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On a relationship between acoustical (more generally scalar) beam shape coefficients and electromagnetic beam shape coefficients of some T-matrix theories for structured beams

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ABSTRACT

On one hand, we consider two T-matrix theories, namely “Generalized Lorenz–Mie theory” (GLMT) and “Extended Boundary Condition Method” (EBCM), in which structured beams may be expanded over Vector Spherical Wave Functions (VSWFs), with expansion coefficients expressed by using electromagnetic Beam Shape Coefficients (BSCs). On the other hand, we consider acoustical (more generally scalar) fields which may be expressed similarly by an expansion over basic functions, with expansion coefficients expressed using acoustical (scalar) BSCs. We establish relationships between electromagnetic and scalar BSCs.

1. Introduction

The Generalized Lorenz–Mie Theory (GLMT) is an analytical theory which describes the interaction between a homogeneous sphere and an arbitrary shaped illuminating beam [1,2], while the Extended Boundary Condition Method (EBCM) is a semi-analytical theory which may deal with a larger set of particles, specifically nonspherical particles [3–5]. In both cases, the illuminating electromagnetic fields may be expanded over a set of Vector Spherical Wave Functions (VSWFs), with the expansion coefficients being expressed using Beam Shape Coefficients (BSCs). The electromagnetic fields may be derived from a vector potential satisfying Helmholtz equation. For entries to a large literature devoted to these issues, the reader may refer to recent reviews [6,7].

We may as well consider acoustical (more generally scalar) fields which satisfy Helmholtz equation and may be expressed in terms of expansions with expansion coefficients expressed as well in terms of acoustical BSCs. As a consequence, there must be a strong analogy between electromagnetic and acoustical scatterings, so that the arsenal developed in the field of electromagnetism may be transferred to the field of acoustics.

In particular, several methods, already developed for electromagnetic BSCs, have been applied, mutatis mutandis, to the case of acoustical BSCs. The electromagnetic quadrature technique [8,9] has been discussed and applied to acoustical fields in [10–14].

Another technique called the localized approximation (with variants) has been developed in electromagnetism, e.g. [15–19] for original papers devoted to Gaussian beams and to laser sheets, [20] for a review including the case of “arbitrary shaped beams”, to be completed with [21–23], and, without being exhaustive, [24,25] for limitations of the localized approximations for beams exhibiting an axicon angle, and [26,27] for limitations of localized approximations for beams exhibiting a topological charge. A variant named Integral Localized Approximation must be mentioned, e.g. [28,29].

Localization techniques have been transferred to the case of acoustical beams. In [10], an acoustical localized approximation is rigorously established in the case of an on-axis acoustical Gaussian beam, defined as a special case of Laguerre–Gauss beams. In [11], the same issue is considered but the on-axis acoustical Gaussian beam is borrowed from the Davis scheme of approximation used to study electromagnetic Gaussian beams, leading to the same result than previously, with however an irrelevant change of prefactor in the expressions of the fields. The case of off-axis Gaussian acoustical beams is available from [13], while the case of Bessel beams, together with an ILA approach, is discussed in [14].

Another technique used for electromagnetic beams is the finite series technique. It has been introduced decades ago in the case of electromagnetic beams, e.g. [30,31], before being essentially forgotten

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for decades due to the advantages in terms of flexibility and computational efficiency of localized approximations, before its recent revival due to the limitations of localized approximations for beams possessing an axicon angle and/or a topological angle, e.g. [32,33] for Laguerre–Gauss beams freely propagating, [34,35] for Laguerre–Gauss beams focused by a lens, see as well [36,37], for the use of different methods and their comparisons.

This technique has been transferred to the case of acoustical beams in [12] for the study of Bessel, Laguerre–Gauss and Gaussian beams under an on-axis configuration, including the use of a modified version recently introduced in the context of electromagnetic fields [38].

Another question, motivated by the strong analogies between the description of electromagnetic and acoustical beams, is to ask whether there would not be any direct relationships between electromagnetic and acoustical BSCs. The present paper provides a first answer, although not complete, to this question. The paper is organized as follows. Section 2 presents a background to expound the strategy used in the present paper, namely (i) a short exposition of the electromagnetic Davis scheme of approximations to the description of electromagnetic Gaussian beams which will be borrowed to a transfer in linear acoustics and (ii) a comparison between basic electromagnetic and acoustical expressions allowing one to emphasize their similarities. Section 3 expresses the radial electromagnetic fields in terms of the acoustical field, for the electric fields in Section 3.1, and for the magnetic field in Section 3.2. Section 4 is the core of the paper, providing relationships between the BSCs of electromagnetic and acoustical fields, for the magnetic field in Section 4.1, and for the electric field in Section 4.2. Section 5 is a discussion and Section 6 is a conclusion.

2. Background

2.1. The Davis scheme of approximations

A popular approach to the description of electromagnetic Gaussian beams is the Davis scheme of approximations [39–41] which is briefly recalled here, limiting ourselves to ingredients required for the sequel. In this scheme, we consider a x-polarized vector potential propagating in the z -direction, with a time dependence of the form $\exp(i\omega t)$, and Cartesian coordinates (x, y, z) reading as:

$$\mathbf{A} = (A_x, 0, 0) \tag{1}$$

while the nonzero x-component of the vector potential is written as:

$$A_x = \psi(x, y, z) \exp(-ikz) \tag{2}$$

The vector potential satisfies the Helmholtz equation, which generates a partial differential equation for ψ which is searched using an expansion reading as:

$$\psi = \sum_{j=0}^{\infty} s^{2j} \psi_{2j} = \psi_0 + s^2 \psi_2 + s^4 \psi_4 + \cdots \tag{3}$$

in which s is a small parameter, called the beam confinement factor (or beam shape factor) describing the level of focusing of the beam, reading as $1/(kw_0)$ in which k is the wavenumber of the light and w_0 the beam waist radius of the beam. The successive approximations in the series of Eq. (3) are called the first-order, third-order, fifth-order... approximations discussed in [41]. The highest-order known, as far as we know, is a ninth-order mode [42], while it is known that the scheme is eventually diverging [43], so that the series of Eq. (3) is actually an asymptotic series [41]. Once an approximation is obtained, the corresponding electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields are deduced from the corresponding approximation of the vector potential using classical expressions which will better be recalled below in a more appropriate context. The contents of this subsection may be summarized as follows: if you use an expression for the x-component of the vector potential, then you may obtain expressions for the electric and magnetic fields.

2.2. Electromagnetic and acoustical basic expressions for beam shape coefficients

On one hand, in GLMT, the BSCs are related to the radial electric and magnetic field components E_r and H_r respectively, according to, e.g. Eqs.(3.39), (3.45), (3.42) and (3.48) in [2]:

$$E_r = kE_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n,TM}^m [\psi_n''(kr) + \psi_n(kr)] P_n^{|m|}(\cos\theta) \exp(im\varphi) \tag{4}$$

$$H_r = kH_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n,TE}^m [\psi_n''(kr) + \psi_n(kr)] P_n^{|m|}(\cos\theta) \exp(im\varphi) \tag{5}$$

in which we used spherical coordinates (r, θ, φ), k is the wavenumber, E_0 and H_0 are electric and magnetic field strengths respectively, $\psi(x) = xj_n(x)$ are Riccati–Bessel functions with $j_n(x)$ being spherical Bessel functions of the first kind, a prime indicates a differentiation with respect to the argument, $P_n^{|m|}$ are associated Legendre functions, $g_{n,TM}^m$ and $g_{n,TE}^m$ are the Transverse Magnetic (TM) and Transverse Electric (TE) electromagnetic BSCs respectively, and c_n^{pw} denote plane wave coefficients reading as:

$$c_n^{pw} = \frac{(-i)^{n+1}}{k} \frac{2n+1}{n(n+1)} \tag{6}$$

We may take advantage of the Riccati–Bessel differential equation, e.g. Eq.(2.82) and (2.87) in [2] to obtain Eq.(3.188) in [2]:

$$\psi_n''(kr) + \psi_n(kr) = \frac{n(n+1)}{k^2 r^2} \psi_n(kr) \tag{7}$$

so that Eqs. (4) and (5) may be rewritten as:

$$E_r = E_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n,TM}^m n(n+1) \frac{j_n(kr)}{r} P_n^{|m|}(\cos\theta) \exp(im\varphi) \tag{8}$$

$$H_r = H_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n,TE}^m n(n+1) \frac{j_n(kr)}{r} P_n^{|m|}(\cos\theta) \exp(im\varphi) \tag{9}$$

On the other hand, a complex scalar acoustical field (velocity potential) ψ_A propagating in a lossless medium, neglecting nonlinear effects, may be written as [44–47]:

$$\psi_A = \psi_{A0} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} c_{n,A}^{pw} g_{n,A}^m j_n(kr) P_n^{|m|}(\cos\theta) \exp(im\varphi) \tag{10}$$

in which ψ_{A0} is an acoustical field strength and $c_{n,A}^{pw}$ denotes acoustical plane wave coefficients reading as:

$$c_{n,A}^{pw} = (-i)^n (2n+1) \tag{11}$$

in which the subscript A stands for “Acoustical”, e.g. [10,11] (with a typo to be corrected in Eq.27).

3. Radial electromagnetic fields expressed in terms of the acoustical field

In the next step, we use the acoustical field ψ_A of Eq. (10) as being the x-component of the vector potential of Eq. (1) and express the radial electromagnetic fields in terms of this component, which is now rewritten as $A_x = \psi_A$ to better enlighten the strategy used. The x-component has been chosen to mimic the Davis scheme of approximations, furthermore due to the fact that it is sufficient to obtain relationships between acoustical and electromagnetic BSCs that we are looking for, omitting possibly the use of more general polarizations which may be postponed to future works. It is furthermore important to keep in mind that both A_x and ψ_A satisfy the same differential equation, namely the Helmholtz equation. Then, interpreting the acoustical field ψ_A as a component of an electromagnetic potential vector will imply a relationship between electromagnetic and acoustical BSCs.

3.1. Electric field

The electric field may be computed using e.g. Eq.(1.120) in [2]:

$$\mathbf{E} = \frac{1}{i\omega\mu\varepsilon}\mathrm{curl}\,\mathrm{curl}\,\mathbf{A} \tag{12}$$

in which μ and ε are the permeability and permittivity respectively of the medium supporting the wave. To evaluate the curl of $\mathbf{A}$, we use a classical expression of the curl in spherical coordinates reading as:

$$(\mathrm{curl}\,\mathbf{A})_r = \frac{1}{r\sin\theta}(\frac{\partial A_\varphi \sin\theta}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}) \tag{13}$$

$$(\mathrm{curl}\,\mathbf{A})_\theta = \frac{1}{r}(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial rA_\varphi}{\partial r}) \tag{14}$$

$$(\mathrm{curl}\,\mathbf{A})_\varphi = \frac{1}{r}(\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}) \tag{15}$$

in which:

$$A_r = \sin\theta\cos\varphi A_x \tag{16}$$

$$A_\theta = \cos\theta\cos\varphi A_x \tag{17}$$

$$A_\varphi = -\sin\varphi A_x \tag{18}$$

leading to:

$$(\mathrm{curl}\,\mathbf{A})_r = \frac{-\sin\varphi}{r}\frac{\partial A_x}{\partial\theta} - \frac{\cos\theta\cos\varphi}{r\sin\theta}\frac{\partial A_x}{\partial\varphi} = R_r \tag{19}$$

$$(\mathrm{curl}\,\mathbf{A})_\theta = \frac{1}{r}\cos\varphi\frac{\partial A_x}{\partial\varphi} + \sin\varphi\frac{\partial A_x}{\partial r} = R_\theta \tag{20}$$

$$(\mathrm{curl}\,\mathbf{A})_\varphi = \cos\theta\cos\varphi\frac{\partial A_x}{\partial r} - \frac{1}{r}\sin\theta\cos\varphi\frac{\partial A_x}{\partial\theta} = R_\varphi \tag{21}$$

in which we conveniently introduce a notation $R_i = (\mathrm{curl}\,\mathbf{A})_i$, so that we can now calculate $\mathrm{curl}\,\mathrm{curl}\,\mathbf{A}$ by using again Eqs. (13)–(15) under the form:

$$(\mathrm{curl}\,\mathrm{curl}\,\mathbf{A})_r = \frac{1}{r\sin\theta}(\frac{\partial R_\varphi \sin\theta}{\partial\theta} - \frac{\partial R_\theta}{\partial\varphi}) \tag{22}$$

$$(\mathrm{curl}\,\mathrm{curl}\,\mathbf{A})_\theta = \frac{1}{r}(\frac{1}{\sin\theta}\frac{\partial R_r}{\partial\varphi} - \frac{\partial rR_\varphi}{\partial r}) \tag{23}$$

$$(\mathrm{curl}\,\mathrm{curl}\,\mathbf{A})_\varphi = \frac{1}{r}(\frac{\partial rR_\theta}{\partial r} - \frac{\partial R_r}{\partial\theta}) \tag{24}$$

from which, after a bit of algebra, we obtain the r-component $(\mathrm{curl}\,\mathrm{curl}\,\mathbf{A})_r$ which is sufficient to our purpose, and hence E_r according to Eq. (12) reading as:

$$\begin{aligned} E_r = \frac{1}{i\omega\varepsilon\mu}(&-\frac{2}{r}\sin\theta\cos\varphi\frac{\partial A_x}{\partial r} - \frac{2}{r^2}\cos\theta\cos\varphi\frac{\partial A_x}{\partial\theta} + \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\frac{\partial A_x}{\partial\varphi} \\ &+ \frac{1}{r}\cos\theta\cos\varphi\frac{\partial^2 A_x}{\partial r\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial^2 A_x}{\partial r\partial\varphi} - \frac{1}{r^2}\sin\theta\cos\varphi\frac{\partial^2 A_x}{\partial\theta^2} \\ &- \frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\frac{\partial^2 A_x}{\partial\varphi^2}) \end{aligned} \tag{25}$$

As a checking, the same result is obtained by using the alternative expression, e.g. Eqs.(1.117) and (1.120) in [2]:

$$\mathbf{E} = \frac{1}{i\omega\varepsilon\mu}(\mathrm{grad}\,\mathrm{div}\,\mathbf{A} - \Delta\mathbf{A}) \tag{26}$$

in which we use classical expressions reading as:

$$(\mathrm{grad}\, f)_r = \frac{\partial f}{\partial r} \tag{27}$$

$$\mathrm{div}\,\mathbf{A} = \frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \tag{28}$$

$$(\Delta\mathbf{A})_r = \Delta A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}(\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{\partial A_\varphi}{\partial\varphi}) \tag{29}$$

$$\Delta A_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial A_r}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\varphi^2} \tag{30}$$

We then express the components A_r, A_θ and A_φ in terms of A_x using Eqs. (16)–(18) to obtain:

$$\mathrm{div}\,\mathbf{A} = \sin\theta\cos\varphi\frac{\partial A_x}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial A_x}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial A_x}{\partial\varphi} \tag{31}$$

$$\begin{aligned} (\mathrm{grad}\,\mathrm{div}\,\mathbf{A})_r = &\sin\theta\cos\varphi\frac{\partial^2 A_x}{\partial r^2} - \frac{1}{r^2}\cos\theta\cos\varphi\frac{\partial A_x}{\partial\theta} \\ &+ \frac{1}{r}\cos\theta\cos\varphi\frac{\partial^2 A_x}{\partial r\partial\theta} + \frac{\sin\varphi}{r^2\sin\theta}\frac{\partial A_x}{\partial\varphi} \\ &- \frac{\sin\varphi}{r\sin\theta}\frac{\partial^2 A_x}{\partial r\partial\varphi} < \end{aligned} \tag{32}$$

$$\begin{aligned} (\Delta\mathbf{A})_r = &\frac{2}{r}\sin\theta\cos\varphi\frac{\partial A_x}{\partial r} + \frac{1}{r^2}\cos\theta\cos\frac{\partial A_x}{\partial\theta} \\ &+ \sin\theta\cos\varphi\frac{\partial^2 A_x}{\partial r^2} + \frac{1}{r^2}\sin\theta\cos\varphi\frac{\partial^2 A_x}{\partial\theta^2} \\ &+ \frac{\cos\varphi}{r^2\sin\theta}\frac{\partial^2 A_x}{\partial\varphi^2} \end{aligned} \tag{33}$$

from which, after assembling, we recover Eq. (25).

3.2. Magnetic field

The magnetic field may be computed using e.g. Eq.(1.110) in [2]:

$$\mathbf{H} = \frac{1}{\mu}\mathrm{curl}\,\mathbf{A} \tag{34}$$

which, by using Eq. (19) immediately leads to:

$$H_r = \frac{-1}{\mu}(\frac{\sin\varphi}{r}\frac{\partial A_x}{\partial\theta} + \frac{\cos\theta\cos\varphi}{r\sin\theta}\frac{\partial A_x}{\partial\varphi}) \tag{35}$$

4. Relationship between electromagnetic and acoustical BSCs

4.1. From the magnetic field

We consider separately the cases $(n-m)$ even and $(n-m)$ odd, in which n and m refer to the subscript and superscript of the BSCs. The distinction between these two cases is typical of the use of the finite series technique and will be found useful as well for the present problem (see quotations in the introduction for references to this technique).

4.1.1. $(n-m)$ even.

We begin with the magnetic field which is simpler to work out. We then start from Eqs. (9) and (35), in which we evaluate the derivatives of $A_x = \psi_A$ using Eq. (10), and obtain:

$$\begin{aligned} &H_0\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} c_n^{pw} g_{n,TE}^m n(n+1)\frac{j_n(kr)}{r}P_n^{|m|}(\cos\theta)\exp(im\varphi) \\ &= \frac{-\psi_{A0}}{\mu}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} c_{n,A}^{pw} g_{n,A}^m \frac{j_n(kr)}{r}[\tau_n^{|m|}(\cos\theta)\sin\varphi \\ &\quad + im\cos\theta\,\pi_n^{|m|}(\cos\theta)\cos\varphi]\exp(im\varphi) \end{aligned} \tag{36}$$

in which we introduce the usual generalized Legendre functions reading as:

$$\tau_n^{|m|}(\cos\theta) = \frac{d P_n^{|m|}(\cos\theta)}{d\theta} \tag{37}$$

$$\pi_n^{|m|}(\cos\theta) = \frac{P_n^{|m|}(\cos\theta)}{\sin\theta} \tag{38}$$

We then get rid of the coordinate φ by multiplying both sides of Eq. (36), integrating over φ, and use classical orthogonality relations to obtain:

$$\begin{aligned} &2H_0\sum_{n=0}^{\infty} c_n^{pw} g_{n,TE}^m n(n+1)\frac{j_n(kr)}{r}P_n^{|m|}(\cos\theta) \\ &= \frac{-i\psi_{A0}}{\mu}\sum_{n=0}^{\infty} c_{n,A}^{pw}\frac{j_n(kr)}{r}[g_{n,A}^{m+1}\tau_n^{|m+1|}(\cos\theta) - g_{n,A}^{m-1}\tau_n^{|m-1|}(\cos\theta) \\ &\quad + (m+1)\cos\theta g_{n,A}^{m+1}\pi_n^{|m+1|}(\cos\theta) + (m-1)\cos\theta g_{n,A}^{m-1}\pi_n^{|m-1|}(\cos\theta)] \end{aligned} \tag{39}$$

Both sides of Eq. (39) depend on the coordinate r with the same function so that, due to the fact that the spherical Bessel functions are linearly independent, we can remove the summation. Then, we simplify the r -dependent term and obtain, after using Eqs. (6) and (11), and rearranging:

$$g_{n,TE}^{m} = \frac{k\psi_{A0}}{2\mu H_0}\frac{1}{P_n^{|m|}(\cos\theta)} \times\{g_{n,A}^{m+1}[\tau_n^{|m+1|}(\cos\theta)+(m+1)\cos\theta\pi_n^{|m+1|}(\cos\theta)] - g_{n,A}^{m-1}[\tau_n^{|m-1|}(\cos\theta)-(m-1)\cos\theta\pi_n^{|m-1|}(\cos\theta)]\} \quad (40)$$

It is here to be noted that the BSCs apparently depend on θ, in contrast with the fact that they should be constant. The coherency of the theory implies that the θ-dependency is only apparent, i.e. Eq. (40) is not in contradiction with the constancy of the BSCs. This is demonstrated in Appendix A.

It may be interesting, from a historical point of view, to remark that a similar situation occurred when we expressed the same BSCs in the electromagnetic context of GLMT, according to Eq.(3.20) of a textbook ([2], 3rd edition), reading as:

$$g_{n,TE}^{m} = \frac{1}{H_0 c_n^{pw}}\frac{2n+1}{4\pi n(n+1)}\frac{(n-|m|)!}{(n+|m|)}\frac{r}{j_n(kr)} \int_0^{\pi}\int_0^{2\pi} H_r(r,\theta,\varphi)P_n^{|m|}(\cos\theta)\exp(-im\varphi)\sin\theta d\theta d\varphi \quad (41)$$

in which c_n^{pw} are constant coefficients. In Eq. (41), the BSCs apparently depend on the radial coordinate r. However, they should be constant and indeed they are, again as a consequence of the coherency of the theory. At the beginning of the development of GLMT, it has then been taken as granted that these BSCs were constant. What happens is that the integral of Eq. (41) whose integrand does depend on r, is actually proportional to $j_n(kr)/r$. This has eventually been demonstrated in several papers, see page 899, second column of [48], and see as well [49–52]. Since equations like the one of Eq. (41) have been published in 1982 [53], about 25 years have been required to obtain a formal proof of the fact that the BSCs $g_{n,TE}^m$ do not depend on r, although they was no doubt about this. It is fortunate that a similar formal proof concerning Eq. (40) was easy to establish. If this did not happen, the coherency of the theory alone is however sufficient to establish the result.

Since $P_n^{|m|}(\cos\theta) \neq 0$ for $(n-m)$ even [54], we may then obtain the TE-BSCs by setting $\theta = \pi/2$ in Eq. (40) to obtain a simple relation reading as:

$$g_{n,TE}^{m} = \frac{k\psi_{A0}}{2\mu H_0}\frac{1}{P_n^{|m|}(0)}[g_{n,A}^{m+1}\tau_n^{|m+1|}(0) - g_{n,A}^{m-1}\tau_n^{|m-1|}(0)],\ (n-m)\ \text{even} \quad (42)$$

It must be here noted that setting $\theta = \pi/2$ is not a way to "resolve" the dependency of Eq. (40) with respect to θ, since this dependency is only apparent. The only requirement is that the denominator $P_n^{|m|}(\cos\theta)$ should not be 0. The choice of $\pi/2$ is then only expedient and in the spirit of the finite series approach developed to evaluate electromagnetic BSCs, e.g. [33,55], and references therein dating back to [30,31]. Apart of that, any value would do since the dependence on θ is only apparent. This has been demonstrated in the electromagnetic context by introducing a modified finite series approach, see [38].

$(n-m)$ odd.

Since $P_n^{|m|}(\cos\theta) = 0$ for $(n-m)$ odd [54], Eq. (40) cannot be used for this case. Instead, we start again from Eq. (35) and derive it with respect to θ to obtain:

$$H_0\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} c_n^{pw} g_{n,TE}^m n(n+1)\frac{j_n(kr)}{r}\tau_n^{|m|}(\cos\theta)\exp(im\varphi) = \frac{-\psi_{A0}}{\mu}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} c_{n,A}^{pw} g_{n,A}^m \frac{j_n(kr)}{r} \times\{\frac{d\tau_n^{|m|}(\cos\theta)}{d\theta}\sin\varphi + im\cos\varphi[\cos\theta\frac{d\pi_n^{|m|}(\cos\theta)}{d\theta} - \sin\theta\pi_n^{|m|}(\cos\theta)]\} \times\exp(im\varphi) \quad (43)$$

We then may use:

$$\frac{d\tau_n^{|m|}(\cos\theta)}{d\theta} = -\sin\theta\tau_n'^{|m|}(\cos\theta) \quad (44)$$

and:

$$\frac{d\pi_n^{|m|}(\cos\theta)}{d\theta} = -\sin\theta\pi_n'^{|m|}(\cos\theta) \quad (45)$$

in which a prime indicates differentiation with respect to the argument, to rewrite Eq. (43) as:

$$H_0\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} c_n^{pw} g_{n,TE}^m n(n+1)\frac{j_n(kr)}{r}\tau_n^{|m|}(\cos\theta)\exp(im\varphi) = \frac{\psi_{A0}}{\mu}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} c_{n,A}^{pw} g_{n,A}^m\frac{j_n(kr)}{r}\sin\theta \times\{\tau_n'^{|m|}(\cos\theta)\sin\varphi + im\cos\varphi[\pi_n^{|m|}(\cos\theta)+\cos\theta\pi_n'^{|m|}(\cos\theta)]\} \times\exp(im\varphi) \quad (46)$$

Next, we use again classical orthogonality relations to obtain an equation with a single summation over n. As previously, we can remove the summation of Eq. (46) to obtain:

$$2H_0 c_n^{pw} g_{n,TE}^m n(n+1)\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta} = \frac{-i\psi_{A0}}{\mu}c_{n,A}^{pw}\{g_{n,A}^{m+1}\tau_n'^{|m+1|}(\cos\theta) - g_{n,A}^{m-1}\tau_n'^{|m-1|}(\cos\theta) + (m+1)g_{n,A}^{m+1}[\pi_n^{|m+1|}(\cos\theta)+\cos\theta\pi_n'^{|m+1|}(\cos\theta)] + (m-1)g_{n,A}^{m-1}[\pi_n^{|m-1|}(\cos\theta)+\cos\theta\pi_n'^{|m-1|}(\cos\theta)]\} \quad (47)$$

As for Eq. (40), the BSCs of Eq. (47) do not actually depend on θ due to the coherency of the theory. We then specify $\theta = \pi/2$, use Eqs. (6) and (11), and rearrange to obtain:

$$g_{n,TE}^{m} = \frac{k\psi_{A0}}{2\mu H_0}\frac{1}{[\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2}} \times\{g_{n,A}^{m+1}[\tau_n'^{|m+1|}(0)+(m+1)\pi_n^{|m+1|}(0)] - g_{n,A}^{m-1}[\tau_n'^{|m-1|}(0)-(m-1)\pi_n^{|m-1|}(0)]\},\ (n-m)\ \text{odd} \quad (48)$$

Eqs. (42) and (48) are the solutions we were looking for, expressing the electromagnetic TE-coefficients in terms of the acoustical BSCs. A more expedient procedure could be to introduce directly $\theta = \pi/2$ in Eq. (35) for the case $(n-m)$ even, and in Eq. (43) for the case $(n-m)$ odd, but we have preferred to expound a somewhat longer procedure which seems more pedagogic because it is closer to the spirit of the finite series approach.

4.2. From the electric field

4.2.1. $(n-m)$ even

The procedure is similar to the one used for the magnetic field. We use Eq. (25) in which we express the ψ_A-derivatives using Eq. (10), to obtain:

$$E_r = \frac{\psi_{A0}}{i\omega\mu\varepsilon}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} c_{n,A}^{pw} g_{n,A}^m e^{im\varphi}\{\frac{dj_n(kr)}{dr}[-\frac{2}{r}\sin\theta\cos\varphi P_n^{|m|}(\cos\theta) + \frac{1}{r}\cos\theta\cos\varphi\tau_n^{|m|}(\cos\theta) - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}imP_n^{|m|}(\cos\theta)] + j_n(kr)[-\frac{2}{r^2}\cos\theta\cos\varphi\tau_n^{|m|}(\cos\theta) + \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}imP_n^{|m|}(\cos\theta) - \frac{1}{r^2}\sin\theta\cos\varphi\frac{d\tau_n^{|m|}(\cos\theta)}{d\theta} + \frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}m^2P_n^{|m|}(\cos\theta)]\} \quad (49)$$

We carry out integrations over φ from the r.h.s. of Eq. (49) using again classical orthogonality relations, achieve the same treatment to Eq. (8), and equate the results to obtain an equation containing a single summation over n. We then obtain:

$$2E_0\sum_{n=0}^{\infty}c_n^{pw}g_{n,TM}^m n(n+1)\frac{j_n(kr)}{r}P_n^{|m|}(\cos\theta) \tag{50}$$
$$=\frac{\Psi_{A0}}{i\omega\mu\varepsilon}\sum_{n=0}^{\infty}c_{n,A}^{pw}\{\frac{dj_n(kr)}{rdr}[-2\sin\theta P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}-2\sin\theta P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}$$
$$+\cos\theta\tau_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}+\cos\theta\tau_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}$$
$$+\frac{1}{\sin\theta}(m+1)P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}-\frac{1}{\sin\theta}(m-1)P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}]$$
$$+\frac{j_n(kr)}{r^2}[-2\cos\theta\tau_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}-2\cos\theta\tau_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}$$
$$-\frac{1}{\sin\theta}(m+1)P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}+\frac{1}{\sin\theta}(m-1)P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}$$
$$-\sin\theta\frac{d\tau_n^{|m+1|}(\cos\theta)}{d\theta}g_{n,A}^{m+1}-\sin\theta\frac{d\tau_n^{|m-1|}(\cos\theta)}{d\theta}g_{n,A}^{m-1}$$
$$+\frac{1}{\sin\theta}(m+1)^2P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}+\frac{1}{\sin\theta}(m-1)^2P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}]\}$$

As for Eq. (40), the BSCs of Eq. (50) again do not actually depend on θ due to the coherency of the theory. We then specify $\theta=\pi/2$, rearrange and obtain:

$$2E_0\sum_{n=0}^{\infty}c_n^{pw}g_{n,TM}^m n(n+1)\frac{j_n(kr)}{r}P_n^{|m|}(0) \tag{51}$$
$$=\frac{\Psi_{A0}}{i\omega\mu\varepsilon}\sum_{n=0}^{\infty}c_{n,A}^{pw}\{\frac{dj_n(kr)}{rdr}[(m-1)P_n^{|m+1|}(0)g_{n,A}^{m+1}-(m+1)P_n^{|m-1|}(0)g_{n,A}^{m-1}]$$
$$+\frac{j_n(kr)}{r^2}[m(m+1)P_n^{|m+1|}(0)g_{n,A}^{m+1}+m(m-1)P_n^{|m-1|}(0)g_{n,A}^{m-1}$$
$$+\tau_n'^{|m+1|}(0)g_{n,A}^{m+1}+\tau_n'^{|m-1|}(0)g_{n,A}^{m-1}]\}$$

in which we have used:

$$[\frac{d\tau_n^p(\cos\theta)}{d\theta}]_{\theta=\pi/2}=[-\sin\theta\frac{d\tau_n^p(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2}=-\tau_n'^p(0) \tag{52}$$

To study Eq. (51), it is convenient to consider separately the l.h.s. and the r.h.s, reading as:

$$LHS=2E_0\sum_{n=1}^{\infty}c_n^{pw}g_{n,TM}^m n(n+1)\frac{j_n(kr)}{r}P_n^{|m|}(0) \tag{53}$$

$$RHS=\frac{\Psi_{A0}}{i\omega\varepsilon}\sum_{n=0}^{\infty}c_{n,A}^{pw}\frac{1}{r}[\frac{dj_n(kr)}{dr}T_{nm}^{(1)}+\frac{j_n(kr)}{r}T_{nm}^{(2)}] \tag{54}$$

in which we found convenient to have n ranging from 1 in Eq. (53), and in which we introduced:

$$T_{nm}^{(1)}=(m-1)P_n^{|m+1|}(0)g_{n,A}^{m+1}-(m+1)P_n^{|m-1|}(0)g_{n,A}^{m-1} \tag{55}$$

$$T_{nm}^{(2)}=[m(m+1)P_n^{|m+1|}(0)+\tau_n'^{|m+1|}(0)]g_{n,A}^{m+1}$$
$$+[m(m-1)P_n^{|m-1|}(0)+\tau_n'^{|m-1|}(0)]g_{n,A}^{m-1} \tag{56}$$

Next, we use classical relations concerning the spherical Bessel functions reading as [56]:

$$\frac{j_n(x)}{x}=\frac{j_{n-1}(x)+j_{n+1}(x)}{2n+1} \tag{57}$$

$$\frac{dj_n(x)}{dx}=\frac{nj_{n-1}(x)-(n+1)j_{n+1}(x)}{2n+1} \tag{58}$$

and, inserting Eqs. (57)–(58) into Eq. (54), we obtain:

$$RHS=\frac{k\Psi_{A0}}{i\omega\mu\varepsilon}\sum_{n=0}^{\infty}c_{n,A}^{pw}\frac{1}{r}[\frac{n}{2n+1}T_{nm}^{(1)}j_{n-1}(kr)+\frac{1}{2n+1}T_{nm}^2j_{n-1}(kr)$$
$$-\frac{n+1}{2n+1}T_{nm}^{(1)}j_{n+1}(kr)+\frac{1}{2n+1}T_{nm}^{(2)}j_{n+1}(kr)] \tag{59}$$

Modifying conveniently the summation indices, Eq. (59) can be rewritten as:

$$RHS=\frac{k\Psi_{A0}}{i\omega\mu\varepsilon}\{\sum_{n=-1}^{\infty}c_{n+1,A}^{pw}\frac{j_n(kr)}{r}[\frac{n+1}{2n+3}T_{n+1,m}^{(1)}+\frac{1}{2n+3}T_{n+1,m}^2] \tag{60}$$
$$+\sum_{n=1}^{\infty}c_{n-1,A}^{pw}\frac{j_n(kr)}{r}[-\frac{n}{2n-1}T_{n-1,m}^{(1)}+\frac{1}{2n-1}T_{n-1,m}^2]\}$$

which may be rewritten as:

$$RHS=\frac{k\Psi_{A0}}{i\omega\mu\varepsilon}\{T_{-10}+\sum_{n=1}^{\infty}\frac{j_n(kr)}{r}[\frac{(n+1)T_{n+1,m}^{(1)}+T_{n+1,m}^{(2)}}{2n+3}c_{n+1,A}^{pw} \tag{61}$$
$$-\frac{nT_{n-1,m}^{(1)}-T_{n-1,m}^{(2)}}{2n-1}c_{n-1,A}^{pw}]\}$$

in which:

$$T_{-10}=\frac{j_{-1}(kr)}{r}c_{0,A}^{pw}T_{0m}^{(2)}+\frac{j_0(kr)}{3r}c_{1,A}^{pw}(T_{1m}^{(1)}+T_{1m}^{(2)}) \tag{62}$$

We may then show that $T_{-10}=0$ (see Appendix B) so that, from Eq. (51), using Eqs. (53) and (54), and Eq. (61) with $T_{-10}=0$, we obtain:

$$2E_0\sum_{n=1}^{\infty}c_n^{pw}g_{n,TM}^m n(n+1)\frac{j_n(kr)}{r}P_n^{|m|}(0) \tag{63}$$
$$=\frac{k\Psi_{A0}}{i\omega\mu\varepsilon}\sum_{n=1}^{\infty}\frac{j_n(kr)}{r}[\frac{(n+1)T_{n+1,m}^{(1)}+T_{n+1,m}^{(2)}}{2n+3}c_{n+1,A}^{pw}-\frac{nT_{n-1,m}^{(1)}-T_{n-1,m}^{(2)}}{2n-1}c_{n-1,A}^{pw}]$$

Both sides of Eq. (63) possess the same r-dependence and, therefore, as for the magnetic case, we can remove the summation, leading to:

$$2E_0c_n^{pw}g_{n,TM}^m n(n+1)P_n^{|m|}(0) \tag{64}$$
$$=\frac{k\Psi_{A0}}{i\omega\mu\varepsilon}[\frac{(n+1)T_{n+1,m}^{(1)}+T_{n+1,m}^{(2)}}{2n+3}c_{n+1,A}^{pw}+\frac{T_{n-1,m}^{(2)}-nT_{n-1,m}^{(1)}}{2n-1}c_{n-1,A}^{pw}]$$

leading to:

$$g_{n,TM}^m=\frac{k\Psi_{A0}}{i\omega\mu\varepsilon 2E_0c_n^{px}n(n+1)P_n^m(0)} \tag{65}$$
$$\times[\frac{(n+1)T_{n+1,m}^{(1)}+T_{n+1,m}^{(2)}}{2n+3}c_{n+1,A}^{pw}+\frac{T_{n-1,m}^{(2)}-nT_{n-1,m}^{(1)}}{2n-1}c_{n-1,A}^{pw}]$$

We finally use $k^2=\omega^2\varepsilon\mu$ [2] and Eqs. (6), (11) to obtain:

$$g_{n,TM}^m=\frac{i\omega\Psi_{A0}}{2(2n+1)E_0P_n^{|m|}(0)}(T_{n-1,m}^{(2)}-T_{n+1,m}^{(2)}-nT_{n-1,m}^{(1)}-(n+1)T_{n+1,m}^{(1)}),$$
$$(n-m)\text{ even} \tag{66}$$

4.2.2. $(n-m)$ odd

We multiply Eq. (50) by r, derive with respect to θ, specify $\theta=\pi/2$, and rearrange to obtain:

$$2E_0\sum_{n=0}^{\infty}c_n^{pw}g_{n,TM}^m n(n+1)j_n(kr)\tau_n^{|m|}(0)= \tag{67}$$
$$=\frac{\Psi_{A0}}{i\omega\mu\varepsilon}\sum_{n=0}^{\infty}c_{n,A}^{pw}\{\frac{dj_n(kr)}{dr}[(m-2)\tau_n^{|m+1|}(0)g_{n,A}^{m+1}-(m+2)\tau_n^{|m-1|}(0)g_{n,A}^{m-1}]$$
$$+\frac{j_n(kr)}{r}[(m^2+m+2)\tau_n^{|m+1|}(0)g_{n,A}^{m+1}+(m^2-m+2)\tau_n^{|m-1|}(0)g_{n,A}^{m-1}$$
$$-\tau_n''^{|m+1|}(0)g_{n,A}^{m+1}-\tau_n''^{|m-1|}(0)g_{n,A}^{m-1}]\}$$

in which we have used:

$$[\frac{dP_n^{|m|}(\cos\theta)}{d\theta}]_{\theta=\pi/2}=\tau_n^{|m|}(0) \tag{68}$$

$$(\frac{d^2\tau_n^{|m|}(\cos\theta)}{d\theta^2})_{\theta=\pi/2}=\tau_n''^{|m|}(0) \tag{69}$$

We then proceed as for the case $(n-m)$ odd, writing:

$$LHS = 2E_0 \sum_{n=1}^{\infty} c_n^{pw} g_{n,TM}^m n(n+1) j_n(kr) \tau_n^{|m|}(0) \tag{70}$$

$$RHS = \frac{\Psi_{A0}}{i\omega\mu\epsilon} \sum_{n=0}^{\infty} c_{n,A}^{pw} \frac{1}{r} [\frac{dj_n(kr)}{dr} V_{nm}^{(1)} + \frac{j_n(kr)}{r} V_{nm}^{(2)}] \tag{71}$$

in which:

$$V_{nm}^{(1)} = (m-2)\tau_n^{|m+1|}(0) g_{n,A}^{m+1} - (m+2)\tau_n^{|m-1|}(0) g_{n,A}^{m-1} \tag{72}$$

$$\begin{aligned} V_{nm}^{(2)} = & [(m^2+m+2)\tau_n^{|m+1|}(0) - \tau_n''^{|m+1|}(0)] g_{n,A}^{m+1} \\ & +[(m^2-m+2)\tau_n^{|m-1|}(0) - \tau_n''^{|m-1|}(0)] g_{n,A}^{m-1} - \end{aligned} \tag{73}$$

to obtain, similarly as for Eq. (66):

$$g_{n,TM}^m = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0 \tau_n^{|m|}(0)} (V_{n-1,m}^{(2)} - V_{n+1,m}^{(2)} - nV_{n-1,m}^{(1)} - (n+1)V_{n+1,m}^{(1)}) , \quad (n-m) \text{ odd} \tag{74}$$

In doing so, we had to introduce (similarly as for T_{-10}) a quantity V_{-10} reading as:

$$V_{-10} = c_{0,A}^{pw} j_{-1}(kr) V_{0m}^2 + \frac{1}{3} c_{1,A}^{pw} j_0(kr)(V_{1m}^{(1)} + V_{1m}^{(2)}) \tag{75}$$

and to show that this quantity is equal to 0. This is done in the Appendix B.

5. Discussion

Eq. (42) for $(n-m)$ even and (48) for $(n-m)$ odd show that the azimuth order "m" for the electromagnetic TE-BSCs is related to the azimuth orders "$m \pm 1$" for the acoustical BSCs. This result may be commented in three different (but related) ways as follows.

(i) For on-axis acoustical Gaussian beams, it has been demonstrated that the only nonzero BSCs are for an azimuth order equal to 0 [10,11], that is to say for $m \pm 1 = 0$ for acoustical BSCs, and therefore $m = \mp 1$ for electromagnetic BSCs. This agrees with the fact demonstrated elsewhere that the only nonzero electromagnetic BSCs of on-axis electromagnetic Gaussian beams are indeed for $m = \pm 1$, e.g. [2].

(ii) As another point of view, considering the same circumstances than for (i) above, the only nonzero electromagnetic BSCs are for $m = \pm 1$, i.e. $g_{n,TE}^1$ and $g_{n,TE}^{-1}$. The BSCs $g_{n,TE}^1$ are then related to the acoustical BSCs $g_{n,A}^2$ and $g_{n,A}^0$. But, since the only nonzero acoustical BSCs are for $g_{n,A}^0$, it happens that $g_{n,TE}^1$'s are related to $g_{n,A}^0$. Similarly, the BSCs $g_{n,TE}^{-1}$ are then related to the acoustical BSCs $g_{n,A}^0$ and $g_{n,A}^{-2}$. But, similarly, since the only nonzero acoustical BSCs are for $g_{n,A}^0$, it happens that $g_{n,TE}^{-1}$'s are related to $g_{n,A}^0$. In other words, both $g_{n,TE}^1$ and $g_{n,TE}^{-1}$ are related uniquely to $g_{n,A}^0$.

(iii) When we design a localized approximation for on-axis acoustical beams, we find that the localization procedure implies the use of a prefactor $(-i)^{|m|}$ to evaluate $g_{n,A}^m$, see again [10,11], while the localization procedure for electromagnetic fields implies the use of a prefactor $(-i)^{|m|-1}$ to evaluate $g_{n,X}^m$, $X = TM$ or TE, e.g. [2]. For the Gaussian beams discussed in [10,11], with the azimuth order $m = 0$ corresponding to the nonzero acoustical BSCs, the prefactor becomes $(-i)^0 = 1$. As we have seen in (ii), the $m = 0$ acoustical BSCs are related only to the electromagnetic BSCs $g_{n,TE}^1$ and $g_{n,TE}^{-1}$. The prefactor $(-i)^{|m|-1}$ to be used for the electromagnetic localization procedure then corresponds to a prefactor $(-i)^{|m=\pm 1|-1} = (-i)^0$ which identifies with the prefactor of the acoustical procedure.

The case of the TM-coefficients is discussed in a similar way since the azimuth order "m" for the electromagnetic TM-BSCs is related to the azimuth orders "$m \pm 1$" for the acoustical BSCs as it was for the TE-BSCs, see Eqs. (55), (56), (65) for $(n-m)$ even and Eqs. (72), (73), (74) for $(n-m)$ odd.

As a last word, it is relevant to compare the results obtained in the present work with those available from [57]. This is postponed to a future research. For the time being, let us just mention that the present work is based on a single vector potential in contrast with [57] which relies on the use of two vector potentials.

6. Conclusion

During the last decades, a vigorous effort has been devoted to the study of electromagnetic scattering by particles, using either analytical methods known as generalized Lorenz–Mie theories or semi-analytical methods like the Extended Boundary Condition Method, both of them being subsumable under the cover of T-matrix methods. Recently, it has been put forward that there are strong analogies between electromagnetic and acoustical scatterings, although one is a vectorial scattering and the other a scalar scattering. In the present paper, we demonstrate that electromagnetic BSCs can be expressed in terms of acoustical BSCs. The interest of this result is threefold.

(i) The relationship between the electromagnetic and acoustical BSCs allows one to insist on the strong similarity between the scalar acoustical scattering theory and the vectorial electromagnetic scattering theory. This analogy has already been indeed exploited to transfer electromagnetic methods of evaluation of electromagnetic BSCs to the evaluation of acoustical BSCs, see [10,11,14,58] for the localized approximation and [12] for the finite series technique.

(ii) Expressing electromagnetic BSCs in terms of acoustical BSCs is a new tool which then augments the arsenal of methods available to the evaluation of electromagnetic BSCs.

(iii) Furthermore, there is only one kind of acoustical BSCs and two kinds of electromagnetic BSCs. These two kinds of electromagnetic BSCs may be therefore expressed in terms of one kind only of acoustical BSCs. Then, roughly speaking, we may expect that the evaluation of electromagnetic BSCs in terms of acoustical BSCs might be twice faster than when working only in the electromagnetic framework.

CRediT authorship contribution statement

Gérard Gouesbet: Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Leonardo A. Ambrosio:** Writing – review & editing, Methodology, Investigation, Funding acquisition, Formal analysis. **Jianqi Shen:** Writing – review & editing, Methodology, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

In this Appendix, we shall demonstrate that Eq. (40) actually does not depend on θ. For this, we shall examine two ratios involved in Eq. (40), namely:

$$T_+ = \frac{\tau_n^{|m+1|}(\cos\theta) + (m+1)\cos\theta\,\pi_n^{|m+1|}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{76}$$

$$T_- = \frac{\tau_n^{|m-1|}(\cos\theta) - (m-1)\cos\theta\,\pi_n^{|m-1|}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{77}$$

From Robin [54], tome 1, p. 102, with a typo corrected, we have, with $\mu = \cos\theta$:

$$(\mu^2 - 1)\frac{dP_n^m}{d\mu} = \sqrt{1-\mu^2}\,P_n^{m+1}(\mu) + m\mu P_n^m(\mu) \tag{78}$$

and, from Arfken et al. [59], Eqs.(15.91)-(15.92), we have:

$$(\mu^2 - 1)\frac{dP_n^m(\mu)}{d\mu} = -(n+m)(n-m+1)\sqrt{1-\mu^2}\,P_n^{m-1}(\mu) - m\mu P_n^m(\mu) \tag{79}$$

From Eqs. (78) and (79), we obtain:

$$\frac{dP_n^m(\cos\theta)}{d\theta} - m\frac{\cos\theta}{\sin\theta}P_n^m(\cos\theta) = P_n^{m+1}(\cos\theta) \tag{80}$$

$$\frac{dP_n^m(\cos\theta)}{d\theta} + m\frac{\cos\theta}{\sin\theta}P_n^m(\cos\theta) = -(n+m)(n-m+1)P_n^{m-1}(\cos\theta) \tag{81}$$

Recalling the definition of the generalized Legendre functions $\tau_n^m(\cos\theta)$ and $\pi_n^m(\cos\theta)$, Eqs. (80) and (81) may be rewritten as:

$$\tau_n^m(\cos\theta) - m\cos\theta\,\pi_n^m(\cos\theta) = P_n^{m+1}(\cos\theta) \tag{82}$$

$$\tau_n^m(\cos\theta) + m\cos\theta\,\pi_n^m(\cos\theta) = -(n+m)(n-m+1)P_n^{m-1}(\cos\theta) \tag{83}$$

From Eq. (82), we may express $\tau_n^{|m-1|}(\cos\theta)$ and, from Eq. (83), we may express $\tau_n^{|m+1|}(\cos\theta)$, leading to:

$$\tau_n^{m-1}(\cos\theta) - (m-1)\cos\theta\,\pi_n^{m-1}(\cos\theta) = P_n^m(\cos\theta) \tag{84}$$

$$\tau_n^{m+1}(\cos\theta) + (m+1)\cos\theta\,\pi_n^{m+1}(\cos\theta) = -(n+m+1)(n-m)P_n^m(\cos\theta) \tag{85}$$

For $m > 0$, we therefore simply obtain:

$$T_+ = \frac{\tau_n^{m+1}(\cos\theta) + (m+1)\cos\theta\,\pi_n^{m+1}(\cos\theta)}{P_n^m(\cos\theta)} = -(n+m+1)(n-m) \tag{86}$$

$$T_- = \frac{\tau_n^{m-1}(\cos\theta) - (m-1)\cos\theta\,\pi_n^{m-1}(\cos\theta)}{P_n^m(\cos\theta)} = 1 \tag{87}$$

which do not depend on θ, as it should since, as stated previously, the θ-dependency in Eq. (40) is only apparent.

For $m = 0$, Eqs. (76) and (77) become:

$$T_+ = T_- = \frac{\tau_n^1(\cos\theta) + \cos\theta\,\pi_n^1(\cos\theta)}{P_n^0(\cos\theta)} \tag{88}$$

while Eq. (85) become:

$$\tau_n^1(\cos\theta) + \cos\theta\,\pi_n^1(\cos\theta) = -n(n+1)P_n^0(\cos\theta) \tag{89}$$

so that Eq. (88) leads to:

$$T_+ = T_- = -n(n+1) \tag{90}$$

which, again does not depend on θ, as it should.

Finally, for $m < 0$, Eqs. (76) and (77) become:

$$T_+ = \frac{\tau_n^{|m|-1}(\cos\theta) - (|m|-1)\cos\theta\,\pi_n^{|m|-1}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{91}$$

$$T_- = \frac{\tau_n^{|m|+1}(\cos\theta) + (|m|+1)\cos\theta\,\pi_n^{|m|+1}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{92}$$

From Eqs. (84) and (85), we then obtain:

$$T_+ = 1 \tag{93}$$

$$T_- = -(n+|m|+1)(n-|m|) \tag{94}$$

which do not depend on θ, again as it should.

Appendix B

We begin with the (-1)-term in T_{-10} of Eq. (62) which is expressed in terms of $T_{0m}^{(2)}$ which, from Eq. (56), reads as:

$$\begin{aligned} T_{0m}^{(2)} &= [m(m+1)P_0^{|m+1|}(0) + \tau_0'^{|m+1|}(0)]g_{0,A}^{m+1} \\ &\quad + [m(m-1)P_0^{|m-1|}(0) + \tau_0'^{|m-1|}(0)]g_{0,A}^{m-1} \end{aligned} \tag{95}$$

We have to consider three cases as follows.

(i) For $|m| \geq 2$, since P_n^m satisfies $m = -n, \ldots, n$, we have $P_0^{|m\pm1|} = 0$ and $\tau_0'^{|m\pm1|} = 0$, so that $T_{0m}^{(2)} = 0$.

(ii) For $|m| = 1$, we similarly have:

$$T_{01}^{(2)} = [2P_0^2(0) + \tau_0'^2(0)]g_{0,A}^2 = 0 \tag{96}$$

$$T_{0-1}^{(2)} = [2P_0^2(0) + \tau_0'^2(0)]g_{0,A}^{-2} = 0 \tag{97}$$

(iii) For $m = 0$, Eq. (95) reduces to:

$$T_{0m}^{(2)} = \tau_0'^1(0)(g_{0,A}^1 + g_{0,A}^{-1}) = 0 \tag{98}$$

because $\tau_0'^1$ depends on P_0^1 which is 0.

Next, we deal with the (0)-term in T_{-10} of Eq. (62) which is expressed in terms of $(T_{1m}^{(1)} + T_{1m}^{(2)})$ which, from Eqs. (55)–(56), reads as:

$$\begin{aligned} T_{1m}^{(1)} + T_{1m}^{(2)} &= [(m^2+2m-1)P_1^{|m+1|}(0) + \tau_1'^{|m+1|}(0)]g_{1,A}^{m+1} \\ &\quad + [(m^2-2m-1)P_1^{|m-1|}(0) + \tau_1'^{|m-1|}(0)]g_{1,A}^{m-1} \end{aligned} \tag{99}$$

which is discussed considering four different cases as follows.

(i) For $|m| \geq 3$, since P_n^m satisfies $m = -n, \ldots, n$, we have $P_0^{|m\pm1|} = 0$ and $\tau_0'^{|m\pm1|} = 0$, so that $T_{1m}^{(1)} + T_{1m}^{(2)} = 0$.

(ii) For $|m| = 2$, we obtain two equations, one for $m = 2$ and the other for $m = -2$, reading as:

$$T_{12}^{(1)} + T_{12}^{(2)} = [-P_1^1(0) + \tau_1'^1(0)]g_{1,A}^1 \tag{100}$$

$$T_{1-2}^{(1)} + T_{1-2}^{(2)} = [-P_1^1(0) + \tau_1'^1(0)]g_{1,A}^{-1} \tag{101}$$

But we have $P_1^1(0) = -1$ and:

$$\tau_1'^1(0) = \left[\frac{d}{d\cos\theta}\frac{dP_1^1(\cos\theta)}{d\theta}\right]_{\theta=\pi/2} = -1 \tag{102}$$

Therefore $T_{1\pm2}^{(1)} + T_{1\pm2}^{(2)} = 0$.

(iii) For $|m| = 1$, we have again to deal with two equations reading as:

$$T_{11}^{(1)} + T_{11}^{(2)} = [2P_1^2(0) + \tau_1'^2(0)]g_{1,A}^2 + [-2P_1^0(0) + \tau_1'^0(0)]g_{1,A}^0 \tag{103}$$

$$T_{1-1}^{(1)} + T_{1-1}^{(2)} = [-2P_1^0(0) + \tau_1'^0(0)]g_{1,A}^0 + [2P_1^2(0) + \tau_1'^2(0)]g_{1,A}^{-2} \tag{104}$$

But we readily find or establish that $P_1^2(0) = \tau_1'^2(0) = P_1^0(0) = \tau_1'^0(0)$ leading to $T_{11}^{(1)} + T_{11}^{(2)} = T_{1-1}^{(1)} + T_{1-1}^{(2)} = 0$.

(iv) For $m = 0$, we have:

$$T_{10}^{(1)} + T_{10}^{(2)} = [-P_1^1(0) + \tau_1'^1(0)](g_{1,A}^1 + g_{1,A}^{-1}) \tag{105}$$

But we have $P_1^1(0) = -1$ and $\tau_1'^1(0) = -1$, see Eq. (102), so that we readily have $T_{10}^{(1)} + T_{10}^{(2)} = 0$.

We now consider the (-1)-term of V_{-10} of Eq. (75) which is expressed in terms of V_{0m}^2 which, from Eq. (73), reads as:

$$\begin{aligned} V_{0m}^{(2)} &= [(m^2+m+2)\tau_0^{|m+1|}(0) - \tau_0''^{|m+1|}(0)]g_{0,A}^{m+1} \\ &\quad + [(m^2-m+2)\tau_0^{|m-1|}(0) - \tau_0''^{|m-1|}(0)]g_{0,A}^{m-1} \end{aligned} \tag{106}$$

which is to be discussed with three cases as follows.

(i) When $|m| \geq 2$, we have $\tau_0^{|m\pm1|}(0) = \tau_0''^{|m\pm1|}(0) = 0$ because $P_0^{|m\pm1|}(\cos\theta) = 0$, hence $V_{0m}^{(2)} = 0$.

(ii) When $|m| = 1$, we have to consider two equations reading as:

$$V_{01}^{(2)} = [4\tau_0^2(0) - \tau_0''^2(0)]g_{0,A}^2 + [2\tau_0^0(0) - \tau_0''^0(0)]g_{0,A}^0 \tag{107}$$

$$V_{0-1}^{(2)} = [2\tau_0^0(0) - \tau_0''^0(0)]g_{0,A}^0 + [4\tau_0^2(0) - \tau_0''^2(0)]g_{0,A}^{-2} \tag{108}$$

which are 0 because it is readily checked that $\tau_0^2(0) = \tau_0''^2(0) = 0$ and $\tau_0^0(0) = \tau_0''^0(0) = 0$.

(iii) When $m = 0$, we have:

$$V_{00}^{(2)} = [2\tau_0^1(0) - \tau_0''^1(0)](g_{0,A}^{+1} + g_{0,A}^{-1}) \tag{109}$$

which is 0 since $P_0^1(0) = 0$.

Finally, we consider the (0)-term of Eq. (75) which is expressed in terms of $V_{1m}^{(1)} + V_{1m}^{(2)}$ which, from Eqs. (72) and (73), reads as:

$$\begin{aligned} V_{1m}^{(1)} + V_{1m}^{(2)} = & [m(m+2)\tau_1^{|m+1|}(0) - \tau_1''^{|m+1|}(0)]g_{1,A}^{m+1} \\ & + [m(m-2)\tau_1^{|m-1|}(0) - \tau_1''^{|m-1|}(0)]g_{1,A}^{m-1} \end{aligned} \tag{110}$$

which is discussed considering four different cases as follows.

(i) When $|m| \geq 3$, we have $\tau_1^{|m+1|}(0) = \tau_1''^{|m+1|}(0)$ because $P_1^{|m+1|} = 0$, hence $V_{1m}^{(1)} + V_{1m}^{(2)} = 0$.

(ii) When $|m| = 2$, we have to deal with two equations reading as:

$$V_{12}^{(1)} + V_{12}^{(2)} = [8\tau_1^3(0) - \tau_1''^3(0)]g_{1,A}^3 - \tau_1''^{|1|}(0)g_{1,A}^1 \tag{111}$$

$$V_{1-2}^{(1)} + V_{1-2}^{(2)} = -\tau_1''^1(0)g_{1,A}^{-1} - +[8\tau_1^3(0) - \tau_1''^1(0)]g_{1,A}^{-3} \tag{112}$$

which are indeed 0 because $\tau_1^3(0) = \tau_1''^3(0) = 0$ and:

$$\tau_1'^1(\cos\theta) = \frac{d\tau_1^1(\cos\theta)}{d\cos\theta} = \frac{d}{d\cos\theta}[\frac{dP_1^1(\cos\theta)}{d\theta}] = -1 \tag{113}$$

and, therefore $\tau_1''^1(0) = \tau_1''^1(\cos\theta) = 0$.

(iii) When $|m| = 1$, we obtain two equations from Eq. (100), reading as:

$$V_{11}^{(1)} + V_{11}^{(2)} = [3\tau_1^2(0) - \tau_1''^2(0)]g_{1,A}^2 + [-\tau_1^0(0) - \tau_1''^0(0)]g_{1,A}^0 \tag{114}$$

$$V_{1-1}^{(1)} + V_{1-1}^{(2)} = [-\tau_1^0(0) - \tau_1''^0(0)]g_{1,A}^0 + [3\tau_1^2(0) - \tau_1''^2(0)]g_{1,A}^{-2} \tag{115}$$

which are 0 because $\tau_1^2(0) = \tau_1''^2(0)$ and $\tau_1^0(0) + \tau_1''^0(0) = 0$.

(iv) When $m = 0$, Eq. (110) reduces to:

$$V_{10}^{(1)} + V_{10}^{(2)} = -\tau_1''^1(0)(g_{1,A}^1 + g_{1,A}^{-1}) \tag{116}$$

which is 0 from Eq. (113).

Data availability

No data was used for the research described in the article.

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