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NORMAL AND FUNCTION SPACES

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In [1] H.H. Corson proved several results concerning relations between spaces of continuous functions and Σ-products of spaces. It seems interesting to prove analogous results for other uniform spaces than metric ones. It is our purpose to give some theorems in this direction.

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All spaces are assumed to be Hausdorff spaces.

Theorem 0. *Let* E *be a completely regular space and* m *be an infinite cardinal number. Then the following statements are equivalent:*

(1) *there is a compatible uniformity for* E *whose uniform weight is at most* m*;*

(2) E *can be embedded into the product of* m *metric spaces;*

(3) X *has an open base* $B = \bigcup_{\alpha < m} B_\alpha$ *such that, for every* $\alpha < m$, B_α *is a discrete collection of functionally open sets.*

Proof 1 ⇒ 2. See, for instance, [2], page 327 (Theorem 7 and the remark following it).

2 ⇒ 1. Since each metrizable space has a uniformity whose weight is at most m, the product of m metrizable spaces has a uniformity whose uniform weight is at most m.

1 ⇒ 3. This is a consequence of Stone's theorem [4] and that if U belongs to a uniformity $\mathcal{U}$ there is $W \in \mathcal{U}$ such that $W \circ W \subset U$.

3 ⇒ 1. Let γ be a discrete collection of functionally open subsets of E. Then, for each $X \in \gamma$ there is a continuous function $f_X: E \to [0, 1]$ such that $f_X^{-1}(]0, 1]) = X$.

We define $d: E \times E \to R$ as follows $d(x, y) = \sum_{X \in J} |f_X(x) - f_X(y)|$.

The function d is a continuous pseudo-metric. We do exactly the same for each $\alpha < m$, getting a pseudo-metric d_α; next we consider the set $P = \{d_\alpha \mid \alpha < m\} \cup \{\sup\{d_{\alpha_1}, \dots, d_{\alpha_n}\} \mid n \geq 2; \alpha_1 < m, \dots, \alpha_n < m\}$ and apply [2] (page 323, Theorem 7).

Theorem 1. *Let E be a completely regular space with uniform weight m, where m is an infinite cardinal number. Then E is homeomorphic to a subspace of a Σ_m-product of copies of the unit interval.*

Proof. Let $B = \bigcup_{\alpha < m} B_\alpha$, be an open base, where each B_α is a discrete collection of functionally open sets. For each $X \in B$ there is a continuous function $f_X: E \to [0, 1]$ such that $f_X^{-1}(]0, 1])^X$.

Put

$$\varphi: E \to [0, 1]^B: \quad x \mapsto (f_X(x))_{X \in B}$$

The function φ is bijective, continuous and open from E onto $\varphi(E)$ and the proof is completed.

As a matter of fact it is possible to prove a deeper result ([3]): If E is a completely regular space with a m-point countable functionally open base, then E is homeomorphic to a subspace of a Σ_m-product of copies of the unit interval. In the proof of Theorem 1 it is sufficient that B be m-point countable.

Let m be an infinite cardinal number, A be a set of cardinality greater than m and let Σ_m denote the following Σ_m-product of copies of the real line:

$$\Sigma_m = \{(x_a)_{a\in A} \mid |\{a \in A \mid x_a \neq 0\}| \leqslant m\}.$$

Theorem 2. *Under these conditions,* Σ_m *is homeomorphic to* $C(X)$, *all the continuous real-valued functions on a* m*-Lindelöf space* X, *with the simple topology.*

Proof. Put $X = A \cup \{w\}$, where w does not belong to A. Consider on X the following topology:

(1) $\{a\}$ is open for every $a \in A$;

(2) $W \subset X$ is an open neighborhood of w if and only if $w \in W$ and the complementary set $A - W$ has cardinality $\leqslant m$.

$C(X)$ will denote the set of all continuous real-valued functions on X and $C_0(X)$ will denote the set of all continuous functions from X into real line which vanish at w. The same notations, $C(X)$ and $C_0(X)$, will be used for the corresponding topological spaces with the simple topology.

First of all let us prove that $C(X)$ is homeomorphic to $C_0(X) \times R$. This is immediate since the function

$$\psi\colon C(X) \to C_0(X) \times R\colon \quad f \mapsto (f', f(w)),$$

where $f'(t) = f(t) - f(w)$ for every $t \in X$, is a homeomorphism.

On the other hand, Σ_m is homeomorphic to $C_0(X)$. Indeed, the function

$$\theta\colon \Sigma_m \to C_0(X),$$

which assigns to $(x_a)_{a\in A}$ the function of $C_0(X)$, such that to each $a \in A$ corresponds the element x_a and to w corresponds the element 0, is a homeomorphism.

Moreover $\Sigma_m \times R$ is homeomorphic to Σ_m. Let us fix $b \in A$; then A is equipotent to $A - \{b\}$; let g be an injective function from $A - \{b\}$ onto A. Consider the homeomorphism from $\Sigma_m \times R$ onto Σ_m which assigns to $((x_a)_{a \in A}, r)$ the point $(y_a)_{a \in A}$ where $y_b = r$ and $y_a = x_{g(a)}$ for every $a \neq b$. Finally we have that

(1) Σ_m is homeomorphic to $C_0(X)$;

(2) $C_0(X) \times R$ is homeomorphic to $C(X)$;

(3) $\Sigma_m \times R$ is homeomorphic to Σ_m.

It thus follows that Σ_m is homeomorphic to $C(X)$ and the proof is completed.

Corollary 1. *Let G be a topological group with local weight $m \geqslant \aleph_0$. Then G is homeomorphic to a subspace of $C(X)$, all the continuous real-valued functions on a m-Lindelöf space X, with the simple topology.*

Proof. It follows immediately from Theorems 1 and 2.

Theorem 3. *Let E be a topological space with local weight $m \geqslant \aleph_0$. Then E is the open continuous image of a subspace of $C(X)$, all the continuous real-valued functions on a m-Lindelöf space X, with the simple topology.*

Proof. Let A be an open basis for the topology on E. Let us consider the topological space A^I, where $|I| = m$ and A has the discrete topology; the topological product space A^I has an open basis. $B = \cup \{B_i \mid i \in I\}$, where each B_i is a discrete open covering of A^I. (The function f_X constructed in the proof of Theorem 1 may be choosen taking values 0 and 1 only.) But, from Theorem 1, A^I is homeomorphic to a subspace of a Σ_m-product of copies of the unit interval (as a matter of fact of copies of $\{0, 1\}$). By Theorem 2, A^I is homeomorphic to a subspace of $C(X)$, all the continuous real-valued functions on a m-Lindelöf space X ($C(X)$ with the simple topology). On the other hand. Let S be the subspace of A^I such that $(A_i)_{i \in I}$ belongs to S if and only if is a fundamental system of open neighborhoods of some point E.

The function $\theta: S \to E$, which assigns to each $(A_i)_{i \in I}$ the point $x \in \in \bigcap \{A_i \mid i \in I\}$, is open, continuous and onto. And the proof is completed, since S is homeomorphic to a subspace of the same $C(X)$ above.

Using the same method of proving Theorem 2 it is possible to show:

Theorem 4. *Let m be an infintie cardinal number and Z be a topological group with local weigth $\leqslant m$. A Σ_m-product of copies of Z is homeomorphic to $C(X)$, the set of all continuous functions from X into Z ($C(X)$ with the simple topology), where X is a m-Lindelöf space.*

Question 1. Does Theorem 4 remains valid if instead of Z, a topological group, we assume that Z is an homogenous space?

Question 2. ([3]). Let m be an infinite cardinal number and E be a completely regular space. If E has a m-point countable open base does it have a m-point countable functionally open base?

REFERENCES

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