Proceedings of the 13th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'25
Budapest, August 25 - 29, 2025

IMMERSIONS OF LARGE CLIQUES IN GRAPHS WITH INDEPENDENCE NUMBER 2 AND BOUNDED MAXIMUM DEGREE*

(EXTENDED ABSTRACT)

Fábio Botler† Cristina G. Fernandes‡ Carla N. Lintzmayer§ Rui A. Lopes¶
Suchismita Mishra‖ Bruno L. Netto** Maycon Sambinelli††

Abstract

An immersion of a graph H in a graph G is a minimal subgraph I of G for which there is an injection $\mathrm{i}\colon V(H) \to V(I)$ and a set of edge-disjoint paths $\{P_e : e \in E(H)\}$ in I such that the end vertices of P_{uv} are precisely $\mathrm{i}(u)$ and $\mathrm{i}(v)$. The immersion analogue of Hadwiger Conjecture (1943), posed by Lescure and Meyniel (1985), asks whether every graph G contains an immersion of $K_{\chi(G)}$. Its restriction to graphs with independence number 2 has received some attention recently, and Vergara (2017) raised the weaker conjecture that every graph with independence number 2 has an immersion of $K_{\chi(G)}$. In this paper, we verify Vergara Conjecture for graphs with bounded maximum degree. Specifically, we prove that if G is a graph with independence number 2 and maximum degree less than $19n/29 - 1$, then G contains an immersion of $K_{\chi(G)}$.

*This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Brazil, Finance Code 001.

†Universidade de São Paulo, São Paulo, Brazil. E-mail: fbotler@ime.usp.br. Partially supported by CNPq (304315/2022-2) and CAPES (88881.973147/2024-01). CNPq is the National Council for Scientific and Technological Development of Brazil.

‡Universidade de São Paulo, São Paulo, Brazil. E-mail: cris@ime.usp.br. Partially supported by CNPq (310979/2020-0 and 404315/2023-2) and by FAPESP (2024/13859-4). FAPESP is the São Paulo Research Foundation, Brazil.

§Univ. Federal do ABC, Santo André, Brazil. E-mail: carla.negri@ufabc.edu.br. Partially supported by CNPq (312026/2021-8 and 404315/2023-2) and by L'ORÉAL-UNESCO-ABC For Women In Science.

¶Univ. Federal do Rio de Janeiro, Rio de Janeiro, Brazil. E-mail: rui@cos.ufrj.br. Supported by CAPES (88887.843699/2023-00).

‖Universidad Andrés Bello, Santiago, Chile. E-mail: suchismitamishra6@gmail.com. Supported by Fondecyt Postdoctoral grant 3220618 of Agencia National de Investigatión y Desarrollo (ANID), Chile.

**Univ. Federal do Rio de Janeiro, Rio de Janeiro, Brazil. E-mail: brunoln@cos.ufrj.br. Supported by CAPES (88887.670803/2022-00).

††Univ. Federal do ABC, Santo André, Brazil. E-mail: m.sambinelli@ufabc.edu.br. Partially supported by CNPq (407970/2023-1).

1 Introduction

In this paper, every graph is simple. We consider the problem of finding special subgraphs in dense graphs. Specifically, we are interested in a problem related to the following well-known conjecture posed by Hadwiger [14].

Conjecture 1 (Hadwiger, 1943)**.** *Every graph G contains $K_{\chi(G)}$ as a minor.*

Conjecture 1 is still open, but it has been verified in many cases, as for graphs with chromatic number at most 6 [20]. An approach that turned out to be fruitful in exploring Conjecture 1 is to parametrize it by the independence number. Indeed, a graph on n vertices with independence number α has chromatic number at least $\lceil n/\alpha \rceil$, and Conjecture 1 implies that such a graph has a minor of $K_{\lceil n/\alpha \rceil}$. In this direction, Duchet and Meyniel [10] proved that such a graph has a minor of $K_{\lceil n/(2\alpha-1) \rceil}$; and, after several partial results (see [11]), Balogh and Kostochka [3] improved this result by proving that every such graph has a minor of $K_{\lceil n/((2-c)\alpha) \rceil}$, for $c \approx 0.0521$.

As pointed out by Quiroz [19], special attention has been given to the case $\alpha = 2$, and important results explore graphs with small clique covering and fractional clique covering numbers. For example, when G is a graph with an even number n of vertices and independence number 2, Blasiak [4] proved that G contains a minor of $K_{n/2}$ if its fractional clique covering number is less than 3; and Chudnovsky and Seymour [9] proved that G contains a minor of $K_{\lceil n/2 \rceil}$ if G contains a clique of size at least $n/4$.

We are interested in the immersion analogue of Conjecture 1 posed by Lescure and Meyniel [17, Problem 2]. An *immersion* of a graph H in a graph G is a minimal subgraph I of G for which there is an injection $\mathrm{i}\colon V(H) \to V(I)$ and a set of edge-disjoint paths $\{P_e : e \in E(H)\}$ in I such that the end vertices of P_{uv} are precisely $\mathrm{i}(u)$ and $\mathrm{i}(v)$. The vertices of G in the image of i are called the *branch vertices* of I, and we say that such an immersion is *strong* if the internal vertices of the paths P_e are not branch vertices.

Problem 2 (Lescure–Meyniel, 1985)**.** *Does every graph G contain a strong immersion of $K_{\chi(G)}$?*

The weaker version of Problem 2 for (not strong) immersions was posed as a conjecture by Abu-Khzam and Langston [1]. Similarly to Conjecture 1, Problem 2 and the Abu-Khzam–Langston Conjecture received special attention in the case of graphs with independence number 2. In particular, in 2017 Vergara [22] posed the following restriction.

Conjecture 3 (Vergara, 2017)**.** *Every graph G with independence number 2 contains an immersion of $K_{\chi(G)}$.*

Observe that if a graph G on n vertices with independence number 2 contains an immersion of $K_{\chi(G)}$, then it contains an immersion of $K_{\lceil n/2 \rceil}$. In fact, Vergara [22] proved that Conjecture 3 is equivalent to the following.

Conjecture 4 (Vergara, 2017)**.** *Every graph on n vertices with independence number 2 contains an immersion of $K_{\lceil n/2 \rceil}$.*

In this direction, Vergara [22] proved that every graph on n vertices with independence number 2 has a strong immersion of $K_{\lceil n/3 \rceil}$. This was improved by Gauthier, Le, and Wollan [13], who proved that such graphs contain strong immersions of $K_{2\lfloor n/5 \rfloor}$. In 2024, Botler, Jiménez, Lintzmayer, Pastine, Quiroz, and Sambinelli [5] proved that such graphs contain immersions of every complete bipartite graph with $\lceil n/2 \rceil$ vertices (see also [8]); in another direction, Quiroz [18] verified Conjecture 3 for graphs that contain a spanning complete-blow-up of C_5, and Conjecture 4 for graphs with special forbidden subgraphs.

One can expect that vertices of high degree help when looking for large clique immersions. For example, one of the first steps of Vergara's proof of the existence of $K_{\lceil n/3 \rceil}$ immersions in graphs with independence number 2 is to show that their minimum degree is at least $\lfloor 2n/3 \rfloor$, and an important step of the proof of existence of an immersion of $K_{2\lfloor n/5 \rfloor}$ is to prove that every vertex contained in an induced C_5 has degree at least $3n/5$. Similarly, it is not hard to prove that any minimum counterexample to Conjecture 4 has no pair of nonadjacent vertices with at least $\lceil n/2 \rceil - 2$ common neighbors. In this paper, we consider graphs without vertices of large degree. Specifically, we answer Problem 2 positively for graphs with independence number 2 and maximum degree bounded as follows.

Theorem 5. *If G be a graph on n vertices with independence number 2 for which $\Delta(G) < 19n/29-1$, then G contains a strong immersion of $K_{\chi(G)}$ and, consequently, a strong immersion of $K_{\lceil n/2 \rceil}$.*

Our proof explores properties of the complement of the studied graph. Specifically, we use the fact that triangle-free graphs with high minimum degree are homomorphic to the well-known Andrásfai graphs (see Section 2). Since C_5 is an Andrásfai graph, our result generalizes the independence-number-2 case of the aforementioned result of Quiroz for graphs containing complements of blow-ups of Andrásfai graphs. Indeed, our result is a consequence of a slightly more general result for graphs with a special 3-clique cover (see Theorem 8 ahead) and the fact that Andrásfai graphs admit a corresponding proper coloring. In fact, we prove the stronger statement that $V(G)$ can be partitioned into two sets A and B such that (i) A induces a clique in G, and (ii) G contains an immersion of a clique whose set of branch vertices is precisely B. One of the main ideas of our proof is to use Hall's Theorem to identify, for each vertex $u \in B$, a vertex $r_u \in A$ such that when u has a missing adjacency, say $uv \notin E(G)$ with $v \in B$, we find a path from u to v through r_v and r_u. The rest of the proof is to show that such paths are edge-disjoint. Due to space constraints, in this paper we present only sketches of some proofs.

2 Andrásfai graphs

Given graphs G and H, a *homomorphism* from G to H is a function $h\colon V(G) \to V(H)$ such that $h(u)h(v) \in E(H)$ for every $uv \in E(G)$. If such a function exists, we say that G is *homomorphic* to H. Let G be a triangle-free graph on n vertices. Andrásfai [2] showed that if $\delta(G) > 2n/5$, then G is bipartite. This result was generalized in many directions, one of which is the following. Häggkvist [15] proved that if $\delta(G) > 3n/8$, then G is 3-colorable, and Jin [16] weakened this minimum degree condition proving that if $\delta(G) > 10n/29$, then G is 3-colorable. Chen, Jin, and Koh [7] strengthened Jin's result showing that G is 3-colorable by exposing the structure of the graph. Specifically, they proved the following result, where Γ_d is the graph (V_d, E_d) for which $V_d = [3d-1]$ and $E_d = \{xy : y = x + i \text{ with } i \in [d, 2d-1]\}$, with arithmetic modulo $3d-1$. The graphs Γ_d for $d \in \mathbb{N}$ are known as the *Andrásfai graphs* (see Figure 1).

Theorem 6 (Chen–Jin–Koh, 1997)**.** *If G is a triangle-free graph on n vertices for which $\delta(G) > 10n/29$, then G is homomorphic to Γ_d for some d.*

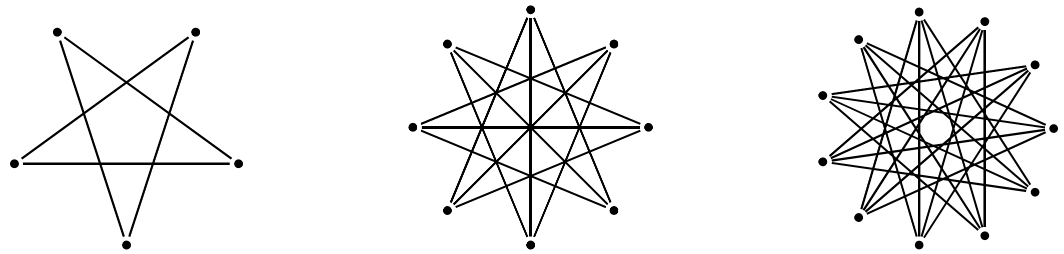

Figure 1: The Andrásfai graphs Γ_2, Γ_3, and Γ_4.

In what follows, given a graph H, we denote by $\overline{H}$ the complement of H. In this paper we use the following property of maximal graphs homomorphic to Γ_d which says, in particular, that an Andrásfai graph with a maximal independent set D admits a 3-coloring having D as one of the color classes.

Lemma 7. *Let G be a maximal graph homomorphic to Γ_d for some $d \in \mathbb{N}$. If I_1 is a maximal independent set of G, then G admits a 3-coloring $\{I_1, I_2, I_3\}$ such that $\overline{G}[I_2 \cup I_3]$ has no induced C_4.*

3 Dense graphs with bounded maximum degree

The proof of Theorem 5 is divided into two steps. First we use that if G is a triangle-free graph on n vertices with independence number 2 and maximum degree less than $19n/29 - 1$, then $\overline{G}$ admits a 3-coloring as in Lemma 7; next, we show that every graph whose complement admits such a 3-coloring contains an immersion of $K_{\chi(G)}$. For that, given a positive integer k, a *k-clique coloring* of a graph G is a partition $\{D_1, \ldots, D_k\}$ of $V(G)$ such that D_i is a clique of G for every $i \in [k]$. For $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, we denote by $G[X, Y]$ the bipartite subgraph of G with vertex set $X \cup Y$ and all edges of G between X and Y. Given $X \subseteq V(G)$ and $u \in V(G)$, we use $N_X(u)$ to denote the set of neighbors of u in X. The proof of the next result uses G and its complement $\overline{G}$ at the same time. To avoid confusion, we write $\overline{N}_X(u)$ to refer to the vertices in $X \setminus \{u\}$ that are not adjacent to u in G, and $\overline{N}_X(Y)$ to refer to the union $\bigcup_{u \in Y} \overline{N}_X(u)$, which is the set of vertices in X that are nonadjacent in G to at least one vertex in Y. Observe that $\overline{N}$ is precisely the neighborhood function in $\overline{G}$.

Theorem 8. *Let G be a graph with independence number 2. If G admits a 3-clique coloring $\{D_1, D_2, D_3\}$ such that (i) D_1 is a maximum clique of G; and (ii) $G[D_2 \cup D_3]$ has no induced C_4, then G contains an immersion of a clique whose set of branch vertices is precisely $D_2 \cup D_3$.*

Sketch of the proof. Let $i \in \{2, 3\}$ and let $C \subseteq D_i$. If $|\overline{N}_{D_1}(C)| < |C|$, then the set $(D_1 \setminus \overline{N}_{D_1}(C)) \cup C$ is a clique in G larger than D_1, a contradiction to the maximality of D_1. So, $|\overline{N}_{D_1}(C)| \geq |C|$ for every subset C of D_i. Hence, by Hall's Theorem, there is a matching M_i in $\overline{G}[D_i, D_1]$ that covers D_i. Note that for each vertex $u \in D_2 \cup D_3$ there is precisely one edge in $M_2 \cup M_3$ that contains u, and let $r_u \in D_1$ be the vertex such that $ur_u \in M_2 \cup M_3$.

Note that $r_u \notin N(u)$, and hence, because $\alpha(G) = 2$, r_u is adjacent in G to every non-neighbor of u, i.e., to every vertex in $V(G) \setminus N[u]$. Now, for every $u \in D_2$ and $v \in D_3$ with $uv \notin E' = E\big(G[D_2 \cup D_3]\big)$, let P_{uv} be the path $\langle u, r_v, r_u, v\rangle \subseteq G$. We can prove that the paths P_{uv} with $u \in D_2$, $v \in D_3$, and $uv \notin E'$ are pairwise edge-disjoint. Then, since D_2 and D_3 are cliques, the graph $G[D_2 \cup D_3] \cup \{P_e : e \notin E'\}$ is the desired immersion. □

The next theorem together with Theorem 6 implies Theorem 5. For its proof, we say that a graph G is *k-critical* if $\chi(G) = k$ and $\chi(G - u) < k$, for every $u \in V(G)$.

Theorem 9. *Let G be a graph with independence number at most 2. If the complement of G is homomorphic to Γ_d for some $d \in \mathbb{N}$, then G contains a strong immersion of $K_{\chi(G)}$.*

Sketch of the proof. Let G be a counterexample that minimizes $|V(G)|$, and let $k = \chi(G)$. We can show that G is k-critical and $\overline{G}$ is connected. Then, by a result of Gallai [12] (see [21, Corollary 2]), we conclude that $|V(G)| \geq 2k - 1$. Next we consider a maximal supergraph H of $\overline{G}$ that is homomorphic to Γ_d and a maximum independent set I in H. As $G[I]$ is a clique, we derive that $|I| \leq k - 1$. Applying Lemma 7 on H and I, and then Theorem 8 on $\overline{H}$, which has independence number 2, we can deduce that there is an immersion in G of a clique with $V(G) \setminus I$ as branch vertices. But $|V(G) \setminus I| \geq (2k-1) - (k-1) = k$, a contradiction. □

4 Concluding remarks

In this paper, we explore Conjecture 3 under a maximum degree constraint that allows us to use structural results on triangle-free graphs, and reveals a connection to the chromatic number of their complement. A natural possible improvement in our result is as follows: Brandt and Thomassé [6] proved that triangle-free graphs on n vertices with minimum degree greater than $n/3$ are homomorphic to Vega graphs (which are 4-colorable). This may be used to weaken the bound on $\Delta(G)$ in Theorem 5 to $\Delta(G) < 2n/3 - 1$. We observe that this improvement can be achieved by extending Theorem 8 to graphs whose complement has fractional chromatic number at most 3. Indeed, a triangle-free graph G with minimum degree δ has fractional chromatic number at most n/δ because the family of neighborhoods $\{N(u) : u \in V(G)\}$ with constant weight function $1/\delta$ is a fractional coloring of G. This would reveal a connection to the result of Blasiak [4] mentioned in Section 1. Another interesting direction is to explore larger chromatic numbers of the complement graph by finding longer paths using a broader notion of representatives.

References

[1] Faisal N. Abu-Khzam and Michael A. Langston. Graph coloring and the immersion order. In *International Computing and Combinatorics Conference*, pages 394–403. Springer, 2003.

[2] Béla Andrásfai. Graphentheoretische Extremalprobleme. *Acta Mathematica Hungarica*, 15(3-4):413–438, 1964.

[3] József Balogh and Alexandr V. Kostochka. Large minors in graphs with given independence number. *Discrete Mathematics*, 311(20):2203–2215, 2011.

[4] Jonah Blasiak. A special case of Hadwiger's conjecture. *Journal of Combinatorial Theory, Series B*, 97(6):1056–1073, 2007.

[5] Fábio Botler, Andrea Jiménez, Carla N. Lintzmayer, Adrián Pastine, Daniel A. Quiroz, and Maycon Sambinelli. Biclique immersions in graphs with independence number 2. *European Journal of Combinatorics*, 122:104042, 2024.

[6] Stephan Brandt and Stéphan Thomassé. Dense triangle-free graphs are four-colorable: A solution to the Erdös-Simonovits problem. Available at `https://perso.ens-lyon.fr/stephan.thomasse/liste/vega11.pdf`, 2011.

[7] Chuan-Chong Chen, Guoping P. Jin, and Khee Meng Koh. Triangle-free graphs with large degree. *Combinatorics, Probability and Computing*, 6(4):381–396, 1997.

[8] Rong Chen and Zijian Deng. A simple proof of the existence of complete bipartite graph immersion in graphs with independence number two. Available at SSRN: `https://ssrn.com/abstract=4882255` or `http://dx.doi.org/10.2139/ssrn.4882255`, 2024.

[9] Maria Chudnovsky and Paul Seymour. Packing seagulls. *Combinatorica*, 32:251–282, 2012.

[10] Pierre Duchet and Henry Meyniel. On Hadwiger's number and the stability number. In *North-Holland Mathematics Studies*, volume 62, pages 71–73. Elsevier, 1982.

[11] Jacob Fox. Complete minors and independence number. *SIAM Journal on Discrete Mathematics*, 24(4):1313–1321, 2010.

[12] Tibor Gallai. Kritische graphen II. *Magyar Tud. Akad. Mat. Kutato Int. Kozl.*, 8:373–395, 1963.

[13] Gregory Gauthier, Tien-Nam Le, and Paul Wollan. Forcing clique immersions through chromatic number. *European Journal of Combinatorics*, 81:98–118, 2019.

[14] Hugo Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88(2):133–142, 1943.

[15] Roland Häggkvist. Odd cycles of specified length in non-bipartite graphs. In *North-Holland Mathematics Studies*, volume 62, pages 89–99. Elsevier, 1982.

[16] Guoping Jin. Triangle-free four-chromatic graphs. *Discrete Mathematics*, 145(1-3):151–170, 1995.

[17] François Lescure and Henry Meyniel. On a problem upon configurations contained in graphs with given chromatic number. *Annals of Discrete Mathematics*, 41:325–331, 1985.

[18] Daniel A. Quiroz. Clique immersions in graphs of independence number two with certain forbidden subgraphs. *Discrete Mathematics*, 344(6):112365, 2021.

[19] Daniel A. Quiroz. Private communication, nov. 2024.

[20] Neil Robertson, Paul Seymour, and Robin Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica*, 13(3):279–361, 1993.

[21] Matěj Stehlík. Critical graphs with connected complements. *Journal of Combinatorial Theory, Series B*, 89(2):189–194, 2003.

[22] Sylvia Vergara. Complete graph immersions in dense graphs. *Discrete Mathematics*, 340(5):1019–1027, 2017.