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EXISTENCE AND REGULARITY OF PERIODIC SOLUTIONS TO CERTAIN FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Existence and regularity of periodic solutions to certain first-order partial differential equations

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Abstract

We present conditions on the coefficients of a class of vector fields on the torus which yield a characterization of global solvability as well as global hypoellipticity, in other words, the existence and regularity of periodic solutions. Diophantine conditions and connectedness of certain sublevel sets appear in a natural way in our results.

1 Introduction

For a positive integer m, let $\mathbb{T}^m \simeq \mathbb{R}^m/2\pi\mathbb{Z}^m$ be the $m-$dimensional torus. We will work on $\mathbb{T}^{N+1}$, where the coordinates are denoted by $(x,t) \in \mathbb{T}^N \times \mathbb{T}^1$, with $x = (x_1, \ldots, x_N) \in \mathbb{T}^N$.

We are interested in the existence of solutions in $\mathcal{C}^\infty(\mathbb{T}^{N+1})$ to first-order partial differential equations given by $Lu = f$, where $f \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ and L is a vector field on $\mathbb{T}^{N+1}$ of the type

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N}(a_j + ib_j)(t)\frac{\partial}{\partial x_j}, \tag{1}$$

where the coefficients a_j and b_j are real-valued smooth functions defined on $\mathbb{T}^1$, for $j = 1, \ldots, N$.

We recall that the existence of local solutions to $Lu = f$ is well-understood and is in fact characterized by the well-known *Nirenberg-Treves* condition $(\mathcal{P})$ (see [12] and [14]).

The problem is still interesting if we consider the existence of solutions on the whole torus, a property which will be refered to as global solvability.

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We say that L is *globally solvable* if the range of the operator $L : \mathcal{C}^\infty(\mathbb{T}^{N+1}) \to \mathcal{C}^\infty(\mathbb{T}^{N+1})$ is closed. Standard arguments of Functional Analysis imply that $\overline{L\mathcal{C}^\infty(\mathbb{T}^{N+1})} = (\ker {}^tL)^\circ$, where ${}^tL$ denotes the transpose operator and $(\ker {}^tL)^\circ$ is the set of functions $\phi \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ such that $\mu(\phi) = 0$, for all $\mu \in \ker {}^tL \subset \mathcal{D}'(\mathbb{T}^{N+1})$. It follows that L is globally solvable if and only if $L\mathcal{C}^\infty(\mathbb{T}^{N+1}) = (\ker {}^tL)^\circ$. We will use this characterization throughout this article.

It is well known that condition $(\mathcal{P})$ is not necessary for global solvability (see [11], and also [5]). On the other hand, it follows from [10] that the equation $Lu = f$ has a solution $u \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ for every $f \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ satisfying a finite number of compatibility conditions, provided L satisfies both condition $(\mathcal{P})$ and the following geometric condition (in which ℓ is the principal symbol of L and q is a complex-valued function):

(GC) every characteristic point of L lies on a compact interval of a bicharacteristic of $\Re(q\ell)$, on which $q \neq 0$, with no characteristic endpoint.

However, as shown in [5], our class of operators (in general) does not satisfy conditions $(\mathcal{P})$ and (GC) simultaneously. Hence, the global solvability of our class of operators cannot be obtained by directly applying results from [10].

When our operator L does not satisfy condition $(\mathcal{P})$, a necessary condition for solvability is given by the connectedness of certain sublevel sets (see Theorem 1.1-(II)). This relation between solvability and connectedness of sublevel sets is due to Treves and it has appeared for instance in [13].

Another question of our interest is the regularity of solutions $\mu \in \mathcal{D}'(\mathbb{T}^{N+1})$ to $L\mu = f$, where L is given by (1) and f belongs to $\mathcal{C}^\infty(\mathbb{T}^{N+1})$. This regularity will be described by the expression global hypoellipticity. We say that L is *globally hypoelliptic* if $\mu \in \mathcal{D}'(\mathbb{T}^{N+1})$ and $L\mu \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ imply that $\mu \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$.

The approach and motivations for this paper are related to those in references such as [1], [2], [3], [4], [5], [7], [11] and [13].

Our results extend those in [5], where global solvability and global hypoellipticity were characterized in the case where $N = 2$ and L had the special form

$$\frac{\partial}{\partial t} + ib_1(t)\frac{\partial}{\partial x_1} + ib_2(t)\frac{\partial}{\partial x_2}.$$

We introduce the following useful notations:

$$a_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(t)dt, \quad \text{for} \quad j = 1, \dots, N,$$

and

$$b_{j0} = \frac{1}{2\pi}\int_0^{2\pi} b_j(t)dt, \quad \text{for} \quad j = 1, \dots, N.$$

Comparing with [5], the presence of the coefficient a_j makes the problem more difficult. For instance, it will be important to use some smooth cutoff functions in order to apply the Laplace method for integrals. Also, we must take into account the influence of the numbers a_{j0} on the Diophantine conditions. As we know, for instance, from [1], [2], [5], and [7], global solvability and global hypoellipticity are related to Diophantine conditions, which, in some cases, are linked with the notion of non-Liouville vector (or number). We say that $(\gamma_1, \gamma_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ is a *Liouville vector* if there exist $C > 0$ and a sequence $(p_n, q_n, j_n) \in \mathbb{Z}^2 \times \mathbb{N}$ so that $j_n \geq 2$, and

$$\left|\gamma_1 - \frac{p_n}{j_n}\right| + \left|\gamma_2 - \frac{q_n}{j_n}\right| < \frac{C}{(j_n)^n}, \quad \text{for all } n \in \mathbb{N}.$$

Motivated by [5], for a pair of vectors $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$ we define the following Diophantine conditions:

$(DC1)$ there exist constants $C > 0$ and $\gamma > 0$ such that

$$|\tau + \langle \xi, \alpha + i\beta \rangle| \geq C(|\xi| + |\tau|)^{-\gamma},$$

for all $(\xi, \tau) = (\xi_1, \ldots, \xi_N, \tau) \in \mathbb{Z}^{N+1} \setminus \{0\}$.

$(DC2)$ there exist constants $C > 0$ and $\gamma > 0$ such that

$$|\tau + \langle \xi, \alpha + i\beta \rangle| \geq C(|\xi| + |\tau|)^{-\gamma},$$

for all $(\xi, \tau) \in \mathbb{Z}^{N+1}$ such that $\tau + \langle \xi, \alpha + i\beta \rangle \neq 0$.

Note that $(DC1)$ implies $(DC2)$, and for a pair (α, β) such that $\tau + \langle \xi, \alpha + i\beta \rangle \neq 0$ for all $(\xi, \tau) \in \mathbb{Z}^{N+1} \setminus \{0\}$, $(DC1)$ and $(DC2)$ are equivalent.

The Diophantine condition $(DC1)$ (respectively, $(DC2)$) is related to the notion of nonresonant (respectively, resonant) condition that appears in [6].

When $b_1 \not\equiv 0$ and b_1 does not change sign, we will see that the operator

$$\frac{\partial}{\partial t} + (a_1(t) + i\lambda b_1(t))\frac{\partial}{\partial x_1} + ib_1(t)\frac{\partial}{\partial x_2}, \quad \lambda \in \mathbb{R}, \tag{2}$$

is globally solvable if and only if $(a_{10}, 0, \lambda b_{10}, b_{10}) \in \mathbb{R}^2 \times \mathbb{R}^2$ satisfies $(DC2)$, which holds if and only if (a_{10}, λ) is not a Liouville vector. Comparing with [5], where a_1 vanishes identically, the operator (2) is globally solvable if and only if λ is not a Liouville number. Recall that, in dimension 2, there is no influence of the real part of the coefficient on the global solvability for the operator

$$\frac{\partial}{\partial t} + (a_1(t) + i\lambda b_1(t))\frac{\partial}{\partial x_1} \quad (\text{with} \quad \lambda b_{10} \neq 0),$$

which is globally solvable if and only if b_1 does not change sign (see [11]-Theorem 3.2).

Finally, we are ready to state our main results.

Define

$$\alpha_0 = (a_{10}, \ldots, a_{N0}), \quad \beta_0 = (b_{10}, \ldots, b_{N0}),$$
$$\alpha(t) = (a_1(t), \ldots, a_N(t)) \quad \text{and} \quad \beta(t) = (b_1(t), \ldots, b_N(t)), \quad t \in \mathbb{T}^1.$$

Theorem 1.1. *Let L be given by* (1).

(I) If $b_j \equiv 0$ for every j, then L is globally solvable if and only if the pair $(\alpha_0, 0)$ satisfies $(DC2)$.

(II) If $b_{j0} = 0$, for every j, and if $b_j \not\equiv 0$ for at least one j, then L is globally solvable if and only if $(a_{10}, \ldots, a_{N0}) \in \mathbb{Z}^N$ and all the sublevel sets

$$\Omega_r^\xi = \left\{ t \in \mathbb{T}^1; \int_0^t \langle \xi, \beta(\tau) \rangle d\tau < r \right\}, \ r \in \mathbb{R} \text{ and } \xi \in \mathbb{Z}^N,$$

are connected.

(III) If $b_{j0} \neq 0$ for some j, then L is globally solvable if and only if the following properties are satisfied:

(III.1) $\dim \operatorname{span}\{b_1, \ldots, b_N\} = 1$;

(III.2) the functions b_j do not change sign;

(III.3) the pair (α_0, β_0) satisfies $(DC2)$.

We now point out an important remark about Theorem 1.1 - case (II). Consider

$$\frac{\partial}{\partial t} + i \sum_{j=1}^{N} b_j(t) \frac{\partial}{\partial x_j}, \tag{3}$$

where $b_{j0} = 0$, for all j, and at least one b_j does not vanish identically. Of course, (3) does not satisfy condition (P); however, a necessary condition for (3) to be globally solvable is that each function $\mathbb{T}^1 \ni t \to \xi_1 b_1(t) + \cdots + \xi_N b_N(t)$ $(\xi \in \mathbb{Z}^N)$ changes sign at most twice, which is weaker than condition $(\mathcal{P})$.

As in Theorem 1.1, Diophantine properties of the coefficients of L are linked to hypoellipticity.

Theorem 1.2. *Let L be given by* (1). *Then, L is globally hypoelliptic if and only if the functions a_j and b_j satisfy the following properties:*

(1) the functions b_j do not change sign;

(2) $\dim \operatorname{span}\{b_1, \ldots, b_N\} \leq 1$;

(3) the pair (α_0, β_0) satisfies $(DC1)$.

Note that, in contrast to Theorem 1.1, condition $(\mathcal{P})$ is a necessary condition for global hypoellipticity.

As a consequence of Theorem 1.2, the operator L given by (2) is globally hypoelliptic if and only if $(a_{10}, \lambda) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ is not a Liouville vector.

When the coefficients of L are constant, theorems 1.1 and 1.2 give us results which are in agreement with those of Greenfield and Wallach in [7]. In fact, when

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} (\alpha_j + i\beta_j) \frac{\partial}{\partial x_j},$$

where α_j and β_j are real numbers, for $j = 1, \ldots, N$, then it follows from simple adaptations of the techniques used in [7] that L is globally solvable if and only if the pair (α, β) satisfies $(DC2)$, where $\alpha = (\alpha_1, \ldots, \alpha_j)$ and $\beta = (\beta_1, \ldots, \beta_N)$. Likewise, this constant coefficient vector field is globally hypoelliptic if and only if the pair (α, β) satisfies $(DC1)$.

We dedicate sections 2 and 3 to the proof of Theorem 1.1, and Section 4 to the proof of Theorem 1.2.

2 Beginning of the proof of Theorem 1.1: cases (I) and (II)

First, recall that

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} (a_j + ib_j)(t) \frac{\partial}{\partial x_j}$$

is globally solvable if and only if $L\mathcal{C}^\infty(\mathbb{T}^{N+1}) = (\ker {}^tL)^\circ$, where ${}^tL$ denotes the transpose operator of L, and $(\ker {}^tL)^\circ$ is the set of the functions $\phi \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ such that $\mu(\phi) = 0$ for all $\mu \in \ker {}^tL$.

As a first step in our proofs, we claim that there is no loss of generality in assuming that the real part of our operator L has constant coefficients. Indeed, by using partial Fourier series in the variables $(x_1, \ldots, x_N)$, we may define the automorphism $T : \mathcal{C}^\infty(\mathbb{T}^{N+1}) \to \mathcal{C}^\infty(\mathbb{T}^{N+1})$ given by

$$\widehat{Tu}(\xi, t) = \hat{u}(\xi, t) \exp\left\{ i \int_0^t \langle \xi, \alpha(\tau) - \alpha_0 \rangle d\tau \right\}, \quad \text{for all} \quad \xi \in \mathbb{Z}^N, \tag{4}$$

with inverse $T^{-1} : \mathcal{C}^\infty(\mathbb{T}^{N+1}) \to \mathcal{C}^\infty(\mathbb{T}^{N+1})$ given by

$$\widehat{T^{-1}v}(\xi, t) = \hat{v}(\xi, t) \exp\left\{ -i \int_0^t \langle \xi, \alpha(\tau) - \alpha_0 \rangle d\tau \right\}, \quad \text{for all} \quad \xi \in \mathbb{Z}^N. \tag{5}$$

The operator L is globally solvable if and only if the new operator TLT^{-1} is globally solvable; the latter has the form

$$TLT^{-1} = \frac{\partial}{\partial t} + \sum_{j=1}^{N}(a_{j0} + ib_j(t))\frac{\partial}{\partial x_j}, \tag{6}$$

where now the real part of the coefficients is constant; moreover, it is easy to see that a_{j0} can be replaced by zero if a_{j0} belongs to $\mathbb{Z}$.

As a consequence, when $b_j \equiv 0$ for all j, we may assume that L is the constant coefficient operator

$$\frac{\partial}{\partial t} + \sum_{j=1}^{N} a_{j0}\frac{\partial}{\partial x_j};$$

hence, as in [7], we can prove that L is globally solvable if and only if $(\alpha_0, 0) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfies $(DC2)$, where $\alpha_0 = (a_{10}, \ldots, a_{N0})$. This concludes the proof of Theorem 1.1 in case (I).

For the remainder of this section, we will proceed with the proof of Theorem 1.1 by analyzing the situation (II), where $b_{j0} = 0$, for all $j = 1, \ldots, N$, and at least one function b_j does not vanish identically. Propositions 2.1 and 2.2 show the necessity of the conditions in (II), while Proposition 2.3 shows the sufficiency.

Proposition 2.1. *Let L be given by the right-hand side of (6). Suppose that $b_{j0} = 0$, for all j, and that at least one b_j does not vanish identically. If $(a_{10}, \ldots, a_{N0}) \notin \mathbb{Z}^N$, then L is not globally solvable.*

Proof: The hypotheses imply the existence of m and ℓ in $\{1, \ldots, N\}$ such that b_m does not vanish identically, $b_{m0} = 0$ and $a_{\ell 0} \notin \mathbb{Z}$. Assume first that $m = \ell$. Without loss of generality we may assume that $m = \ell = 1$. In this case, it follows from Theorem 3.2 of [11] that the vector field

$$L_1 = \frac{\partial}{\partial t} + (a_{10} + ib_1(t))\frac{\partial}{\partial x_1},$$

viewed as a vector field on $\mathbb{T}^2$, is not globally solvable. By using partial Fourier series in the variables $(x_2, \ldots, x_N)$, we can verify that the fact that L_1 is not globally solvable easily implies that L itself is not globally solvable.

Now, assume that $m \neq \ell$. Again, there is no loss of generality in assuming that $m = 1$ and $\ell = 2$; hence b_1 does not vanish identically, $b_{10} = 0$, $a_{10} \in \mathbb{Z}$, b_2 vanishes identically and $a_{20} \notin \mathbb{Z}$. As above, in order to prove that L is not globally solvable it is enough to prove that

$$\frac{\partial}{\partial t} + (a_{10} + ib_1(t))\frac{\partial}{\partial x_1} + a_{20}\frac{\partial}{\partial x_2}$$

is not globally solvable. Recall that, since $a_{10} \in \mathbb{Z}$, it is enough to prove that

$$L_{12} \doteq \frac{\partial}{\partial t} + ib_1(t)\frac{\partial}{\partial x_1} + a_{20}\frac{\partial}{\partial x_2}$$

is not globally solvable.

We will construct a function f in $(\ker {}^tL_{12})^\circ \setminus L_{12}\mathcal{C}^\infty(\mathbb{T}^3)$ by using partial Fourier series in the variables (x_1, x_2). Note that, since $a_{20} \notin \mathbb{Z}$, there is an increasing sequence $(k_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ such that $k_n \geq n > 0$ and $a_{20}k_n \notin \mathbb{Z}$, for all $n \in \mathbb{N}$. We will construct the function f of the form

$$f(x_1, x_2, t) = \sum_{n=1}^{\infty} \hat{f}(k_n, k_n, t)e^{ik_n(x_1+x_2)}. \tag{7}$$

The next step is to construct the sequence $\hat{f}(k_n, k_n, \cdot)$ (the Fourier coefficients of f).

Define

$$H(s,t) = \int_{t-s}^{t} b_1(\tau)d\tau,\ 0 \leq s, t \leq 2\pi,$$

and set

$$A = H(s_0, t_0) = \max_{0\leq s,t\leq 2\pi} H(s,t).$$

Since b_1 changes sign we have $A > 0$; also, we may assume that $0 < s_0,\ t_0,\ t_0 - s_0 < 2\pi$. Set $\sigma_0 = t_0 - s_0$ and take a function $\phi \in \mathcal{C}_c^\infty((\sigma_0 - \delta, \sigma_0 + \delta))$ which satisfies $0 \leq \phi(t) \leq 1$ and $\phi(t) \equiv 1$ in a neighborhood of $[\sigma_0 - \delta/2, \sigma_0 + \delta/2]$, where $\delta > 0$ is small enough so that $(\sigma_0 - \delta, \sigma_0 + \delta) \subset (0, t_0)$.

We finally define $\hat{f}(k_n, k_n, \cdot) \in \mathcal{C}^\infty(\mathbb{T}^1)$ to be the 2π- periodic extension of

$$\phi(t)e^{ik_na_{20}(t_0-t)}e^{-k_nA}, \quad t \in [0, 2\pi],$$

for all $n \in \mathbb{N}$.

Since $A > 0$, the sequence $(\hat{f}(k_n, k_n, \cdot))_{n\in\mathbb{N}} \subset \mathcal{C}^\infty(\mathbb{T}^1)$ decays rapidly, which implies that the function f given by (7) belongs to $\mathcal{C}^\infty(\mathbb{T}^3)$. Moreover, our assumption $k_na_{20} \notin \mathbb{Z}$ implies that $\hat{\mu}(-k_n, -k_n, \cdot) \equiv 0$, for all $\mu \in \ker {}^tL_{12}$ and $n \in \mathbb{N}$. Hence $f \in (\ker {}^tL_{12})^\circ$.

Next, it will be shown that there is no $u \in \mathcal{C}^\infty(\mathbb{T}^3)$ such that $L_{12}u = f$. Indeed, suppose that $u \in \mathcal{C}^\infty(\mathbb{T}^3)$ is such that $L_{12}u = f$. Then the partial Fourier series in the variables (x_1, x_2) gives

$$\frac{d}{dt}\hat{u}(j,k,t) + (-jb_1(t) + kia_{20})\hat{u}(j,k,t) = \hat{f}(j,k,t),$$

for all $(j,k) \in \mathbb{Z}^2$ and $t \in \mathbb{T}^1$. In particular, for each $n \in \mathbb{N}$ we have

$$\frac{d}{dt}\hat{u}(k_n,k_n,t) + k_n(-b_1(t) + ia_{20})\hat{u}(k_n,k_n,t) = \hat{f}(k_n,k_n,t),$$

which has a unique solution, since $k_na_{20} \notin \mathbb{Z}$. This solution can be written as

$$\hat{u}(k_n,k_n,t) = (1 - e^{-2\pi ik_na_{20}})^{-1}\int_0^{2\pi} \hat{f}(k_n,k_n,t-s)e^{-isk_na_{20}}e^{k_n\int_{t-s}^{t} b_1(\tau)d\tau}ds.$$

For $t = t_0$, the definition of $\hat{f}(k_n, k_n, \cdot)$ implies that

$$|\hat{u}(k_n, k_n, t_0)| = |1 - e^{-2\pi i k_n a_{20}}|^{-1} \int_{|s-s_0|<\delta} \phi(t_0 - s) e^{-k_n(A-H(s,t_0))} ds \geq$$

$$\frac{1}{2} \int_{|s-s_0|<\delta/2} e^{-k_n(A-H(s,t_0))} ds.$$

Since $A - H(s_0, t_0) = 0$, s_0 is a point of minimum of $A - H(s, t_0) \geq 0$. In particular, s_0 is either a zero of infinite order or a zero of even order. By proceeding as in the proof of Proposition 3.3 in [5], we may apply the Laplace method for integrals (see [8]) so that we obtain a constant $C > 0$, which does not depend on n, such that

$$|\hat{u}(k_n, k_n, t_0)| \geq C(k_n)^{-1/2},$$

contradicting the fact that $\hat{u}(k_n, k_n, t_0)$ decays rapidly.

Therefore, there is no $u \in \mathcal{C}^\infty(\mathbb{T}^3)$ such that $L_{12}u = f$.

This completes the proof of Proposition 2.1. □

Proposition 2.2. Under the assumptions of *Proposition 2.1*, but now supposing that $(a_{10}, \ldots, a_{N0}) \in \mathbb{Z}^N$, the existence of a disconnected sublevel set Ω_r^ξ implies that L is not globally solvable.

Proof: Recall that, since $a_{j0} \in \mathbb{Z}$, for $j = 1, \ldots, N$, we may assume that

$$L = \frac{\partial}{\partial t} + i \sum_{j=1}^{N} b_j(t) \frac{\partial}{\partial x_j}. \tag{8}$$

The proof will be by contradiction. Suppose that L is globally solvable. Due to Lemma 6.1.2 of [9], it is well known that the solvability of L implies an inequality involving the transpose operator. As in the proof of necessity in Proposition 2.1 of [5], a convenient version of this lemma yields the existence of constants $m \in \mathbb{N}$ and $C > 0$ such that

$$\left| \int_{\mathbb{T}^{N+1}} fv \right| \leq C \cdot \left(\sum_{|\alpha| \leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha f| \right) \cdot \left(\sum_{|\alpha| \leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha ({}^tLv)| \right), \tag{9}$$

for every $f \in (\ker {}^tL)^\circ$ and every $v \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$.

The proof is then reduced to the study of inequality (9). By making use of a disconnected sublevel set Ω_r^ξ, we will construct sequences of functions, $\{f_n\}$ and $\{v_n\}$, which will produce a contradiction with (9).

Take a disconnected sublevel

$$\Omega_r^{\xi'} = \left\{ t \in \mathbb{T}^1; \int_0^t \langle \xi', \beta(\tau) \rangle d\tau < r \right\}, \text{ where } \beta(\tau) = (b_1(\tau), \ldots, b_N(\tau)).$$

As in [11] and [5], we can find a real number $r_0 < r$ such that

$$\Omega_{r_0}^{\xi'} = \left\{ t \in \mathbb{T}^1; \int_0^t \langle \xi', \beta(\tau) \rangle d\tau < r_0 \right\}$$

has two connected components with disjoint closure. Also, we can construct functions $f_0, v_0 \in \mathcal{C}^\infty(\mathbb{T}^1)$, satisfying the following conditions:

$$\int_0^{2\pi} f_0(t)dt = 0, \ \ supp(f_0) \cap \Omega_{r_0}^{\xi'} = \emptyset,$$

$$supp(v_0') \subset \Omega_{r_0}^{\xi'},$$

and

$$\int_0^{2\pi} f_0(t) v_0(t) dt > 0.$$

For $n \in \mathbb{N}$, define the functions $f_n, v_n : \mathbb{T}^{N+1} \to \mathbb{C}$ by

$$f_n(x,t) = \exp\left\{ -n \int_0^t \langle \xi', \beta(\tau) \rangle d\tau \right\} f_0(t) e^{-in\langle \xi', x \rangle},$$

and

$$v_n(x,t) = \exp\left\{ n \int_0^t \langle \xi', \beta(\tau) \rangle d\tau \right\} v_0(t) e^{in\langle \xi', x \rangle}.$$

Since $b_{j0} = 0$, for all j, it follows that f_n and v_n belong to $\mathcal{C}^\infty(\mathbb{T}^{N+1})$. Moreover, we claim that $f_n \in (\ker {}^tL)^\circ$. In fact, for each $\mu \in \ker {}^tL \subset \mathcal{D}'(\mathbb{T}^{N+1})$, by using partial Fourier series in the variables $(x_1, \dots, x_N)$ we can write

$$\mu = \sum_{\xi \in \mathbb{Z}^N} c_\xi \exp\left\{ \int_0^t \langle \xi, \beta(\tau) \rangle d\tau \right\} e^{i\langle \xi, x \rangle}, \text{ with } c_\xi \in \mathbb{C},$$

from which we obtain

$$\langle \mu, f_n \rangle = (2\pi)^N c_n \int_0^{2\pi} f_0(t) dt = 0, \quad \text{for every} \quad n \in \mathbb{N},$$

where c_n is a complex number. Thus $f_n \in (\ker {}^tL)^\circ$, for all $n \in \mathbb{N}$.

The inequality (9) for f_n and v_n yields

$$\left| \int_{\mathbb{T}^{N+1}} f_n v_n \right| \leq C \cdot \left(\sum_{|\alpha| \leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha f_n| \right) \cdot \left(\sum_{|\alpha| \leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha ({}^tL v_n)| \right), \tag{10}$$

for every $n \in \mathbb{N}$.

Notice that the left-hand side of the inequality (10) satisfies

$$\int_{\mathbb{T}^{N+1}} f_n(t)v_n(t)dt = (2\pi)^N \int_0^{2\pi} f_0(t)v_0(t)dt > 0.$$

On the other hand, since $(supp(f_0)) \cap \Omega^{\xi'}_{r_0} = \emptyset$, there exists $C_1 > 0$, which does not depend on n, such that

$$\sum_{|\alpha|\leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha f_n| \leq C_1 n^m \sup_{t\in\mathbb{T}^1\setminus\Omega^{\xi'}_{r_0}} e^{-n\int_0^t \langle\xi',\beta(\tau)\rangle d\tau} \leq C_1 n^m e^{-nr_0}; \tag{11}$$

also, since $(supp(v_0')) \subset \Omega^{\xi'}_{r_0}$, there exists $C_2 > 0$, which does not depend on n, such that

$$\begin{aligned}\sum_{|\alpha|\leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha({}^tLv_n)| &= \sum_{|\alpha|\leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha[e^{n\int_0^t\langle\xi',\beta(\tau)\rangle d\tau} v_0'(t)e^{in\langle\xi',x\rangle}]| \\ &\leq C_2 n^m \sup_{t\in supp(v_0')} e^{n\int_0^t\langle\xi',\beta(\tau)\rangle d\tau} \\ &\leq C_2 n^m \left(\sup_{t\in supp(v_0')} e^{\int_0^t\langle\xi',\beta(\tau)\rangle d\tau}\right)^n \\ &= C_2 n^m e^{n\int_0^{t_1}\langle\xi',\beta(\tau)\rangle d\tau}, \end{aligned} \tag{12}$$

where t_1 is a maximum point of the function $t \mapsto e^{\int_0^t\langle\xi',\beta(\tau)\rangle d\tau}$ restricted to $supp(v_0') \subset \Omega^{\xi'}_{r_0}$.

We now set $c \doteq \int_0^{t_1} \langle\xi',\beta(\tau)\rangle d\tau - r_0$. Notice that $c < 0$ and that c does not depend on $n \in \mathbb{N}$. Moreover, from (11) and (12) we have

$$\left(\sum_{|\alpha|\leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha f_n|\right) \cdot \left(\sum_{|\alpha|\leq m} \sup_{\mathbb{T}^{N+1}} |\partial^\alpha({}^tLv_n)|\right) \leq C_1C_2 n^{2m} e^{nc}, \tag{13}$$

for all $n \in \mathbb{N}$.

Finally, from (10) and (13) we have

$$0 < (2\pi)^N \int_0^{2\pi} f_0(s)v_0(s)ds \leq CC_1C_2 n^{2m}e^{nc}, \text{ for all } n \in \mathbb{N},$$

which is a contradiction, since $\lim_{n\to\infty} n^{2m}e^{nc} = 0$.

The proof of Proposition 2.2 is complete. □

The next result shows that L is globally solvable under the conditions given in (II) - Theorem 1.1. In fact, the correct choice of the point t_ξ in (17) will imply that we can construct a continuous right inverse for L, namely, $L^{-1} : Im(L) \to \mathcal{C}^\infty(\mathbb{T}^{N+1})$, given by means of formula (19).

Proposition 2.3. Let L be the operator given by *(6)*. Suppose that $b_{j0} = 0$, for all j, and that at least one b_j does not vanish identically. If $(a_{10}, \ldots, a_{N0}) \in \mathbb{Z}^N$ and, moreover, all the sublevel sets Ω_r^ξ are connected, then L is globally solvable.

Proof: Given a function $f \in (\ker {}^tL)^\circ$, we will find $u \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ such that $Lu = f$. As in the proof of Proposition 2.2, we may assume that L is given by (8). By using partial Fourier series in the variables $(x_1, \ldots, x_N)$, we are led to solve

$$\frac{d}{dt}\hat{u}(\xi, t) - \langle \xi, \beta(t) \rangle \hat{u}(\xi, t) = \hat{f}(\xi, t), \quad t \in \mathbb{T}^1, \quad \xi \in \mathbb{Z}^N, \tag{14}$$

where we recall that $\beta(t) = (b_1(t), \ldots, b_N(t))$.

Our assumption $b_{j0} = 0$ (for all $j = 1, \ldots, N$) implies that, for each $\xi \in \mathbb{Z}^N$, the function

$$\phi_\xi(x, t) = \exp\left\{-\int_0^t \langle \xi, \beta(\tau) \rangle d\tau\right\} e^{-i\langle \xi, x \rangle}$$

belongs to $\mathcal{C}^\infty(\mathbb{T}^{N+1})$. Moreover, ${}^tL(\phi_\xi) = -L(\phi_\xi) = 0$; that is, $\phi_\xi \in \ker {}^tL$. Since $f \in (\ker {}^tL)^\circ$ we have

$$\int_0^{2\pi} e^{-\int_0^t \langle \xi, \beta(\tau) \rangle d\tau} \hat{f}(\xi, t) dt = 0, \quad \text{for all} \quad \xi \in \mathbb{Z}^N. \tag{15}$$

It follows from (15) that, for an arbitrary $t_\xi \in \mathbb{T}^1$, the function $\hat{u}(\xi, \cdot)$ defined by

$$\hat{u}(\xi, t) = e^{\int_0^t \langle \xi, \beta(\tau) \rangle d\tau} \int_{t_\xi}^t e^{-\int_0^s \langle \xi, \beta(\tau) \rangle d\tau} \hat{f}(\xi, s) ds, \tag{16}$$

with $t_\xi \in \mathbb{T}^1$ to be chosen later, is well defined and belongs to $\mathcal{C}^\infty(\mathbb{T}^1)$. Simple calculations show that $\hat{u}(\xi, \cdot)$ is a solution to (14) (in particular, there are infinitely many solutions to (14)).

In order to obtain a sequence of solutions $\hat{u}(\xi, \cdot)$ which decays rapidly, we will choose a convenient point t_ξ in (16). For each $\xi \in \mathbb{Z}^N$, choose $t_\xi \in \mathbb{T}^1$ such that

$$\int_0^{t_\xi} \langle \xi, \beta(\tau) \rangle d\tau = \sup_{t \in \mathbb{T}^1} \left\{ \int_0^t \langle \xi, \beta(\tau) \rangle d\tau \right\},$$

or equivalently

$$-\int_0^{t_\xi} \langle \xi, \beta(\tau) \rangle d\tau = \inf_{t \in \mathbb{T}^1} \left\{ -\int_0^t \langle \xi, \beta(\tau) \rangle d\tau \right\}. \tag{17}$$

For t_ξ satisfying (17), we claim that the sequence $\hat{u}(\xi, \cdot)$, given by (16), decays rapidly. In fact, for fixed $\xi \in \mathbb{Z}^N$ and for fixed $t \in \mathbb{T}^1$, define

$$r_t \doteq -\int_0^t \langle \xi, \beta(\tau) \rangle d\tau$$

and

$$\Omega_t \doteq \left\{ s \in \mathbb{T}^1; -\int_0^s \langle \xi, \beta(\tau) \rangle d\tau \le r_t \right\}.$$

It follows from our assumptions that Ω_t is connected. Since t and t_ξ belong to Ω_t, there is an arc Γ_t entirely contained in Ω_t and joining t_ξ to t. Hence

$$-\int_0^s \langle \xi, \beta(\tau) \rangle d\tau \le -\int_0^t \langle \xi, \beta(\tau) \rangle d\tau, \quad \text{for all} \quad s \in \Gamma_t. \tag{18}$$

Again, by using (15) we can write

$$\hat{u}(\xi, t) = \int_{\Gamma_t} \exp\left\{ \int_0^t \langle \xi, \beta(\tau) \rangle d\tau - \int_0^s \langle \xi, \beta(\tau) \rangle d\tau \right\} \hat{f}(\xi, s) ds. \tag{19}$$

Finally, the conjunction of (19) with the estimate (18) implies that

$$|\hat{u}(\xi, t)| \le 2\pi \sup_{t \in \mathbb{T}^1} |\hat{f}(\xi, t)|; \tag{20}$$

hence, from the rapid decay of $\hat{f}(\xi, \cdot)$, for every $n \in \mathbb{Z}_+$, we obtain a constant $C = C(n) > 0$ such that

$$|\hat{u}(\xi, t)| \le \frac{C}{|\xi|^n}, \text{ for all } t \in \mathbb{T}^1.$$

By proceeding as above we can prove analogous estimates for all the derivatives $\hat{u}^{(m)}(\xi, \cdot)$, $m \in \mathbb{N}$.

Therefore, $\hat{u}(\xi, \cdot)$ decays rapidly and

$$u(x, t) = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(\xi, t) e^{i\langle \xi, x \rangle}$$

is a solution to $Lu = f$ in $\mathcal{C}^\infty(\mathbb{T}^{N+1})$.

This concludes the proof of Proposition 2.3. □

3 End of the proof of Theorem 1.1: case (III)

We proceed with the proof of Theorem 1.1 and we now deal with case (III), in which $b_{j0} \neq 0$ for at least one $j \in \{1, \ldots, N\}$. Proposition 3.2 shows the necessity of conditions (III.1)-(III.3), while Proposition 3.3 shows the sufficiency. First, we will prove the following equivalence:

Lemma 3.1. *For a pair of vectors $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\beta = (\beta_1, \ldots, \beta_N)$ in $\mathbb{R}^N$, the condition $(DC2)$ is equivalent to the following Diophantine condition:*

$(DC3)$ *there exist constants $C > 0$ and $\gamma > 0$ such that*

$$|1 - e^{2\pi\langle \xi, \beta - i\alpha\rangle}| \geq C|\xi|^{-\gamma},$$

for all $\xi \in \mathbb{Z}^N$ such that $\langle \xi, \beta - i\alpha\rangle \notin i\mathbb{Z}$.

Proof: Suppose that (α, β) does not satisfy $(DC3)$. Thus we can find a sequence $(\xi(n))_{n\in\mathbb{N}} \subset \mathbb{Z}^N$ such that $|\xi(n)| \geq n$, $\langle \xi(n), \beta - i\alpha\rangle \notin i\mathbb{Z}$, and

$$|1 - e^{2\pi\langle \xi(n), \beta - i\alpha\rangle}| < |\xi(n)|^{-n}, \tag{21}$$

for all $n \in \mathbb{N}$. In particular, $|\langle \xi(n), \beta\rangle| \to 0$, as $n \to \infty$, which implies that

$$|1 - e^{2\pi\langle \xi(n), \beta\rangle}| \geq e^{-1}2\pi|\langle \xi(n), \beta\rangle|, \tag{22}$$

for $n \in \mathbb{N}$ sufficiently large.

It follows from (21) that there is a sequence of integers (τ_n) such that $\tau_n + \langle \xi(n), \alpha\rangle \to 0$, as $n \to \infty$ (in fact, it is enough, for each $n \in \mathbb{N}$, to choose τ_n to be either the greatest integer not exceeding $-\langle \xi(n), \alpha\rangle$ or the least integer greater than $-\langle \xi(n), \alpha\rangle$). Hence, if $n \in \mathbb{N}$ is sufficiently large, we have

$$|\sin(2\pi[\tau_n + \langle \xi(n), \alpha\rangle])| \geq \pi|\tau_n + \langle \xi(n), \alpha\rangle|. \tag{23}$$

Inequalities (21), (22) and (23) imply that, for $n \in \mathbb{N}$ sufficiently large,

$$e^{-1}2\pi|\langle \xi(n), \beta\rangle| \leq |1 - e^{2\pi\langle \xi(n), \beta\rangle}| \leq |1 - e^{2\pi\langle \xi(n), \beta - i\alpha\rangle}| < |\xi(n)|^{-n}$$

and

$$\pi|\tau_n + \langle \xi(n), \alpha\rangle| \leq |\sin(2\pi[\tau_n + \langle \xi(n), \alpha\rangle])| \leq 2e^{2\pi\langle \xi(n), \beta\rangle}|\sin(2\pi[\tau_n + \langle \xi(n), \alpha\rangle])| \leq$$
$$2|1 - e^{2\pi\langle \xi(n), \beta - i\alpha\rangle}| < 2|\xi(n)|^{-n}.$$

Finally, from these last inequalities we obtain

$$|\tau_n + \langle \xi(n), \alpha + i\beta\rangle| \leq |\tau_n + \langle \xi(n), \alpha\rangle| + |\langle \xi(n), \beta\rangle| < |\xi(n)|^{-n}\frac{2}{\pi} < \frac{2n^{-n}}{\pi},$$

for all $n \in \mathbb{N}$ sufficiently large, which implies that (α, β) does not satisfy $(DC2)$.

Conversely, suppose that (α, β) does not satisfy $(DC2)$. In particular, we can find a sequence $(\xi(n), \tau_n) \in \mathbb{Z}^N \times \mathbb{Z}$ such that $|\xi(n)| \geq n$, $\tau_n + \langle \xi(n), \alpha + i\beta\rangle \neq 0$, $\langle \xi(n), \beta\rangle \leq 0$ and

$$|\tau_n + \langle \xi(n), \alpha + i\beta\rangle| < (|\xi(n)| + |\tau_n|)^{-n} < |\xi(n)|^{-n},$$

for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$ sufficiently large we have $\cos(2\pi\langle\xi(n), \alpha\rangle) \neq 0$, and since $\langle\xi(n), \beta\rangle \leq 0$, it follows that

$$\begin{aligned}|1 - e^{2\pi\langle\xi(n),\beta - i\alpha\rangle}| \leq |1 - e^{2\pi\langle\xi(n),\beta\rangle}\cos(2\pi\langle\xi(n), \alpha\rangle)| + |\sin(2\pi\langle\xi(n), \alpha\rangle)| \leq \\ |1 - \cos(2\pi\langle\xi(n), \alpha\rangle)| + |1 - e^{2\pi\langle\xi(n),\beta\rangle}| + |\sin(2\pi\langle\xi(n), \alpha\rangle)| \leq \\ |1 - \cos(2\pi[\tau_n + \langle\xi(n), \alpha\rangle])| + |1 - e^{2\pi\langle\xi(n),\beta\rangle}| + |\sin(2\pi[\tau_n + \langle\xi(n), \alpha\rangle])| \leq \\ 4\pi|\tau_n + \langle\xi(n), \alpha\rangle| + 2\pi|\langle\xi(n), \beta\rangle| \leq 6\pi|\tau_n + \langle\xi(n), \alpha + i\beta\rangle| < |\xi(n)|^{-n} < n^{-n},\end{aligned}$$

which implies that (α, β) does not satisfy $(DC3)$. □

We now prove the necessity of conditions (III.1)-(III.3) in Theorem 1.1.

Proposition 3.2. The operator L given by *(6)* is not globally solvable in the case where $b_{j0} \neq 0$, for some $j = 1, \dots, N$, and at least one of the following conditions fails:

(III.1) $\dim span\{b_1, \dots, b_N\} = 1$;

(III.2) the functions b_j do not change sign;

(III.3) the pair (α_0, β_0) satisfies $(DC2)$.

Proof: Under the general assumption that $b_{j0} \neq 0$ for some j, the proof can be divided into three parts.

First, suppose that (III.1) fails; that is, there exist m and ℓ in $\{1, \dots, N\}$ such that $b_{m0} \neq 0$, and $b_m(t)$ and $b_\ell(t)$ are $\mathbb{R}-$linearly independent functions. There is no loss of generality in assuming that $m = 1$ and $\ell = 2$. We claim that

$$L_{12} = \frac{\partial}{\partial t} + (a_{10} + ib_1(t))\frac{\partial}{\partial x_1} + (a_{20} + ib_2(t))\frac{\partial}{\partial x_2}$$

is not globally solvable. The arguments to prove our claim are similar to that of Proposition 2.1. We will sketch the proof.

Recall that Lemma 3.1 of [5] implies the existence of integers p and q such that

$$\mathbb{T}^1 \ni t \mapsto \psi(t) = pb_1(t) + qb_2(t)$$

changes sign and $\psi_0 \doteq (2\pi)^{-1}\int_0^{2\pi}\psi(t)dt < 0$. We now define

$$H(s,t) = \int_{t-s}^{t}\psi(\tau)d\tau,\ 0 \leq s, t \leq 2\pi,$$

and

$$A = H(s_0, t_0) = \max_{0\leq s,t\leq 2\pi} H(s,t),$$

where, without loss of generality, we can assume that $0 < s_0, t_0, t_0 - s_0 < 2\pi$. Set $\sigma_0 \doteq t_0 - s_0$ and let $\delta > 0$ small enough so that $(\sigma_0 - \delta, \sigma_0 + \delta) \subset (0, t_0)$. Next, let

$\phi \in \mathcal{C}_c^\infty((\sigma_0 - \delta, \sigma_0 + \delta))$ such that $\phi(t) \equiv 1$ in a neighborhood of $[\sigma_0 - \delta/2, \sigma_0 + \delta/2]$ and $0 \leq \phi(t) \leq 1$, for all $t \in \mathbb{T}^1$.

As in the proof of Proposition 2.1, by using partial Fourier series we may consider a function $f \in (\ker {}^t L_{12})^\circ \setminus L_{12}\mathcal{C}^\infty(\mathbb{T}^3)$, given by

$$f(x_1, x_2, t) = \sum_{n=1}^{\infty} \hat{f}(np, nq, t) e^{in(px_1 + qx_2)},$$

where $\hat{f}(np, nq, \cdot)$ is the function in $\mathcal{C}^\infty(\mathbb{T}^1)$ which is the 2π-periodic extension of

$$\phi(t) e^{in(pa_{10} + qa_{20})(t_0 - t)} e^{-nA}, \quad t \in [0, 2\pi],$$

for all $n \in \mathbb{N}$.

By proceeding as in the proof of Proposition 2.1, we can verify that f belongs to $(\ker {}^t L_{12})^\circ \setminus L_{12}\mathcal{C}^\infty(\mathbb{T}^3)$, which implies that L is not globally solvable.

Now, assume that (III.1) satisfied and that (III.2) fails. In this case, there exists $m \in \{1, \ldots, N\}$ such that $b_{m0} \neq 0$ and $b_m(t)$ changes sign. Thus, it follows from [11]-Theorem 3.2 that

$$L_m = \frac{\partial}{\partial t} + (a_{m0} + ib_m(t)) \frac{\partial}{\partial x_m}$$

is not globally solvable, which implies that L is not globally solvable.

To complete the proof, we now assume that (III.1) and (III.2) are satisfied, but (III.3) fails. In this last case, we can write the operator L as

$$L = \frac{\partial}{\partial t} + \sum_{j=0}^{N} (a_{j0} + i\lambda_j b(t)) \frac{\partial}{\partial x_j}, \tag{24}$$

where $b \in \mathcal{C}^\infty(\mathbb{T}^1, \mathbb{R})$, b does not change sign, $b_0 \doteq (2\pi)^{-1} \int_0^{2\pi} b(t)dt \neq 0$, and $\lambda \doteq (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \setminus \{0\}$.

Recall that $\alpha_0 = (a_{10}, \ldots, a_{N0})$, $\beta_0 = (b_{10}, \ldots, b_{N0}) = b_0(\lambda_1, \ldots, \lambda_N) = b_0\lambda$, and $\beta(t) = (b_1(t), \ldots, b_N(t)) = b(t)(\lambda_1, \ldots, \lambda_N) = b(t)\lambda$. Since we are supposing that (α_0, β_0) does not satisfy $(DC2)$, Lemma 3.1 implies that (DC3) does not hold. Thus, one of the following situations must occur:

(i) there exists a sequence $(\xi(n))_{n\in\mathbb{N}} \subset \mathbb{Z}^N$ such that $|\xi(n+1)| > |\xi(n)| \geq n$, $\langle \xi(n), \beta_0 \rangle > 0$, and

$$|1 - \exp\{2\pi \langle \xi(n), \beta_0 - i\alpha_0 \rangle\}| < |\xi(n)|^{-n}, \quad \text{for all} \quad n \in \mathbb{N}, \tag{25}$$

or

(ii) there exists a sequence $(\xi(n))_{n\in\mathbb{N}} \subset \mathbb{Z}^N$ such that $|\xi(n+1)| > |\xi(n)| \geq n$, $\langle \xi(n), \beta_0 \rangle = 0$, $\langle \xi(n), \alpha_0 \rangle \notin \mathbb{Z}$, and

$$|1 - \exp\{-2\pi i \langle \xi(n), \alpha_0 \rangle\}| < |\xi(n)|^{-n}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{26}$$

In order to finish this proof we must show that L given by (24) is not globally solvable when either (i) or (ii) occurs.

Assume first that (i) occurs. For L given by (24) we will construct a function f belonging to $(\ker {}^tL)^\circ \setminus L\mathcal{C}^\infty(\mathbb{T}^{N+1})$. By using partial Fourier series in the variables $(x_1, \ldots, x_N)$, we consider the function

$$f(x,t) = \sum_{n=1}^{\infty} \hat{f}(\xi(n), t) e^{i\langle \xi(n), x\rangle}, \tag{27}$$

with partial Fourier coefficients, $\hat{f}(\xi(n), \cdot)$, given by the $2\pi-$periodic extension of

$$|\xi(n)|^{-n/2} \varphi(t) e^{i(\pi - t)\langle \xi(n), \alpha_0\rangle}, \quad t \in [0, 2\pi],$$

where $\varphi(t) \in \mathcal{C}_c^\infty((\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta))$, $\varphi(t) \equiv 1$ in a neighborhood of $[\frac{\pi-\delta}{2}, \frac{\pi+\delta}{2}]$, $0 \le \varphi(t) \le 1$, and $\delta > 0$ is small enough so that $(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta) \subset (0, \pi)$. The factor $|\xi(n)|^{-n/2}$ implies that $\hat{f}(\xi(n), \cdot)$ decays rapidly. Hence $f \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$. Moreover, for all $\mu \in \ker {}^tL$ and for $\xi \in \mathbb{Z}^N$ satisfying $\langle \xi, \beta_0\rangle \neq 0$, we have $\hat{\mu}(\xi, \cdot) = 0$. Thus $f \in (\ker {}^tL)^\circ$.

We claim that $f \notin L\mathcal{C}^\infty(\mathbb{T}^{N+1})$. Indeed, suppose by contradiction that there exists $u \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ such that $Lu = f$. The partial Fourier series in the variables $(x_1, \ldots, x_N)$ gives

$$\hat{f}(\xi(n), t) = \frac{d}{dt}\hat{u}(\xi(n), t) + i\langle \xi(n), \alpha_0 + i\beta(t)\rangle \hat{u}(\xi(n), t) = \\ \frac{d}{dt}\hat{u}(\xi(n), t) + i\langle \xi(n), \alpha_0 + ib(t)\lambda\rangle \hat{u}(\xi(n), t), \tag{28}$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{T}^1$. Since $\langle \xi(n), \beta_0\rangle > 0$, equation (28) has a unique solution, which can be written as

$$\hat{u}(\xi(n), t) = \\ (1 - e^{2\pi c_n})^{-1} \int_0^{2\pi} \hat{f}(\xi(n), t - s) \exp\left\{\langle \xi(n), \lambda\rangle \int_{t-s}^{t} b(\tau) d\tau\right\} e^{-is\langle \xi(n), \alpha_0\rangle} ds,$$

where, to simplify notation, we have defined

$$c_n \doteq \langle \xi(n), \beta_0 - i\alpha_0\rangle = \langle \xi(n), b_0\lambda - i\alpha_0\rangle \quad (n \in \mathbb{N}).$$

The definition of the function $\hat{f}(\xi(n), \cdot)$ implies that

$$|\hat{u}(\xi(n), \pi)| \ge |1 - e^{2\pi c_n}|^{-1} |\xi(n)|^{-n/2} \int_{|s - \pi/2| \le \delta/2} e^{\langle \xi(n), \lambda\rangle \int_{\pi-s}^{\pi} b(\tau) d\tau} ds.$$

Recall that b does not change sign and that $\langle \xi(n), \lambda b_0\rangle > 0$. Hence

$$\langle \xi(n), \lambda\rangle \int_{\pi-s}^{\pi} b(\tau) d\tau \ge 0, \quad \text{for all} \quad |s - \pi/2| \le \delta/2.$$

Moreover, (25) implies that

$$|1 - e^{2\pi c_n}|^{-1} = |1 - \exp\{2\pi\langle \xi(n), \beta_0 - i\alpha_0\rangle\}|^{-1} > |\xi(n)|^n.$$

Summarizing, we obtain

$$|\hat{u}(\xi(n), \pi)| \geq |1 - e^{2\pi c_n}|^{-1}|\xi(n)|^{-n/2}\delta > |\xi(n)|^n|\xi(n)|^{-n/2}\delta \geq n^{n/2}\delta,$$

which is a contradiction, since $|\hat{u}(\xi(n), \pi)|$ should decay rapidly.

Therefore, f given by (27) belongs to $(\ker {}^tL)^\circ \setminus L\mathcal{C}^\infty(\mathbb{T}^{N+1})$, which implies that L is not globally solvable when (i) occurs.

To finish the proof we just mention that similar computations show that L is not globally solvable when (ii) occurs. Indeed, under conditions (ii), the same f, given by (27), belongs to $(\ker {}^tL)^\circ \setminus L\mathcal{C}^\infty(\mathbb{T}^{N+1})$.

The proof of Proposition 3.2 is complete. □

We finish the proof of Theorem 1.1 by showing that conditions (III.1)-(III.3) are sufficient for global solvability.

Proposition 3.3. Let L be given by (6) and suppose that $b_{j0} \neq 0$, for some $j = 1, \ldots, N$. Then L is globally solvable if the following conditions are verified:

(III.1) $\dim \mathit{span}\{b_1, \ldots, b_N\} = 1$;

(III.2) the functions b_j do not change sign;

(III.3) the pair (α_0, β_0) satisfies $(DC2)$.

Proof: Given $f \in (\ker {}^tL)^\circ$, our goal is to seek $u \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ solution of $Lu = f$.

Notice that our assumptions imply that L can be written as

$$\frac{\partial}{\partial t} + \sum_{j=0}^{N}(a_{j0} + i\lambda_j b(t))\frac{\partial}{\partial x_j},$$

where $b \in \mathcal{C}^\infty(\mathbb{T}^1, \mathbb{R})$, b does not change sign, $b_0 \doteq (2\pi)^{-1}\int_0^{2\pi} b(t)dt \neq 0$, and $\lambda \doteq (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \setminus \{0\}$.

Partial Fourier series in the variables $(x_1, \ldots, x_N)$ leads us to solve the equations

$$\frac{d}{dt}\hat{u}(\xi, t) + i\langle \xi, \alpha_0 + ib(t)\lambda\rangle \hat{u}(\xi, t) = \hat{f}(\xi, t), \quad t \in \mathbb{T}^1, \quad \xi \in \mathbb{Z}^N, \tag{29}$$

where we recall that $\alpha_0 = (a_{10}, \ldots, a_{N0})$.

We now divide the task of finding solutions to (29) into three cases.

First: $\xi \in \mathbb{Z}^N$ is such that $\langle \xi, b_0\lambda\rangle = 0$ and $\langle \xi, \alpha_0\rangle \in \mathbb{Z}$.

In this case, the function $\phi_\xi(x,t) \doteq e^{it\langle\xi,\alpha_0\rangle}e^{-i\langle\xi,x\rangle}$ belongs to $\ker {}^tL$, which implies that

$$0 = \langle\phi_\xi, f\rangle = (2\pi)^N \int_0^{2\pi} \hat{f}(\xi,t)e^{it\langle\xi,\alpha_0\rangle}dt.$$

Hence, equation (29) has infinitely many solutions and we fix

$$\hat{u}(\xi,t) = \int_0^t \hat{f}(\xi,s)e^{i(s-t)\langle\xi,\alpha_0\rangle}ds, \quad t \in \mathbb{T}^1, \tag{30}$$

as a solution of (29). Notice that

$$|\hat{u}(\xi,t)| \leq \int_0^t |\hat{f}(\xi,s)|ds. \tag{31}$$

Second: $\xi \in \mathbb{Z}^N$ is such that $\langle\xi, b_0\lambda\rangle \neq 0$.

Now equation (29) has a unique solution in $\mathcal{C}^\infty(\mathbb{T}^1)$. It is worth pointing out that this solution can be written in two different ways.

When $\langle\xi,\lambda\rangle b_0 < 0$, we write

$$\hat{u}(\xi,t) = (1-e^{2\pi c_\xi})^{-1}\int_0^{2\pi} \hat{f}(\xi,t-s)e^{\langle\xi,\lambda\rangle\int_{t-s}^t b(\tau)d\tau}e^{-is\langle\xi,\alpha_0\rangle}ds, \tag{32}$$

where $c_\xi = \langle\xi, b_0\lambda - i\alpha_0\rangle$. Since b does not change sign and since $\langle\xi,\lambda\rangle b_0 < 0$ we have

$$\langle\xi,\lambda\rangle\int_{t-s}^t b(\tau)d\tau \leq 0, \quad \text{for all} \quad t,s \in [0,2\pi]. \tag{33}$$

Recall now that the validity of (III.3) and Lemma 3.1 imply that $(\alpha_0, b_0\lambda)$ satisfies $(DC3)$. It follows from (33) and $(DC3)$ that the solution given by (32) satisfies

$$|\hat{u}(\xi,t)| \leq \frac{1}{C}|\xi|^\gamma \int_0^{2\pi} |\hat{f}(\xi,t-s)|ds. \tag{34}$$

On the other hand, when $\langle\xi,\lambda\rangle b_0 > 0$ we write

$$\hat{u}(\xi,t) = (e^{-2\pi c_\xi}-1)^{-1}\int_0^{2\pi} \hat{f}(\xi,t+s)e^{-\langle\xi,\lambda\rangle\int_t^{t+s} b(\tau)d\tau}e^{is\langle\xi,\alpha_0\rangle}ds. \tag{35}$$

Since b does not change sign and, also, $\langle\xi,\lambda\rangle b_0 > 0$, we have

$$\langle\xi,\lambda\rangle\int_t^{t+s} b(\tau)d\tau \geq 0, \quad \text{for all} \quad t,s \in [0,2\pi]. \tag{36}$$

As before, (36) and ($DC3$) imply that the solution given by (35) satisfies

$$|\hat{u}(\xi,t)| \leq \frac{1}{C}|\xi|^{\gamma}\int_0^{2\pi}|\hat{f}(\xi,t+s)|ds. \tag{37}$$

Third: $\xi \in \mathbb{Z}^N$ is such that $\langle \xi, b_0\lambda\rangle = 0$ and $\langle \xi, \alpha_0\rangle \notin \mathbb{Z}$. Again equation (29) has a unique solution in $\mathcal{C}^\infty(\mathbb{T}^1)$, which can be written as

$$\hat{u}(\xi,t) = (1-e^{-2\pi i\langle\xi,\alpha_0\rangle})^{-1}\int_0^{2\pi}\hat{f}(\xi,t-s)e^{-is\langle\xi,\alpha_0\rangle}ds. \tag{38}$$

By using estimate ($DC3$) we obtain

$$|\hat{u}(\xi,t)| \leq \frac{1}{C}|\xi|^{\gamma}\int_0^{2\pi}|\hat{f}(\xi,t-s)|ds. \tag{39}$$

Up to this point we have defined a sequence $(\hat{u}(\xi,\cdot))_{\xi\in\mathbb{Z}^N}$ of solutions of the equations (29). The proof will be concluded if we show that this sequence decays rapidly. Since $\hat{f}(\xi,\cdot)$ decays rapidly, it follows from (31), (34), (37) and (39) that, for each $n \in \mathbb{Z}_+$ there exists $C(n) > 0$ such that

$$|\hat{u}(\xi,t)| \leq \frac{C(n)}{|\xi|^n}, \quad \text{for all} \quad t\in\mathbb{T} \quad \text{and} \quad \xi\in\mathbb{Z}^N.$$

Similar estimates can be found for each derivative $\hat{u}^{(m)}(\xi,\cdot)$, $m \in \mathbb{Z}_+$. Therefore, $(\hat{u}(\xi,\cdot))_{\xi\in\mathbb{Z}^N}$ decays rapidly, which completes the proof of Proposition 3.3. □

4 Proof of Theorem 1.2

In this section we will deal with global hypoellipticity by proving Theorem 1.2. As in the preceding section (in which we studied global solvability), by using an automorphism we may assume that L is given by (6), that is,

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N}(a_{j0} + ib_j(t))\frac{\partial}{\partial x_j}. \tag{40}$$

We begin by proving the necessity of the conditions (1)-(3).

Suppose that b_j changes sign, for some $j \in \{1,\ldots,N\}$. Without loss of generality we may assume that $j = 1$. It follows from Theorem 2.2 of [11] that there exists $\nu \in \mathcal{D}'(\mathbb{T}^2_{(x_1,t)}) \setminus \mathcal{C}^\infty(\mathbb{T}^2_{(x_1,t)})$ for which

$$f \doteq \left(\frac{\partial}{\partial t} + (a_{10} + ib_1(t))\frac{\partial}{\partial x_1}\right)\nu$$

belongs to $\mathcal{C}^\infty(\mathbb{T}^2_{(x_1,t)})$. Hence, by setting $x' = (x_2, \ldots, x_N)$ it follows that $\omega \doteq \nu \otimes 1_{x'} \in \mathcal{D}'(\mathbb{T}^{N+1}) \backslash \mathcal{C}^\infty(\mathbb{T}^{N+1})$ and, also, $L\omega \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$. Therefore, L is not globally hypoelliptic.

Hence we have the right to restrict ourselves to the case where b_j does not change sign, for $j = 1, \ldots, N$.

Suppose now that $\dim \operatorname{span}\{b_1, \ldots, b_N\} > 1$. Then there exist functions b_m and b_ℓ which are $\mathbb{R}-$linearly independent in $\mathcal{C}^\infty(\mathbb{T}^1, \mathbb{R})$. Since $b_m \not\equiv 0$ and b_m does not change sign, we have $b_{m0} \neq 0$. Similarly, $b_{\ell 0} \neq 0$. Again, we may assume that $m = 1$ and $\ell = 2$. Under these assumptions, recall that in the proof of Proposition 3.2 we exhibited a smooth function $f \in (\ker {}^tL)^\circ$, for which the equation

$$L_{12}u = \left(\frac{\partial}{\partial t} + (a_{10} + ib_1(t))\frac{\partial}{\partial x_1} + (a_{20} + ib_2(t))\frac{\partial}{\partial x_2}\right) u = f$$

does not have smooth solutions u in $\mathbb{T}^3$. However, we claim that for such f we can find a distribution solution, μ, for $L_{12}\mu = f$. Indeed, with the same notation used in the proof of Proposition 3.2, we define

$$\mu = \sum_{n=1}^{\infty} \hat{\mu}(np, nq, t) e^{in(px_1 + qx_2)},$$

where

$$\hat{\mu}(np, nq, t) = (1 - e^{2\pi d_n})^{-1} \int_0^{2\pi} \hat{f}(np, nq, t - s) e^{nH(s,t)} e^{-ins(pa_{10} + qa_{20})} ds$$

and $d_n \doteq n(\psi_0 - i(pa_{10} + qa_{20})) = n(pb_{10} + qb_{20}) - in(pa_{10} + qa_{20})$. Recall that $\hat{f}(np, nq, \cdot)$ is the function belonging to $\mathcal{C}^\infty(\mathbb{T}^1)$ which is the 2π-periodic extension of

$$\phi(t) e^{in(pa_{10} + qa_{20})(t_0 - t)} e^{-nA}, \quad t \in [0, 2\pi],$$

for all $n \in \mathbb{N}$. Since $\psi_0 < 0$ and $0 \leq \phi(t) \leq 1$, for each $n \in \mathbb{N}$ we have

$$| < \hat{\mu}(np, nq, t), \theta(t) > | \leq (2\pi)^2 \|\theta\|_\infty |1 - e^{2\pi d_n}|^{-1} \leq (2\pi)^2 \|\theta\|_\infty (1 - e^{2\pi\psi_0})^{-1},$$

for every $\theta \in \mathcal{C}^\infty(\mathbb{T}^1)$. Hence $\mu \in \mathcal{D}'(\mathbb{T}^3) \setminus \mathcal{C}^\infty(\mathbb{T}^3)$ and $L_{12}\mu = f$; consequently, L_{12} is not globally hypoelliptic, which implies that L is not globally hypoelliptic.

Up to this point we have shown that the global hypoellipticity of L implies that each b_j does not change sign and that $\dim \operatorname{span}\{b_1, \ldots, b_N\} \leq 1$. Notice that, under these assumptions the operator L can be written in the form

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} (a_{j0} + i\lambda_j b(t)) \frac{\partial}{\partial x_j},$$

where $b \in \mathcal{C}^\infty(\mathbb{T}^1, \mathbb{R})$, b does not change sign and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \setminus \{0\}$.

In order to complete the proof of necessity in Theorem 1.2 we must verify that (DC1) is also a necessary condition for the global hypoellipticity of L.

When $b \equiv 0$, the necessity of the condition $(DC1)$ follows by applying the techniques of [7].

Suppose now that $b \not\equiv 0$. In particular, $b_0 \neq 0$. If now $(DC1)$ does not hold, then one of the following situations must occur:

(i) there exists a sequence $(\xi(n), \tau_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^{N+1}$ such that $|\xi(n)|+|\tau_n| \geq n$, $\langle \xi(n), b_0\lambda\rangle = 0$, and $\tau_n = -\langle \xi(n), \alpha_0\rangle \in \mathbb{Z}$.

(ii) the pair $(\alpha_0, b_0\lambda)$ does not satisfy $(DC2)$.

Suppose first that (i) occurs. Notice that $|\xi(n)|$ is unbounded. By taking a subsequence if necessary we can assume that $|\xi(n)|$ is increasing. We now set

$$\mu \doteq \sum_{n=1}^{\infty} e^{-it\langle \xi(n), \alpha_0\rangle} e^{i\langle \xi(n), x\rangle},$$

which belongs to $\mathcal{D}'(\mathbb{T}^{N+1}) \setminus \mathcal{C}^\infty(\mathbb{T}^{N+1})$ and satisfies $L\mu = 0$. Therefore, L is not globally hypoelliptic.

Now, assume that (ii) occurs. We will proceed as in the proof of Proposition 3.2 (the part in which (III.1) and (III.2) are true, but (III.3) fails) in order to construct a function f in $\mathcal{C}^\infty(\mathbb{T}^{N+1})$ so that $L\mu = f$ has a solution in $\mathcal{D}'(\mathbb{T}^{N+1}) \setminus \mathcal{C}^\infty(\mathbb{T}^{N+1})$. Since $(\alpha_0, \beta_0) = (\alpha_0, b_0\lambda)$ does not satisfy $(DC2)$, Lemma 3.1 implies that (DC3) does not hold; hence, either (25) or (26) occurs. When (25) occurs, we have, with the same notations as in the proof of Proposition 3.2, $|1 - e^{2\pi c_n}| \leq |\xi(n)|^{-n}$ (where $c_n \doteq \langle \xi(n), b_0\lambda - i\alpha_0\rangle$) and $\langle \xi(n), b_0\lambda\rangle > 0$. It follows that the sequence $\hat{f}(\xi(n), \cdot)$ given by the 2π-periodic extension of

$$(1 - e^{2\pi c_n}) e^{-\langle \xi(n), 2\pi b_0\lambda\rangle} \varphi(t) e^{i(\pi - t)\langle \xi(n), \alpha_0\rangle}, \quad t \in [0, 2\pi],$$

is rapidly decreasing (recall that $\varphi(t) \in \mathcal{C}_c^\infty((\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta))$, $\varphi(t) \equiv 1$ in a neighborhood of $[\frac{\pi-\delta}{2}, \frac{\pi+\delta}{2}]$, $0 \leq \varphi(t) \leq 1$, and $\delta > 0$ is small enough so that $(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta) \subset (0, \pi)$). Thus we may define a function $f \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ by

$$f(x,t) = \sum_{n=1}^{\infty} \hat{f}(\xi(n), t) e^{i\langle \xi(n), x\rangle}.$$

For this f, we define the distribution

$$\mu = \sum_{n=1}^{\infty} \hat{\mu}(\xi(n), t) e^{i\langle \xi(n), x\rangle},$$

where

$$\hat{\mu}(\xi(n),t) =$$

$$(1-e^{2\pi c_n})^{-1}\int_0^{2\pi}\hat{f}(\xi(n),t-s)e^{-is\langle\xi(n),\alpha_0\rangle}e^{\langle\xi(n),\lambda\rangle\int_{t-s}^{t}b(\tau)d\tau}ds \in \mathcal{C}^\infty(\mathbb{T}^1).$$

Notice that, since $\langle\xi(n),b_0\lambda\rangle > 0$ and since $\langle\xi(n),b_0\lambda\rangle \to 0$ as $n\to\infty$ (see (25)), the definition of $\hat{f}(\xi(n),\cdot)$ implies

$$|\hat{\mu}(\xi(n),\pi)| \geq \delta e^{-\langle\xi(n),2\pi b_0\lambda\rangle} \geq \delta/2;$$

hence $\mu \notin \mathcal{C}^\infty(\mathbb{T}^{N+1})$. On the other hand, for every $\theta\in\mathcal{C}^\infty(\mathbb{T}^1)$ we have

$$|\langle\hat{\mu}(\xi(n),t),\theta(t)\rangle| \leq (2\pi)^2\|\theta\|_\infty,$$

which implies that $\mu\in\mathcal{D}'(\mathbb{T}^{N+1})\setminus\mathcal{C}^\infty(\mathbb{T}^{N+1})$. A simple computation shows that $L\mu = f$. Therefore, L is not globally hypoelliptic when (25) occurs.

Finally, suppose that (26) occurs. In this case, we consider the function $f\in\mathcal{C}^\infty(\mathbb{T}^{N+1})$ given by

$$f(x,t) = \sum_{n=1}^{\infty}\hat{f}(\xi(n),t)e^{i\langle\xi(n),x\rangle},$$

where $\hat{f}(\xi(n),\cdot)$ is the 2π-periodic extension of

$$(1-e^{-2\pi i\langle\xi(n),\alpha_0\rangle})\varphi(t)e^{i(\pi-t)\langle\xi(n),\alpha_0\rangle}, \quad t\in[0,2\pi].$$

Notice that f is well defined since the estimate $|1-e^{-2\pi i\langle\xi(n),\alpha_0\rangle}| \leq |\xi(n)|^{-n}$ (see (26)) implies that $\hat{f}(\xi(n),\cdot)$ decays rapidly.

For such f, consider the distribution

$$\mu = \sum_{n=1}^{\infty}\hat{\mu}(\xi(n),t)e^{i\langle\xi(n),x\rangle},$$

where

$$\hat{\mu}(\xi(n),t) = (1-e^{-2\pi\langle\xi(n),\alpha_0\rangle})^{-1}\int_0^{2\pi}\hat{f}(\xi(n),t-s)e^{-is\langle\xi(n),\alpha_0\rangle}ds \in \mathcal{C}^\infty(\mathbb{T}^1).$$

As before, simple computations show that $\mu\in\mathcal{D}'(\mathbb{T}^{N+1})\setminus\mathcal{C}^\infty(\mathbb{T}^{N+1})$, and that $L\mu = f$. Therefore, L is not globally hypoelliptic when (26) occurs.

The proof of necessity in Theorem 1.2 is complete.

We now move on to the proof of sufficiency. Assuming that (1)-(3) hold, let μ be a distribution in $\mathcal{D}'(\mathbb{T}^{N+1})$ and suppose that $L\mu = f\in\mathcal{C}^\infty(\mathbb{T}^{N+1})$. Our goal is to prove that $\mu\in\mathcal{C}^\infty(\mathbb{T}^{N+1})$.

By using partial Fourier series in the variables $x = (x_1, \ldots, x_N)$ we can write

$$f(x,t) = \sum_{\xi \in \mathbb{Z}^N} \hat{f}(\xi, t) e^{i\langle \xi, x \rangle}$$

and

$$\mu = \sum_{\xi \in \mathbb{Z}^N} \hat{\mu}(\xi, t) e^{i\langle \xi, x \rangle)},$$

where a priori $\hat{\mu}(\xi, \cdot)$ belongs to $\mathcal{D}'(\mathbb{T}^1)$. The equation $L\mu = f$ imply that, for each $\xi \in \mathbb{Z}^N$, $\hat{\mu}(\xi, \cdot)$ is a solution of

$$\frac{d}{dt}\hat{\mu}(\xi, t) + \langle \xi, i\alpha_0 - b(t)\lambda \rangle \hat{\mu}(\xi, t) = \hat{f}(\xi, t). \tag{41}$$

Now, recall that $(DC1)$ implies $(DC2)$. Hence, Lemma 3.1 implies that $(DC3)$ is satisfied. Also, $(DC1)$ implies that we simultaneously have $\langle \xi, \alpha_0 \rangle \in \mathbb{Z}$ and $\langle \xi, b_0 \lambda \rangle = 0$ only for $\xi = 0$. Thus, by proceeding as in the proof of Proposition 3.3 we see that $\hat{\mu}(\xi, \cdot)$ is a smooth function on $\mathbb{T}^1$, which is uniquely determined for all indices $\xi \in \mathbb{Z}^N \setminus \{0\}$. Moreover, since $(DC3)$ holds we see that $\hat{\mu}(\xi, \cdot)$ is rapidly decreasing. Therefore, $\mu \in \mathcal{C}^\infty(\mathbb{T}^{N+1})$ and the proof of sufficiency in Theorem 1.2 is complete.

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