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LETTER

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Improved ring approximation for the free energy in thermal scalar field theory

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Abstract – In a d-dimensional field theory at finite temperature T, with a $g^2\phi^4$ interaction, we compute the contributions to the free energy in the ring approximation. These are calculated in a consistent manner, by evaluating the thermal self-energy Π_T in a similar approximation. The complete result can be expressed in closed form in terms of the Haber and Weldon functions $h_{d+1}(\sqrt{\Pi_T/T^2})$. This result exhibits, to leading order, a non-analyticity in the coupling constant of the form $(g^2)^{(d-1)/2}$ or $\log^N(g^2)$ when d is even or odd, respectively.

It is well known that the occurrence of infrared singularities in field theories at finite temperature leads to a breakdown of the usual perturbative series [1,2]. Then, it is necessary to perform a resummation of the thermal loops, which yields an effective action allowing for a consistent perturbative expansion [3,4]. This problem has been further studied in the context of the free energy in 4-dimensional field theories, where the infrared divergences induce non-analiticities in the calculations at higher loop order [5–11]. The scalar ϕ^4 model provides a simple context to study this and other important aspects of field theories at finite temperature, such as the nature of phase transitions [12]. Moreover, this model is also relevant in $d \neq 4$ space-time dimensions. For example, in lower dimensions, the d-dimensional Ising model is described, close to its critical point, by the ϕ^4 action [13]. On the other hand, in higher dimensions, such scalar theories exhibit some interesting similarities to gauge theories like QCD [14] and quantum gravity [15,16].

The purpose of this letter is to evaluate, in the d-dimensional $g^2\phi^4$ theory at finite temperature T, the contributions to the free energy which arise from the set of ring diagrams shown in fig. 1. To this end, we find convenient to employ the imaginary time formalism, where the energies take the discrete values $\omega_l = 2\pi l T$ [1,2,17]. Furthermore, we assume that T is much greater than the zero temperature mass, which we consequently neglect. The set of ring diagrams contains the leading infrared divergent contributions to the free energy, which come from the static mode $l = 0$ [18]. In consequence of this behavior, the complete result which arises from the summation over all Matsubara modes involves non-analytic contributions

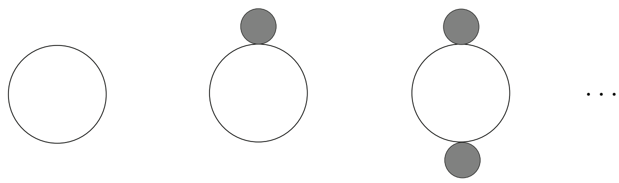

Fig. 1: Ring diagrams in the ϕ^4 model. Small blobs represent the thermal self-energies of scalar particles.

in the coupling constant. This result can be expressed in closed form (see eqs. (11)–(13)) in terms of the functions $h_{d+1}(\sqrt{\Pi_T/T^2})$ discussed by Haber and Weldon [19]. Here Π_T denotes the thermal self-energy of the scalar particle which will be evaluated to higher orders in a consistent approximation. Assuming that $\Pi_T/T^2 \ll 1$, these functions can be expanded in a power series, which leads to the expressions given in eqs. (16) and (18). As can be seen from eq. (16), when $d = 2n$ is even, the free energy exhibits a leading non-analyticity in the coupling constant of the form $(g^2)^{n-1/2}$. On the other hand, when $d = 2n+1$ is odd, we can see from eq. (18) that the non-analyticity is logarithmic, being to leading order of the form $\log(g^2)$. An exception occurs for $d = 3$ ($n = 1$), when the leading non-analyticity is of the form $\log^2(g^2)$.

In the following, we will compute the contributions to the free energy which are generated by the class of ring diagrams in fig. 1. The first diagram, which corresponds to the free boson gas in d space-time dimensions, gives

$$\Omega_0(d) = V \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \left[\frac{|\vec{p}|}{2} + T \log\left(1 - \mathrm{e}^{-\frac{|\vec{p}|}{T}}\right) \right]. \quad (1)$$

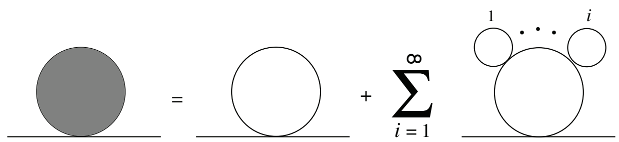

Fig. 2: Thermal self-energy Π_T in the chain approximation.

The sum of the contributions to the free energy coming from all other ring diagrams of fig. 1 may be written as [1,2]

$$\Omega_r(d) = \frac{VT}{2}\sum_{l=-\infty}^{\infty}\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \times \log\left[1+\frac{\Pi_T}{(2\pi lT)^2+|\vec{p}|^2}\right], \tag{2}$$

The thermal self-energy Π_T can be determined in a way consistent with the ring approximation, from the equation

$$\Pi_T = \frac{g^2}{2}T\sum_{j=-\infty}^{\infty}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\frac{1}{\omega_j^2+|\vec{k}|^2+\Pi_T^1}, \tag{3}$$

where

$$\Pi_T^1 = g^2T^{d-2}\frac{\Gamma(d-2)\zeta(d-2)}{(4\pi)^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}. \tag{4}$$

is the thermal self-energy at one loop order. Here Γ and ζ denote, respectively, the gamma and zeta functions [20] (we point out that g^2 has canonical mass dimension $4-d$). Equation (3) involves the scalar field propagator evaluated in the chain approximation, arising from the sum of all graphs obtained by chains of one-loop self-energy insertions as shown in fig. 2. Performing in (3) the summation over j and an expansion in powers of g^2, leads to a result of the form

$$\Pi_T = \Pi_T^1 + C_d(T)\left(\frac{\Pi_T^1}{T^2}\right)^{\frac{d-1}{2}} + \mathcal{O}\left(g^4\right), \tag{5}$$

where

$$C_d(T) = \begin{cases} \dfrac{(-1)^{\frac{d-1}{2}}T^2}{2\Gamma(d-2)\zeta(d-2)}\log\left(\dfrac{\Pi_T^1}{T^2}\right), & \text{for } d \text{ odd},\\ \dfrac{(-1)^{\frac{d-2}{2}}\pi T^2}{2\Gamma(d-2)\zeta(d-2)}, & \text{for } d \text{ even}.\end{cases} \tag{6}$$

For d even we see that Π_T^1/T^2 appears with a non-integer exponent in the second term of (5). On the other hand, when d is odd, it follows from (6) that a logarithmic factor $\log\left(\frac{\Pi_T^1}{T^2}\right)$ occurs in the second term of (5). Since $\Pi_T^1/T^2 \sim g^2T^{d-4}$, these features reflect the breakdown of the naïve perturbation theory due to infrared divergences which arise in individual diagrams in fig. 2.

We now turn to the calculation of the sum over l in eq. (2), where it is useful to relate, with the help of an auxiliary integral, the logarithm to a simple propagator

$$\begin{aligned}\Omega_r(d) &= \frac{VT}{2}\int_0^{\Pi_T}\mathrm{d}m^2\sum_{l=-\infty}^{\infty} \times\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}}\frac{1}{(2\pi lT)^2+|\vec{p}|^2+m^2}\\ &= \frac{V}{2}\int_0^{\Pi_T}\mathrm{d}m^2 \times\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}}\frac{1+2N_B\left(\frac{\sqrt{|\vec{p}|^2+m^2}}{T}\right)}{2\sqrt{|\vec{p}|^2+m^2}},\end{aligned} \tag{7}$$

where $N_B(\omega/T)$ is the Bose-Einstein distribution

$$N_B\left(\frac{\omega}{T}\right) = \frac{1}{\mathrm{e}^{\frac{|\omega|}{T}}-1}. \tag{8}$$

One can now perform the m^2 integration in eq. (7), which leads to

$$\Omega_r(d) = V\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \times\left[\frac{\sqrt{|\vec{p}|^2+\Pi_T}}{2}+T\log\left(1-\mathrm{e}^{-\frac{\sqrt{|\vec{p}|^2+\Pi_T}}{T}}\right) -\frac{|\vec{p}|}{2}-T\log\left(1-\mathrm{e}^{-\frac{|\vec{p}|}{T}}\right)\right]. \tag{9}$$

Hence, combining the contributions from eqs. (1) and (9), we get the result

$$\Omega(d) = V\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}}\left[\frac{\sqrt{|\vec{p}|^2+\Pi_T}}{2} +T\log\left(1-\mathrm{e}^{-\frac{\sqrt{|\vec{p}|^2+\Pi_T}}{T}}\right)\right]. \tag{10}$$

This has the simple interpretation of replacing the free propagator which occurs in the first diagram of fig. 1, with the propagator corrected by the thermal self-energy.

The first integral in eq. (10) is easily evaluated and gives the contribution

$$\Omega'(d) = -\frac{V}{2}\Gamma\left(-\frac{d}{2}\right)\left(\frac{\Pi_T}{4\pi}\right)^{\frac{d}{2}}. \tag{11}$$

The integral over the logarithm in eq. (10) can be done after integration by parts and one gets

$$\Omega''(d) = -\frac{VT^d\Gamma(d+1)}{(4\pi)^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)}h_{d+1}\left(\sqrt{\frac{\Pi_T}{T^2}}\right). \tag{12}$$

Here $h_{d+1}\left(\sqrt{\Pi_T/T^2}\right)$ denote the Haber-Weldon functions which are defined, at zero chemical potential, as [19]

$$h_{d+1}\left(\sqrt{\frac{\Pi_T}{T^2}}\right) = \frac{1}{\Gamma(d+1)}\int_0^\infty \mathrm{d}x \frac{x^d}{\sqrt{x^2+\frac{\Pi_T}{T^2}}} \times N_B\left(\sqrt{x^2+\frac{\Pi_T}{T^2}}\right), \tag{13}$$

where N_B is the Bose-Einstein distribution. Since $\Omega'(d)$ and $\Omega''(d)$ have a different behavior when d is even or odd, we will discuss these cases separately.

For even values of d, $\Omega'(d)$ in eq. (11) contains simple poles. To regularize these, we set $d = 2n - 2\epsilon$ and use dimensional regularization which introduces a mass scale μ. In the limit $\epsilon \to 0$, we then obtain

$$\Omega'(d=2n-2\epsilon) = -\frac{V}{2\Gamma(n+1)} \times \left[\frac{1}{\epsilon} - \log\left(\frac{\Pi_T}{4\pi\mu^2}\right) - \gamma + \sum_{k=1}^{n}\frac{1}{k}\right] \times \left(-\frac{\Pi_T}{4\pi}\right)^n. \tag{14}$$

The pole term $1/\epsilon$ is cancelled by contributions coming from other diagrams [5], after performing the ultraviolet renormalization of the coupling constant. With this understanding, this term will be henceforth omitted. On the other hand, since the contributions to $\Omega''(d)$ in eq. (12) are well behaved in the ultraviolet region, we may set here directly $\epsilon = 0$. Assuming that $\Pi_T/T^2 \ll 1$, we can expand the functions $h_{d+1}\left(\sqrt{\Pi_T/T^2}\right)$ according to the formulas given in [19]. Proceeding in this way, and using the identity [20]

$$\Gamma(2n+1) = \frac{2^{2n}}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1), \tag{15}$$

one can show that the $\log(\Pi_T)$ terms cancel between eqs. (12) and (14), in the sum $\Omega' + \Omega''$. Thus, the total contribution to the free energy may be written in this case as

$$\Omega(d{=}2n){=}-VT^d\left\{\frac{1}{\pi^n}\sum_{k=0}^{n-1}(-1)^k\frac{\Gamma(n{-}k)\zeta(2n{-}2k)}{\Gamma(k{+}1)}\left(\frac{\Pi_T}{4T^2}\right)^k + \frac{(-1)^n\pi}{2\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{\Pi_T}{4\pi T^2}\right)^{n-\frac{1}{2}} + \frac{1}{2\Gamma(n+1)} \times\left[\gamma - \log\left(\frac{4\pi T^2}{\mu^2}\right)\right]\left(-\frac{\Pi_T}{4\pi T^2}\right)^n + \mathcal{O}\left[\left(\frac{\Pi_T}{T^2}\right)^{n+1}\right]\right\}. \tag{16}$$

From the relations (5) and (6), it follows that higher-order corrections in eq. (16) can generate sub-leading non-analiticities of the form $(g^2)^{k+n-3/2}$ $(k \geq 2)$. Moreover, one can see that the $(\Pi_T)^{n-1/2}$ term in eq. (16) exhibits a leading non-analyticity in the coupling constant of the form $(g^2)^{n-1/2}$. Similarly, when $k=1$, the first term in eq. (16) will also yield a non-analyticity of the same order. For example, when $d=4$, the above expression for the free energy takes the form

$$\Omega(d=4) = -\frac{\pi^2 VT^4}{90}\left\{1 - \frac{15}{8}\left(\frac{g^2}{24\pi^2}\right) + \left(1+\frac{3}{2}\right)\frac{15}{2}\left(\frac{g^2}{24\pi^2}\right)^{\frac{3}{2}} + \mathcal{O}(g^4)\right\}. \tag{17}$$

We point out that the factor 1 in the last term of eq. (17) corresponds to the usual result, obtained by considering only one-loop self-energy insertions in the ring diagrams [1, 2]. On the other hand, the factor 3/2 in eq. (17) arises from multi-loop contributions to Π_T, which lead to the non-analytical term appearing in eq. (5).

We now turn to the case when $d = 2n+1$ is odd. Then, although $\Omega'(d=2n+1)$ in eq. (11) does not have poles, it leads to a non-analyticity in the coupling constant due to the factor $(\Pi_T)^{n+1/2}$. However, one can show with the help of the identity (15), that such terms cancel in the sum $\Omega' + \Omega''$. We then get the following contribution to the free energy:

$$\Omega(d=2n+1) = -VT^d\left\{\frac{1}{\pi^{n+\frac{1}{2}}}\sum_{k=0}^{n-1}(-1)^k \times \frac{\Gamma\left(n+\frac{1}{2}-k\right)\zeta(2n+1-2k)}{\Gamma(k+1)}\left(\frac{\Pi_T}{4T^2}\right)^k + \frac{(-1)^{n+1}}{2\Gamma(n+1)}\left[\log\left(\frac{\Pi_T}{T^2}\right) - \gamma - \Psi(n+1)\right] \times \left(\frac{\Pi_T}{4\pi T^2}\right)^n + \mathcal{O}\left[\left(\frac{\Pi_T}{T^2}\right)^{n+1}\right]\right\}, \tag{18}$$

where Ψ is the Euler psi function [20]. Using the relations (5) and (6), we see that due to the presence of a $\log(\Pi_T^1)$ term in Π_T and of the $\log(\Pi_T)$ factor in eq. (18), $\Omega(d = 2n+1)$ exhibits a logarithmic non-analyticity in the coupling constant. To leading order, the non-analytic contributions from eq. (18) have the form $(g^2)^n\log(g^2)$. In addition, the $\log(\Pi_T)\Pi_T^n$ term in eq. (18) can also generate sub-leading non-analytic contributions like $(g^2)^{2n-1}\log^2(g^2)$. An interesting exception occurs in the case $n=1$, when the last contribution is actually leading being of the form $g^2\log^2(g^2)$. In fact, from eqs. (5), (6) and (18) we readily find that for $d=3$, the main non-analytic contribution to the free energy is given by

$$\Omega^\star(d=3) = \frac{VT^3}{64\pi^2}\frac{g^2}{T}\log^2\left(\frac{g^2}{T}\right), \tag{19}$$

where we note that the coupling constant g^2 now carries the dimension of mass.

In conclusion, we remark that the non-analyticities present in eqs. (16) and (18) arise in consequence of the infrared singularities. These come from the static mode $l = 0$ in eq. (2) as well as from the $j = 0$ mode in the thermal self-energy (3). Indeed, as shown in [18], such infrared divergent contributions induce, when d is even, a leading non-analyticity in the coupling constant of the form $(g^2)^{(d-1)/2}$, whereas when d is odd the non-analyticity is only logarithmic. The results given in eqs. (16) and (18) may be useful approximations for the free energy in thermal ϕ^4 field theory in d space-time dimensions. The set of ring diagrams can be relevant also in thermal fermionic field theory, describing a system of interacting electrons. For example, for a high-density electron gas in two space-time dimensions, this class of graphs gives, to all orders, the dominant contributions to the free energy [13]. It would be interesting to extend the present study to QED in a d-dimensional space time [21].

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REFERENCES

[1] Kapusta J. I., *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England) 1989.

[2] Bellac M. L., *Thermal Field Theory* (Cambridge University Press, Cambridge, England) 1996.

[3] Frenkel J. and Taylor J. C., *Nucl. Phys. B*, **334** (1990) 199; **374** (1992) 156.

[4] Braaten E. and Pisarski R. D., *Nucl. Phys. B*, **337** (1990) 569; **339** (1990) 310.

[5] Frenkel J., Saa A. V. and Taylor J. C., *Phys. Rev. D*, **46** (1992) 3670.

[6] Arnold P. and Zhai C.-X., *Phys. Rev. D*, **50** (1994) 7603.

[7] Parwani R. and Singh H., *Phys. Rev. D*, **51** (1995) 4158.

[8] Kajantie K., Laine M., Rummukainen K. and Schröder Y., *Phys. Rev. Lett.*, **86** (2001) 10; *Phys. Rev. D*, **67** (2003) 105008.

[9] Blaizot J., Iancu E. and Rebhan A., *Phys. Rev. D*, **68** (2003) 025011.

[10] Vuorinen A., *Phys. Rev. D*, **67** (2003) 074032.

[11] Brandt F. T., Frenkel J. and Muramoto C., *Nucl. Phys. B*, **754** (2006) 146; *Phys. Lett. B*, **649** (2007) 483.

[12] Carrington M., *Phys. Rev. D*, **45** (1992) 2933.

[13] Altland A. and Simons B., *Condensed Matter Field Theory* (Cambridge University Press, Cambridge, England) 2006.

[14] Muta T., *Foundations of Quantum Chromodynamics* (World Scientific, Singapore) 1987.

[15] Altherr T., Grandou T. and Pisarski R. D., *Phys. Lett. B*, **271** (1991) 183.

[16] Brandt F. T., Das A. and Frenkel J., *Phys. Rev. D*, **78** (2008) 065030.

[17] Das A., *Finite Temperature Field Theory* (World Scientific) 1997.

[18] Brandt F. T., Frenkel J. and Siqueira J. B., *Phys. Rev. D*, **86** (2012) 107701.

[19] Haber H. E. and Weldon H. A., *J. Math. Phys.*, **23** (1982) 1852.

[20] Gradshteyn I. S. and Ryzhik M., *Tables of Integral Series and Products* (Academic Press, New York) 1980.

[21] Brandt F. T., Frenkel J. and Siqueira J. B., *Phys. Rev. D*, **89** (2014) 045011, arXiv:1311.2519.