Locating-dominating sets on infinite grids with finite height

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Abstract. *A locating-dominating set of a graph G is a dominating set C of G such that, for each pair of distinct vertices u and v not in C, the neighborhoods of u and v in C are distinct. We focus on locating-dominating sets of minimum density in the infinite square, triangular and king grids with finite height. Optimal results on these grids were known only for height up to 3. We extend these results showing optimal solutions with density $1/3$ for square grids with height 4, 5 and 6, and show how locating-dominating sets with density $1/3$ can be obtained for any square grid with finite height. We also show optimal solutions for the infinite triangular and king grids with heights 4 and 5.*

1. Introduction

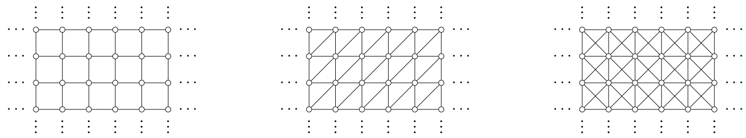
In this work, we are concerned with locating-dominating sets of connected infinite graphs. This concept was first introduced by [Slater 1975] with the motivation of detecting faulty processors in a network. Other interesting applications are also known. A *locating-dominating set* (for short, LDS) of a connected graph $G = (V, E)$ is a set $C \subseteq V$ such that each vertex v not in C has a neighbor in C (that is, C is a dominating set of G) and for each pair of distinct vertices u, v not in C, we have that $N(u) \cap C \neq N(v) \cap C$, where $N(v)$ is the (open) neighborhood of vertex v. It is immediate that every graph has an LDS, as the vertex set of the graph is such a set. However, finding an LDS of minimum cardinality is an NP-hard problem, even for very specific graph classes such as bipartite [Charon et al. 2003], interval and intersection graphs [Foucaud et al. 2016].

For infinite graphs, instead of cardinality of an LDS, we use the concept of *density* of an LDS that captures the measure of the "ratio" of the elements in the LDS with respect to the whole graph. This parameter is defined in the next section.

Locating-dominating sets have been studied both on finite and infinite graphs. A well-updated and comprehensive source on this topic and related ones is the bibliography maintained by [Jean 2024]. For infinite graphs, studies have focused mostly on square, triangular, king and hexagonal grids (see Figure 1), all of which have vertex set $\mathbb{Z} \times \mathbb{Z}$, and are regular. For these grids, LDS of minimum density have already been obtained [Slater 2002, Honkala 2006, Honkala and Laihonen 2006]. In this work we are interested in a special subclass of these infinite grids, all of which are infinite, but have a finite height (or equivalently, a finite number of rows).

We denote by S_k (resp. T_k and K_k) the *square* (resp. *triangular* and *king*) grid with k rows. These grids are subgraphs of their corresponding infinite grids, and have vertex set $[k] \times \mathbb{Z}$, where $[k] = \{1, \ldots, k\}$. While optimal solutions are known for the unrestricted infinite case, for S_k, T_k and K_k optimal solutions have been found only for $k \leq 3$ [Bouzinif et al. 2019]. We extend these results showing optimal solutions for S_4,

S_5 and S_6; and for $k \geq 7$ we prove that $1/3$ is an upper bound for the minimum density of an LDS in S_k. For the grids T_k and K_k, we show optimal solutions when $k = 4$ and $k = 5$.



(a) Square grid $\mathcal{G}_S$ $d^*(\mathcal{G}_S) = 3/10$ **(b)** Triangular grid $\mathcal{G}_T$ $d^*(\mathcal{G}_T) = 13/57$ **(c)** King grid $\mathcal{G}_K$ $d^*(\mathcal{G}_K) = 1/5$

Figure 1. Square, triangular and king grids

2. Locating-dominating sets of minimum density on grids with finite height

We define first the concept of density of an LDS, and establish the notation we shall use.

Let G be an infinite graph. If v is a vertex of G and $r \geq 1$ is a natural number, then the *r-open neighborhood* of v is defined as $N_r(v) := \{w \in V : \text{dist}(v, w) \leq r\}$, where $\text{dist}(v, w)$ denotes the distance between vertices v and w in G. The *density of an* LDS C in G, denoted $d(C, G)$, is defined as $d(C, G) := \inf\{d_w(C, G) : w \in V\}$, where $d_w(C, G) := \limsup_{r\to\infty} \frac{|C\cap N_r(w)|}{|N_r(w)|}$. The *minimum density* of an LDS in G, denoted $d^*(G)$, is defined as $d^*(G) := \inf\{d(C, G) : C \text{ is an LDS of } G\}$.

In this section we describe the algorithm we have implemented to find a minimum density LDS in the grids S_k, T_k and K_k. This algorithm is based on the notion of a *configuration digraph* and was first introduced by [Jiang 2018] for the identifying code problem in S_k. Later, [Bouzinif et al. 2019] have used this approach for both the identifying code and LDS problems in S_k, T_k and K_k.

As the size of the configuration digraph (to be defined in what follows) grows exponentially with k (the number of rows), implementations of this technique have led to optimal solutions only for grids with small number of rows. For the identifying code problem, [Jiang 2018] was able to solve for S_k, when $k \leq 5$. For the LDS problem, [Bouzinif et al. 2019] proved that $d^*(S_2) = 3/8$, $d^*(S_3) = 1/3$; $d^*(T_2) = 1/3$, $d^*(T_3) = 3/10$; $d^*(K_2) = 1/2$ and $d^*(K_3) = 4/15$. The grids S_1, T_1 and K_1 are all isomorphic to an infinite path P_∞, and it is easy to prove that $d^*(P_\infty) = 2/5$.

In their solution, [Bouzinif et al. 2019] used the periodicity of the powers of a matrix associated with the configuration digraph to find the mean of a minimum mean cycle in it. Due to the size of the matrices, they were able to find solutions only for grids with height at most 3. For our solution, we construct a digraph with weights on the arcs, and find a minimum mean cycle in this digraph. This description is given in what follows.

Let $\mathcal{G}_k$ denote one of the grids S_k, T_k or K_k, $k \geq 2$. The following two concepts are fundamental: *ℓ-bar* and a *barcode*. Let ℓ be a natural number, $\ell \geq 4$. An *ℓ-bar* of $\mathcal{G}_k$ is a subgraph of it induced by the vertices in the set $[k] \times \{j_1, \ldots, j_\ell\}$, where $j_1, \ldots, j_\ell$ are ℓ consecutive columns of the grid. Let R be an ℓ-bar of $\mathcal{G}_k$, and let C be a subset of vertices of R (those which are candidates to be in an LDS of the grid – indicated in black in the figures). We say that C is a *barcode* of R if the following holds for the $(\ell - 2)$-bar, say R', indexed by the columns 2 to $\ell - 1$ (i.e., the interior columns of the ℓ-bar R): for each vertex v in $R' \setminus C$, we have $N(v) \cap C \neq \emptyset$, and for each pair of distinct vertices u, v in $R' \setminus C$, we have $N(u) \cap C \neq N(v) \cap C$. (Note that a barcode is an ℓ-bar together with a subset C of selected vertices satisfying the mentioned properties; but for simplicity, we refer only to those vertices in C, as the subjacent ℓ-bar is fixed).

Construction of (G, w), the configuration digraph G with weight $w(.)$ on the arcs.

(i) Each vertex v_B of G represents a distinct barcode B of a 4-bar;

(ii) There is an arc $(v_B, v_{B'})$ in G if the *last three columns* of barcode B are identical to the *first three columns* of barcode B', and the 5-bar formed by their overlap (on the identical columns) is a barcode; the weight of this arc is the number of vertices in C present in the last column of the barcode B'.

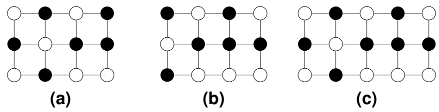


Figure 2. Two barcodes ((a) and (b)), and the barcode ((c)) formed by their overlap.

We can see that the number of vertices (resp. arcs) in G is at most 2^{4k} (resp. 2^{5k}). Let C^* be a *minimum (weight) mean cycle* of the configuration digraph. The mean of C^* is defined as the total weight of the arcs in C^* divided by the number of arcs in C^*. The next theorem shows that a minimum mean cycle of the configuration digraph G is intrinsically related to a minimum-density LDS in the corresponding grid. We observe that [Jiang 2018] proved such a result for identifying codes on square grids. Our proof for LDS on grids is different. Owing to space limitation, we provide only a sketch of the proof. For polynomial-time algorithms to find minimum mean cycle on weighted digraphs we refer to [Karp 1978] and [Hartmann and Orlin 1993].

We use the term *pattern* to refer to each feasible periodic solution (that is, an LDS of a grid). And we say that a pattern is optimal when it corresponds to a minimum-density LDS. In the grids shown in the next figures, two red consecutive vertical dashed bars indicate the beginning and the end of the corresponding optimal periodic pattern.

Theorem 1. *For a fixed integer $k \geq 2$, let $\mathcal{G}_k$ denote one of the grids S_k, T_k or K_k, and let (G, w) be the weighted configuration digraph constructed as defined previously. A pattern P defined by a set $C \subseteq V(\mathcal{G}_k)$ is a periodic* LDS *of $\mathcal{G}_k$ with n columns if, and only if, G has a cycle $\widehat{C}$ with n arcs and weight $w(\widehat{C})$ that is equal to the number of vertices of C in the pattern P, and the overlapping of consecutive barcodes associated with the vertices in $\widehat{C}$ makes the pattern P. Moreover, if λ is the mean of a minimum mean cycle of G, then the minimum density of an* LDS *in $\mathcal{G}_k$ is given by $d^*(\mathcal{G}_k) = \lambda/k$.*

(Sketch of a proof of Theorem 1.) The way we defined the graph G (its vertices and arcs), gives us immediately the correspondence between a periodic pattern P in $\mathcal{G}_k$ and a cycle in G. If we take the pattern of a cycle with mean λ, it is immediate that it has density λ/k. To prove that λ/k is a lower bound for any LDS, we notice that if there were a pattern with smaller density, then there would be a cycle in G with mean smaller than λ.

Thus, to find a minimum-density LDS in $\mathcal{G}_k$, all we have to do is find a minimum mean cycle in the configuration digraph G (it may not be unique). We implemented in C++ the algorithm that we have described. As the digraph for K_k is much larger than the digraph for S_k, we were able to find an optimal solution for S_6 but not for K_6.

The next theorems summarize the results we obtained with our code.

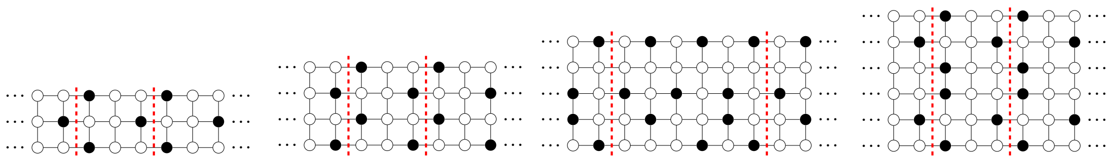


Figure 3. Optimal LDS for S_3, S_4, S_5 and S_6, all of them with density $1/3$.

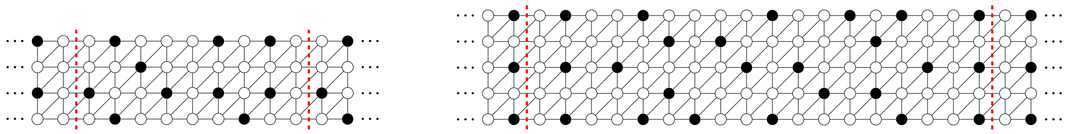


Figure 4. Optimal LDS for T_4 and T_5, with $d^*(T_4) = 5/18$ and $d^*(T_5) = 4/15$.

Theorem 2. *The locating-dominating sets shown in Figure 3 are of minimum density on the square grids S_4, S_5, and S_6. Their densities are $d^*(S_4) = d^*(S_5) = d^*(S_6) = 1/3$.*

Theorem 3. *The locating-dominating sets shown in Figure 4 are of minimum density on the triangular grids T_4 and T_5. Their densities are $d^*(T_4) = 5/18$ and $d^*(T_5) = 4/15$.*

Theorem 4. *The locating-dominating sets shown in Figure 5 are of minimum density on the king grids K_4 and K_5. Their densities are $d^*(K_4) = 1/4$ and $d^*(K_5) = 6/25$.*

3. An upper bound for the density of an LDS on the square grid with any bounded height

Figure 3 shows an optimal LDS for S_3. We may notice that the optimal pattern shown for S_6 consists of two stacked copies of the pattern for S_3. From this, it is clear that we can stack any number of these S_3 patterns to construct a pattern for any square grid S_{3m} with $m \geq 3$, maintaining a density of $1/3$, and thus, $d^*(S_{3m}) \leq 1/3$. To obtain LDS for square grids with different number of rows, we may combine (vertically and/or horizontally) optimal patterns, whenever such a combination gives a feasible solution. More formally, we can proceed as follows.

Let P_i and P_j be patterns of LDS C_i and C_j for grids S_i and S_j, with periods p and q, respectively. Let $t = \text{lcm}(p, q)$. Construct patterns P_i^* and P_j^* by repeating P_i and P_j horizontally t/p and t/q times, respectively. Clearly, the densities of P_i^* and P_j^* are the same as those of P_i and P_j, respectively. Let P^* be the pattern formed by taking P_i^* on the first i rows of S_{i+j} and P_j^* on the next j rows. If P^* keeps the desired properties for an LDS in the transition from the ith row to the $(i+1)$th row, then P^* is a pattern of an LDS on S_{i+j} with density given by $d(P^*, S_{i+j}) = \big(i \cdot d(P_i^*, S_i) + j \cdot d(P_j^*, S_j)\big)/(i + j)$.

We note that we can stack the optimal pattern shown for S_4 in Figure 3 on top of the optimal pattern shown for S_3, and obtain a pattern for S_7. Similarly, we can stack the optimal pattern for S_5 on top of the optimal pattern for S_3, and obtain a pattern for S_8. See Figure 6. Thus, we have that $d^*(S_7) \leq 1/3$ and $d^*(S_8) \leq 1/3$. Furthermore, since we can stack as many patterns for S_3 beneath these newly obtained patterns, we can construct patterns for S_k, $k \geq 10$, with density $1/3$. Thus, we have the following result.

Theorem 5. *Let S_k be the square grid with k rows. Then $d^*(S_k) = 1/3$ for $3 \leq k \leq 6$, and $d^*(S_k) \leq 1/3$ for $k \geq 7$.*

Figure 5. Optimal LDS for K_4 and K_5, with $d^*(K_4) = 1/4$ and $d^*(K_5) = 6/25$.

Figure 6. Feasible LDS for the grids S_7 and S_8, with density $1/3$.

Conclusion. We are currently working on the triangular and king grids with height at least 6, and also on the hexagonal grid with bounded height. In the moment, we only have some preliminary results, but we hope to be able to extend them soon.

References

Bouzinif, M., Darlay, J., Moncel, J., and Preissmann, M. (2019). Exact values for three domination-like problems in circular and infinite grid graphs of small height. *Discrete Mathematics & Theoretical Computer Science*, Vol. 21 no. 3.

Charon, I., Hudry, O., and Lobstein, A. (2003). Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard. *Theor. Comp. Sci.*, 290(3):2109–2120.

Foucaud, F., Mertzios, G., Naserasr, R., Parreau, A., and Valicov, P. (2016). Algorithms and complexity for metric dimension and location-domination on interval and permutation graphs. In *Graph-Theoretic Concepts in C.S.*, volume 9224, pages 456–471.

Hartmann, M. E. and Orlin, J. B. (1993). Finding minimum cost to time ratio cycles with small integral transit times. *Networks*, 23:567–574.

Honkala, I. (2006). An optimal locating-dominating set in the infinite triangular grid. *Discrete Mathematics*, 306(21):2670–2681.

Honkala, I. and Laihonen, T. (2006). On locating–dominating sets in infinite grids. *European Journal of Combinatorics*, 27(2):218–227.

Jean, D. (2024). Watching systems, identifying, locating-dominating and discriminating codes in graphs. `https://dragazo.github.io/bibdom/main.pdf`.

Jiang, M. (2018). Periodicity of identifying codes in strips. *Information Processing Letters*, 135:77–84.

Karp, R. M. (1978). A characterization of the minimum cycle mean in a digraph. *Discrete Mathematics*, 23(3):309–311.

Slater, P. J. (1975). Leaves of trees. In *Proc. of the 6th Southw. Conf. on Comb., Graph Theory, and Comput., Congressus Numerantium*, volume 14, pages 549–559.

Slater, P. J. (2002). Fault-tolerant locating-dominating sets. *Discrete Mathematics*, 249(1):179–189. Combinatorics, Graph Theory, and Computing.