where $\mathcal{B}$ is the absorbing set given in Lemma 3.4.

Proof The proof is analogous to the proof of [3, Lemma 5.2]. □

Theorem 5.5 *The uniform attractor $\mathcal{A}_\Sigma$ (in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$) of system (3.3) is bounded in $X^1 \times X^1$.*

Proof Let us denote $D = \mathcal{A}_\Sigma$. If $x \in D = \mathcal{A}_\Sigma = \omega_{0,\Sigma}(\mathcal{B})$ is arbitrary, then there are sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$, $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$ such that $\lim_{n\to\infty} S_{\sigma_n}(t_n,0)x_n = x$ with $\lim_{n\to\infty} t_n = \infty$. Using the fact that $S_{\sigma_n}(t_n,0)x_n = L_{\sigma_n}(t_n,0)x_n + U_{\sigma_n}(t_n,0)x_n$, it follows by Lemma 5.1 that

$$\lim_{n\to\infty} \left\| U_{\sigma_n}(t_n,0)x_n - x \right\|_{X^{\frac{1}{2}} \times X^{\frac{1}{2}}} = 0. \tag{5.1}$$

According to Lemma 5.4,

$$\left\| U_{\sigma_n}(t_n,0)x_n \right\|_{X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}}} \leqslant C,$$

for all $n \in \mathbb{N}$, with constant $C = C(\mathcal{B}) > 0$ independent of x. As $X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}}$ is reflexive we may assume without loss of generality that

$$U_{\sigma_n}(t_n,0)x_n \xrightarrow{w} y \text{ in } X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}},$$

for some $y \in X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}}$. Since the embedding $X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}} \hookrightarrow X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ is continuous in the strong topology, if follows that (see [9, Theorem 3.10])

$$U_{\sigma_n}(t_n,0)x_n \xrightarrow{w} y \text{ in } X^{\frac{1}{2}} \times X^{\frac{1}{2}},$$

and from (5.1), $U_{\sigma_n}(t_n,0)x_n \xrightarrow{w} x$ in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$, that is, $y = x$. Consequently,

$$\|x\|_{X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}}} \leqslant \liminf_{n\to\infty} \left\| U_{\sigma_n}(t_n,0)x_n \right\|_{X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}}} \leqslant C,$$

and, therefore, $\mathcal{A}_\Sigma$ is bounded in $X^{\frac{1+\widetilde{\sigma}}{2}} \times X^{\frac{1+\widetilde{\sigma}}{2}}$.

Note that if $\widetilde{\sigma} = 1$ then the result is proved. Now let us suppose that $\widetilde{\sigma} < 1$ and set $r_0 := \widetilde{\sigma}$ and $r_1 := \min\{1, (\frac{N+2}{N-2})r_0\}$. By condition (1.4) and the continuous embeddings

$$X^{\frac{1+r_0}{2}} \hookrightarrow L^{\frac{2N\rho}{N+2(1-r_1)}}(\Omega) \hookrightarrow L^{\frac{2N}{N+2(1-r_1)}}(\Omega) \hookrightarrow X^{\frac{r_1-1}{2}},$$

we obtain for all $\phi \in X^{\frac{1+r_0}{2}}$ that (denote $r' := \frac{2N}{N+2(1-r_1)}$)

$$\begin{aligned}\|\widetilde{f}^e(\phi)\|_{X^{\frac{r_1-1}{2}}} &\leqslant c\|f(\phi)\|_{L^{r'}(\Omega)} \leqslant c\big\|1+|\phi|^\rho\big\|_{L^{r'}(\Omega)}\\ &\leqslant c\Big(1+\|\phi\|^\rho_{L^{r'\rho}(\Omega)}\Big) \leqslant c\Big(1+\|\phi\|^\rho_{X^{\frac{1+r_0}{2}}}\Big),\end{aligned} \tag{5.2}$$

for some constant $c > 0$.

Let $x \in D = \mathcal{A}_\Sigma$. As done in the first case, we have $x = \lim_{n\to\infty} S_{\sigma_n}(t_n, 0)x_n$ for some $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$, $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\lim_{n\to\infty} t_n = \infty$. We know that

$$\lim_{n\to\infty} \big\|U_{\sigma_n}(t_n, 0)x_n - x\big\|_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}} = 0.$$

Now, by Lemma 2.12, Proposition 3.2 and relation (5.2), we obtain

$$\begin{aligned}&\Big\|U_{\sigma_n}(t_n,0)x_n\Big\|_{X^{\frac{1+r_1}{2}}\times X^{\frac{1+r_1}{2}}}\\ &= \Big\|\Phi_{-1-\frac{r_1}{2}}\circ\Phi_1\int_0^{t_n} L_{-1/2,\sigma_n}(t_n,\tau)\mathcal{F}_{\sigma_n}\Big(\tau,\Phi_{-1}S_{\sigma_n}(\tau,0)x_n\Big)d\tau\Big\|_{X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}}\\ &\leqslant \int_0^{t_n} Ke^{-\alpha(t_n-\tau)}\Big(\big\|\widetilde{f}^e\big(u^{\sigma_n}(\tau,0,x_n)\big)\big\|_{X^{\frac{r_1-1}{2}}} + \|g_n(\tau)\|_{X^{\frac{r_1-1}{2}}}\Big)d\tau\\ &\leqslant \int_0^{t_n} ce^{-\alpha(t_n-\tau)}\Big(1+\|u^{\sigma_n}(\tau,0,x_n)\|^\rho_{X^{\frac{1+r_0}{2}}}\Big)d\tau + c\int_0^{t_n} e^{-\alpha(t_n-\tau)}\|g_n(\tau)\|_{L^2(\Omega)}d\tau\\ &\leqslant \int_0^{t_n} ce^{-\alpha(t_n-\tau)}\Big(1+\|u^{\sigma_n}(\tau,0,x_n)\|^\rho_{X^{\frac{1+r_0}{2}}}\Big)d\tau + \frac{c}{(1-e^{-\alpha})}\|g\|^2_{W_b^{1,2}(\mathbb{R},L^2(\Omega))},\end{aligned}$$

and since by Remark 5.2 we conclude that the set $\gamma_\Sigma(D,0) = \bigcup_{\sigma\in\Sigma}\gamma_\Sigma(D,0)$ is bounded in $X^{\frac{1+r_0}{2}} \times X^{\frac{1+r_0}{2}}$, then

$$\big\|u^{\sigma_n}(\tau,0,x_n)\big\|^\rho_{X^{\frac{1+r_0}{2}}} \leqslant \sup_{\tilde{u}\in\gamma_{\sigma_n}(D,0)} \|\tilde{u}\|^\rho_{X^{\frac{1+r_0}{2}}\times X^{\frac{1+r_0}{2}}} \leqslant \sup_{\tilde{u}\in\gamma_\Sigma(D,0)} \|\tilde{u}\|^\rho_{X^{\frac{1+r_0}{2}}\times X^{\frac{1+r_0}{2}}} \leqslant C_1,$$

and this yields

$$\Big\|U_{\sigma_n}(t_n,0)x_n\Big\|_{X^{\frac{1+r_1}{2}}\times X^{\frac{1+r_1}{2}}} \leqslant C_2, \qquad \text{for all } \; n \in \mathbb{N},$$

for some constant $C_2 = C_2(\mathcal{B}) > 0$ (independent of n). Since $X^{\frac{1+r_1}{2}} \times X^{\frac{1+r_1}{2}}$ is reflexive, we may assume without loss of generality that

$$U_{\sigma_n}(t_n, 0)x_n \xrightarrow{w} y \text{ in } X^{\frac{1+r_1}{2}} \times X^{\frac{1+r_1}{2}},$$

for some $y \in X^{\frac{1+r_1}{2}} \times X^{\frac{1+r_1}{2}}$. Since the embedding $X^{\frac{1+r_1}{2}} \times X^{\frac{1+r_1}{2}} \hookrightarrow X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ is continuous, we have

$$U_{\sigma_n}(t_n, 0)x_n \xrightarrow{w} y \text{ in } X^{\frac{1}{2}} \times X^{\frac{1}{2}},$$

and then $x = y$. Finally,

$$\|x\|_{X^{\frac{1+r_1}{2}} \times X^{\frac{1+r_1}{2}}} \leqslant \liminf_{n\to\infty} \left\| U_{\sigma_n}(t_n, 0)x_n \right\|_{X^{\frac{1+r_1}{2}} \times X^{\frac{1+r_1}{2}}} \leqslant C_2,$$

proving that $\mathcal{A}_\Sigma$ is bounded in $X^{\frac{1+r_1}{2}} \times X^{\frac{1+r_1}{2}}$.

If $r_1 = \left(\frac{N+2}{N-2}\right) r_0 < 1$, then we continue with the previous process and we conclude that

$$\mathcal{A}_\Sigma \text{ is bounded in } X^{\frac{1+r_2}{2}} \times X^{\frac{1+r_2}{2}},$$

where $r_2 := \min\left\{1, (\frac{N+2}{N-2})^2 r_0\right\}$. Following the same steps and after a finite number of them we reach $r_k := \min\left\{1, \left(\frac{N+2}{N-2}\right)^k r_0\right\} = 1$ for some $k \in \mathbb{N}$ (that is, $(\frac{N+2}{N-2})^k r_0 \geqslant 1$), which concludes this proof. $\square$

6 Characterization of the uniform attractor

Let us now assume that Σ is compact in Ξ in order to apply the Theorem A.14 and obtain a characterization of the uniform attractor, as described in the cited result. For that we may suppose for instance that $\{g_h : h \in \mathbb{R}\}$ is precompact in Ξ_2 (in other words, g is translation compact in Ξ_2). Then, by [14, Chapter V, Section 5], Σ is compact in Ξ. Let us present some auxiliary results.

Lemma 6.1 *The family of processes $\{S_\sigma(t, s)\}_{\sigma\in\Sigma}$ is $\left(\Sigma \times X^{\frac{1}{2}} \times X^{\frac{1}{2}}, X^{\frac{1}{2}} \times X^{\frac{1}{2}}\right)$–continuous.*

Proof Let u^i be a solution of system (3.3) with symbols $\sigma_i = (\eta_i, g_i) \in \Sigma$, and initial condition (u_0^i, v_0^i), $i = 1, 2$. Making a variable change $z^i := (I - \Delta)u^i$, where z^i is a solution of (3.4) with symbol σ_i and initial condition (z_0^i, w_0^i), $i = 1, 2$. The difference

$z = z^1 - z^2$ satisfies the system

$$\begin{cases} z_{tt} + \eta_1(t)\Lambda z + (\eta_1(t) - \eta_2(t))\Lambda z_t^2 + \Lambda z = f^e(z^1) - f^e(z^2) \\ +g_1(t) - g_2(t), t > s, x \in \Omega, \\ z = 0, \quad t \geqslant s, \quad x \in \partial\Omega, \\ z(s) = z_0^1 - z_0^2 \quad \text{and} \quad z_t(s) = w_0^1 - w_0^2. \end{cases} \tag{6.1}$$

Taking the inner product of (6.1) with z_t in $X^{-\frac{1}{2}}$, we obtain from (3.10) the estimate

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}} + \|z\|^2_{X^{-\frac{1}{2}}} - \|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right)\right] \\ &\quad \leqslant \left\langle f^e(z^1) - f^e(z^2), z_t\right\rangle_{X^{-\frac{1}{2}}} + \big(\eta_2(t) - \eta_1(t)\big)\left\langle \Lambda z_t^2, z_t\right\rangle_{X^{-\frac{1}{2}}} \\ &\qquad + \langle g_1(t) - g_2(t), z_t\rangle_{X^{-\frac{1}{2}}}. \end{aligned} \tag{6.2}$$

Using Lemma 3.5, the Young inequality, and the continuous embeddings $X^{\frac{1}{2}} \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ and $X^{-\frac{1}{2}} \hookrightarrow X^{\frac{-1-\gamma}{2}}$, we obtain

$$\begin{aligned} \left\langle f^e(z^1) - f^e(z^2), z_t\right\rangle_{X^{-\frac{1}{2}}} &= \int_\Omega \left[f^e(z^1) - f^e(z^2)\right] A^{-1} z_t \, dx \\ &\leqslant \left\| f(\tilde{A}^{-1}z^1) - f(\tilde{A}^{-1}z^2)\right\|_{L^{\frac{2N}{N+2}}(\Omega)} \|\tilde{A}^{-1}z_t\|_{L^{\frac{2N}{N-2}}(\Omega)} \\ &\leqslant c_0 \left\|A^{-1}z^1 - A^{-1}z^2\right\|_{H^{1-\gamma}(\Omega)} \|\tilde{A}^{-1}z_t\|_{X^{\frac{1}{2}}} \\ &\leqslant c_0 \left\|z\right\|_{X^{\frac{-1-\gamma}{2}}} \|z_t\|_{X^{-\frac{1}{2}}} \\ &\leqslant c_0 \|z\|^2_{X^{-\frac{1}{2}}} + c_0 \|z_t\|^2_{X^{-\frac{1}{2}}}, \end{aligned}$$

for some constant $c_0 > 0$.

By Lemma 3.4, the solution z_t^2, z_t are bounded in $X^{-\frac{1}{2}}$ and then

$$\begin{aligned} \big(\eta_2(t) - \eta_1(t)\big)\langle \Lambda z_t^2, z_t\rangle_{X^{-\frac{1}{2}}} &\leqslant |\eta_2(t) - \eta_1(t)| \|\Lambda z_t^2\|_{X^{-\frac{1}{2}}} \|z_t\|_{X^{-\frac{1}{2}}} \\ &\leqslant c_1 |\eta_2(t) - \eta_1(t)|, \end{aligned}$$

for some constant $c_1 > 0$.

Additionally, by Schwarz inequality, Young inequality, and the embedding $X^0 \hookrightarrow X^{-\frac{1}{2}}$,

$$\langle g_1(t) - g_2(t), z_t\rangle_{X^{-\frac{1}{2}}} \leqslant c_2 \|g_1(t) - g_2(t)\|^2_{X^0} + c\|z_t\|^2_{X^{-\frac{1}{2}}},$$

for some constant $c_2 > 0$.

Replacing the previous estimates in (6.2) and using Lemma 2.8, we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}}+\|z\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right)\right]\\
&\quad\leqslant c\left[\frac{1}{2}\left(\|z_t\|^2_{X^{-\frac{1}{2}}}+\|z\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z\|^2_{X^{-\frac{1}{2}}}\right)\right]\\
&\qquad+c|\eta_1(t)-\eta_2(t)|+c\|g_1(t)-g_2(t)\|^2_{L^2(\Omega)},
\end{aligned}
$$

for some constant $c>0$. Consequently, we have

$$
\begin{aligned}
&\|z_t(t)\|^2_{X^{-\frac{1}{2}}}+\|z(t)\|^2_{X^{-\frac{1}{2}}}\\
&\quad\leqslant \tilde{c}(\|z_t(t)\|^2_{X^{-\frac{1}{2}}}+\|z(t)\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z(t)\|^2_{X^{-\frac{1}{2}}})\\
&\quad\leqslant \tilde{\tilde{c}}\bigg(\|z_0^1-z_0^2\|^2_{X^{-\frac{1}{2}}}+\|w_0^1-w_0^2\|^2_{X^{-\frac{1}{2}}}-\|\tilde{A}^{-\frac{1}{2}}z_0^1-\tilde{A}^{-\frac{1}{2}}z_0^2\|^2_{X^{-\frac{1}{2}}}\\
&\qquad+\int_s^t\|g_1(r)-g_2(r)\|^2_{L^2(\Omega)}dr+\int_s^t|\eta_1(r)-\eta_2(r)|dr\bigg)e^{C(t-s)}\\
&\quad\leqslant \tilde{\tilde{c}}\bigg(\|z_0^1-z_0^2\|^2_{X^{-\frac{1}{2}}}+\|w_0^1-w_0^2\|^2_{X^{-\frac{1}{2}}}+\int_s^t\|g_1(r)-g_2(r)\|^2_{L^2(\Omega)}dr\\
&\qquad+\int_s^t|\eta_1(r)-\eta_2(r)|dr\bigg)e^{C(t-s)},
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\|u^1(t)-u^2(t)\|^2_{X^{\frac{1}{2}}}+\|u_t^1(t)-u_t^2(t)\|^2_{X^{\frac{1}{2}}}\leqslant\\
&\quad\leqslant \tilde{\tilde{c}}\bigg(\|u_0^1-u_0^2\|^2_{X^{\frac{1}{2}}}+\|v_0^1-v_0^2\|^2_{X^{\frac{1}{2}}}+\int_s^t\|g_1(r)-g_2(r)\|^2_{L^2(\Omega)}dr\\
&\qquad+\int_s^t|\eta_1(r)-\eta_2(r)|dr\bigg)e^{C(t-s)}.
\end{aligned}
$$

Hence, for any fixed t and s, $t\geqslant s$, $s\in\mathbb{R}$, if

$$
\eta_n\xrightarrow[n\to\infty]{\Xi_1}\eta,\qquad g_n\xrightarrow[n\to\infty]{\Xi_2}g\qquad\text{and}\qquad\lim_{n\to\infty}\|(u_0^n,v_0^n)-(u_0,v_0)\|_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}=0,
$$

then

$$
\lim_{n\to\infty}\left\|(u^n(t),u_t^n(t))-(u(t),u_t(t))\right\|_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}=0,
$$

and the proof is complete. $\square$

Theorem 6.2 (Characterization of the uniform attractor) *The family of processes* $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ *associated with system* (3.3) *admits a uniform attractor* $\mathcal{A}_\Sigma$ *in* $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ *given by*

$$\mathcal{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau), \quad \textit{for all} \quad \tau \in \mathbb{R},$$

where $\mathcal{K}_\sigma(\tau)$ *is the kernel section at* τ *of process* $S_\sigma(t,s)$ *with symbol* σ.

Proof It follows by Theorem A.14. □

Corollary 6.3 *The skew product flow associated with the problem* (3.3) *is a semigroup with a global attractor and the conclusions of Theorem* A.17 *are valid.*

A Appendix

In this appendix, we bring some fundamental definitions and concepts related to the theory of evolution processes and uniform attractors. Additionally, we provide a result concerning the existence and uniqueness of solutions.

A.1 Uniform attractors for systems of evolution processes

Let (Ξ, d_Ξ) be a complete metric space and $\{\theta_s\}_{s\in\mathbb{R}}$ be a group of continuous operators acting on Ξ, that is, $\theta_0\sigma = \sigma$ and $\theta_t(\theta_s\sigma) = \theta_{t+s}\sigma$ for all $\sigma \in \Xi$, $t, s \in \mathbb{R}$, and for each $s \in \mathbb{R}$, $\theta_s : \Xi \longrightarrow \Xi$ is a continuous map in Ξ. Let $\Sigma \subseteq \Xi$ be a complete subset of Ξ which is invariant under $\{\theta_s\}_{s\in\mathbb{R}}$, that is, $\theta_s\Sigma = \Sigma$ for all $s \in \mathbb{R}$.

Remark A.1 In applications, the family of operators $\{\theta_s\}_{s\in\mathbb{R}}$ is typically defined as the translations

$$\theta_s\sigma(\cdot) = \sigma(\cdot + s), \quad \text{for all} \quad s \in \mathbb{R},$$

for time-dependent functions σ. Consequently, these are often referred to as *translation operators*.

For a given Banach space $(Y, \|\cdot\|_Y)$ and for each $\sigma \in \Sigma$, let $\{S_\sigma(t,s) : t, s \in \mathbb{R},\ t \geqslant s\}$ be an evolution process in Y, namely,

$$\begin{aligned} S_\sigma(t,\tau)\, S_\sigma(\tau,s) = S_\sigma(t,s), &\quad \text{for all} \quad t,\tau,s \in \mathbb{R}, \quad t \geqslant \tau \geqslant s, \\ S_\sigma(s,s) = Id_Y, &\quad \text{for all} \quad s \in \mathbb{R}, \end{aligned}$$

where $Id_Y : Y \longrightarrow Y$ denotes the identity map on Y. For simplicity, we will denote $\{S_\sigma(t,s) : t, s \in \mathbb{R},\ t \geqslant s\}$ simply by $\{S_\sigma(t,s)\}$.

Definition A.2 A family $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ of evolution processes is called a *system of evolution processes* (or *system* for short), if the following *translation-identity* is satisfied:

$$S_{\theta_h\sigma}(t,s) = S_\sigma(t+h, s+h), \quad \text{for all } \sigma \in \Sigma,\ t \geqslant s,\ t, s, h \in \mathbb{R}. \tag{A.1}$$

In this case, the parameter σ is called the *symbol* of the process $\{S_\sigma(t,s)\}$ and the set Σ is called the *symbol space* of the system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$.

Remark A.3 The translation-identity (A.1) is satisfied, for instance, provided the underlying nonautonomous evolution equation has a unique solution.

Let $\sigma \in \Sigma$ be given. A map $u\colon \mathbb{R} \longrightarrow Y$ is said to be a *complete trajectory* for the process $\{S_\sigma(t,s)\}$ if $S_\sigma(t,s)u(s) = u(t)$ for all $t \geqslant s$, with $t, s \in \mathbb{R}$. The *kernel* $\mathcal{K}_\sigma$ of $\{S_\sigma(t,s)\}$ is defined by

$$\mathcal{K}_\sigma = \Big\{u(\cdot) : u(\cdot) \text{ is a bounded complete trajectory for } S_\sigma(t,s)\Big\}.$$

On the other hand, for each $s \in \mathbb{R}$, the set

$$\mathcal{K}_\sigma(s) = \big\{u(s) : u(\cdot) \in \mathcal{K}_\sigma\big\}$$

stands for the *kernel section at moment* s. Clearly, kernel sections satisfy the invariance property

$$S_\sigma(t,s)\mathcal{K}_\sigma(s) = \mathcal{K}_\sigma(t), \quad \text{for all } t \geqslant s,\ t, s \in \mathbb{R}.$$

By $\mathscr{B}(Z)$, we denote the collection of all bounded subsets of a Banach space Z. Next, we present the definition of uniform attractors for systems of processes.

Definition A.4 A compact set $\mathcal{A}_\Sigma$ in Y is said to be a *uniform* (*w.r.t.* $\sigma \in \Sigma$) *attractor* of a system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ of evolution processes in Y, if:

(i) $\mathcal{A}_\Sigma$ is uniformly attracting, that is, for any $s \in \mathbb{R}$ and for any $D \in \mathscr{B}(Y)$ it holds

$$\lim_{t\to\infty}\left[\sup_{\sigma\in\Sigma} \operatorname{dist}_Y\big(S_\sigma(t,s)D, \mathcal{A}_\Sigma\big)\right] = 0, \tag{A.2}$$

where $\operatorname{dist}_Y(\cdot,\cdot)$ is the usual Hausdorff semidistance in Y;

(ii) (Minimality) if $\mathcal{A}'_\Sigma$ is a closed set in Y uniformly attracting, then $\mathcal{A}_\Sigma \subset \mathcal{A}'_\Sigma$.

Remark A.5 Since the symbol space Σ is invariant under translations, it follows by the translation-identity (A.1) that

$$\begin{aligned}\sup_{\sigma\in\Sigma} \operatorname{dist}_Y\big(S_\sigma(t,0)D, \mathcal{A}_\Sigma\big) &= \sup_{\sigma\in\Sigma} \operatorname{dist}_Y\big(S_{\theta_\tau\sigma}(t,0)D, \mathcal{A}_\Sigma\big)\\ &= \sup_{\sigma\in\Sigma} \operatorname{dist}_Y\big(S_\sigma(t+\tau,\tau)D, \mathcal{A}_\Sigma\big),\end{aligned}$$

for all $\tau \in \mathbb{R}$. Consequently, the uniform attracting property (A.2) is equivalent to

$$\lim_{t\to\infty}\left[\sup_{\sigma\in\Sigma}\operatorname{dist}_Y\big(S_\sigma(t,0)D,\mathcal{A}_\Sigma\big)\right]=0,$$

which implies that, under the conditions considered involving limits, the initial time can be set to $s = 0$.

In the sequel, we exhibit sufficient conditions to ensure the existence of uniform attractors. Before that, we present some definitions and auxiliary results.

Definition A.6 A set $\mathcal{B} \subset Y$ is called a *uniformly absorbing set* for a system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ in Y if for any $\tau \in \mathbb{R}$ and any $D \in \mathscr{B}(Y)$, there exists a time $t_0 = t_0(\tau, D) \geqslant \tau$ such that

$$\bigcup_{\sigma\in\Sigma} S_\sigma(t,\tau)D \subseteq \mathcal{B}, \quad \text{for all } \ t \geqslant t_0.$$

Definition A.7 A system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ in Y is said to be *uniformly* (*w.r.t.* $\sigma \in \Sigma$) *asymptotically compact*, if for any fixed $s \in \mathbb{R}$, any sequence $\{t_n\}_{n\in\mathbb{N}} \subset [s,\infty)$ with $t_n \to \infty$, and any bounded sequences $\{u_n\}_{n\in\mathbb{N}} \subset Y$ and $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$, then the sequence $\left\{S_{\sigma_n}(t_n,s)u_n\right\}_{n\in\mathbb{N}}$ has a convergent subsequence in Y.

The Definition A.8 in the following concerns the concept of contractive functions, which is an important tool for providing sufficient conditions for a system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ in Y to be uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact.

Definition A.8 Let $B \in \mathscr{B}(Y)$. A function $\Psi(\cdot,\cdot,\cdot,\cdot)$ defined on $Y \times Y \times \Sigma \times \Sigma$ is called a *contractive function* on $B \times B$, if for any sequences $\{x_n\}_{n\in\mathbb{N}} \subset B$ and $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$, there are subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{\sigma_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\Psi(x_{n_k},x_{n_l},\sigma_{n_k},\sigma_{n_l})=0.$$

The set of all contractive functions on $B \times B$ is denoted by $\mathrm{Contr}(B,\Sigma)$.

Lemma A.9 [25, Theorem 4.2] *Let $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system in Y which admits a bounded uniformly (w.r.t $\sigma \in \Sigma$) absorbing set $\mathcal{B} \subset Y$. Moreover, assume that for every $\epsilon > 0$, there exist $T = T(\mathcal{B},\epsilon) > 0$ and $\Psi_T \in$ Contr$(\mathcal{B},\Sigma)$ such that*

$$\|S_{\sigma_1}(T,0)x - S_{\sigma_2}(T,0)y\|_Y \leqslant \epsilon + \Phi_T(x,y,\sigma_1,\sigma_2),$$

for all $x, y \in \mathcal{B}$ and all $\sigma_1,\sigma_2 \in \Sigma$. Then $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact in Y.

Definition A.10 The *uniform ω-limit set* of a subset $D \subset Y$ at initial time $\tau \in \mathbb{R}$ is represented by

$$\omega_{\tau,\Sigma}(D)=\bigcap_{t\geqslant\tau}\overline{\bigcup_{\sigma\in\Sigma}\bigcup_{r\geqslant t}S_\sigma(r,\tau)D}.$$

For a given $D \in \mathscr{B}(Y)$, it follows by [25, Lemma 3.2] that $y \in \omega_{\tau,\Sigma}(D)$, if and only if there exist sequences $\{x_n\}_{n\in\mathbb{N}} \subset D$, $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$, $\{t_n\}_{n\in\mathbb{N}} \subset [\tau,\infty)$ with $\lim_{n\to\infty} t_n = \infty$, such that $\lim_{n\to\infty} S_{\sigma_n}(t_n,\tau)x_n = y$.

In the next result, we give necessary and sufficient conditions for a system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ to admit a uniform attractor.

Theorem A.11 [25, Theorem 3.4] *A system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ in Y is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact and has a bounded uniformly absorbing set $\mathcal{B}$, if and only if it admits a compact uniform attractor $\mathcal{A}_\Sigma$ given by*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(\mathcal{B}) = \omega_{\tau,\Sigma}(\mathcal{B}) = \bigcup_{D\in\mathscr{B}(Y)} \omega_{\tau,\Sigma}(D),$$

for all $\tau \in \mathbb{R}$.

Remark A.12 If a system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ admits a uniform attractor $\mathcal{A}_\Sigma$, then any neighborhood $\mathcal{B}$ of $\mathcal{A}_\Sigma$ is a uniformly absorbing set, that is, for any $\tau \in \mathbb{R}$ and any $D \in \mathscr{B}(Y)$, there exists a time $t_0 = t_0(\tau, D) \geqslant \tau$ such that

$$\bigcup_{\sigma\in\Sigma} S_\sigma(t,\tau)D \subseteq \mathcal{B}, \quad \text{for all} \quad t \geqslant t_0.$$

Definition A.13 A system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ in Y is called $(\Sigma \times Y, Y)$-*continuous*, if for each $t, s \in \mathbb{R}$ with $t \geqslant s$ the mapping $\Sigma \times Y \ni (\sigma, x) \longmapsto S_\sigma(t,s)x \in Y$ is continuous.

Under the $(\Sigma \times Y, Y)$-continuity and the compactness of the symbol space Σ, we may characterize the uniform attractor by kernel sections.

Theorem A.14 [25, Theorem 3.8] *Let Y be a Banach space and Σ be a compact metric space. Assume that a family of processes $\{S_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, satisfies the translation identity (A.1), as well as the following conditions:*

(i) *The translation semigroup $\{\theta_t\}_{t\geqslant 0}$ is continuous on Σ;*
(ii) *$\{S_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is norm-to-weak continuous on Y;*
(iii) *$\{S_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0 in Y;*
(iv) *$\{S_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is uniform (w.r.t. $\sigma \in \Sigma$) asymptotically compact in Y.*

Then, $\{S_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, has a uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathscr{A}_\Sigma$ satisfying

$$\mathscr{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(s), \quad \text{for all} \ \ s \in \mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ is the section at $t = s$ of the kernel $\mathcal{K}_\sigma$ of the process $\{S_\sigma(t,\tau)\}$ with symbol σ.

The characterization of a uniform attractor of a $(\Sigma \times Y, Y)$-continuous system via kernel sections allows us to obtain a lifted negative semi-invariance

$$\mathcal{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(t) = \bigcup_{\sigma\in\Sigma} S_\sigma(t,0)\mathcal{K}_\sigma(0) \subset \bigcup_{\sigma\in\Sigma} S_\sigma(t,0)\mathcal{A}_\Sigma, \quad \text{for all } \ t \geqslant 0.$$

A.2 Skew product flow: global attractors and uniform attractors

Let $(Y, \|\cdot\|_Y)$ be a Banach space, Σ be a compact symbol space with $\{\theta_s\}_{s\in\mathbb{R}}$ as being a group of continuous operators acting on Σ, and let $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system in Y.

Definition A.15 A *skew product flow* associated with the system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ is a family of maps $\{T(t)\colon Y \times \Sigma \to Y \times \Sigma,\ t \geqslant 0\}$ defined by

$$T(t)(y,\sigma) = (S_\sigma(t,0)y, \theta_t\sigma),$$

for every $(y,\sigma) \in Y \times \Sigma$ and $t \geqslant 0$.

Remark A.16 If a system $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ is $(Y \times \Sigma, Y)$-continuous, then the associated skew product flow is a semigroup.

Theorem A.17 [14, Theorem 5.1] *Let $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$ be uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact system which is $(\Sigma \times Y, Y)$-continuous. The following properties hold:*

(i) *The skew product $\{T(t)\colon Y \times \Sigma \to Y \times \Sigma,\ t \geqslant 0\}$ has a global attractor. Moreover, this global attractor $\mathcal{A}$ satisfies*

$$\mathcal{A} = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau) \times \{\sigma\} \quad \textit{for all } \tau \in \mathbb{R}.$$

(ii) *The projections Π_Y and Π_Σ defined by*

$$\begin{aligned} \Pi_\Sigma \ Y \times \Sigma &\longrightarrow \qquad \Sigma \\ (u,\sigma) &\longmapsto \Pi_\Sigma(u,\sigma) = \sigma \end{aligned}$$

and

$$\begin{aligned} \Pi_Y \ Y \times \Sigma &\longrightarrow \qquad Y \\ (u,\sigma) &\longmapsto \Pi_Y(u,\sigma) = u \end{aligned}$$

satisfy $\Pi_\Sigma(\mathcal{A}) = \Sigma$ and $\Pi_Y(\mathcal{A}) = \mathcal{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau)$, where $\mathcal{A}_\Sigma$ is the uniform attractor of $\{S_\sigma(t,s)\}_{\sigma\in\Sigma}$.

A.3 Existence and uniqueness of solutions

This last appendix section concerns with an existence and uniqueness result that will be applied to solve problem (1.1). Let us consider $(Y, \|\cdot\|_Y)$ a Banach space.

Definition A.18 A map $F\colon \mathbb{R}\times Y \longrightarrow Y$ is called *locally Lipschitz continuous* in Y uniformly in t on bounded intervals, if for any $B\in\mathscr{B}(Y)$ and any interval $I=[t_1,t_2]\subset\mathbb{R}$ there exists $M=M(B,I)>0$ such that

$$\big\|F(t,x)-F(t,y)\big\|_Y \leqslant M\|x-y\|_Y, \qquad \text{for all } \; x,y\in B,\; t\in I.$$

Definition A.19 A map $F\colon \mathbb{R}\times Y \longrightarrow Y$ is called *locally bounded* in Y uniformly in t on bounded intervals, if for any $B\in\mathscr{B}(Y)$ and any interval $I=[t_1,t_2]\subset\mathbb{R}$, there exists $M=M(B,I)>0$ such that

$$\big\|F(t,x)\big\|_Y \leqslant M, \qquad \text{for all } \; x\in B,\; t\in I.$$

Theorem A.20 (Existence and uniqueness of solutions) *Let $F\in\mathcal{C}(\mathbb{R}\times Y, Y)$ and $Q(t)\colon Y\longrightarrow Y$ be a bounded linear operator defined for all $t\in\mathbb{R}$. If $\mathbb{R}\ni t\longmapsto Q(t)\in\mathcal{L}(Y)$ is continuous in the uniform operator topology and F is locally Lipschitz continuous in Y and locally bounded in Y (both uniformly in t on bounded intervals), then for any $s\in\mathbb{R}$ and any $y_0\in Y$ there exists $T^{max}=T^{max}(y_0,s)>s$ such that the initial value problem*

$$\begin{cases} \dfrac{dy}{dt}+Q(t)y=F(t,y), & t>s,\\ y(s)=y_0, \end{cases}$$

admits a unique solution $y(\cdot)=y(\cdot,s;y_0)\in\mathcal{C}\big([s,T^{max}),Y\big)\cap\mathcal{C}^1\big((s,T^{max}),Y\big)$ which satisfies in Y the variation of constants formula

$$y(t,s;y_0)=L(t,s)y_0+\int_s^t L(t,\tau)F\big(\tau,y(\tau,s;y_0)\big)\,d\tau,$$

where $L(t,s)=I-\int_s^t Q(\tau)L(\tau,s)\,d\tau$, $t\geqslant s$. Moreover, either $T^{max}=\infty$ or $\lim\limits_{t\to(T^{max})^-}\|y(t)\|_Y=\infty$.

Proof The proof of the theorem is adapted from [3, Theorem 2.6]. For further details on this, we refer the reader to [2, 21, 23]. □

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Declarations

Conflict of interest The authors report that there are no conflict of interest to declare.

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