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# On non-contractible periodic orbits and bounded deviations

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## Abstract

We present a dichotomy for surface homeomorphisms in the isotopy class of the identity. We show that, in the absence of a degenerate fixed point set, either there exists a uniform bound on the diameter of orbits of non-wandering points for the lifted dynamics in the universal covering space, or the map has non-contractible periodic orbits. We then use this new tool to characterize the dynamics of area preserving homeomorphisms of the torus without non-contractible periodic orbits, showing that if the fixed point set is non-degenerate, then either the lifted dynamics is uniformly bounded, or it has a single strong irrational dynamical direction.

Keywords: dynamical systems, surface dynamics, rotation theory

Mathematics Subject Classification numbers: 37E45, 37E30

## 1. Introduction

In this work we study the dynamical consequences of the existence or absence of non-contractible periodic orbits in conservative surface dynamics. This is a direction that has been garnering increased attention in recent years, with new results due to mostly the increasingly developed field of Brouwer-homeomorphisms like techniques, and that has drawn increased interest due not only to its applications in the study of some relevant classes of torus homeomorphisms, but also to its connection to symplectic dynamics, where the subject has been

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much more largely exploited (see for example [8] as well as references herein, see also [17]), albeit through very different techniques.

Let  $S$  denote a closed oriented surface, endowed with a metric of constant curvature, and let  $\tilde{S}$  be its universal covering, with the associated metric, which we simply denote  $\|\cdot\|$ . Assume that we have a homeomorphism  $f$  of  $S$  in the isotopy class of the identity. Taking an isotopy  $I = (f_t)_{t \in [0,1]}$  between the identity  $f_0 = Id$  and  $f_1 = f$ , one can naturally associate to each point  $x \in S$  a curve  $\gamma_x : [0, 1] \rightarrow S$ , defined by  $\gamma_x(t) = f_t(x)$ . If  $x \in \text{Fix}(f)$ , then  $\gamma_x$  is a closed loop. A point  $x \in \text{Fix}(f)$  is called *contractible* for the isotopy  $I$  if the loop  $\gamma_x$  is null-homotopic; otherwise  $x$  is called *non-contractible* for  $I$ . Likewise, if  $x$  is a periodic point with minimal period  $k$ , then by concatenating the curves one arrives again at a closed loop  $\prod_{i=0}^{k-1} \gamma_{f^i(x)}$ . One defines, analogously to the fixed point case, contractible and non-contractible periodic orbits for the isotopy. The isotopy  $I$  lifts in a unique way to an isotopy  $\tilde{I} = (\tilde{f}_t)_{t \in [0,1]}$  in  $\tilde{S}$ , such that  $\tilde{f}_0 = Id$ , and  $\tilde{f} = \tilde{f}_1$  is a lift of  $f$  commuting with all deck transformations in  $\tilde{S}$ . Note for an  $f$ -periodic point  $x \in S$  with minimal period  $k$ , the  $f$ -orbit containing  $x$  is a contractible periodic orbit for the isotopy  $I$  if, and only if, any lift  $\tilde{x}$  of  $x$  is also an  $\tilde{f}$ -periodic point.

Many recent progresses were made on relating the existence of non-contractible periodic orbits with boundedness of some orbits for the lifted dynamics. The main heuristics is usually the following idea. In the absence of a very degenerate condition involving the fixed points set of the dynamics, either the diameter of an orbit of points with some sort of recurrent property in the lift is uniformly bounded, or one can find non-contractible periodic orbits with arbitrarily large minimal period. In [18] this was shown when the ‘recurrent property in the lift’ meant to be periodic points in the lift, and in [15] the condition was weakened for points which are recurrent for the lift. In the present paper, our first theorem improves both of these works, extending the result to the class of non-wandering points of the lifted dynamics.

**Theorem 1.1.** *Let  $S$  be a compact orientable surface, with the universal covering space  $\tilde{S}$ . Let  $f$  be a homeomorphism of  $S$  isotopic to the identity, with a lift  $\tilde{f}$  on  $\tilde{S}$  commuting with the deck transformations. Assume that the fixed point set of  $f$  is non-empty and is contained in a topological open disk. Then*

- (1) *either  $f$  admits non-contractible periodic orbits of arbitrarily large prime period,*
- (2) *or there exists a constant  $M > 0$ , such that, for any  $\tilde{f}$ -non-wandering point  $\tilde{z}$  and all integer  $n$ ,*

$$\|\tilde{f}^n(\tilde{z}) - \tilde{z}\| < M. \tag{1.1}$$

Another remark is that the hypothesis on the fixed point set cannot be removed, as there are examples where the dichotomy fails (see for example [11]). However, failing to satisfy the hypotheses can only happen if there exists a homotopically non-trivial continuum of fixed points, which is a very degenerate condition.

In both [15, 18], results on the existence of non-contractible periodic orbits were used to understand some torus homeomorphisms isotopic to the identity. Denote by  $\text{Homeo}_0(\mathbb{T}^2)$  the set of homeomorphisms on  $\mathbb{T}^2$  which are isotopic to the identity, and therefore orientation-preserving, and let  $\widetilde{\text{Homeo}}_0(\mathbb{T}^2)$  be the set of lifts of homeomorphisms in  $\text{Homeo}_0(\mathbb{T}^2)$  to the plane. Denote by  $\text{Homeo}_{0,\text{nw}}(\mathbb{T}^2)$  the subset of  $\text{Homeo}_0(\mathbb{T}^2)$  of homeomorphisms having no wandering points, and by  $\widetilde{\text{Homeo}}_{0,\text{nw}}(\mathbb{T}^2)$  the set of their lifts to  $\mathbb{R}^2$ . Next,  $f \in \text{Homeo}_0(\mathbb{T}^2)$  is called *Hamiltonian* if it preserves the Lebesgue measure and it has a lift  $\tilde{f}$  to the universal covering space such that the rotation vector of  $f$  with respect to the Lebesgue measure is null (we refer to the next section for more notation). In this case, we call  $\tilde{f}$  the Hamiltonian lift.

Say that  $f \in \text{Homeo}_0(\mathbb{T}^2)$  has a *uniformly bounded lift*  $\tilde{f}$  if there exists  $M > 0$  such that, for any  $\tilde{z} \in \mathbb{R}^2$  and all integer  $n$ ,  $\|\tilde{f}^n(\tilde{z}) - \tilde{z}\| < M$ . In the previous existing work on conditions ensuring the existence of non-contractible periodic orbits, it was usually fundamental to show that, if  $f$  is a Hamiltonian homeomorphism with non-degenerate fixed point set (that is, the fixed point set is contained in a topological disk), then its Hamiltonian lift is a uniformly bounded lift (see corollary I of [15]).

In order to see a uniformly bounded lift, the hypothesis that the dynamics is Hamiltonian is crucial, since clearly one can consider an ergodic rigid translation in the torus. Moreover, if we assume that the dynamics has at least a periodic point there are examples of area-preserving homeomorphisms without any non-contractible periodic orbits, while at the same time, almost all points in the lift have unbounded orbits (see [2] for an example, even in the smooth setting). However, these examples are very particular, as we will see more clearly below.

For any  $v \in \mathbb{R}^2 \setminus \{(0,0)\}$ , denote by  $\text{pr}_v : w \mapsto \langle w, v \rangle / \|v\|$  the projection of a vector  $w$  to the oriented line passing through the origin in the direction  $v$ . We say  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$  has *strong dynamical direction*  $v$ , if it is not a uniformly bounded lift, and if there exists a constant  $M > 0$  such that, for any point  $\tilde{z}$  and any  $n \geq 0$ , it holds that  $-M < \text{pr}_v(\tilde{f}^n(\tilde{z}) - \tilde{z})$  and  $-M < \text{pr}_{v^\perp}(\tilde{f}^n(\tilde{z}) - \tilde{z}) < M$  for any  $v^\perp \in \mathbb{R}^2 \setminus \{(0,0)\}$  which is perpendicular to  $v$ . If additionally  $v$  has irrational slope, we say  $f$  has *strong irrational dynamical direction*  $v$ . For all known examples of area-preserving, non-Hamiltonian homeomorphisms with contractible periodic orbits but without non-contractible periodic orbits, the lifted map had strong irrational dynamical direction. Our next theorem shows that this is in fact the only possibility:

**Theorem 1.2.** *Let  $f$  be an area-preserving homeomorphism with a lift  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$ . Assume that  $f$  has contractible periodic orbits but no non-contractible periodic orbits, and that the fixed point set of  $f$  is contained in a topological disk. Then either  $f$  is Hamiltonian and  $\tilde{f}$  is a uniformly bounded lift, or  $f$  has a strong irrational dynamical direction.*

Theorem 1.2 will be deduced as a consequence of several previous results on torus homeomorphisms in the isotopy class of the identity, as well as a result that is the fundamental new contribution in the second part of the present paper. The critical step needed to prove theorem 1.2 is to understand the dynamics of a particular class of homeomorphisms, a class whose rotation set is a line segment, such that one endpoint is a rational vector, and the other one is a irrational vector  $(\alpha, \beta)$ , where  $\alpha/\beta$  is irrational. We postpone the introduction of the concept of rotation set  $\rho(\tilde{f})$  to the section 2.5. Moreover, in section 2.5 we will introduce the concept of bounded deviation. Informally,  $\tilde{f}$  is said to have bounded deviation along some direction  $v$  if, for all  $n \geq 1$ , the difference  $\tilde{f}^n(\tilde{x}) - \tilde{x}$ , when projected onto a line in the direction of  $v$ , is uniformly bounded from above for all  $\tilde{x}$ .

Here we also comment this special rotation set was studied in depth in the paper [1], where discussions about the stable and unstable behaviour were given. In particular, it was shown in [1] that, if  $\tilde{f}$  is a lift of a non-wandering diffeomorphism, then under some broadly satisfied conditions  $\tilde{f}$  admits bounded deviation along the direction  $-(\alpha, \beta)$  (see theorem 1.3 of [1]). Here, using similar ideas as in the proof of theorem 1.1, we are able to extend the bounded deviations result for all non-wandering homeomorphisms.

**Theorem 1.3.** *Let  $\tilde{f} \in \widetilde{\text{Homeo}}_{0,nw}(\mathbb{T}^2)$ , whose rotation set  $\rho(\tilde{f})$  is a line segment from  $(0,0)$  to some vector  $(\alpha, \beta)$  with  $\alpha/\beta$  irrational. Then  $\tilde{f}$  has bounded deviation along the direction  $-(\alpha, \beta)$ .*

Examples of homeomorphisms having rotation sets as the one described in the theorem are well known. In [9], one finds a construction using reparametrization of the flows, attributed to Katok. In [2], one obtains an example which is area preserving and smooth.

This result, together with the main theorem (theorem A) of [19] will imply  $\tilde{f}$  has strong irrational dynamical direction. We stress that, the same as in [19], the proof of theorems 1.1 and 1.3 rely on the newly developed forcing theory of transverse trajectories for surface homeomorphisms (see [15]), and a crucial step in both is the use of the technical proposition 4.4, which may be useful elsewhere. Let us refer to [19] for more references, backgrounds and other related discussions.

The rest of the paper will be organized as follows. In section 2 we introduce notation and preliminaries and then give a quick proof that theorem 1.3 implies theorem 1.2. In section 3, we prove theorem 1.1. In section 4, we show proposition 4.4. In section 5, we prove theorem 1.3.

## 2. Generalities and the forcing theory

### 2.1. Topological dynamics on surfaces and torus

Denote by  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  the unit circle and by  $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$  the 2-torus. Let  $S$  be an oriented surface. We will denote by  $\tilde{S}$  the universal covering space of  $S$ , and by  $\pi_S : \tilde{S} \rightarrow S$  the covering projection. The set of *Deck Transformations* of  $S$  are the isometries of  $\tilde{S}$  that lift the identity map on  $S$ . A loop in  $S$  is a continuous function  $B : \mathbb{S}^1 \rightarrow S$ . Given a loop  $B$ , let us call the natural extension of  $B$  to the curve  $\beta : \mathbb{R} \rightarrow S$  defined by  $\beta(t) = B(\pi_{\mathbb{S}^1}(t))$ , where  $\pi_{\mathbb{S}^1} : \mathbb{R} \rightarrow \mathbb{S}^1$  is the canonical projection from the line on the circle. In particular  $\beta$  is a 1-periodic function.

If furthermore  $S$  is a closed surface, a set  $K \subset S$  is called *inessential* if it is contained in some topological disk  $D \subset S$ , otherwise it is called *essential*. If the complement of  $K$  is inessential, then  $K$  is called *fully essential*, in which case it must intersect any homotopically non-trivial loop in  $S$ . An open connected essential set  $U$  is called *annular* if it is homeomorphic to an open annulus.

For any closed surface  $S$ , denote by  $\text{Homeo}_0(S)$  (respectively,  $\text{Homeo}_{0,\text{nw}}(S)$ ) the set of orientation-preserving homeomorphisms (respectively, non-wandering orientation-preserving homeomorphisms) on  $S$  which are isotopic to the identity. Also denote by  $\widetilde{\text{Homeo}}_0(S)$  (respectively,  $\widetilde{\text{Homeo}}_{0,\text{nw}}(S)$ ) the set of homeomorphisms on  $\mathbb{R}^2$  which are lifts of homeomorphisms in  $\text{Homeo}_0(S)$  (respectively,  $\text{Homeo}_{0,\text{nw}}(S)$ ). For any  $f \in \text{Homeo}_0(S)$ , a point  $x \in S$  is called *f-essential* if, for any open set  $U$  containing  $x$ , the set  $W = \bigcup_{n \in \mathbb{Z}} f^n(U)$  is an essential set. Otherwise,  $x$  is called *f-inessential*. Denote by  $\text{Ess}(f)$  the set of *f-essential* points, which is *f*-invariant and closed. Write  $\text{Ine}(f) = S \setminus \text{Ess}(f)$ , which is *f*-invariant and open. We refer to [12] for more information on these notions.

### 2.2. Oriented singular foliation and transverse paths

We call  $\mathcal{F}$  an *oriented singular foliation* of  $S$ , if there exists a closed set  $\text{Sing}(\mathcal{F}) \subset S$ , called the *singular set*, and if  $\mathcal{F}$  is a partition of  $\text{Dom}(\mathcal{F}) := S \setminus \text{Sing}(\mathcal{F})$  into immersed real lines or circles, called *leaves*, with a continuous choice of orientation. We also lift  $\mathcal{F}$  to an oriented singular foliation  $\tilde{\mathcal{F}}$  on  $\tilde{S}$ , where the singular set is  $\text{Sing}(\tilde{\mathcal{F}}) = \pi_S^{-1}(\text{Sing}(\mathcal{F}))$ , and therefore  $\text{Dom}(\tilde{\mathcal{F}}) = \pi_S^{-1}(\text{Dom}(\mathcal{F}))$ . Given any  $z \in S \setminus \text{Sing}(\mathcal{F})$ , there exists a local chart  $h : W \rightarrow \mathbb{R}^2$ , where  $W$  is a neighbourhood of  $z$ , such that  $h(\mathcal{F}|_W)$  is the foliation into vertical lines, oriented downward. In this case, we call  $h$  a local *trivialization chart around z*.

A path  $\gamma : J \rightarrow \text{Dom}(\mathcal{F})$  is called *transverse to  $\mathcal{F}$*  (or,  $\gamma$  is an  *$\mathcal{F}$ -transverse path*) if for any  $t_0$  in the interior of  $J$ , there exist  $\delta > 0$  and a local trivialization chart  $h$  around the point  $\gamma(t_0)$ , such that the function

$$t \mapsto \text{pr}_1 \circ h \circ \gamma(t), \tag{2.1}$$

is strictly increasing for  $t \in (t_0 - \delta, t_0 + \delta)$ , where  $\text{pr}_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the projection in the first canonical coordinate. A loop  $B : \mathbb{S}^1 \rightarrow \text{Dom}(\mathcal{F})$  is said to be  *$\mathcal{F}$ -transverse* if its natural extension is  *$\tilde{\mathcal{F}}$ -transverse*.

### 2.3. Maximal isotopy and transverse foliation

Let  $f : S \rightarrow S$  be a homeomorphism isotopic to the identity. Given an isotopy  $I = (f_t)_{t \in [0,1]}$  from  $f_0 = \text{Id}$  to  $f_1 = f$ , denote by  $\text{Fix}(I)$  the set of points which are fixed by  $f_t$  for all  $t \in [0, 1]$ . Write  $\text{Dom}(I) = S \setminus \text{Fix}(I)$ . For any point  $z \in \text{Dom}(I)$ , consider the isotopy path

$$\begin{aligned} \gamma_z = I^{[0,1]}(z) : [0, 1] &\rightarrow \text{Dom}(I), \\ t &\mapsto f_t(z). \end{aligned} \tag{2.2}$$

Then we can extend the isotopy path in the following way. For any  $t \in \mathbb{R}$ , write  $t = n_0 + r$ , where  $n_0$  is an integer and  $r \in [0, 1)$ . Then define

$$f_t(z) = f_r \circ f^{n_0}(z). \tag{2.3}$$

By the *whole isotopy trajectory* we mean the following.

$$I^{\mathbb{R}}(z) : \mathbb{R} \rightarrow \text{Dom}(I), \tag{2.4}$$

$$t \mapsto f_t(z). \tag{2.5}$$

In particular,  $I^{\mathbb{R}}(z)(0) = f_0(z) = z$ . By abusing notation, sometimes we also call the image of a path by the same name. For example, denote by  $I^{[a,b]}(z) = I^{\mathbb{R}}(z)([a, b])$  the image of a finite isotopy trajectory. We also call *whole isotopy trajectory* for  $I^{\mathbb{R}}(z)(\mathbb{R})$ .

Given isotopies  $I' = (f'_t)_{t \in [0,1]}$  and  $I = (f_t)_{t \in [0,1]}$ , we say that  $I' \preceq I$  if  $\text{Fix}(I') \subset \text{Fix}(I)$ . An isotopy is said to be a *maximal isotopy* if it is a maximum element for this partial order. It was shown in [3] that given any isotopy  $I'$  one could always find a maximal isotopy larger than it (see also [10] for a previous existence result which was already enough for most applications). A crucial property of a maximal isotopy  $I = (f_t)_{t \in [0,1]}$  is that, for every  $z \in \text{Fix}(f_1) \setminus \text{Fix}(I)$ , the closed curve  $\gamma_z$  is not contractible in  $\text{Dom}(I)$ .

Recall we have defined that a path is *transverse to  $\mathcal{F}$*  (or  *$\mathcal{F}$ -transverse*) if locally it crosses through each leaf from its right to its left. An oriented singular foliation  $\mathcal{F}$  on  $S$  is called a *transverse foliation for the maximal isotopy  $I$* , if the singular set  $\text{Sing}(\mathcal{F})$  coincides with the fixed point set  $\text{Fix}(I)$ , and if for any  $z \in \text{Dom}(I)$ , the isotopy path  $\gamma_z$  is homotopic in  $\text{Dom}(I)$ , relative to its endpoints, to a path which is transverse to the foliation  $\mathcal{F}$ . Write  $\text{Dom}(\mathcal{F}) = S \setminus \text{Sing}(\mathcal{F})$  and note that  $\text{Dom}(\mathcal{F}) = \text{Dom}(I)$ .  $\mathcal{F}$  is also called a *Brouwer–Le Calvez foliation*.

For any  $z \in \text{Dom}(I)$ , denote by  $I_{\mathcal{F}}(z)$  the set of  $\mathcal{F}$ -transverse paths that are homotopic in  $\text{Dom}(I)$ , relative to its endpoints, to  $\gamma_z$ . Also, for  $n \geq 1$ , denote by  $I_{\mathcal{F}}^n(z) := \prod_{k=0}^{n-1} I_{\mathcal{F}}(f^k z)$  the set of transverse paths obtained by concatenating the corresponding transverse paths. Similarly, denote  $I_{\mathcal{F}}^{\mathbb{R}}(z)$  the set of infinite transverse path  $\prod_{k=-\infty}^{\infty} I_{\mathcal{F}}(f^k z)$ . By abusing notation, when the context is clear, we sometimes use  $I_{\mathcal{F}}^n(z)$  or  $I_{\mathcal{F}}^{\mathbb{R}}(z)$  to denote one specific transverse path contained in the corresponding sets of paths.

The following important theorem, stated here as a lemma, was a breakthrough result and it provides a completely new tool for understanding surface dynamics in general.

**Lemma 2.1 (see [13]).** *Let  $f \in \text{Homeo}_0(S)$  and let  $I = (f_t)_{t \in [0,1]}$  be a maximal isotopy such that  $f_0 = \text{Id}$  and  $f_1 = f$ . Then there exists a transverse foliation  $\mathcal{F}$  for  $I$ .*

Note that  $f, I$  and  $\mathcal{F}$  are naturally lifted to  $\tilde{f}, \tilde{I}$  and  $\tilde{\mathcal{F}}$  respectively in the covering space  $\tilde{S}$  of  $S$ . These objects are compatible in the following sense.  $\tilde{I}$  is a maximal isotopy from  $\text{Id}$  to  $\tilde{f}$ , and  $\tilde{\mathcal{F}}$  is a transverse foliation to  $\tilde{I}$ . We find that  $\text{Dom}(\tilde{\mathcal{F}}) = \text{Dom}(\tilde{I})$  is the complement of the corresponding singular set for both the isotopy  $\tilde{I}$  and the foliation  $\tilde{\mathcal{F}}$ . Furthermore, as the restriction of  $\tilde{I}$  to  $\text{Dom}(\tilde{I})$  is also isotopic to the identity, one can lift all these objects to the universal covering of this latter surface, namely  $\text{Dom}(\mathcal{F})^{\text{uni}}$ , which is homeomorphic to a union of disjoint copies of  $\mathbb{R}^2$ . The lifted objects will be denoted  $f^{\text{uni}}, I^{\text{uni}}$  and  $\mathcal{F}^{\text{uni}}$ . The restriction of  $f^{\text{uni}}$  to each connected component of  $\text{Dom}(\mathcal{F})^{\text{uni}}$  is a Brouwer homeomorphism, which is, by definition, an orientation-preserving homeomorphism on  $\mathbb{R}^2$  without fixed points. Finally,  $\mathcal{F}^{\text{uni}}$  is a non-singular foliation, and each leaf  $\ell^{\text{uni}}$  of  $\mathcal{F}^{\text{uni}}$  is an  $f^{\text{uni}}$ -Brouwer line, that is,  $f^{\text{uni}}(\ell^{\text{uni}})$  is contained in the left of  $\ell^{\text{uni}}$ , and  $(f^{\text{uni}})^{-1}(\ell^{\text{uni}})$  is contained in the right of  $\ell^{\text{uni}}$ .

We need the following definitions from [15]:

**Definition 2.2.** Let  $\mathcal{F}$  be an oriented foliation on  $\mathbb{R}^2$ , either singular or non-singular. Let  $\gamma_0 : [a, b] \rightarrow S \setminus \text{Sing}(\mathcal{F}), \gamma_1 : [a', b'] \rightarrow S \setminus \text{Sing}(\mathcal{F})$  be two  $\mathcal{F}$ -transverse paths. We say  $\gamma_0$  and  $\gamma_1$  are  $\mathcal{F}$ -equivalent if there is a homotopy  $H : [a, b] \times [0, 1] \rightarrow S \setminus \text{Sing}(S)$  and an increasing homeomorphism  $\phi : [a, b] \rightarrow [a', b']$ , such that for all  $c \in [a, b], H(c, 0) = \gamma_0(c), H(c, 1) = \gamma_1(\phi(c))$ , and the image of  $H(c, t)|_{t \in [0,1]}$  is contained in a single leaf of  $\mathcal{F}$ .

**Definition 2.3.** Let  $J$  be an interval (possibly unbounded) in  $\mathbb{R}$ . We say an  $\mathcal{F}$ -transverse path  $\gamma : J \rightarrow \text{Dom}(I)$  is admissible if for any compact subinterval  $J_1 \subset J$ , there exists  $z \in \text{Dom}(I)$  such that the restriction of  $\gamma$  to  $J_1$  is  $\mathcal{F}$ -equivalent to a subpath of  $I_{\mathcal{F}}^{\mathbb{R}}(z)$ . Furthermore, if there exists  $z \in \text{Dom}(I)$  and a positive integer  $n$  such that  $\gamma$  is  $\mathcal{F}$ -equivalent to an element in  $I_{\mathcal{F}}^n(z)$ , then  $\gamma$  is said to be *admissible of order  $n$* . If  $\gamma$  is  $\mathcal{F}$ -equivalent with a subpath of such an element, then it is said to be admissible of order  $\leq n$ . A loop is said to be admissible if its natural extension is admissible.

The following continuity property for admissible  $\mathcal{F}$ -transverse paths will be useful.

**Lemma 2.4 (lemma 17 of [15]).** *Let  $I$  be a maximal isotopy and  $\mathcal{F}$  is a transverse foliation for  $I$ . For any  $z \in \text{Dom}(I)$  and any  $n \geq 1$ , there exists a neighbourhood  $W$  of  $z$  such that, for every  $z', z'' \in W$ , the path  $I_{\mathcal{F}}^n(z')$  is  $\mathcal{F}$ -equivalent to a subpath of  $I_{\mathcal{F}}^{n+2}(f^{-1}(z''))$ . Also,  $I_{\mathcal{F}}^2(f^{-1}(z''))$  intersects the leaf of  $\mathcal{F}$  that contains  $z'$ .*

The following result focuses on the torus case.

**Lemma 2.5 (lemma 4.8 of [12], essentially dating back to section 10 of [13]).** *Suppose we are working with a maximal isotopy  $I$  and the transverse foliation  $\mathcal{F}$ , which are lifted to  $\tilde{I}$  and  $\tilde{\mathcal{F}}$  in  $\mathbb{R}^2$ . Assume an  $\tilde{\mathcal{F}}$ -transverse loop  $\Sigma$  is written as a concatenation of finitely many transverse loops  $\{\beta_i\}_{i=1}^p$ , with a same base point, which induce the homology directions  $\{\beta_i^*\}_{i=1}^p$ . Assume these directions  $\{\beta_i^*\}_{i=1}^p$  can linearly represent any element in the first homology group  $H_1(\mathbb{T}^2, \mathbb{Z})$ , only using scalars which are positive integers. Assume also  $\sum_{i=1}^p \beta_i^* = 0$ . Then every leaf of  $\tilde{\mathcal{F}}$  is uniformly bounded.*

2.4. Forcing results

Consider an oriented non-singular foliation  $\mathcal{F}$  of  $\mathbb{R}^2$ , and denote by  $\phi_0, \phi_1$  and  $\phi$  three distinct leaves of  $\mathcal{F}$ , parameterized according to their orientation, and satisfying that none of which separate the other two. We say  $\phi_1$  is above  $\phi_0$  relative to  $\phi$  (and  $\phi_0$  is below  $\phi_1$  relative to  $\phi$ ), if for some  $t_0 < t_1$ , writing  $z_0 = \phi(t_0)$  and  $z_1 = \phi(t_1)$ , there exist disjoint paths  $\lambda_0$  and  $\lambda_1$ , such that for  $i = 0, 1$ ,  $\lambda_i$  joins  $z_i$  to  $x_i \in \phi_i$ , and moreover  $\lambda_0$  and  $\lambda_1$  do not intersect any of the three leaves except at the endpoints.

For any  $z \in \text{Dom}(\mathcal{F})$ , denote by  $\phi_z$  the leaf in  $\mathcal{F}$  which contains  $z$ . Let  $\gamma_0 : [a_0, b_0] \rightarrow \mathbb{R}^2$  and  $\gamma_1 : [a_1, b_1] \rightarrow \mathbb{R}^2$  be two  $\mathcal{F}$ -transverse paths. Suppose also  $\gamma_0(t_0)$  and  $\gamma_1(t_1)$  belong to a same leaf  $\phi \in \mathcal{F}$ , for certain times  $t_0 \in (a_0, b_0)$  and  $t_1 \in (a_1, b_1)$ . We say  $\gamma_0|_{[a_0, b_0]}$  and  $\gamma_1|_{[a_1, b_1]}$  intersect  $\mathcal{F}$ -transversely (at the leaf  $\phi$ ), if  $\phi_{\gamma_0(t_0)}$  is below  $\phi_{\gamma_1(t_1)}$  relative to  $\phi$ , and  $\phi_{\gamma_0(b_0)}$  is above  $\phi_{\gamma_1(b_1)}$  relative to  $\phi$ .

More generally, suppose  $\mathcal{F}$  is an oriented singular foliation on a surface  $S$ . Let again  $\gamma_0 : [a_0, b_0] \rightarrow \text{Dom}(\mathcal{F}), \gamma_1 : [a_1, b_1] \rightarrow \text{Dom}(\mathcal{F})$  be  $\mathcal{F}$ -transverse paths, and assume there exists a leaf  $\phi$  containing both  $\gamma_0(t_0)$  and  $\gamma_1(t_1)$  for some times  $t_0 \in (a_0, b_0)$  and  $t_1 \in (a_1, b_1)$ . We say  $\gamma_0$  and  $\gamma_1$  intersect  $\mathcal{F}$ -transversely (at the leaf  $\phi$ ) if,  $\gamma_0$  and  $\gamma_1$  can be lifted to the paths  $\gamma_0^{\text{uni}}, \gamma_1^{\text{uni}}$  on the universal covering space  $\text{Dom}(\mathcal{F})^{\text{uni}}$  (see the paragraph after lemma 2.1), such that  $\gamma_0^{\text{uni}}(t_0)$  and  $\gamma_1^{\text{uni}}(t_1)$  belong to the same leaf  $\phi^{\text{uni}}$ , and such that,  $\gamma_0^{\text{uni}}$  and  $\gamma_1^{\text{uni}}$  intersect  $\mathcal{F}^{\text{uni}}$ -transversely at the leaf  $\phi^{\text{uni}}$ . Note that it is possible for a path to have a  $\mathcal{F}$ -transverse intersection with itself by considering two distinct lifts of the path to  $\text{Dom}(\mathcal{F})^{\text{uni}}$ . In this case we say that the path has a  $\mathcal{F}$ -transverse self-intersection.

The central result of [15], often referred to as the *Forcing Proposition*, is stated in the following lemma.

**Lemma 2.6 (proposition 20 of [15]).** *Let  $f$  be a homeomorphism of a surface  $S$  isotopic to the identity,  $I$  a maximal isotopy for  $f$ ,  $\mathcal{F}$  a singular oriented foliation transverse to  $I$ . Let  $\gamma_0 : [a_0, b_0] \rightarrow \text{Dom}(\mathcal{F})$  and  $\gamma_1 : [a_1, b_1] \rightarrow \text{Dom}(\mathcal{F})$  be two  $\mathcal{F}$ -transverse paths and assume they intersect  $\mathcal{F}$ -transversely at  $\phi = \phi_{\gamma_0(t_0)}$  and that  $\gamma_0(t_0) = \gamma_1(t_1)$ . If  $n_0, n_1$  are positive integers such that  $\gamma_0$  is admissible of order  $n_0$  and  $\gamma_1$  is admissible of order  $n_1$ , then the concatenations  $\gamma_0|_{[a_0, t_0]} \gamma_1|_{[t_1, b_1]}$  and  $\gamma_1|_{[a_1, t_1]} \gamma_0|_{[t_0, b_0]}$  are both admissible of order  $n_0 + n_1$ .*

We remark that sometimes  $\gamma_0(t_0)$  and  $\gamma_1(t_1)$  are not the same point but they belong to the same leaf of the foliation. In such cases, the path we obtain ‘concatenates’ the two segments only up to  $\tilde{\mathcal{F}}$ -equivalence. We also refer to these final paths as concatenations.

We will also need the following result. We stress that in the statement of the next result, the existence of the resulting  $\gamma'$  is achieved by ‘cutting’ the original path  $\gamma$  through an application of the above main forcing proposition. The complete proof would essentially replicate the proof of corollary 24 in [15], so we omit the full details.

**Lemma 2.7.** *Let  $\gamma : [a, b] \rightarrow \text{Dom}(\mathcal{F})$  be an  $n$ -admissible  $\mathcal{F}$ -transverse path. Given  $a \leq a' < b' \leq b$ , there exists an  $n$ -admissible transverse path  $\gamma'$  that is the same as  $\gamma$  in  $[a, a']$  and in  $[b', b]$  and such that  $\gamma'|_{[a, b]}$  has no  $\mathcal{F}$ -transverse self-intersection, and the image of  $\gamma'|_{[a', b']}$  is contained in that of  $\gamma|_{[a', b']}$ .*

Consider a singular transverse foliation  $\tilde{\mathcal{F}}$  on  $\mathbb{R}^2$ , and fix an  $\tilde{\mathcal{F}}$ -transverse simple loop  $\tilde{B}$ , whose natural extension is denoted by  $\tilde{\beta}$ . The set of leaves that are intersected by  $\tilde{\beta}$  is

a topological open annulus, which we denote by  $U_{\tilde{\beta}}$ . Assume  $\tilde{\gamma} : J \rightarrow \text{Dom}(\tilde{\mathcal{F}})$  is an  $\tilde{\mathcal{F}}$ -transverse path. Following [14], let us make the following useful definitions.

**Definition 2.8 (drawing loops and drawing components for transverse paths).** We say  $\tilde{\gamma}$  draws the loop  $\tilde{B}$  whose natural extension is  $\tilde{\beta}$ , if there exist times  $a < b$  such that  $\tilde{\gamma}|_{[a,b]}$  is  $\tilde{\mathcal{F}}$ -equivalent to  $\tilde{\beta}|_{[t,t+1]}$  for some  $t$ . A connected component  $I$  of the set  $\{t \in J \mid \tilde{\gamma}(t) \in U_{\tilde{\beta}}\}$  is called a drawing component if  $\tilde{\gamma}|_I$  draws  $\tilde{B}$ .

**Definition 2.9 (crossing loops and crossing components for transverse paths).** We say  $\tilde{\gamma}$  crosses  $\tilde{B}$ , if there are times  $a, b \in J$  such that  $\tilde{\gamma}(a)$  and  $\tilde{\gamma}(b)$  belong to two different connected components of the complement of  $U_{\tilde{\beta}}$ , one of which is bounded and the other which is unbounded. A connected component  $I$  of the set  $\{t \in J \mid \tilde{\gamma}(t) \in U_{\tilde{\beta}}\}$  is called a crossing component if both ends of  $I$  belong to  $J$  and if  $\tilde{\gamma}|_I$  crosses  $\tilde{B}$ .

We refer to the original paper [14] for more details, but let us stress that drawing/crossing components are subintervals of the domain of a transverse path, and are always defined in relation to a given simple loop  $\tilde{B}$ , although this dependency will not be remarked when it is clear from the context. The following lemma is a reformulation of proposition 24 of [14].

**Lemma 2.10.** *Assume  $S$  has genus 0. Denote by  $\gamma_0, \gamma_1$  two  $\mathcal{F}$ -transverse paths, and  $\beta$  an  $\mathcal{F}$ -transverse loop. Suppose an interval  $J_0 = [a_0, b_0]$  is such that  $(a_0, b_0)$  is a drawing component of  $\gamma_0$ , and an interval  $J_1 = [a_1, b_1]$  is such that  $(a_1, b_1)$  is either a drawing component or a crossing component of  $\gamma_1$ . Assume also  $\gamma_0(b_0)$  and  $\gamma_1(a_1)$  belong to the same connected component of the complement of  $U_{\beta}$ . Then  $\gamma_0|_{J_0}$  and  $\gamma_1|_{J_1}$  have an  $\mathcal{F}$ -transverse intersection. Note that  $J_0$  and  $J_1$  may be the same.*

**Proof.** The conclusion follows from the proof of proposition 24 of [14]. □

The following result inspired our main theorem, which was already described in the introduction. We say an  $\mathcal{F}$ -transverse path  $\gamma : \mathbb{R} \rightarrow \text{Dom}(\mathcal{F})$  is  $\mathcal{F}$ -recurrent if for any compact interval  $J \subset \mathbb{R}$  and any  $t \in \mathbb{R}$ , there exists some interval  $J' \subset [t, \infty)$ , such that  $\gamma|_{J'}$  and  $\gamma|_J$  are  $\mathcal{F}$ -equivalent. If it also satisfies that for any  $t$  there is  $J'' \in (-\infty, t]$  such that  $\gamma|_{J''}$  and  $\gamma|_J$  are  $\mathcal{F}$ -equivalent, then we say  $\gamma$  is  $\mathcal{F}$ -bi-recurrent.

**Lemma 2.11 (proposition 43 of [15]).** *Let  $S$  be a surface,  $f$  a homeomorphism of  $S$  isotopic to the identity,  $I$  a maximal isotopy for  $f$  and  $\mathcal{F}$  a singular transverse foliation for  $I$ . Let  $\tilde{S}$  be the universal covering space of  $S$ , and let  $\tilde{f}, \tilde{I}, \tilde{\mathcal{F}}$  be, respectively, lifts of  $f, I, \mathcal{F}$  to  $\tilde{S}$ . Assume an  $\tilde{\mathcal{F}}$ -transverse path  $\tilde{\gamma} : \mathbb{R} \rightarrow \text{Dom}(\tilde{\mathcal{F}})$  is admissible and is  $\tilde{\mathcal{F}}$ -bi-recurrent, and there are three distinct deck transformations  $T_1, T_2, T_3$  and certain leaf  $\tilde{\phi}$  of  $\tilde{\mathcal{F}}$ , such that  $\tilde{\gamma}$  intersects all three translates  $T_i(\tilde{\phi})$ ,  $i = 1, 2, 3$ . Then there exists non-contractible periodic orbits for  $f$  of arbitrarily large prime period.*

The previous lemma is dependent of the next one, which we also use in this work:

**Lemma 2.12 (proposition 26 of [15]).** *Suppose an admissible  $\mathcal{F}$ -transverse loop  $\Gamma$  has an  $\mathcal{F}$ -transverse self-intersection. In other words, its natural extension is an admissible path  $\tilde{\gamma}$*

and there is a non-trivial deck translation  $T$  so that  $T\tilde{\gamma}$  and  $\tilde{\gamma}$  intersect  $\tilde{F}$ -transversely. Then  $f$  admits non-contractible periodic orbits of arbitrarily large prime period.

2.5. Torus rotation set theory and bounded deviations

This subsection is specific for dynamics on the two torus  $\mathbb{T}^2$ . Given  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$  which is a lift of  $f \in \text{Homeo}_0(\mathbb{T}^2)$ , the Misiurewicz–Ziemann rotation set (as introduced in [16]) is defined as follows.

$$\rho(\tilde{f}) := \left\{ \rho \mid \rho = \lim_{i \rightarrow \infty} \frac{1}{n_i} (\tilde{f}^{n_i}(\tilde{z}_i) - \tilde{z}_i), \text{ for some } \tilde{z}_i \in \mathbb{R}^2 \text{ and some } n_i \rightarrow +\infty \right\}. \tag{2.6}$$

One can also define a rotation vector for a fixed point  $z \in \mathbb{T}^2$ ,

$$\rho(\tilde{f}, z) := \lim_{n \rightarrow \infty} \frac{1}{n} (\tilde{f}^n(\tilde{z}) - \tilde{z}), \tilde{z} \in \pi_{\mathbb{T}^2}^{-1}(z), \tag{2.7}$$

which is only well-defined when the limit exists. Note that in this definition, the limit does not depend on the choice of the lift  $\tilde{z}$  of  $z$  when the limit exists. Moreover, we can denote  $\mathcal{M}_f(\mathbb{T}^2)$  the set of  $f$ -invariant Borel probability measures on  $\mathbb{T}^2$ . Also denote by  $\mathcal{M}_{f,E}(\mathbb{T}^2) \subseteq \mathcal{M}_f(\mathbb{T}^2)$  the subset of ergodic measures. For the lift  $\tilde{f}$ , consider the displacement function  $\Delta_{\tilde{f}}: \mathbb{T}^2 \rightarrow \mathbb{R}^2$ , such that  $\Delta_{\tilde{f}}(z) = \tilde{f}(\tilde{z}) - \tilde{z}$ . Note again here the choice of the lift  $\tilde{z}$  does not affect this value, since  $\tilde{f}$  commutes with all deck transformations (i.e. integer translations). For any  $\mu \in \mathcal{M}_f(\mathbb{T}^2)$ , define  $\rho_\mu(\tilde{f}) := \int_{\mathbb{T}^2} \Delta_{\tilde{f}}(z) d\mu(z)$ . Then define  $\rho_{\text{mes}}(\tilde{f}) := \{ \rho_\mu(\tilde{f}) \mid \mu \in \mathcal{M}_f(\mathbb{T}^2) \}$ . A rotation vector  $v \in \mathbb{R}^2$  is said to be realized by a measure  $\mu$  if  $\rho_\mu(\tilde{f}) = v$ . One has:

**Lemma 2.13 (see [16], especially section 3).** *For any  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$ ,  $\rho(\tilde{f})$  and  $\rho_{\text{mes}}(\tilde{f})$  coincide, and it is always a compact and convex subset of  $\mathbb{R}^2$ . Moreover, the extremal points of  $\rho(\tilde{f})$  are always realizable by some  $\mu \in \mathcal{M}_f(\mathbb{T}^2)$ .*

The natural question of realizing a rotation set has historically attracted much attention. Several realization results are recorded in the following lemma, which are relevant with the pursuit of non-contractible periodic orbits.

**Lemma 2.14 (theorem 3.2 of [6]).** *Given  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$  and suppose  $\rho(\tilde{f})$  has non-empty interior. Then every rational vector  $v$  in the interior of  $\rho(\tilde{f})$  is realizable by a periodic orbit, that is, if  $v = (\frac{p_1}{q}, \frac{p_2}{q})$  is written in irreducible form, there exists a periodic point  $x$  with period  $q$  such that  $\tilde{f}^q(\tilde{x}) = \tilde{x} + (p_1, p_2)$ . In particular, if  $v$  is not null,  $x$  lies in a non-contractible periodic orbit.*

**Lemma 2.15 (main result of [7]).** *Let  $\tilde{f} \in \widetilde{\text{Homeo}}_{0,nw}(\mathbb{T}^2)$  and  $\rho(\tilde{f})$  is a line segment. If  $v \in \rho(\tilde{f})$  is a rational vector, then it is realizable by a periodic orbit. In particular, if  $\rho(\tilde{f})$  contains at least two distinct rational vectors, then  $f$  admits non-contractible periodic orbits.*

A homeomorphism  $f \in \text{Homeo}_0(\mathbb{T}^2)$  is called irrotational if  $\rho(\tilde{f}) = \{(0, 0)\}$  for some lift  $\tilde{f}$  of  $f$ , which we call the the irrotational lift.  $f \in \text{Homeo}_0(\mathbb{T}^2)$  is called *Hamiltonian* if it preserves

a Borel probability measure  $\mu$  with full support and no atoms, and  $\rho_\mu(\tilde{f}) = \{(0, 0)\}$ , for some lift  $\tilde{f}$  of  $f$ , which we call the *Hamiltonian lift*. Now we can restate the the motivating result from [15].

**Lemma 2.16 (corollary I of [15]).** *Let  $f \in \text{Homeo}_0(\mathbb{T}^2)$  be Hamiltonian with Hamiltonian lift  $\tilde{f}$ . Assume its fixed point set is contained in a topological disk. Then either  $\tilde{f}$  is a uniformly bounded lift, or  $f$  admits non-contractible periodic orbits.*

Another relevant result, also from [15], resolved one case of the famous Franks–Misiurevicz conjecture.

**Lemma 2.17 (theorem C of [15]).** *Let  $f \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$ . If the boundary of  $\rho(\tilde{f})$  includes a line segment with irrational slope and contains exactly one rational point, then the rational point must be of the endpoints of this segment.*

Recall that for any  $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ ,  $\text{pr}_v : w \mapsto \langle w, v \rangle / \|v\|$  denotes the projection of a vector  $w$  into the line passing through the origin in the direction  $v$ . We say  $\tilde{f}$  has *bounded deviation* (from its rotation set  $\rho(\tilde{f})$ ) along a direction  $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , if there exists a uniform constant  $C > 0$ , such that for any  $n \geq 1$ , and for any lift  $\tilde{x}$  of  $x$ ,

$$\sup_{w \in \rho(\tilde{f})} \left\{ \text{pr}_v \left( \tilde{f}^n(\tilde{x}) - \tilde{x} - nw \right) \right\} < C. \tag{2.8}$$

We will be mainly interested in the case when  $\rho(\tilde{f})$  is a line segment containing  $(0, 0)$  and some non-null vector  $(\alpha, \beta)$ . In this case we say that the rotation set has direction  $(\alpha, \beta)$ .

We next state several recent results on establishing bounded deviation under different contexts. The first case we consider is when the rotation set has interior.

**Lemma 2.18 (theorem D of [15]).** *Assume  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$  and  $\rho(\tilde{f})$  has non-empty interior. Then  $\tilde{f}$  admits bounded deviation along every direction  $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , and the bound is uniform in all direction in the unit circle  $\mathbb{S}^1 = \{v \in \mathbb{R}^2 \mid \|v\| = 1\}$ .*

The following two results focus on perpendicular directions to rotation sets which are line segments.

**Lemma 2.19 (theorem A of [5]).** *Suppose  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$  and  $\rho(\tilde{f})$  is a line segment with rational slope, containing  $(0, 0)$  and with direction  $v$ . Then  $\tilde{f}$  admits bounded deviation along both perpendicular directions  $v^\perp$  and  $-v^\perp$ .*

**Lemma 2.20 (theorem A of [19]).** *For some  $\tilde{f} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$ , assume  $\rho(\tilde{f})$  is a line segment from  $(0, 0)$  to some vector  $v = (\alpha, \beta)$ , where  $\alpha/\beta$  is irrational, then  $\tilde{f}$  admits bounded deviation along the perpendicular directions  $v^\perp$  and  $-v^\perp$ .*

As a summary of all these results on rotation sets, let us finish this section proving that theorem 1.3 implies theorem 1.2.

**Proof of theorem 1.2 based on theorem 1.3.** If  $f$  is Hamiltonian the result follows directly from lemma 2.16. Assume then that  $f$  is not Hamiltonian. By assumption,  $\tilde{f}$  admits fixed point and thus the rotation set  $\rho(\tilde{f})$  contains  $(0, 0)$ . Recall that  $\rho(\tilde{f})$  is convex compact subset of  $\mathbb{R}^2$ . If it has interior, then it contains some rational vector in its interior. Therefore, by lemma 2.14  $\tilde{f}$  admits non-contractible periodic orbit, which is excluded by our assumption. A similar argument, using the fact that  $f$  preserves area and lemma 2.15, shows that  $\rho(\tilde{f})$  cannot be a segment with two rational points, therefore it is either just the origin or a line segment with irrational

slope containing  $(0, 0)$ . Since we have also assumed that  $f$  is not Hamiltonian, the possibility that  $\rho(\tilde{f}) = \{(0, 0)\}$  is also excluded. By lemma 2.17,  $\rho(\tilde{f})$  must be a segment whose two end-points are  $(0, 0)$  and some vector  $(\alpha, \beta)$  with  $\alpha/\beta$  irrational. Then, theorem 1.3 and lemma 2.20 together show that  $\tilde{f}$  has a strong irrational dynamical direction  $(\alpha, \beta)$ .  $\square$

### 3. Non-wandering points and non-contractible periodic orbits

Assume  $S$  is a closed connected surface of genus  $g \geq 1$ ,  $f \in \text{Homeo}_0(S)$ ,  $I$  is a maximal isotopy for  $f$  and  $\mathcal{F}$  is a foliation transverse to  $I$ . Further assume that  $\tilde{\mathcal{F}}$  and  $\tilde{I}$  are the lifts of  $\mathcal{F}$  and  $I$  to  $\tilde{S}$ , the universal covering space of  $S$ , which is homeomorphic to  $\mathbb{R}^2$ . The main result of this section is the following proposition, which states that, if an admissible path intersects twice five distinct deck images of one leaf in a nice way, then  $f$  admits non-contractible periodic orbits.

**Proposition 3.1.** *Let  $0 = t_0 < t_1 < t_2 < t_3 < t_4 < 1$ , and assume there exists  $\tilde{\beta} : [0, 1 + t_4] \rightarrow \text{Dom}(\tilde{\mathcal{F}})$  which is an admissible  $\tilde{\mathcal{F}}$ -transverse path, satisfying the following properties:*

- *There exists a leaf  $\tilde{\phi}_0$  of  $\tilde{\mathcal{F}}$ , and five pairwise distinct deck transformations  $u_i, 0 \leq i \leq 4$ , such that  $\tilde{\beta}(t_i) = \beta(1 + t_i) \in u_i(\tilde{\phi}_0)$  for  $i = 0, 1, 2, 3, 4$ .*
- *If  $t \in (0, t_1)$  then both  $\tilde{\beta}(t)$  and  $\tilde{\beta}(1 + t)$  do not belong to  $u(\tilde{\phi}_0)$  for any deck transformation  $u$ .*

*Then  $f$  admits non-contractible periodic orbits of arbitrarily large prime period, and the path  $\tilde{\beta}$  has a  $\tilde{\mathcal{F}}$ -transverse self-intersection.*

The proof of this proposition is postponed to section 3.2.

#### 3.1. Proof of theorem 1.1 using proposition 3.1

Let us first show an immediate consequence of proposition 3.1.

**Corollary 3.2.** *Let  $\tilde{x}_0 \in \Omega(\tilde{f})$ , and assume that there are distinct deck transformations  $u_i, 0 \leq i \leq 4$ , and a leaf  $\tilde{\phi}_0$  of  $\tilde{\mathcal{F}}$ , such that the transverse trajectory  $\tilde{I}_{\tilde{\mathcal{F}}}^{\mathbb{R}}(\tilde{x}_0)$  intersects  $u_i(\tilde{\phi}_0)$  for each  $0 \leq i \leq 4$ . Then,  $\tilde{f}$  admits non-contractible periodic orbits.*

**Proof.** Up to replacing  $\tilde{x}_0$  with some  $\tilde{f}^{-k}(\tilde{x}_0)$  and renaming the deck transformations if necessary, we can assume that the transverse trajectory  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{x}_0)$  intersects each of  $u_0(\tilde{\phi}_0), u_1(\tilde{\phi}_0), \dots, u_4(\tilde{\phi}_0)$  (in that order), for some large integer  $T$ . We can also assume, otherwise we rename again the deck transformations, that if  $t_1 > 0$  is the first instant such that  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{x}_0)(t_1)$  belongs to  $u_1(\tilde{\phi}_0)$ , then  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{x}_0)$  does not meet any nontrivial deck translate of  $u_0(\tilde{\phi}_0)$  for  $0 \leq t < t_1$ . Let also  $t_0 = \max_{t \in [0, t_1]} \{t \mid \tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{x}_0)(t) \in u_0(\tilde{\phi}_0)\}$ . By lemma 2.4, we can find a small neighbourhood  $\tilde{V}$  of  $\tilde{x}_0$ , with the following properties.

- (1) for any  $\tilde{y}, \tilde{y}' \in \tilde{V}$ , the transverse path  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{y})$  is  $\tilde{\mathcal{F}}$ -equivalent to a subpath of the trajectory  $\tilde{I}_{\tilde{\mathcal{F}}}^{T+2}(\tilde{f}^{-1}(\tilde{y}'))$ .
- (2) for any point  $\tilde{y} \in \tilde{V}$ , the transverse path  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{y})$  intersects the same 5 deck images of  $\tilde{\phi}_0$  as above, in the same order.

Since  $\tilde{x}_0$  is non-wandering, we can find  $\tilde{y}_* \in \tilde{V}$  and some integer  $M > T + 2$ , such that  $\tilde{f}^M(\tilde{y}_*)$  is also in  $\tilde{V}$ . By the choice of  $\tilde{V}$ , the paths  $\tilde{I}_{\tilde{\mathcal{F}}}^{T+2}(\tilde{f}^{-1}(\tilde{y}_*))$  and  $\tilde{I}_{\tilde{\mathcal{F}}}^{T+2}(\tilde{f}^{M-1}(\tilde{y}_*))$  both contain subpaths that are  $\tilde{\mathcal{F}}$ -equivalent to  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{x}_0)$ , and moreover contains paths that are  $\tilde{\mathcal{F}}$ -equivalent to the restriction of  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{x}_0)$  to  $[t_0, T]$  and that therefore meets the same five deck images of  $\tilde{\phi}_0$ , in the same order. By construction, one finds therefore a subpath of  $\tilde{I}_{\tilde{\mathcal{F}}}^{M+T+1}(\tilde{f}^{-1}(\tilde{y}_*))$ . At its beginning, this subpath is equivalent to the restriction of  $\tilde{I}_{\tilde{\mathcal{F}}}^T(\tilde{x}_0)$  to  $[t_0, T]$ . It satisfies the two conditions required of  $\tilde{\beta}$  in proposition 3.1, and is also admissible. It suffices then to take a reparametrization of this subpath and the conclusion follows.  $\square$

In order to show theorem 1.1, let us prove a covering lemma first.

**Lemma 3.3.** *Under the hypotheses of theorem 1.1, there exists a bounded fundamental domain  $\tilde{S}_0$  and a finite family of bounded open disks  $\{\tilde{E}_j\}_{j=1}^\kappa$  in  $\tilde{S}$ , satisfying the following properties.*

- (1) *The union  $\bigcup_{j=1}^\kappa \tilde{E}_j$  projects to a fully essential open set  $O$ , whose complement  $O^c$  in  $S$  contains  $\text{Fix}(I)$ .*
- (2) *Any  $\tilde{f}$ -orbit segment intersecting both  $\tilde{S}_0$  and its complement in  $\tilde{S}$ , must intersect some  $\tilde{E}_j$  from the above list.*
- (3) *For any  $j = 1, \dots, \kappa$ , there exists some leaf  $\tilde{\phi}_j$  of  $\tilde{\mathcal{F}}$  such that, for any  $\tilde{z}_0 \in \tilde{E}_j$ , the transverse path  $\tilde{I}_{\tilde{\mathcal{F}}}^2(\tilde{f}^{-1}(\tilde{z}_0))$  intersects  $\tilde{\phi}_j$ .*

**Proof.** By the assumption that  $\text{Fix}(f)$  is inessential, we also know that  $\text{Fix}(I)$  is a closed and inessential set. This means that  $\text{Fix}(I)$  is contained in the interior of an open topological disk  $U_0$ . Let  $\tilde{U}_0$  be a connected component of the lift of  $U_0$ , and let  $\tilde{S}_0$  be a (closed) fundamental domain for the surface  $S$  in its universal covering space, which is compact and contains  $\tilde{U}_0$ . Consider the compact set

$$\tilde{K} = \tilde{S}_0 \setminus \left( \tilde{f}^{-1}(\tilde{U}_0) \cap \tilde{U}_0 \right). \tag{3.1}$$

Since  $I$  is a maximal isotopy, the set  $\tilde{f}^{-1}(\tilde{U}_0) \cap \tilde{U}_0$  contains  $\text{Fix}(\tilde{I}) \cap \tilde{S}_0$ . Moreover, for any  $\tilde{f}$ -orbit entering the set  $\tilde{f}^{-1}(\tilde{U}_0) \cap \tilde{U}_0$ , its next iterate belongs to  $\tilde{U}_0$  and therefore does not leave  $\tilde{S}_0$ . This property also holds for any subset of  $\tilde{f}^{-1}(\tilde{U}_0) \cap \tilde{U}_0$  and it will eventually ensure item (2) in the conclusion. Since each point in  $\text{Fix}(I)$  has a lift in the interior of  $\tilde{f}^{-1}(\tilde{U}_0) \cap \tilde{U}_0$ , it follows that  $\tilde{K}$  is a subset of  $\text{Dom}(\tilde{\mathcal{F}})$ . By lemma 2.4, for each  $\tilde{y} \in \tilde{K}$ , there exists an open topological disk  $\tilde{E}_{\tilde{y}}$  containing  $\tilde{y}$ , such that if  $\tilde{z}_0 \in \tilde{E}_{\tilde{y}}$ , then  $\tilde{I}_{\tilde{\mathcal{F}}}^2(\tilde{f}^{-1}(\tilde{z}_0))$  intersects  $\tilde{\phi}_{\tilde{y}}$ . We also need these disks to be small enough to not intersect  $\text{Fix}(\tilde{I})$ . The result now follows from the compactness of  $\tilde{K}$ .  $\square$

The proof of theorem 1.1 reduces to the following lemma. Indeed, observing corollary 3.2, in order to obtain non-contractible periodic orbits, we only need to show the following.

**Lemma 3.4.** *There exists a leaf  $\tilde{\phi}$ , and some point  $\tilde{z}_0$ , so that  $\tilde{I}_{\tilde{\mathcal{F}}}^{\mathbb{R}}(\tilde{z}_0)$  intersects five distinct deck translations of  $\tilde{\phi}$ .*

**Proof.** Obtain  $\tilde{S}_0$ ,  $\kappa$  and  $\tilde{E}_j, 1 \leq j \leq \kappa$  from lemma 3.3. Let  $L_0$  be the diameter of  $\tilde{S}_0$ , and let  $L_1 = \max_{\tilde{z} \in \tilde{S}} \|\tilde{f}(\tilde{z}) - \tilde{z}\|$ , which is well defined since  $\tilde{f}$  commutes with the deck transformations. Recall we have assumed that the  $\tilde{f}$ -orbits are not uniformly bounded. Then we choose

$M \geq (4\kappa + 2)(L_0 + L_1)$ , and for some  $\tilde{z}_0 \in \Omega(\tilde{f})$ , and some positive integer  $N_1$ ,  $\|\tilde{f}^{N_1}(\tilde{z}_0) - \tilde{z}_0\| > M$ . Then one can find  $(4\kappa + 2)$  integers  $k_l$ , with  $0 \leq k_1 < k_2 < \dots < k_{4\kappa+2} \leq N_1$  such that  $\|\tilde{f}^{k_{l_1}}(\tilde{z}_0) - \tilde{f}^{k_{l_2}}(\tilde{z}_0)\| > L_0$  if  $l_1 < l_2$  are two elements chosen from  $\{1, \dots, 4\kappa + 2\}$ . Therefore one can find distinct deck transformations  $u_l$ ,  $1 \leq l \leq (4\kappa + 2)$  such that  $\tilde{f}^{k_l}(\tilde{z}_0) \in u_l(\tilde{S}_0)$ .

Again by lemma 3.3, one finds  $(j_l)_{1 \leq l \leq 4\kappa+1}$  and  $(k'_l)_{1 \leq l \leq 4\kappa+1}$ , with  $k_l \leq k'_l < k_{l+1}$  such that  $\tilde{f}^{k'_l}(\tilde{z}_0) \in u_l(\tilde{E}_{j_l})$ . So by pigeonhole principle, there exists some disk  $\tilde{E}$  from the list  $\{\tilde{E}_j\}_{j=1}^\kappa$ , such that the trajectory  $\tilde{I}_{\tilde{F}}^{[0, N_1]}(\tilde{z}_0)$  intersects at least five distinct translates of  $\tilde{E}$ . Then lemma 3.3, point (3) shows one can find a corresponding leaf  $\tilde{\phi}$  so that  $\tilde{I}_{\tilde{F}}^{\mathbb{R}}(\tilde{z}_0)$  intersects five distinct deck translations of this leaf.  $\square$

### 3.2. Proving proposition 3.1

This subsection is devoted to the proof of proposition 3.1. The proof is relatively long. We start with some initial preparations.

We assume that  $\tilde{S}$  is connected (homeomorphic to  $\mathbb{R}^2$ ), otherwise we work with the connected component of it that contains  $\tilde{\phi}_0$ . Without loss of generality and to lighten the notation, we assume that  $u_0$  is the identity, otherwise we just rename  $u_0(\tilde{\phi}_0)$  as  $\tilde{\phi}_0$ . Let  $u_1 \neq id$  be a deck transformation, and let  $\tilde{\gamma} : [0, 1] \rightarrow \text{Dom}(\tilde{F}) \subset \tilde{S}$  be an  $\tilde{F}$ -transverse path such that  $\tilde{\gamma}(0) \in \tilde{\phi}_0$  and  $\tilde{\gamma}(1) = u_1(\tilde{\gamma}(0)) \in u_1(\tilde{\phi}_0)$ . Assume further that the projection of  $\tilde{\gamma}|_{(0,1)}$  does not intersect the projection of  $\tilde{\phi}_0$ , and that  $\tilde{\gamma}$  has no  $\tilde{F}$ -transverse intersection with  $u_1^j(\tilde{\gamma})$  for all  $j \in \mathbb{Z}$ , which implies it has no transverse self-intersection. The path  $\tilde{\gamma}$  can be taken as a simple path whose image is contained in the image of the restriction of  $\tilde{\beta}$  to the interval  $[0, t_1]$ , by merely cutting away some loops. Note that we did not require  $\tilde{\gamma}$  to be admissible; we only required that it be  $\tilde{F}$ -transverse. Then, we can look at the quotient  $\check{S} = \tilde{S}/u_1$ , which is homeomorphic to the open annulus. Denote by  $\check{\gamma}$  the projection of the path  $\tilde{\gamma}|_{[0,1]}$  to  $\check{S}$ , and observe that it is  $\check{F}$ -equivalent to a closed  $\check{F}$ -transverse loop, where  $\check{F}$  is the projection of  $\tilde{F}$  to  $\check{S}$ . Extend the domain of  $\check{\gamma}$  to the whole line  $\mathbb{R}$  by defining

$$\check{\gamma}(t + 1) = u_1(\check{\gamma}(t)) \text{ for all } t \in \mathbb{R}. \tag{3.2}$$

Recall that a line is a proper and injective map from  $\mathbb{R}$  to  $\tilde{S}$ . The next lemma is very similar to previously known results as lemma 27 and corollary 28 of [19], but we include a proof here for the convenience of the readers.

**Lemma 3.5.**  *$\check{\gamma}$  is a line.*

**Proof.** That  $\check{\gamma}$  is proper follows directly from the fact that  $u_1$  is a non-trivial deck transformation. We want to show that  $\check{\gamma}$  is injective. From standard Brouwer theory we know that if  $\check{\gamma}|_{[0,1]}$  is a translation arc, that is, if  $\check{\gamma}|_{[0,1]}$  is disjoint from  $u_1(\check{\gamma}|_{[0,1]})$ , then it holds that  $(u_1)^{n_1}(\check{\gamma}|_{[0,1]})$  is disjoint from  $(u_1)^{n_2}(\check{\gamma}|_{[0,1]})$  whenever  $n_1 \neq n_2$ . Therefore we need only to consider three cases:

- (a) There exists  $0 < t_0 < t_1 < 1$  such that  $\check{\gamma}(t_0) = \check{\gamma}(t_1)$ .
- (b) There exists  $0 < t_0 < t_1 < 1$  such that  $\check{\gamma}(t_0) = u_1(\check{\gamma}(t_1))$ .
- (c) There exists  $0 < t_0 < t_1 < 1$  such that  $\check{\gamma}(t_0) = (u_1)^{-1}(\check{\gamma}(t_1))$ .

In case (a), one finds that there exists some  $t_0 \leq s_0 < s_1 \leq t_1$  such that  $\check{\gamma}(s_0)$  and  $\check{\gamma}(s_1)$  lie in the same leaf of  $\check{F}$ , but such that for all  $s_0 \leq s < s' < s_1$  one has that  $\check{\gamma}(s)$  and  $\check{\gamma}(s')$  belong to

different leaves. This implies that  $\tilde{\gamma}|_{[s_0, s_1]}$  is  $\tilde{\mathcal{F}}$ -equivalent to a simple closed loop  $\Gamma$ , and since  $\Gamma$  does not intersect any translate of  $\check{\phi}_0$ , one has that there exists a drawing interval  $J \subset [0, 1]$  for  $\Gamma$ . In case both  $\tilde{\gamma}(0)$  and  $\tilde{\gamma}(1)$  lie in the same connected component of the complement of  $\Gamma$ , we have a contradiction using lemma 2.10, because we assumed  $\tilde{\gamma}$  has no  $\tilde{\mathcal{F}}$ -transverse self-intersection. If not, and we assume that  $\tilde{\gamma}(1)$  is in the bounded connected component of  $\Gamma^c$ , then there exists a smallest  $n > 1$  such that  $\tilde{\gamma}(n)$  is in the unbounded connected component of  $\Gamma^c$ , which is finite since  $\tilde{\gamma}$  is unbounded. In this case  $\tilde{\gamma}$  has a crossing component contained in  $[n - 1, n]$ . In other words, by lemma 2.10,  $\tilde{\gamma}|_{[0, 1]}$  has an  $\tilde{\mathcal{F}}$ -transverse intersection with  $\tilde{\gamma}|_{[n-1, n]} = (u_1)^{n-1}\tilde{\gamma}|_{[0, 1]}$ , which again contradicts our assumptions.

In case (b), proceeding as in the above paragraph we find  $t_0 \leq s_0 < s_1 \leq t_1$  such that  $\tilde{\gamma}|_{[s_0, s_1]}$  is  $\tilde{\mathcal{F}}$ -equivalent to a simple transverse loop  $\tilde{\Gamma}$  which is essential in  $\tilde{S}$ , and which does not intersect the leaf  $\check{\phi}_0$ , the projection of  $\tilde{\phi}_0$ . We can apply here again lemma 2.10 to obtain that  $\tilde{\gamma}|_{[0, 1]}$  has a  $\tilde{\mathcal{F}}$ -transverse self-intersection, which implies that  $\tilde{\gamma}|_{[0, 1]}$  must have an  $\tilde{\mathcal{F}}$ -transverse intersection with  $(u_1)^l(\tilde{\gamma}|_{[0, 1]})$  for some  $l \in \mathbb{Z}$ , again a contradiction. Case (c) is done almost exactly as case (b).  $\square$

**Definition 3.6.** Suppose  $\tilde{\gamma}, u_1, \tilde{\phi}_0$  are as in the previous paragraph. Let  $A_0 = A_0(\tilde{\gamma})$  denote the union of all leaves met by  $\tilde{\gamma}|_{[0, 1]}$ , and call it the *foliated block* for  $\tilde{\gamma}([0, 1])$ , or simply the *block*. For each  $k \in \mathbb{Z}$ , call the  $k$ th translated block as follows

$$A_k = A_k(\tilde{\gamma}) = (u_1)^k(A_0). \tag{3.3}$$

Now define

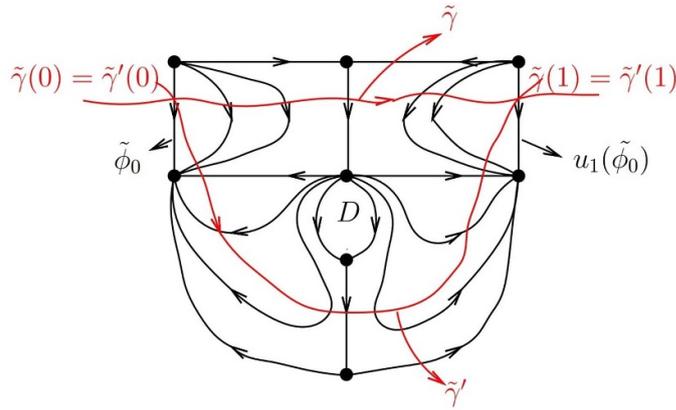
$$A = A(\tilde{\gamma}) = \bigcup_{k \in \mathbb{Z}} A_k = \bigcup_{k \in \mathbb{Z}} (u_1)^k(A_0). \tag{3.4}$$

In other words,  $A(\tilde{\gamma})$  is the infinite strip consisting of all the leaves intersected by  $\tilde{\gamma}|_{\mathbb{R}}$ . Denote by  $\mathcal{L}(A)$  the union of all the connected components of the complement of  $A$  that lie in the left of  $\tilde{\gamma}$ . Similarly, let  $\mathcal{R}(A)$  denote the union of all connected components of the complement of  $A$  that lie in the right of  $\tilde{\gamma}$ .

If an  $\tilde{\mathcal{F}}$ -transverse path  $\tilde{\beta}|_{[a, a+\varepsilon]}$  satisfies that  $\tilde{\beta}(a) \in \mathcal{L}(A)$  (respectively,  $\mathcal{R}(A)$ ) and  $\tilde{\beta}((a, a + \varepsilon]) \subset A_k$  for some  $k \in \mathbb{Z}$ , then we say the segment  $\tilde{\beta}|_{[a, a+\varepsilon]}$  enters  $A$  from  $\mathcal{L}(A)$  at the block  $A_k$ . (respectively, from  $\mathcal{R}(A)$  at the block  $A_k$ ). Similarly, we say  $\tilde{\beta}|_{[a-\varepsilon, a]}$  leaves  $A$  from  $\mathcal{L}(A)$  at the block  $A_k$  (respectively, from  $\mathcal{R}(A)$  at the block  $A_k$ ), if  $\tilde{\beta}([a - \varepsilon, a]) \subset A_k$  and  $\tilde{\beta}(a) \in \mathcal{L}(A)$  (respectively,  $\mathcal{R}(A)$ ).

**Definition 3.7.** Suppose two  $\tilde{\mathcal{F}}$ -transverse paths  $\tilde{\gamma}$  and  $\tilde{\gamma}' : [0, 1] \rightarrow \text{Dom}(\tilde{\mathcal{F}})$  are as described at the beginning of this subsection, and they satisfy that  $\tilde{\gamma}(0) = \tilde{\gamma}'(0) \in \tilde{\phi}_0$  and  $\tilde{\gamma}(1) = \tilde{\gamma}'(1) = u_1(\tilde{\gamma}(0)) \in u_1(\tilde{\phi}_0)$ , where  $u_1$  is a nontrivial deck transformation. Then one can obtain the blocks  $A_0 = A_0(\tilde{\gamma})$  and  $A'_0 = A_0(\tilde{\gamma}')$  as in definition 3.6 with respect to the paths  $\tilde{\gamma}$  and  $\tilde{\gamma}'$ , respectively. Also define the translated blocks  $A_k$  and  $A'_k$  as in (3.3) for all  $k \in \mathbb{Z}$ , as well as the infinite strips  $A = A(\tilde{\gamma})$  and  $A' = A(\tilde{\gamma}')$  as in (3.4), with respect to the extended infinite paths  $\tilde{\gamma}$  and  $\tilde{\gamma}'$ , respectively. We say  $A'$  is below  $A$  if  $A' \subset \mathcal{R}(A) \cup A$ . We say  $A'$  is above  $A$  if  $A' \subset \mathcal{L}(A) \cup A$ . If neither case above happens, we say  $A'$  and  $A$  are in mixed position.

Note that in particular, if  $A = A'$ , then they are both true that  $A$  is below  $A'$  and  $A$  is above  $A'$ . See figure 1 for an illustration of another more typical case of  $A'$  below  $A$ . In this case, the



**Figure 1.** Here is the case where  $A'$  is below  $A$ . We only depict the blocks  $A_0$  and  $A'_0$  above. The leaves of the foliation are in black and the transverse paths  $\tilde{\gamma}$  and  $\tilde{\gamma}'$  are in red.

closure of the bounded region  $D$  is the bounded connected component of the complement of  $A_0 \cup A'_0$ .

With the preparations done above, we can now start the proof of proposition 3.1. Let us make several first reductions. **Claim** that without loss of generality, one can assume that  $\tilde{\beta}|_{[0,t_1]}$  has no  $\tilde{\mathcal{F}}$ -transverse self-intersection. Suppose otherwise, that is,  $\tilde{\beta}|_{[0,t_1]}$  has an  $\tilde{\mathcal{F}}$ -transverse self-intersection at times  $t$  and  $t'$ , with  $0 < t < t' < t_1$ . Then we can apply lemma 2.7 to replace the transverse path by a new transverse path, also admissible, and still denoted by  $\tilde{\beta}$ , whose restriction to  $[0, t_1]$  shortens and avoids  $\tilde{\mathcal{F}}$ -transverse self-intersections, and moreover  $\tilde{\beta}$  satisfies the rest of the conditions in proposition 3.1.

Similarly, using forcing property, we can also assume that  $\tilde{\beta}|_{[0,t_1]}$  and  $\tilde{\beta}|_{[1,1+t_1]}$  have no  $\tilde{\mathcal{F}}$ -transverse intersection. To see this, suppose an  $\tilde{\mathcal{F}}$ -transverse intersection between the above two transverse paths happens at the time  $t \in [0, t_1]$  and  $t' \in [1, 1 + t_1]$ , respectively. Then we apply lemma 2.6 to obtain a new  $\tilde{\mathcal{F}}$ -transverse admissible path, which concatenates  $\tilde{\beta}|_{[0,1+t']}$  with  $\tilde{\beta}|_{[t,t_4+\varepsilon]}$ , which, after appropriate re-parametrization, avoids  $\tilde{\mathcal{F}}$ -transverse intersection and still satisfies the original properties.

We can also assume that  $\tilde{\beta}$  has no  $\tilde{\mathcal{F}}$ -transverse intersection with  $u(\tilde{\beta})$  for any deck transformation  $u$  other than the identity, because otherwise by lemma 2.12 we already know there exist non-contractible periodic orbits and we have nothing to prove.

With all the above reductions, we look at the path  $\tilde{\beta}$ . The path  $\tilde{\beta}|_{[0,t_1]}$  is  $\tilde{\mathcal{F}}$ -equivalent to some path  $\tilde{\gamma} : [0, 1] \rightarrow \text{Dom}(\tilde{\mathcal{F}})$ , with  $\tilde{\gamma}(1) = u_1(\tilde{\gamma}(0))$  and  $\tilde{\gamma}$  has the same properties as we have assumed in earlier discussion at the beginning of this sub-section. Likewise, the path  $\tilde{\beta}|_{[1,1+t_1]}$  is  $\tilde{\mathcal{F}}$ -equivalent to some path  $\tilde{\gamma}' : [0, 1] \rightarrow \text{Dom}(\tilde{\mathcal{F}})$ , with  $\tilde{\gamma}'(0) = \tilde{\gamma}(0)$  and  $\tilde{\gamma}'(1) = \tilde{\gamma}(1) = u_1(\tilde{\gamma}'(0))$ . Then by definition 3.7, they induce the foliated blocks  $A_0 = A_0(\tilde{\gamma}), A'_0 = A_0(\tilde{\gamma}')$ , and moreover the infinite strips  $A, A'$ , respectively. Note that if  $A$  and  $A'$  are in mixed position, then it follows that  $\tilde{\beta}|_{[0,t_1]}$  and  $\tilde{\beta}|_{[1,1+t_1]}$  intersect  $\tilde{\mathcal{F}}$ -transversely. Indeed, one can find two instants  $0 < s_0 < s_1 < 1$  such that  $\tilde{\gamma}'(s_0)$  and  $\tilde{\gamma}'(s_1)$  lie in different connected components of the strip  $A$ , while the restriction of  $\tilde{\gamma}|_{(s_0,s_1)}$  lies in  $A_0$ . This implies that  $\tilde{\gamma}|_{[0,1]}$  and  $\tilde{\gamma}|_{[s_0,s_1]}$  intersect  $\tilde{\mathcal{F}}$ -transversely. By previous reductions, we have assumed it is not the case. Thus, let

us suppose  $A'$  is below  $A$  and the other case is symmetric. Then we will look at a bigger region defined as follows. We remind the readers that as in (3.2), we can extend  $\tilde{\gamma}$  and  $\tilde{\gamma}'$  to curves defined on the whole line.

**Definition 3.8.** Define  $U$  to be the union of  $A$  and  $A'$  and all the closures of the bounded connected components of the complement of  $\tilde{\gamma}|_{[k,k+1]} \cup \tilde{\gamma}'|_{[k,k+1]}$  for all  $k \in \mathbb{Z}$ . Indeed, in the case that  $A'$  is below  $A$ ,  $U$  is the complement of  $\mathcal{R}(A') \cup \mathcal{L}(A)$ .

Viewing figure 1, we see  $U$  is simply the union of  $A_0, A'_0$ , the closure of  $D$  and all their  $u_1^j$  translates.

**Lemma 3.9.** *There exists some  $b \in (t_1, 1)$  such that  $\tilde{\beta}(b) \notin U$ .*

**Proof.** Suppose for contradiction that  $\tilde{\beta}|_{[t_1,1]} \subset U$ . The leaf  $\tilde{\phi}_0$  divides each of the strips  $A$  and  $A'$  into two disconnected subsets. Let us write  $A \setminus \tilde{\phi}_0 = A(1) \cup A(2)$  and  $A' \setminus \tilde{\phi}_0 = A'(1) \cup A'(2)$ , respectively. Since  $\tilde{\beta}|_{[0,1]}$  is  $\tilde{\mathcal{F}}$ -transverse, it follows that  $\tilde{\beta}|_{(0,1]} \subset A(2) \cup A'(2)$ . However, because  $\tilde{\beta}(0) = \tilde{\beta}(1)$ , we can find some small number  $\delta > 0$  so that  $\tilde{\beta}(1 - \delta) \in A(1) \cap A'(1)$ , which is absurd.  $\square$

After the Lemma, we can write

$$b_0 = \inf \left\{ b > t_1 \mid \tilde{\beta}(b) \notin U \right\}, \tag{3.5}$$

$$c_0 = \sup \left\{ c < 1 \mid \tilde{\beta}(c) \notin U \right\}. \tag{3.6}$$

Note that if a path leaves or enters  $U$ , it either does this from  $\mathcal{L}(A)$  or from  $\mathcal{R}(A')$ , due to the assumption that  $A'$  is below  $A$ . Thus, it suffices to discuss the following cases,

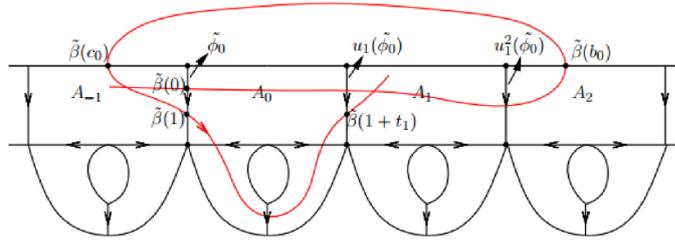
- Case (1).  $\tilde{\beta}(b_0) \in \mathcal{L}(A), \tilde{\beta}(c_0) \in \mathcal{L}(A)$ .
- Case (2).  $\tilde{\beta}(b_0) \in \mathcal{L}(A), \tilde{\beta}(c_0) \in \mathcal{R}(A')$ .
- Case (3).  $\tilde{\beta}(b_0) \in \mathcal{R}(A'), \tilde{\beta}(c_0) \in \mathcal{L}(A)$ .
- Case (4).  $\tilde{\beta}(b_0) \in \mathcal{R}(A'), \tilde{\beta}(c_0) \in \mathcal{R}(A')$ .

**End of the proof of proposition 3.1.** We try to discuss all four cases listed above.

**Case (1).**  $\tilde{\beta}(b_0) \in \mathcal{L}(A), \tilde{\beta}(c_0) \in \mathcal{L}(A)$ . (See figure 2 for an illustration of this case.)

We can suppose for some  $\varepsilon > 0$ ,  $\tilde{\beta}|_{[b_0-\varepsilon, b_0]}$  leaves  $A$  from  $\mathcal{L}(A)$  at the block  $A_{n_1}$  for some  $n_1 \geq 1$ .  $\tilde{\beta}|_{[c_0, c_0+\varepsilon]}$  enters  $A$  from  $\mathcal{L}(A)$  at the block  $A_{n_0}$ , for some  $n_0 < 0$ . Let  $\ell = n_1 - n_0 - 1 \geq 1$ . Then  $(u_1)^\ell \cdot \tilde{\beta}([c_0, 1 + t_1])$  enters  $A$  from  $\mathcal{L}(A_{n_1-1})$  and stays in  $U$  until reaching  $(u_1)^{n_1+1}(\tilde{\phi}_0)$ . On the other hand,  $\tilde{\beta}|_{[0, b_0]}$  starts at  $\tilde{\phi}_0$  and remains in  $U$  intersecting  $(u_1)^i(\tilde{\phi}_0), 0 < i \leq n_1$ , before it leaves  $A$  from  $\mathcal{L}(A)$  at the block  $A_{n_1}$ . Therefore, both  $\tilde{\mathcal{F}}$ -transverse paths  $(u_1)^\ell \cdot \tilde{\beta}([c_0, 1 + t_1])$  and  $\tilde{\beta}|_{[0, b_0]}$  intersect the leaf  $(u_1)^{n_1}(\tilde{\phi}_0)$ .

Now we want to apply lemma 2.10. The interval  $[0, b_0]$  is part of some drawing component of  $\tilde{\beta}$ . Also, the interval  $[c_0, 1 + t_1]$  is part of some drawing component of  $(u_1)^\ell \cdot \tilde{\beta}$ . They all draw the block  $A_{n_1}$ . Moreover, since both  $\tilde{\beta}(b_0)$  and  $\tilde{\beta}(c_0)$  belong to  $\mathcal{L}(A)$ , their projections to the surface  $S$  satisfies the conditions of lemma 2.10. Therefore they must have an  $\tilde{\mathcal{F}}$ -transverse intersection.



**Figure 2.** Here depicted is one possible example of Case (1). We have  $n_1 = 2, n_0 = -1$ , and therefore  $\ell = 2$ .

**Case (2).**  $\tilde{\beta}(b_0) \in \mathcal{L}(A), \tilde{\beta}(c_0) \in \mathcal{R}(A')$ .

Note in this case there exists a crossing component for  $U$ , by which we mean the following. There exists an open interval  $(d, d^*) \subset (b_0, c_0)$ , such that,  $\tilde{\beta}(d) \in \mathcal{L}(A), \tilde{\beta}(d^*) \in \mathcal{R}(A')$ , and  $\tilde{\beta}((d, d^*)) \subset U$ . Similar with previous case, there are integers  $n_1 \geq 1, n_0 \leq -1$ , such that  $\tilde{\beta}|_{[0, b_0]}$  leaves  $A$  from  $\mathcal{L}(A)$  at the block  $A_{n_1}$ , and  $\tilde{\beta}|_{[c_0, 1]}$  enters  $A'$  from  $\mathcal{R}(A')$  at the block  $A'_{n_0}$ . There is also  $n_2 \in \mathbb{Z}$  such that for some  $\varepsilon > 0$ ,  $\tilde{\beta}|_{[d, d+\varepsilon]}$  enters  $U$  from  $\mathcal{L}(A)$  at the block  $A_{n_2}$ , and note that  $\tilde{\beta}|_{[d, d^*]}$  leaves  $U$  from  $\mathcal{R}(A')$ .

Similar to the argument in the final part of Case (1), we know that if  $0 \leq n_2 + \ell < n_1$  for some  $\ell \in \mathbb{Z}$ , then similar to the previous case, an application of lemma 2.10 gives that

$$(u_1)^\ell \cdot \tilde{\beta}|_{[d, d^*]} \text{ and } \tilde{\beta}|_{[0, b_0]} \text{ intersect } \tilde{\mathcal{F}}\text{-transversely.} \tag{3.7}$$

We further consider three subcases.

**subcase (2.1):**  $n_1 \geq 2$ . In this case, depending on  $n_2 \geq 1$  or  $n_2 \leq 0$ , one can choose either a strictly negative or a strictly positive integer  $\ell$ . It follows statement (3.7) holds for non-zero  $\ell$ .

**subcase (2.2):**  $n_2 \neq 0$ . In this case, we simply choose  $\ell = -n_2$ . Then statement (3.7) holds for non-zero  $\ell$ .

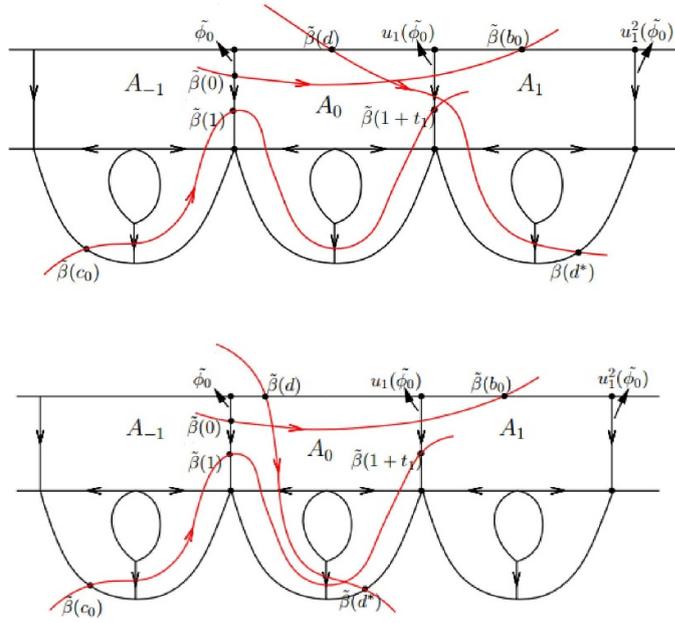
**subcase (2.3):**  $n_2 = 0$  and  $n_1 = 1$ . (See figure 3 for an illustration of this subcase.) Now there is  $\ell \geq n_2 = 0$  and some  $\varepsilon > 0$ , such that  $\tilde{\beta}|_{[d^* - \varepsilon, d^*]}$  leaves  $A'$  from  $\mathcal{R}(A')$ . If  $\ell \geq 1$ , then similar to Case (1), we see  $(u_1)^\ell \cdot \tilde{\beta}|_{[c_0, 1+t_1]}$  and  $\tilde{\beta}|_{[d, d^*]}$  intersect  $\tilde{\mathcal{F}}$ -transversely. If  $\ell = 0$ , then  $\tilde{\beta}|_{[d, d^*]}$  and  $\tilde{\beta}|_{[0, t_1]}$  intersect  $\tilde{\mathcal{F}}$ -transversely. But in this case,  $\tilde{\beta}|_{[d, d^*]}$  and  $\tilde{\beta}|_{[1, 1+t_1]}$  also intersect  $\tilde{\mathcal{F}}$ -transversely. Note that in this case, since  $\tilde{\beta}|_{[d, d^*]}$  lies in the union of  $A_0 \cup A'_0$  with the bounded connected components of its complement, and since neither  $\tilde{\beta}|_{[0, t_1]}$  nor  $\tilde{\beta}|_{[1, 1+t_1]}$  intersect any copy of  $\tilde{\phi}_0$ , we deduce that  $t_i \notin [d, d^*], i \in \{2, 3, 4\}$ .

**Case (3).**  $\tilde{\beta}(b_0) \in \mathcal{R}(A), \tilde{\beta}(c_0) \in \mathcal{L}(A)$ .

This case is very similar with Case (2). We also split it into subcases (3.1) (3.2) and (3.3) similarly. We omit the details of the discussion because they will mostly repetitions.

**Case (4).**  $\tilde{\beta}(b_0) \in \mathcal{R}(A), \tilde{\beta}(c_0) \in \mathcal{R}(A)$ .

This case is very similar to Case (1) and we omit the details of the discussion.



**Figure 3.** Here depicted are two possible situations of subcase (2.3). We have  $n_1 = 1, n_0 = -1, n_2 = 0$ . In the figure in the first line,  $\tilde{\beta}|_{[d, d^]}$  leaves  $U$  from  $\mathcal{R}(A_1')$ , while in the figure in the second line,  $\tilde{\beta}|_{[d, d^]}$  leaves  $U$  from  $\mathcal{R}(A_0')$ .

To sum up, in Case (1), Case (4), subcases (2.1), (2.2), (3.1) and (3.2), we find some integer  $\ell \neq 0$ , such that  $\tilde{\beta}$  and  $(u_1)^\ell \cdot \tilde{\beta}$  intersect  $\tilde{\mathcal{F}}$ -transversely, then we can conclude with non-contractible periodic orbit, with the help of lemma 2.12.

On the other hand, we need to deal with cases (2.3) or (3.3). Recall we have obtained interval  $[d, d^*]$ , which contains no  $t_i$  for  $i = 2, 3, 4$ . If  $d$  is larger than  $t_3$ , then the  $\tilde{\mathcal{F}}$ -transverse paths  $\tilde{\beta}|_{[0, t_1]}$  and  $\tilde{\beta}|_{[d, d^]}$  intersect  $\tilde{\mathcal{F}}$ -transversely at moments  $T_1 < t_1$  and  $T'_1 > t_3$ , respectively. It means  $\tilde{\beta}(T_1)$  and  $\tilde{\beta}(T_2)$  belongs to the same leaf. Therefore  $\tilde{\beta}|_{[T_1, T'_1]}$  is  $\tilde{\mathcal{F}}$ -equivalent to a loop. Observe that this loop is admissible and intersects  $u_1(\tilde{\phi}_0), u_2(\tilde{\phi}_0)$  and  $u_3(\tilde{\phi}_0)$ . Note that the natural extension of an admissible loop is a  $\tilde{\mathcal{F}}$ -bi-recurrent path, as it is periodic. Therefore we can apply lemma 2.11 and get the result. If  $d < t_3$ , which implies that  $d^* < t_3$ . Then the  $\tilde{\mathcal{F}}$ -transverse paths  $\tilde{\beta}|_{[1, 1+t_1]}$  and  $\tilde{\beta}|_{[d, d^]}$  intersect  $\tilde{\mathcal{F}}$ -transversely at moments  $T'_2 > 1$  and  $T_2 < t_3$ , respectively. In this case, similar to the first one,  $\tilde{\beta}|_{[T_2, T'_2]}$  is  $\tilde{\mathcal{F}}$ -equivalent to a loop, which is admissible. Moreover, it intersects  $u_3(\tilde{\phi}_0), u_4(\tilde{\phi}_0)$  and  $\tilde{\phi}_0$ . We end the proof by applying lemma 2.11.  $\square$

#### 4. Unbounded deviation and unbounded orbit

In this section, we show proposition 4.4, which guarantees existence of orbits which are unbounded in certain direction, under some mild assumptions. We start with the following statement, originally proved in [18], which is now seen as a corollary of theorem 1.1.

**Corollary 4.1.** *Let  $\tilde{f} \in \widetilde{\text{Homeo}}_{0,mw}(\mathbb{T}^2)$ , and assume  $(0,0) \in \rho(\tilde{f})$ . Then one of the following cases is true.*

- (1)  $\pi(\text{Fix}(\tilde{f}))$  is essential.
- (2)  $f$  has some non-contractible periodic orbit.
- (3) There exists some  $M > 0$ , with the following properties. For any  $x \in \mathbb{T}^2$ , which is either an  $f$ -periodic point, or is  $f$ -inessential, the orbit  $\{\tilde{f}^n(\tilde{x})\}_{n \in \mathbb{Z}}$  has diameter bounded from above by  $M$ , for any lift  $\tilde{x}$  of  $x$ .

**Proof.** Let us assume the first two items do not happen. Then if  $x$  is periodic for  $f$ , any lift  $\tilde{x}$  of  $x$  is also periodic and therefore non-wandering. On the other hand, if  $x$  is  $f$ -inessential, for some open neighbourhood  $U$  of  $x$ , the invariant set  $\bigcup_{n \in \mathbb{Z}} f^n(U)$  is inessential. Since  $f$  is non-wandering, it follows that  $x$  is contained in an  $f$ -periodic open disk  $D$ , and if  $k$  is the period of  $D$ , then we can assume  $f^j(D)$  is disjoint from  $D$  for  $1 \leq j \leq k - 1$ . Of course since  $f^k$  preserves  $D$  and it is non-wandering, the Brouwer’s plane translation theorem implies that there is a fixed point for  $f^k$  in  $D$ . Then, if  $\tilde{D}$  is a connected component of  $\pi^{-1}(D)$ , then  $f^k(\tilde{D}) = \tilde{D} + v$  for some  $v \in \mathbb{Z}^2$ . Since there is a periodic point in  $\tilde{D}$  whose orbit projects to a contractible periodic orbit, we get that  $v$  is null. Since  $x$  is non-wandering, it follows that  $\tilde{x}$  belongs to  $\Omega(\tilde{f})$ . Then item (3) follows immediately from theorem 1.1. □

Let us prove two elementary lemmas, which will be used several times below.

**Lemma 4.2.** *Let  $W$  be a Baire space, and suppose  $W = \bigcup_{k=1}^{\infty} B_k$ . For each  $k \geq 1$ , write  $I_k = \text{Int}(B_k)$  to denote the interior of  $B_k$ , and write  $U = \bigcup_{k \geq 1} I_k$ . Then  $U$  is dense in  $W$ .*

**Proof.** Suppose otherwise. Then there is some open subset  $O$  of  $W$  such that  $O \cap U = \emptyset$ . It follows that  $O \cap B_k$  intersects no interior in  $B_k$ . Now Baire category theorem implies that the set  $O \cap (\bigcup_{k \geq 1} B_k)$  intersects no interior in  $W$ . This is a contradiction since  $\bigcup_{k \geq 1} B_k = W$  and  $O$  is a subset of  $W$ . □

**Lemma 4.3.** *Let  $\tilde{D}$  be a disk in the universal covering space, with diameter  $\text{diam}(\tilde{D})$ . Suppose for points in the boundary of  $\tilde{D}$ , the  $\tilde{f}$ -deviations in the  $v$ -direction are (strictly) bounded from above by  $M$ , then for every point in  $\tilde{D}$ , its  $\tilde{f}$ -deviation in the  $v$ -direction is (strictly) bounded from above by  $M + \text{diam}(\tilde{D})$ .*

**Proof.** Note that for any  $n \geq 0$ , since  $\tilde{f}^n(\tilde{x}) \in \tilde{f}^n(\tilde{D})$ , with respect to a fixed direction  $v$ , the  $n$ th deviation of the point  $\tilde{x}$  is at most  $\text{diam}(\tilde{D})$  plus the maximum of the  $n$ th deviations for points in the boundary of  $\tilde{D}$ , for every  $n$ . The conclusion follows. □

**Proposition 4.4.** *Suppose  $\tilde{f} \in \widetilde{\text{Homeo}}_{0,mw}(\mathbb{T}^2)$  satisfies that its rotation set  $\rho(\tilde{f})$  contains  $(0,0)$ . Assume that for some  $v \in \mathbb{S}^1$ ,*

$$\sup_{\tilde{x} \in \mathbb{R}^2, n \geq 0} pr_v(\tilde{f}^n(\tilde{x}) - \tilde{x}) = +\infty. \tag{4.1}$$

*Then, one of the followings is true.*

- (a)  $\pi(\text{Fix}(\tilde{f}))$  is essential.
- (b) There is some  $f$ -invariant annulus, whose homological direction is not perpendicular to  $v$ .

- (c)  $f$  has a non-contractible periodic orbit.
- (d) There exists  $\tilde{x}_0 \in \mathbb{R}^2$ , such that,

$$\sup_{n \geq 0} \text{pr}_v \left( \tilde{f}^n(\tilde{x}_0) - \tilde{x}_0 \right) = +\infty. \tag{4.2}$$

**Proof.** Fix the direction  $v \in \mathbb{S}^1$ . For any  $K > 0$ , denote

$$B_K = \left\{ x \mid \text{pr}_v \left( \tilde{f}^n(\tilde{x}) - \tilde{x} \right) \leq K \text{ for any } n \geq 0 \text{ and any lift } \tilde{x} \text{ of } x \right\}. \tag{4.3}$$

Then  $B_K$  is a closed subset for any  $K > 0$ . Now we write  $I_K = \text{Int}(B_K)$  to denote its interior, which is possibly an empty set, and then write  $U = \bigcup_{K>0} I_K$ .

From now on, let us suppose item (d) is not true. By definition, it means that

$$\bigcup_{K>0} B_K = \mathbb{T}^2. \tag{4.4}$$

Then we will discuss all the possible cases under this assumption. We will demonstrate that if none of the items (a), (b), or (c) holds true, then a contradiction arises to either (4.1) or (4.4).

The set  $U$  is  $f$ -invariant, and  $U$  is also open because it is a union of open sets. We can take  $K$  to be positive integer  $k$ 's and look at all the sets  $B_k$ 's and  $I_k$ 's. It follows immediately from lemma 4.2 that  $U$  is dense in  $\mathbb{T}^2$ .

If  $U$  is essential, there are two subcases. Either  $U$  is essential and not fully essential, or  $U$  is fully essential.

If  $U$  is essential and not fully essential, then it has a connected component  $U'$  that includes some loop representing an element in the first homology group  $H_1(\mathbb{T}^2) \simeq \mathbb{Z}^2$ , which is identified with some rational direction  $(p, q)$ . We can call this direction the homological direction of  $U'$ . Any connected component  $\tilde{U}'$  of  $\pi^{-1}(U')$  is invariant under translation by  $(p, q)$ . Also, it must be  $\tilde{f}$ -invariant, otherwise every point in the rotation set of  $\tilde{f}$  would have a non-zero projection in the direction perpendicular to  $(p, q)$ . But since  $\tilde{U}'$  is invariant, then  $\tilde{f}$  have bounded deviation along the perpendicular direction of  $(p, q)$ . It follows from assumption (4.1) that,  $v$  is not perpendicular to  $(p, q)$ . One can then consider  $\text{Fill}(U')$ , the filling of  $U'$ , obtained by taking the union of  $U$  and all inessential connected components of its complement. Then  $\text{Fill}(U')$  is the annulus in item (b) of the Proposition.

If  $U$  is fully essential, then for any  $z \notin U$ , there exists some topological disk  $D$  containing  $z$ , such that  $\partial D \subset U$ . Since  $\partial D$  is compact, it is included in  $I_K$  for some sufficiently large  $K > 0$ . It follows from lemma 4.3 that the whole disk  $D$  is included in  $I_{K+\text{diam}(D)}$ , where  $\text{diam}(D)$  denotes the diameter of some connected component of the lift  $\pi^{-1}(D) \subset \mathbb{R}^2$ . In particular,  $z \in I_{K+\text{diam}(D)} \subset U$ , which is a contradiction. This contradiction shows  $U = \mathbb{T}^2$  if  $U$  is fully essential. However, since  $\mathbb{T}^2$  is compact, it follows  $\mathbb{T}^2 = I_K$  for some large  $K$ , and this is a contradiction with the assumption (4.1).

We are left now to the case when  $U$  is inessential. Since  $f$  is non-wandering, this means  $U$  is a union of  $f$ -periodic disks. Note that the assumption that item (d) does not happen implies that, the rotation set  $\rho(\tilde{f})$  is contained in the closed half plane whose boundary is perpendicular to  $v$ . By corollary 4.1, there are several subcases.

Subcase (1).  $\pi(\text{Fix}(\tilde{f}))$  is essential. This gives item (a) of the proposition.

Subcase (2).  $f$  admits non-contractible periodic orbits. This is item (c).

Subcase (3). There exists some  $M > 0$ , such that for any point  $x \in U$  and any lift  $\tilde{x}$  of  $x$ , the orbit  $\{\tilde{f}^n(\tilde{x})\}_{n \in \mathbb{Z}}$  has diameter uniformly bounded by  $M$  from above. Then, since  $U$

is dense, it follows for all point  $x \in \mathbb{T}^2$ , the same holds true. This is a contradiction with assumption (4.1).  $\square$

**Remark 4.5.** In the case of annular dynamics, an interesting example was provided in proposition 1.1 in [4], where the dynamics admits unbounded displacement in the negative direction, but there is no single orbit which is unbounded in the negative direction. In our context, this means it is possible that item (b) happens and item (c) and (d) do not happen. The example is not non-wandering, but satisfies a weaker condition, called the curve intersection property. There are also examples of torus homeomorphisms with unbounded deviation along every direction, which satisfies item (a) of the proposition and does not satisfy items (b) and (c). See theorem 3 of [11].

### 5. Non-contractible periodic orbits and bounded deviations

This section is mainly devoted to proving theorem 1.3, which will be split into two subsections. Finally, the proof of theorem 1.2 is at the end of the section. Recall the conclusion of theorem 1.3 claims about the bounded deviation of certain homeomorphism. We will prove the theorem by contradiction. Thus, throughout, let us assume  $\tilde{f} \in \widetilde{\text{Homeo}}_{0, \text{nw}}(\mathbb{T}^2)$ , whose rotation set  $\rho(\tilde{f})$  is the line segment connecting  $(0, 0)$  and some irrational vector  $(\alpha, \beta)$  so that  $\alpha/\beta$  is also an irrational number. We will also fix a maximal isotopy  $I$  for  $f$  that lifts to a maximal isotopy  $\tilde{I}$  for  $\tilde{f}$ . Then we find  $\mathcal{F}, \tilde{\mathcal{F}}$  which are Brouwer-Le Calvez foliations for  $I$  and  $\tilde{I}$  respectively.

Assume, for a contradiction, that  $\tilde{f}$  admits unbounded deviations along the direction  $-(\alpha, \beta)$ .

#### 5.1. Unbounded trajectories and bounded leaves

The first lemma provides abundance of points whose orbits are unbounded in the direction  $-(\alpha, \beta)$ . Without loss of generality, we also assume that  $\alpha > 0$ .

**Lemma 5.1.** *There exists some point  $\tilde{x}_0 \in \mathbb{R}^2$  whose positive half-orbit  $\{\tilde{f}^n(\tilde{x}_0)\}_{n \geq 0}$  is unbounded in the direction  $-(\alpha, \beta)$ . Moreover, the set of points satisfying this property is dense in  $\text{Ess}(f)$ .*

**Proof.** Take  $v = -(\alpha, \beta)$ . By proposition 4.4, we only need to show with respect to the direction  $v$ , item (a), item (b) or item (c) do not happen. Note that items (b) and (c) imply either the rotation set contains a line segment with rational slope or is reduced to a simple vector  $(0, 0)$ , which is contradictory to the assumption on the shape of  $\rho(\tilde{f})$ . So they can not happen.

Let us suppose for contradiction that item (a) in proposition 4.4 is true. It means that  $\text{Fix}(f)$  is not contained in a topological disk. Then, its complement  $U$  either has a connected component which has a non-trivial homological direction, or all of its connected components are disks. If the first case happens, the rotation set must be contained in a line with rational slope, which is a contradiction. If the second case happens, by [16], we can consider a typical point  $x$  for some ergodic measure  $\mu$  whose average rotation number is  $(\alpha, \beta)$ . Such a typical point  $x$  is  $f$ -recurrent. Moreover,

$$\rho(\tilde{f}, x) = (\alpha, \beta). \tag{5.1}$$

However,  $x$  is clearly an  $f$ -inessential point. By corollary 4.1, item (3), we see that for a lift  $\tilde{x}$ , the whole orbit  $\{f^n(\tilde{x})\}_{n \in \mathbb{Z}}$  is uniformly bounded, which is absurd. This contradiction concludes the first assertion.

Suppose there is some open set  $U$  intersecting  $\text{Ess}(f)$ , consisting of points for whose lift the positive half-orbit is bounded in the direction  $-(\alpha, \beta)$ . By definition of  $\text{Ess}(f)$ ,

$$W = \bigcup_{n \in \mathbb{Z}} f^n(U) \tag{5.2}$$

is essential. It follows for every point in the essential set  $W$ , its lift has positive half-orbit is bounded in the direction  $-(\alpha, \beta)$ . Note the essential set  $W$  is either fully essential, or it contains some annular component. The second case can be excluded, since as argued before, it will lead to a contradiction with the shape of  $\rho(\tilde{f})$ . We are left with the case when  $W$  is fully essential, which means its complement is inessential. Now there are two subcases.

In subcase one, there is a uniform bound  $R$ , such that, for all point in  $W$ , the  $\tilde{f}$ -positive half-orbit of its lift is bounded in the direction  $-(\alpha, \beta)$  by  $R$ . In this case, for any point  $x \in W^c$ , there is some topological disk  $D$  containing  $x$ , whose boundary circle  $\ell$  is included in  $W$ . Choose  $\tilde{x}$  and denote by  $\tilde{D}$  the connected component of  $\pi^{-1}(D)$  which contains  $\tilde{x}$ , with boundary circle  $\tilde{\ell}$ . By lemma 4.3, the  $\tilde{f}$ -deviation of  $\tilde{x}$  in the direction  $-(\alpha, \beta)$  is bounded by  $R + \text{diam}(D)$ . This shows that every point has bounded deviation in the direction  $-(\alpha, \beta)$ , a contradiction to the assumption.

In subcase two,  $\tilde{f}$  restricted to  $\pi^{-1}(W)$  has unbounded deviation along the direction  $-(\alpha, \beta)$ . In this case, we define the set  $B_K \subset W$  to be the set of points, for whose lifts the  $v$ -directional deviation are bounded by  $K$ , and then define  $I_K$  to be the interior of  $B_K$ . These definitions are similar to those given at the beginning paragraphs in the proof of proposition 4.4. Then applying lemma 4.2 we get that for a dense subset of  $W$ , the positive  $\tilde{f}$ -half orbit of their lifts are all unbounded in the direction of  $-(\alpha, \beta)$ , which is a contradiction to the assumption. So we have shown the second assertion.  $\square$

We also consider unbounded forward orbits in the direction  $(\alpha, \beta)$  next.

**Lemma 5.2.** *Define*

$$Q := \left\{ x \in \mathbb{T}^2 \mid \left\{ \text{pr}_{(\alpha, \beta)} \left( \tilde{f}^n(\tilde{x}) - \tilde{x} \right) \right\}_{n \geq 0} \text{ is unbounded to the right} \right\}. \tag{5.3}$$

*Then  $Q$  is dense in  $\text{Ess}(f)$ .*

**Proof.** This proof is similar with the previous lemma. Note by the shape of the rotation set  $\rho(\tilde{f})$ , the following holds.

$$\sup_{\tilde{x} \in \mathbb{R}^2, n \geq 0} \text{pr}_{(\alpha, \beta)} \left( \tilde{f}^n(\tilde{x}) - \tilde{x} \right) = \infty. \tag{5.4}$$

Then the rest argument goes exactly as in the proof of previous lemma.  $\square$

Finally, we show all the leaves of  $\mathcal{F}$  has uniformly bounded diameter.

**Lemma 5.3.** *There exists  $M_{\tilde{\mathcal{F}}} > 0$  such that every leaf of  $\tilde{\mathcal{F}}$  has diameter smaller than  $M_{\tilde{\mathcal{F}}}$ .*

**Proof.** For this proof, let us write  $v = (\alpha, \beta)$ , not to be confused with the symbol  $v$  we used in the statement of proposition 4.4. With respect to the direction  $v$ , let  $v^\perp$  be one of its perpendicular directions, say,  $v^\perp = (-\beta, \alpha)$ . Write the ‘first quadrant’  $Q_1 := \{w \in \mathbb{R}^2 \mid \langle w, v \rangle >$

0 and  $\langle w, v^\perp \rangle > 0\}$  consisting of vectors with positive projections in both  $v$  and  $v^\perp$ . Write the ‘fourth quadrant’  $Q_4 = \{w \in \mathbb{R}^2 \mid \langle w, v \rangle > 0 \text{ and } \langle w, v^\perp \rangle < 0\}$ . Lemma 21 of [19] says that one can find two  $\tilde{\mathcal{F}}$ -transverse paths, not necessarily admissible, namely  $\alpha_1, \alpha_2 : [0, 1] \rightarrow \text{Dom}(\tilde{\mathcal{F}})$ , such that,  $\alpha_1(1) = \alpha_1(0) + w_1$ ,  $\alpha_2(1) = \alpha_2(0) + w_2$ , where  $w_1 \in \mathbb{Z}^2 \cap Q_1$  and  $w_2 \in \mathbb{Z}^2 \cap Q_4$ .

Recall the contradiction assumption says that there exists some point  $\tilde{x}_0$  whose positive half-orbit is unbounded in the  $-(\alpha, \beta)$  direction. A standard compactness argument shows that one can find a point  $\tilde{z}$  which is not fixed by  $\tilde{f}$ , an increasing sequence of  $(n_k)_{k \in \mathbb{N}}$  of positive integers, and a sequences of vectors  $(u_k)_{k \in \mathbb{N}}$  in  $\mathbb{Z}^2$ , such that  $\lim_{k \rightarrow \infty} \tilde{f}^{n_k}(\tilde{x}_0) - u_k = \tilde{z}$ , and such that  $\lim_{k \rightarrow \infty} \text{pr}_{-(\alpha, \beta)}(u_k) = \infty$ . Such a point  $\tilde{z}$  exists because a small neighborhood of the fixed points of  $\tilde{f}$  cannot trap the orbit of  $\tilde{x}_0$ . Therefore, we can consider only those iterates away from the  $\tilde{f}$ -fixed points and then identify some accumulation point  $\tilde{z}$ . By lemma 2.20, the projections of the vectors  $u_k$ ’s in the two directions perpendicular to  $(\alpha, \beta)$  are uniformly bounded by a constant  $L_0$ .

We get, using lemma 2.4, that if  $k_1 < k_2$  are sufficiently large, then the path  $\tilde{I}_{\tilde{\mathcal{F}}}^{[n_{k_1}-1, n_{k_2}+1]}(\tilde{x}_0)$  contains a subpath that is  $\tilde{\mathcal{F}}$ -equivalent to an  $\tilde{\mathcal{F}}$ -transverse path joining  $\tilde{z} + u_{k_1}$  to  $\tilde{z} + u_{k_2}$ . Note that, with two vectors  $w_1, w_2$  as above fixed, there exists some  $L_1 > 0$  such that, if any vector  $w$  is such that  $\text{pr}_{-(\alpha, \beta)}(w) > L_1$  while  $|\text{pr}_{(\beta, -\alpha)}(w)| < L_0$ , then the triangle whose vertices are  $w, w_1$  and  $w_2$  has the origin in its interior. Clearly, if we choose  $k_1$  large and  $k_2 \gg k_1$ , then  $\text{pr}_{-(\alpha, \beta)}(u_{k_2} - u_{k_1}) > L_1$ . This way, we find a third transverse path  $\alpha_3 : [0, 1] \rightarrow \text{Dom}(\tilde{\mathcal{F}})$ , such that  $\alpha_3(1) = \alpha_3(0) + w_3$ , with  $w_3 \in \mathbb{Z}^2$ , and such that the triangle whose vertices are  $w_1, w_2$  and  $w_3$  has the origin in its interior. This implies that there are positive integers  $s_1, s_2, s_3$ , such that  $\sum_{i=1}^3 s_i w_i = 0$ . Finally, one can apply lemma 2.5 to conclude that the diameter of every leaf of  $\tilde{\mathcal{F}}$  is bounded by some uniform constant  $M_{\tilde{\mathcal{F}}}$ .  $\square$

5.2. Setting the barrier and end of proof

In the rest of the argument, the main idea is to put an infinite strip as a barrier, which is bounded in the horizontal direction. Then we use lemma 5.1 and lemma 5.2 alternatively to construct ‘oscillating’ isotopy path, which passes the barrier as many times as we like, which will eventually become too crowded and satisfy the conditions for proposition 3.1.

We remind the readers that, under the contradiction hypothesis of unbounded deviation, we have already demonstrated in the proof of lemma 5.1 that the set  $\text{Fix}(f)$  is contained in a topological disk. Then we show the next lemma, which is simply a restatement of lemmas 3.3 and 3.4.

**Lemma 5.4.** *There exists some  $N_* > 0$  and a finite set of leaves  $\{\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_k\}$  of  $\tilde{\mathcal{F}}$  such that, if  $n > 0$  and  $\|\tilde{f}^n(\tilde{x}) - \tilde{x}\| > N_*$ , then there exists some  $j \in \{1, \dots, k\}$  and five different vectors  $w_i \in \mathbb{Z}^2$  for  $1 \leq i \leq 5$ , such that  $\tilde{I}_{\tilde{\mathcal{F}}}^n(\tilde{x})$  intersects  $\tilde{\phi}_j + w_i, 1 \leq i \leq 5$ .*

Next, we show that there are trajectories that largely ‘oscillate’ in the horizontal direction as many times as necessary. Recall that we have assumed that  $\alpha > 0$ .

**Lemma 5.5.** *Given any  $L > 0, \varepsilon > 0$  and  $\tilde{x}_0 \in (-1, 0)^2$  which is a lift of an  $f$ -essential point  $x_0$ , there exists some  $\tilde{y}_L$  with  $\|\tilde{x}_0 - \tilde{y}_L\| < \varepsilon$  and integers  $n_i, m_i, 1 \leq i \leq L$  with  $n_i < m_i < n_{i+1}$  and such that:*

- $\tilde{f}^{n_i}(\tilde{y}_L)$  is in  $(-\infty, 0) \times \mathbb{R}$  and  $\tilde{f}^{m_i}(\tilde{y}_L)$  is in  $(N_*, \infty) \times \mathbb{R}$ ;
- For  $n_i < n < m_i, \tilde{f}^n(\tilde{y}_L)$  is in  $[0, N_*] \times \mathbb{R}$ .

**Proof.** The proof is elementary so we will be a bit sketchy. Fix  $\varepsilon > 0$ . We look for each  $y_L$  inductively. We can assume that  $\varepsilon$  is sufficiently small such that  $B_\varepsilon(\tilde{x}_0) \subset (-1, 0)^2$ . For  $L = 1$ , note that, by lemma 5.2, we can find an essential point  $y_1$  which is  $\varepsilon/2$  close to  $\pi(\tilde{x}_0)$  such that, if  $\tilde{y}_1$  is a lift of  $y_1$  that is  $\varepsilon/2$  close to  $\tilde{x}_0$ , then  $\text{pr}_{(\alpha, \beta)}(\tilde{f}^n(\tilde{y}_1) - \tilde{y}_1)$  is unbounded from above. The existence of  $n_1$  and  $m_1$  follows easily from here. Now, given an integer  $L$ , a point  $\tilde{y}_L$  that projects to an essential point and that is at distance at most  $(1 - 2^{-L})\varepsilon$  of  $\tilde{x}_0$  and the sequences of integer  $n_i, m_i$ , with  $1 \leq i \leq L$ , satisfying the desired properties, we first find, using lemma 5.1 some point  $\tilde{z}_L$ , projecting to an essential point, and having a future orbit unbounded in the  $-(\alpha, \beta)$  direction. Furthermore, we can assume  $\tilde{z}_L$  is sufficiently close to  $\tilde{y}_L$  so that  $\tilde{f}^{m_i}(\tilde{z}_L)$  is in  $(-\infty, 0) \times \mathbb{R}$  and  $\tilde{f}^{n_i}(\tilde{z}_L)$  is in  $(N_*, \infty) \times \mathbb{R}$ , for  $1 \leq i \leq L$ . There exists some integer  $s_L > m_L$  such that  $\tilde{f}^{s_L}(\tilde{z}_L)$  is in  $(-\infty, 0) \times \mathbb{R}$ . We get the point  $\tilde{y}_{L+1}$  by applying again lemma 5.2 to  $\tilde{z}_L$  and finding a point whose orbit follows that of  $\tilde{z}_L$  sufficiently close until at least the iterate  $s_L$ , but for which the forward orbit is unbounded in the direction  $(\alpha, \beta)$ . The induction step finishes and the conclusion follows because this induction process never ends.  $\square$

The next lemma is elementary and states that, for a finite isotopy path and any leaf mentioned in lemma 5.4, the number of intersecting integer translates of that leaf is uniformly bounded. This is due to the boundedness of leaves and compactness. We extract this fact as a lemma to avoid it becoming cumbersome within the final proof.

**Lemma 5.6.** *There exists some  $M > 0$  so that the following holds. For any  $\tilde{z} \in [-1, 0]^2$ , and any  $1 \leq j \leq k$ , suppose there are  $\bar{n} < \bar{m}$  such that,*

- $\tilde{f}^{\bar{n}}(\tilde{z})$  is in  $(-\infty, 0) \times \mathbb{R}$  and  $\tilde{f}^{\bar{m}}(\tilde{z})$  is in  $(N_*, \infty) \times \mathbb{R}$ ;
- For  $\bar{n} < n < \bar{m}$ ,  $\tilde{f}^n(\tilde{z})$  is in  $[0, N_*] \times \mathbb{R}$ .
- $\tilde{I}_{\tilde{\mathcal{F}}}^{[\bar{n}, \bar{m}]}(\tilde{z})$  intersects  $\tilde{\phi}_j + w$  with  $w \in \mathbb{Z}^2$ .

Then  $\|w\| < M$ .

**Proof.** Note first that, by lemma 2.20, there exists some constant  $M_1$  such that the whole orbit of any point  $\tilde{z}$  that lies in  $[-1, 0]^2$  lies in a strip whose projection into a direction perpendicular to  $(\alpha, \beta)$  has value less than  $M_1$ . This, and the hypotheses on  $\bar{n}$  and  $\bar{m}$  imply that there exists a compact set  $K_1$  that contains  $\tilde{f}^n(\tilde{z})$ ,  $\bar{n} \leq n \leq \bar{m}$  if  $\tilde{z} \in [-1, 0]^2$ ,  $\bar{n}$  and  $\bar{m}$  are as in the hypothesis. In particular, from the continuity of the isotopy  $I$ , we have that there exists a compact set  $K \supset K_1$  such that, if  $\tilde{y}$  lies in  $K_1$ , then  $\tilde{I}^{[0, 1]}(\tilde{y})$  lies in  $K$ . We deduce that  $\tilde{I}^{[\bar{n}, \bar{m}]}(\tilde{z})$  is contained in  $K$ .

Furthermore, since  $\tilde{I}_{\tilde{\mathcal{F}}}^{[\bar{n}, \bar{m}]}(\tilde{z})$  is homotopic with fixed endpoints in  $\text{Dom}(\tilde{\mathcal{F}})$  to  $\tilde{I}^{[\bar{n}, \bar{m}]}(\tilde{z})$  and since  $\tilde{I}_{\tilde{\mathcal{F}}}^{[\bar{n}, \bar{m}]}(\tilde{z})$  is transverse to  $\tilde{\mathcal{F}}$ , one has that every leaf that is intersected by  $\tilde{I}_{\tilde{\mathcal{F}}}^{[\bar{n}, \bar{m}]}(\tilde{z})$  must also be intersected by  $\tilde{I}^{[\bar{n}, \bar{m}]}(\tilde{z})$ . But since the leaves of  $\tilde{\mathcal{F}}$  are uniformly bounded, and only a finite number of integer translates of each  $\tilde{\phi}_j$  can intersect  $K$ , and the result follows.  $\square$

**End of proof of theorem 1.3.** In summary, we have obtained from lemma 5.4 a collection of  $k$  leaves and a constant  $M$  from lemma 5.6 which bounds the norms of the translates  $w \in \mathbb{Z}^2$  that we are concerned about. Now it reduces to a simple application of the pigeonhole principle. The total number of sequences of five different elements in  $\mathbb{Z}^2$  that have norm smaller than  $M$  is bounded from above by some constant  $k_1$ . Pick  $\tilde{x}_0$  as in lemma 5.5, and pick  $\varepsilon$  so that  $B_\varepsilon(\tilde{x}_0) \subset (-1, 0)^2$ . Pick  $L > k_1 \cdot k + 1$ , where  $k$  is given by lemma 5.4, and pick  $\tilde{y}_L, n_i, m_i, 1 \leq i \leq L$  again as in lemma 5.5. We deduce that there exists some  $1 \leq j_0 \leq k$  and a sequence of five distinct  $\mathbb{Z}^2$  vectors  $w_1, w_2, w_3, w_4, w_5$  and some  $1 \leq i_1 < i_2 \leq L$  such that both  $\tilde{I}_{\tilde{\mathcal{F}}}^{[n_{i_1}, m_{i_1}]}(\tilde{y}_L)$

and  $\tilde{I}_{\tilde{\mathcal{F}}}^{[n_1, m_1]}(\tilde{y}_L)$  intersect, in order,  $\phi_{j_0} + w_i, 1 \leq i \leq 5$ . The theorem then follows by applying proposition 3.1 to a re-parametrization of  $\tilde{I}_{\tilde{\mathcal{F}}}^{[n_1, m_1]}(\tilde{y}_L)$ . Then we conclude with the existence of non-contractible periodic orbits, which is contradictory to our rotation set assumption.  $\square$

### Data availability statement

No new data were created or analysed in this study.

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