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THE THURSTON OPERATOR FOR SEMI-FINITE COMBINATORICS

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Abstract. Given a continuous l-modal map g of the interval $[0,1]$ we prove the existence of a polynomial P with modality $\leq l$ such that g is strongly semi-conjugate to P in $[0,1]$. This is an improvement of a result in [4]. We do a modification on the Thurston operator in order to control the semi-finite combinatorial case. It turns out that all the essential attractors of P have the same local topological type as those of g. This allows to construct the strong semi-conjugacy. We also present some examples agreeing with the results.

1. Introduction. A full family of maps is a family which describes essentially all the dynamics of a certain class of transformations. For instance, the quadratic family $f_\mu(x) = \mu x(1-x)$, $\mu \in [0,4]$, describes essentially all the possible dynamics of C^1 unimodal maps of the interval and, therefore, is called a full family. This means that given a C^1 unimodal map g of the interval, it is possible to find $\mu \in [0,4]$ such that g is strongly semi-conjugate[1] to f_μ.

In [4], W. de Melo and S. van Strien develop a theory to construct full families of polynomial maps which describe essentially all the possible dynamics of C^1 l-modal maps of the interval. They obtain in some cases a conjugacy between a given l-modal map and a polynomial map after collapsing some particular intervals. In this paper we improve their construction to get the strong semi-conjugacy as stated in Theorem 1. The proportions associated to the Thurston operator are modified in order to control the combinatoric and the local topological type of the essential attractors. This turns out to be crucial for constructing the desired strong semi-conjugacy. We also present a method for interpolating critical values which is used to iterate the Thurston operator. Many examples are shown in order to illustrate the results.

2. The Main Statement. Let's start with some definitions.

Definition 2.1. We say that a continuous map $g : [0,1] \to [0,1]$ is l-modal if there exist points $c_0 = 0 < c_1 < \ldots < c_l < c_{l+1} = 1$ such that g is alternatingly strictly increasing and strictly decreasing in each interval $[c_i, c_{i+1}]$, $i = 0, \ldots, l$. Moreover, we assume that $g(\{0,1\}) \subset \{0,1\}$. The points $c_1, \ldots, c_l$ are called the turning points of g.

Definition 2.2. We say that $g : [0,1] \to [0,1]$ is topologically semi-conjugate to $f : [0,1] \to [0,1]$ if there exists a continuous monotone increasing and surjective map $h : [0,1] \to [0,1]$ such that $h \circ g = f \circ h$. The map h is called the semi-conjugacy

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[1]See Section 3 for definitions.

between g and f. If h is a homeomorphism we say that g and f are topologically conjugate and h is called the conjugacy between g and f.

By combinatoric of an l-modal map g we mean the order of the forward orbits of its l turning points and, roughly speaking, the dynamics of g is determined by its combinatoric.

Definition 2.3. We say that two l-modal maps f, g of the interval $[0,1]$ are combinatorially equivalent if they have the same combinatoric, that is, if the following map

$$h : \bigcup_{i=1}^{l} \bigcup_{n \geq 0} f^n(c_i) \to \bigcup_{i=1}^{l} \bigcup_{n \geq 0} g^n(\tilde{c}_i)$$

defined by $h(f^n(c_i)) = g^n(\tilde{c}_i)$ is an order preserving bijection. Here, $c_1 < \ldots < c_l$ and $\tilde{c}_1 < \ldots < \tilde{c}_l$ are the turning points of f and g, respectively.

The following hypothesis will hold throughout the paper:

[H1] every periodic turning point is attracting[2].

Note that this condition is satisfied for all C^1 maps. The main result of this paper is the following

Theorem 1. *Let $g : [0,1] \to [0,1]$ be a continuous l-modal map satisfying [H1]. Then there exists a polynomial P with modality $l_0 \leq l$ and degree $l_0 + 1$ such that g is strongly semi-conjugate to P in $[0,1]$, i.e., there exists a semi-conjugacy between g and P which collapses only the following intervals: wandering intervals, intervals of periodic points of the same period and their pre-images, and the components of the attracting basins of the inessential periodic attractors.*

In order to prove Theorem 1 we use the theory developed in [4]. The proof depends on the combinatoric of g. More precisely, 3 different types of combinatorics are considered:

Finite Combinatoric. The orbits of the turning points of g are finite, i.e., all turning points are periodic or eventually periodic.

Semi-finite Combinatoric. Each turning point of g is periodic, eventually periodic or has an infinite orbit converging to an essential periodic orbit (at least one of the turning points satisfies this last condition).

Infinite Combinatoric. All the other cases, i.e., there exists a turning point of g with an infinite orbit which does not converge to an essential periodic orbit.

In the case of finite combinatoric, Theorem 1 is proved in [4] and the essential conjugacy obtained there is in fact a strong semi-conjugacy. In this paper we treat the cases of semi-finite and infinite combinatoric where a modification on the Thurston operator presented in [4] may be necessary to guarantee the existence of h. This modification is the main contribution of this paper. A method is presented in Section 8 to iterate the Thurston operator in order to obtain the desired polynomial.

3. Preliminary definitions and facts. Let g be a continuous l-modal map and p be a periodic point of g, i.e., $g^n(p) = p$ for some $n > 0$.

[2]see Section 3 for definitions.

Definition 3.1. We say that the periodic orbit $O(p) := \{g^i(p) : i \in N\}$ is attracting if its basin $B(p) := \{x \in I : g^i(x) \stackrel{i\to\infty}{\longrightarrow} O(p)\}$ contains an open interval. The union of the components of $B(p)$ which intersect $O(p)$ is called the immediate basin and denoted by $B_0(p)$. If $B_0(p)$ contains a neighborhood of $O(p)$ then we say that $O(p)$ is a two-sided periodic attractor. Otherwise the points of $O(p)$ are contained in the end points of $B_0(p)$ and it is called a one-sided periodic attractor.

Definition 3.2. An attracting periodic orbit $O(p)$ is called essential if there is a turning point in its immediate basin $B_0(p)$. Otherwise it is called inessential. To simplify notations, if $O(p)$ is essential then we can assume that there exists a turning point in the connected component of $B_0(p)$ which contains p.

Definition 3.3. An interval $J \subset I$ is called a wandering interval if its iterates are pairwise disjoint and do not converge to a periodic orbit.

We introduce the following equivalence relation in I: $x \sim y$ if and only if the closed interval connecting x and y is contained in a wandering interval, an interval or a pre-image of an interval of periodic points with constant period, or the closure of a component of the basin of an inessential periodic attractor. It follows that the quotient map $\tilde{g} = g/\sim: I/\sim \to I/\sim$ is a continuous l_0-modal map of the interval $I/\sim$ where $l_0 \leq l$. Note that $\tilde{g}$ has none of the intervals above.

To construct the semi-conjugacy h in Theorem 1 it is enough to construct a conjugacy $\tilde{h}$ between $\tilde{g}$ and a polynomial P with degree $l_0 + 1$ and modality l_0. The conjugacy $\tilde{h}$ is the desired semi-conjugacy h after collapsing points in the same equivalence class. So from now on we consider the quotient map $\tilde{g}$ which will also be denoted by g. We therefore assume that besides [H1] g also satisfies the following hypothesis

[H2] there is no wandering interval.
[H3] there is no non-trivial interval of periodic points with the same period.
[H4] all the attracting periodic orbits are essential.

4. Polynomial families. Let P_{l+1}, $l+1 \geq 2$, be the set of polynomials P of degree $l+1$ such that its restriction to the interval $[0,1]$ is an l-modal map[3] of $[0,1]$. We denote by P^0_{l+1} and P^1_{l+1} the subsets of P_{l+1} of polynomials satisfying $P(0) = 0$ and $P(0) = 1$ respectively[4]. For an arbitrary polynomial we set $||P(x)|| = max_{x\in[0,1]}|P(x)|$.

We have the following inequality due to A.A.Markov

Theorem 2. *Let P be a polynomial of degree $l+1$. If $0 \leq P(x) \leq 1$ for all $x \in [0,1]$ then*

$$||P'|| \leq (l+1)^2. \tag{2}$$

The equality in (2) holds if and only if $P(x) = \frac{1\pm T_{l+1}(2x-1)}{2}$, where T_{l+1} is the Tchebychev polynomial defined by $\cos(l+1)x = T_{l+1}(\cos x)$. Moreover $|P'(x)| = (l+1)^2$ is attained only at $x = 0$ and $x = 1$.

It is clear from Theorem 2 that if $P \in P_{l+1}$ then

$$||P^{(k)}|| \leq C_{l,k}, \tag{3}$$

[3]this implies that all turning points of P are non-degenerate.
[4]in each case $P(1)$ is determined.

where the constant $C_{l,k} > 0$ depends only on l and k. For a proof of Theorem 2, see [2] and [5].

4.1. Realizing critical values. Let $P \in P^0_{l+1}$ where $l+1$ is even[5]. Since $P(0) = P(1) = 0$ we have $P(x) = P_{\underline{a}}(x) = a_1x + a_2x^2 + \ldots + a_lx^l + (-a_1 - \ldots - a_l)x^{l+1}$ where $\underline{a} = (a_1, \ldots, a_l)$. Let $\Delta = \{\underline{a} : P_{\underline{a}} \in P^0_{l+1}\}$ and $V = \{\underline{v} = (v_1, \ldots, v_l) \in [0,1]^l : (-1)^i(v_{i+1} - v_i) > 0, \quad 1 \le i \le l-1\}$. We define the map $F : \Delta \to V$ by

$$F(\underline{a}) = (P_{\underline{a}}(c_1(\underline{a})), \ldots, P_{\underline{a}}(c_l(\underline{a}))), \tag{4}$$

which assigns to each parameter $\underline{a}$ the critical values[6] of the corresponding polynomial $P_{\underline{a}}$. ¿From Theorem 2 we know that Δ is bounded. The boundary $\partial\Delta$ is composed by parameters where at least one of the following possibilities occurs:

1. two or more turning points coincide.
2. some of the critical values are 0 or 1.

In [4], it is shown that F is a diffeomorphism. Here we show how to obtain $\underline{a}$ which realizes an arbitrary $\underline{v} \in V$. This is equivalent to solve for c_i and a_i, $1 \le i \le l$, the following system

$$P_{\underline{a}}(c_i) = v_i, \quad P'_{\underline{a}}(c_i) = 0, \quad P_{\underline{a}}(0) = 0, \quad P_{\underline{a}}(1) = 0. \tag{5}$$

Let $P_i(\underline{a}) = P_{\underline{a}}(c_i(\underline{a}))$. Using Vandermonde's formula, we can see that F is a local diffeomorphism. This implies that given $\underline{a} \in \Delta$, $T_{\underline{a}}\Delta = E_1 \oplus \ldots \oplus E_l$ where E_i is one-dimensional, $DP_i(\underline{a})v \neq 0$ and $DP_j(\underline{a})v = 0$ for all $v \in E_i$ and $j \neq i$.

For a given $\underline{v} = (v_1, \ldots, v_l) \in V$, let $S : \Delta \to R$ be defined by $S(\underline{a}) = \frac{1}{2}\Sigma^l_{i=1}(v_i - P_i(\underline{a}))^2$. Now consider the gradient vector field X in Δ given by

$$\dot{\underline{a}} = X(\underline{a}) = -\nabla S(\underline{a}). \tag{6}$$

A point $\underline{a} \in \Delta$ is an equilibrium point of (6) (or a critical point of S) if and only if $F(\underline{a}) = \underline{v}$ (this means $S(\underline{a}) = 0$). Solutions of (6) decrease S and therefore approach the desired $\underline{a}$. However it may hit $\partial\Delta$ and the solution may not be continued. To avoid that let $\tilde{X}_i$ be the vector field in Δ given by $\tilde{X}_i(\underline{a}) = -\nabla S_i(\underline{a})$, where $S_i(\underline{a}) = \frac{1}{2}(v_i - P_i(\underline{a}))^2$, $1 \le i \le l$, and let X_i be the canonical projection of $\tilde{X}_i$ onto E_i. A point $\underline{a} \in \Delta$ is an equilibrium point of

$$\dot{\underline{a}} = X_i(\underline{a}) \tag{7}$$

if and only if $P_i(\underline{a}) = v_i$. Moreover, solutions of (7) preserve P_j for any $j \neq i$. We can now construct the desired $\underline{a}$ using step by step each X_i. Starting from an arbitrary $\underline{a}_0 \in \Delta$ we do the following: if $P_1(\underline{a}_0) < v_1$ we use X_1 and then X_2 to reach in the limit the critical values v_1 and v_2. Otherwise we first use X_2 to reach v_2 and then use X_1 to reach v_1. This avoids hitting $\partial\Delta$. After that we proceed in the same way with X_3 and X_4 and so on. In the end we adjust v_l and find $\underline{a}$ such that $F(\underline{a}) = \underline{v}$.

This proves that F is a surjective map. Moreover, P^0_{l+1} has only one connected component otherwise there would exist $P \in P^0_{l+1}$, $P \neq P^*$, with the same critical

[5] everything in this section is analogous if $P \in P^1_{l+1}$ and/or $l+1$ is odd.
[6] $0 < c_1(\underline{a}) < \ldots < c_l(\underline{a}) < 1$ are the turning points of $P_{\underline{a}}$.

values of P^*, the corresponding Tchebychev polynomial defined in Theorem 2. The extremal properties of P^* and a topological argument on the graphs of P and P^* imply that $P - P^*$ has more than $l+1$ zeros, a contradiction.

We summarize this section with the following

Theorem 3. *P_{l+1} has two connected components P^0_{l+1} and P^1_{l+1}. The component P^0_{l+1} can be parameterized by Δ and the map $F : \Delta \to V$ defined in (4) is a diffeomorphism. Given l and k, there is a constant $C_{l,k} > 0$ such that $||P^{(k)}|| \le C_{l,k}$ for any $P \in P^0_{l+1}$. An analogous statement is true for P^1_{l+1}.*

4.2 Schwarzian derivative. Let $P \in P_{l+1}$ and $0 < c_1 < \ldots < c_l < 1$ be its turning points. The Schwarzian derivative $S_P : [0,1] \backslash \{c_1, \ldots, c_l\} \to \mathbb{R}$ of P is defined by $S_P(x) = \frac{P'''(x)}{P'(x)} - \frac{3}{2}(\frac{P''(x)}{P'(x)})^2$. Since all zeros of P' are real and distinct, it is not difficult to see that $S_P(x)$ is negative for all x. This implies that

Proposition 4.1. *Let $P : [0,1] \to [0,1]$ where $P \in P_{l+1}$. Then P satisfies [H1]-[H4]. Moreover P' does not have a positive local minimum or a negative local maximum.*

See [4] for a proof.

5. The Thurston Operator for finite combinatorics. In [4], it is introduced the Thurston operator associated to the combinatoric of a given l-modal map $g : [0,1] \to [0,1]$. We assume that we are in the finite combinatorial case, i.e., all turning points are periodic or eventually periodic. Let $C(g) = \{c_1(g), \ldots, c_l(g)\}$ be the turning points of g and $C^*(g) = \cup_{i \ge 0} g^i(C(g))$. By assumption $C^*(g) = \{z_1 < \ldots < z_k\}$ for some $k > 0$. Now define $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ by $\pi(i) = j \Leftrightarrow g(z_i) = z_j$. Moreover, let $t_1 < \ldots < t_l$ be such that $z_{t_i} = c_i(g)$. Note that π and the numbers $t_1, \ldots, t_l$ carry all the combinatorial information of g. This is enough to define the Thurston operator $T : W \to W$ where $W = \{x = (x_1, \ldots, x_k) \in [0,1]^k : 0 < x_1 < \ldots < x_k < 1\}$ is an open simplex.

Given $x \in W$, let $\underline{a} \in \Delta$ be such that $P_{\underline{a}} \in P^{g(0)}_{l+1}$ and all the critical values associated to x are realized by $P_{\underline{a}}$, i.e., $F(\underline{a}) = (x_{\pi(t_1)}, \ldots, x_{\pi(t_l)})$ where F is defined in (4). By Theorem 3 there is only one such $\underline{a}$.

We set $y_{t_i} = c_i(\underline{a})$. For $j \notin \{t_1, \ldots, t_l\}$ let m be such that $z_j \in [z_{t_m}, z_{t_{m+1}}]$, assuming that $z_{t_0} = 0$ and $z_{t_{l+1}} = 1$. We then define y_j to be the point in $[y_{t_m}, y_{t_{m+1}}]$ such that $P_{\underline{a}}(y_j) = x_{\pi(j)}$. This defines y_i for $1 \le i \le k$. Now set $T((x_1, \ldots, x_k)) = (y_1, \ldots, y_k)$.

This is the pull-back construction for finite combinatorics in the real case. The Thurston operator T can be continuously extended to ∂W and it is possible to prove that depending on the combinatoric of g there exists a fixed point of T in the interior of W (see [1], [4] and [6]). In fact, this is the case when g satisfies hypotheses [H3]-[H4]. This fixed point corresponds to a parameter $\underline{a} \in \Delta$ such that the combinatorics of g and $P_{\underline{a}}$ coincide. Moreover, this operator is a contraction of the Teichmüller metric and, therefore, the fixed point can be easily obtained just by iterating an initial condition. It also implies uniqueness of this combinatoric in P_{l+1}.

Remark 5.1. If $z_1 = 0$ we fix $x_1 = 0$ and work in the open simplex $W = \{x = (x_2, \ldots, x_k) \in [0,1]^{k-1} : 0 < x_2 < \ldots < x_k < 1\}$. It is analogous in the case $z_k = 1$.

More about the Thurston operator can be found in [1], [3], [4] and [6].

5.1 Existence condition The condition for the existence of the fixed point of T is the following:

[C1] given z_i and z_j, $i \neq j \in \{1, \dots, k\}$, there exists $m \geq 0$ and $k \in \{1, \dots, l\}$ such that c_k is contained in the closed interval connecting $g^m(z_i)$ and $g^m(z_j)$.

In terms of the combinatoric of g this condition is equivalent to

[C2] Given $i \neq j \in \{1, \dots, k\}$ there exists $m \geq 0$ and $k \in \{t_1, \dots, t_l\}$ such that $\pi^m(i) \leq t_k \leq \pi^m(j)$ or $\pi^m(j) \leq t_k \leq \pi^m(i)$.

Condition [C1] is sufficient as it can be seen in the proof of Lemma 4.1 in Chapter II of [4]. It is also necessary since otherwise this would imply that $P_{\underline{a}}$ does not satisfies one of the hypotheses [H3]-[H4], which is impossible by Proposition 4.1.

It is interesting to note that if [C1] is not satisfied then the Thurston operator T is still a contraction and iterates will accumulate in a point at ∂W, which is the union of simplexes with dimension smaller than k. Some experiments suggest that the polynomial associated to this limit point has a simpler combinatoric which can be obtained from π after an appropriate quotient. This quotient is obtained by identifying i and j not satisfying condition [C2].

We conclude that given an l-modal map g satisfying properties [H1]-[H4] and with finite combinatoric, there exists a unique $P_{\underline{a}} \in P_{l+1}^{g(0)}$ with the same combinatoric of g.

6. The Thurston operator for semi-finite combinatorics. Now we introduce the Thurston operator for the case where either the orbit of a turning point is finite or it is infinite converging to an essential periodic orbit. Assume that at least one turning point has infinite orbit. In order to get the strong semi-conjugacy as in Theorem 1 we introduce a modification on its associated proportions. We continue assuming hypotheses [H1]-[H4].

To define the finite-dimensional simplex W where the Thurston operator acts, we first "cut" the tails of the orbits of those turning points with infinite orbit. Before doing that, we analyze the local topological types of periodic attractors.

Let p be an attracting periodic point with prime period $n > 0$. Locally, the topological type of $O(p)$ can be one of the following cases:

A1. g^n is monotone increasing in p and p is a one-sided attractor.

A2. g^n is monotone increasing in p and p is a two-sided attractor. In Section 6.1, this case is divided in two sub-cases A2a and A2b depending, respectively, if there are turning points approaching p through only one or both sides of p.

A3. p is a turning point of $g^n(p)$ and therefore is a two-sided attractor.

A4. g^n is monotone decreasing in p and therefore p is a two-sided attractor.

6.1 "Cutting" infinite orbits. We deal with each case separately. Let p be as in case $A1$ and c be the nearest turning point to p which is in the connected component of $B_0(p)$ that contains p. The turning point c converges monotonically to p by g^n. So we can find $k > 0$ such that the interval $I = [g^{kn}(c), g^{(k+1)n}(c)]$ is a fundamental domain for the attracting periodic orbit $O(p)$ in the following sense: every turning point c' with infinite orbit converging to $O(p)$ intersects I at least once and at most twice[7]. For all turning points converging to $O(p)$ we will only consider their iterates until hitting I for the last time, forgetting the iterates after that. In this case we

[7] the latter case happens when an iterate of c' coincides with $g^{kn}(c)$.

define the interval $J = [g^{kn}(c), p]$ which will be important in the definition of the Thurston operator.

Let $O(p)$ be an attracting periodic orbit as in case $A2$ and, as before, let c be the nearest turning point to p which is in the connected component of $B_0(p)$ that contains p. Assume first that all turning points converging to p by g^n intersect the interval connecting c and p (case $A2a$). So we can find $k > 0$ such that all turning points with infinite orbit converging to $O(p)$ intersect the fundamental domain $I = [g^{kn}(c), g^{(k+1)n}(c)]$. We only consider the iterates of the turning points with infinite orbit converging to $O(p)$ until hitting I for the last time, forgetting the iterates after that. In this case let $J = [g^{kn}(c), p]$. Now assume we are still in case $A2$ but there are turning points with infinite orbits converging by g^n through both sides of p (case $A2b$). In the side that contains c we construct the fundamental domain $I_1 = [g^{k_1 n}(c), g^{(k_1+1)n}(c)]$ as in the previous case for an appropriate k_1. In the other side, we consider $k_2 > 0$ and a turning point c' such that any other turning point converging to p by g^n through the same side of c' intersects the fundamental domain $I_2 = [g^{(k_2+1)n}(c'), g^{k_2}(c')]$. Forget all the iterates of the turning points with infinite orbit converging to $O(p)$ after hitting $I_1 \cup I_2$. Let J be the smallest closed interval containing $I_1 \cup I_2$.

In case $A3$ it may happen that there is no other turning point in $B_0(p)$. There is one side of p such that orbits converge to p by g^n monotonically. So if necessary we choose a turning point c and $k > 0$ such that every turning point with infinite orbit converging to $O(p)$ intersects the fundamental domain $I = [g^{kn}(c), g^{(k+1)n}(c)]$ which is in that side of p. Forget the iterates of the turning points with infinite orbit converging to $O(p)$ after hitting I for the last time and let $J = [g^{kn}(c), p]$.

In the last case $A4$ let c be the nearest turning point to p in the connected component of $B_0(p)$ that contains p and $k > 0$ be such that every turning point with infinite orbit converging to $O(p)$ intersects the interval $I = [g^{2kn}(c), g^{2(k+1)n}(c)] \subset [c, p]$. Also we do not consider the iterates of the turning points after hitting I for the last time. Let $J = [g^{2kn}(c), p]$.

To define the simplex W we consider the following points in the interval:

1. all the iterates of the turning points which are periodic or eventually periodic.

2. the iterates of the turning points with infinite orbit only until hitting for the last time the corresponding fundamental domain I as explained before. Note that both endpoints of I are included.

3. all the periodic orbits which attract turning points.

So we have $k > 0$ points $z_1 < \ldots < z_k$ corresponding to the orbits of the turning points of g (after cutting the tails of the infinite orbits as described above) and the essential periodic attractors. Denote by $z_{t_1} < \ldots < z_{t_l}$ the corresponding turning points of g. Let $\pi : \{1, \ldots, k\} \backslash A \to \{1, \ldots, k\}$ be defined by $\pi(i) = j \Leftrightarrow g(z_i) = z_j$ whenever it is well defined. The set A consists of indexes where π is not defined and corresponds to the points in the interior of the intervals J constructed above. The Thurston map will act on $W = \{x = (x_1, \ldots, x_k) \in [0,1]^k : 0 < x_1 < \ldots < x_k < 1\}$ in the following way[8]: first find $\underline{a} \in \Delta$ realizing the critical values associated to x, that is, $F(\underline{a}) = (x_{\pi(t_1)}, \ldots, x_{\pi(t_l)})$. The turning points of $P_{\underline{a}}$ are the points $y_{t_1} < \ldots < y_{t_l}$, that is, $y_{t_i} = c_i(\underline{a})$ for all $1 \le i \le l$. For each j in the domain of π, $j \notin \{t_1, \ldots, t_l\}$ let m be such that $z_j \in [z_{t_m}, z_{t_{m+1}}]$ ($z_{t_0} = 0$ and $z_{t_{l+1}} = 1$). We

[8] see remark 5.1.

define y_j to be the point in $[y_{t_m}, y_{t_{m+1}}]$ such that $P_{\underline{a}}(y_j) = x_{\pi(j)}$. For $j \in A$ we do the following: z_j corresponds to the last point considered in the infinite orbit of a turning point or to a periodic point in the case A2b. It is in the interior of the interval J where $J = [z_r, z_s]$ is one of the intervals defined above. Now note that the points y_r and y_s are already defined so let $J' = [y_r, y_s]$ and $S : J \to J'$ be an affine map such that $S(z_r) = y_r$ and $S(z_s) = y_s$. We now define $y_j = S(z_j)$. We have defined y_j for all $1 \le j \le k$ so the Thurston operator $T : W \to W$ is defined by $T((x_1, \dots, x_k)) = (y_1, \dots, y_k)$.

A topological argument proves that since g satisfies hypotheses [H3]-[H4], then T has a fixed point in the interior of W which corresponds to a polynomial $P_{\underline{a}}$ with the same modality as g, but in the case A2b it may have a slightly different combinatoric. See [4].

6.2 Changing proportions. To construct a conjugacy between g and the polynomial $P_{\underline{a}}$ obtained in Section 6.1 it is necessary that not only the combinatorics of g and $P_{\underline{a}}$ are the same but also that each essential periodic attractor of g has the same local topological type of the corresponding periodic attractor of $P_{\underline{a}}$. We now show how to change the Thurston operator to get this correspondence.

Let $y = (y_1, \dots, y_k)$ be the fixed point of T and $P_{\underline{a}}$ be the corresponding polynomial. Let $J' = [z_r, z_s]$ be one of the intervals associated to an essential periodic orbit constructed in Section 6.1 and let $J = [y_r, y_s]$ be the corresponding interval for y. Then the following holds: for any $r \le i, j \le s$ the corresponding points in J' and J have the same proportions, i.e, $\frac{|z_i - z_j|}{|J'|} = \frac{|y_i - y_j|}{|J|}$. This follows from the fact that the Thurston operator is an affine map from J' to J.

Now note that if we change the relative positions of the points $\{z_r, \dots, z_s\}$ without changing their order, then we find a different fixed point for T corresponding to these new proportions. We now explain how to change these proportions in order to find $P_{\underline{a}}$ such that the corresponding attracting periodic orbits of g and $P_{\underline{a}}$ have the same type.

Let's start with case A4 and show that depending on the proportions chosen either we find the "correct" attractor or a "wrong" one. Let $h = P_{\underline{a}}^n$. Let $J = [y_r, y_s]$ where $y_r = h^{2k}(c)$ is as above, y_s is a fixed point of h and c is a turning point of $P_{\underline{a}}$ which is in the immediate basin of y_s and $[y_r, y_s] \subset [c, y_s]$. Let $y_m = h^{2(k+1)}(c) \in (y_r, y_s)$. Note that the position of y_m in J is determined by the position of z_m in $J' = [z_r, z_s]$, i.e., $d = \frac{|y_s - y_m|}{|y_s - y_r|} = \frac{|z_s - z_m|}{|z_s - z_r|}$. The following Lemma gives a criterion to choose d in order to get the same type of periodic attractor

Lemma 6.1 (Case A4). *There are constants $\bar{d} > 0$ and $\tilde{d} > 0$ which depend only on l (the modality of g), n (the period of y_s) and k (which defines the fundamental domain) such that (i) If $0 < d < \bar{d}$ then y_s is an essential periodic attractor of $P_{\underline{a}}$ with period n and $[c, y_s] \subset B_0(y_s)$. This implies that the periodic attractors of g and $P_{\underline{a}}$ have the same local type. Moreover, (ii) if $\tilde{d} < d < 1$ then y_s is a repelling periodic point and there is an attracting periodic point $\tilde{y} \in (y_m, y_s)$ with period $2n$ such that $[y_r, y_s) \subset B_0(\tilde{y})$. This implies that the corresponding periodic attractors of g and $P_{\underline{a}}$ are different and, therefore, g and $P_{\underline{a}}$ cannot be topologically conjugate.*

Proof. Let $f = h^2$. Then y_s is a fixed point of f and $f(y_r) = y_m$. Let $I_1 = [y_r, y_s]$ and $I_2 = [y_m, y_s]$. Note that $d = \frac{|I_2|}{|I_1|}$. For each $0 < d < 1$ we find a polynomial f_d corresponding to the fixed point of the Thurston operator T_d, points $c(d)$, $y_r(d)$,

$y_m(d)$, $y_s(d)$, and intervals $I_1(d)$ and $I_2(d)$. To simplify, we omit d in the notation. By construction, f is always strictly increasing in I_1 and to prove part (i) we must show that if d is small enough then y_s is a two-sided periodic attractor and $c \in B_0(y_s)$.

Let $I_3 = [y_m - 2|I_2|, y_m]$. For d small, $I_3 \subset I_1$ and since f is an increasing function in I_1 and $f(I_3) \subset I_2$ there exists a point $w \in I_3$ such that $0 \leq f'(w) < \frac{1}{2}$. If we take $d \to 0$ then $w \to y_s$. Now we recall from (3) that f'' is uniformly bounded. So we find $\bar{d} > 0$ such that w is near enough y_s and this implies that $|f'(y_s)| < 1$ for any $0 < d < \bar{d}$.

If part (ii) is not true we can assume by contradiction that for d arbitrarily close to 1, $f'(y_s) \leq 1$, $f([c, y_s]) \subset [c, y_s]$ and there is no fixed point of f in $[c, y_s)$. We enumerate the claims

1. We first claim that $|I_1| \to 0$ as $d \to 1$ otherwise from a subsequence of $d \to 1$ we would find in the limit a non-affine polynomial map $f_{\underline{a}}$ with an interval of periodic points with the same period which is impossible.

2. the turning point c cannot approach y_s for a subsequence of $d \to 1$ since $f'(y_s)$ would converge to 0 and this would imply that d would approach 0.

3. $f'(y_s) \to 1$ as $d \to 1$ otherwise if for a subsequence of $d \to 1$, we have $f'(y_s) \to \lambda$, $0 < \lambda < 1$, then, since $|I_1| \to 0$, $d \to \lambda < 1$, a contradiction.

4. $f(c)$ also cannot approach y_s for a subsequence of $d \to 1$ since this would imply the existence of points with derivative less than $\frac{1}{2}$ approaching y_s which is in contradiction with claim 3 because f'' is uniformly bounded.

5. claims 2,3,4 and the uniform bound of f'' imply that $f^k(c)$ cannot approach y_s as $d \to 1$ which is in contradiction with claim 1.

This shows that $f'(y_s) > 1$ for d close enough to 1. Since $f(c) > c$ and there exist at most 2 fixed points of f in $[c, y_s]$ we conclude that for d close enough to 1, y_s is repelling and there exists an attracting fixed point of f in (c, y_s) which contains $[c, y_s)$ in its basin, finishing the proof of the lemma. □

For instance, suppose g is unimodal ($l = 1$) and its turning point converges to an essential fixed point which reverses orientation. As explained in Section 6.1 the points considered are $0 < z_1 < z_2 < z_3 < z_4 < 1$ where z_1 is the turning point of g, $g(z_1) = z_4$, $g(z_4) = z_2$ and $z_3 = g(z_3)$ is the essential attracting fixed point of g. The simplex is defined by $W = \{x = (x_1, x_2, x_3, x_4) \in [0,1]^4 : 0 < x1 < x2 < x3 < x4 < 1\}$ and $\pi : \{1, 3, 4\} \to \{1, 2, 3, 4\}$ is given by $\pi(1) = 4$, $\pi(3) = 3$ and $\pi(4) = 2$. In this case $t_1 = 1$ and the Thurston operator $T : W \to W$ is defined as follows: given $x = (x_1, x_2, x_3, x_4) \in W$, let $a_1 = 4x_4$ and $P_{a_1}(x) = a_1 x(1 - x)$ (note that $P_{a_1}(\frac{1}{2}) = x_4$). Then $y_1 = \frac{1}{2}$, $y_4 \in (\frac{1}{2}, 1)$ is such that $P_{a_1}(y_4) = x_2$ and $y_3 \in (\frac{1}{2}, 1)$ satisfies $P_{a_1}(y_3) = x_3$. To define y_2 let $S : [z_1, z_3] \to [y_1, y_3]$ be an affine map with $S(z_1) = y_1$ and $S(z_3) = y_3$. Let $y_2 := S(z_2)$. Now $T(x) = (y_1, y_2, y_3, y_4)$. Iterating an initial condition we find a good approximation for the fixed point P of T which has the same combinatoric of g. This fixed point depends on the proportion $d = \frac{|z_3 - z_2|}{|z_3 - z_1|}$. For each value of d we find a different P and if we take d small enough as stated in Lemma 6.1 then, agreeing with g, P has a fixed point which attracts the turning point $c = \frac{1}{2}$. Otherwise, if $1 - d$ is small the corresponding fixed point is repelling and c converges to an attracting periodic orbit with prime period 2, so g and P cannot be conjugate. See Figure 1.

Now we consider case A2a where y_s is a two-sided attracting periodic orbit with period n, $h = P_{\underline{a}}^n$ is strictly increasing in p and orbits of the turning points con-

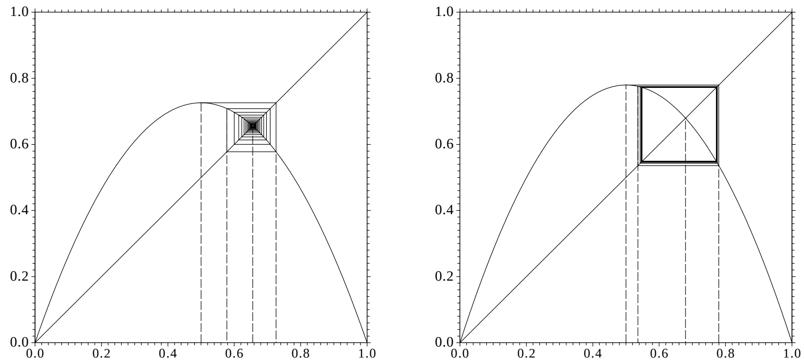

Figure. 1. An example of case A4. The dashed lines correspond to the fixed point of the Thurston operator. In (a) we obtain $P_{\mu_1}(x) \approx 2.903x(1-x)$ corresponding to $d = 0.5$; the critical point converges to the attracting fixed point. In (b), $P_{\mu_2}(x) \approx 3.118x(1-x)$ and $d = 0.8$; the critical point converges to the attracting periodic orbit of period 2. Although combinatorially equivalent, P_{μ_1} and P_{μ_2} are not conjugate.

verging to $O(p)$ approach through the same side of p. As before let $J = [y_r, y_s]$ where $y_r = h^k(c)$ is as above, y_s is a fixed point of h and c is a turning point of $P_{\underline{a}}$ which is in the immediate basin of y_s. Let $y_m = h^{k+1}(c) \in (y_r, y_s)$. The topological properties of h depend on the choice of the proportion $d = \frac{|z_s - z_m|}{|z_s - z_r|} = \frac{|y_s - y_m|}{|y_s - y_r|}$ as stated in the following

Lemma 6.2. (Case A2a). *There are constants $\bar{d} > 0$ and $\tilde{d} > 0$ which depend only on l (the modality of g), n (the period of y_s) and k (which defines the fundamental domain) such that (i) If $0 < d < \bar{d}$ then y_s is an essential periodic attractor of $P_{\underline{a}}$ with period n and $[c, y_s] \subset B_0(y_s)$. Moreover, (ii) if $\tilde{d} < d < 1$ then y_s is a repelling periodic point with prime period n and there is an attracting periodic point $\tilde{y} \in (y_m, y_s)$ with prime period n such that $[y_r, y_s) \subset B_0(\tilde{y})$.*

Proof. The proof is the same as in Lemma 6.1. □

Case A1 can be obtained by a dichotomy-like procedure of case A2a. In fact, varying d from 0 to 1 in a continuous path in the space of parameters Δ, the eigenvalue of h at y_s changes from less than 1 to bigger than 1. We have the following

Lemma 6.3. (Case A1). *There exists $0 < d^* < 1$ such that y_s is a one-sided attracting periodic orbit of $P_{\underline{a}}$ and the turning point c is in the connected component of $B_0(y_s)$ which contains y_s.*

Proof. By dichotomy we find $0 < d^* < 1$, such that $h'(y_s) = 1$. In fact, following the same ideas as those of Section 4.1, we can find a continuous path in the space of parameters Δ, corresponding to the fixed points of T_d, so that d varies continuously from close to 0 to close to 1. This gives the desired d^*. For such d^*, c is in the connected component of $B_0(y_s)$ which contains y_s. Since y_s attracts turning points only through one side and since h has negative Schwarzian derivative, y_s must be repelling in the other side of y_s. This implies that y_s is locally topologically equivalent to the corresponding one-sided attractor z_s of g. □

The following example covers cases A1 and A2a. Let g be a 2-modal map of the interval [0,1]. Suppose $g(0) = 0$, $g(1) = 1$ and $g(x) > x$ for all $x \in]0, 1[$. Assume that both turning points $0 < c_1 < c_2 < 1$ converge to the fixed point $x = 1$. Assume that $I = [g(c_2), g^2(c_2)]$ is a fundamental domain and $g(c_1) \in]g(c_2), g^2(c_2)[$. So we consider only the points $0 < z_1 < \ldots < z_5 < z_6 = 1$ where $z_1 = c_1$, $z_2 = c_2$, $z_3 = g(c_2)$, $z_4 = g(c_1)$ and $z_5 = g^2(c_2)$. In the cases A1 and A2a, the fixed point $x = 1$ is a one-sided and two-sided attractor, respectively. The function $\pi : \{1, 2, 3, 6\} \to \{1, \ldots, 6\}$ is defined by $\pi(1) = 4$, $\pi(2) = 3$, $\pi(3) = 5$ and $\pi(6) = 6$. Also $t_1 = 1$ and $t_2 = 2$. In this case $J = [z_3, 1]$ and $d = \frac{|1-z_5|}{|1-z_3|}$. The Thurston operator T acts on the open simplex $W = \{0 < x_1 < x_2 < \ldots < x_5 < 1\}$ in the following way: given $(x_1, \ldots, x_5)$ let $\underline{a} = (a_1, a_2)$ be such that the polynomial $P_{\underline{a}}(x) = a_1 x + a_2 x^2 + (1 - a_1 - a_2)x^3$ has critical values $v_1 = x_4$ and $v_2 = x_3$. Let $y_1 < y_2$ be the critical points of $P_{\underline{a}}$ and y_3 be the only point in $]y_2, 1[$ such that $P_{\underline{a}}(y_3) = x_5$. To determine y_4 and y_5 let $S : J \to [y_3, 1]$ be the affine map satisfying $S(z_3) = y_3$ and $S(1) = 1$. Finally, let $y_4 := S(z_4)$ and $y_5 := S(z_5)$. Iterations of this map approach the desired fixed point which corresponds to a polynomial also denoted by $P_{\underline{a}}$. According to Lemmas 6.2 and 6.3 there exists d^* such that if the initial choice of d satisfies $0 < d < d^*$ then $x = 1$ is a two-sided attractor of $P_{\underline{a}}$ attracting both its turning points and we are in case A2a. If $d = d^*$ we are in case A1 and $x = 1$ is a one-sided attracting fixed point with both turning points in its immediate basin. Also, if d is sufficiently close to 1, then $x = 1$ is repelling and there exists an attracting fixed point $\tilde{x} < 1$ which contains both turning points in its immediate basin. Note that in this particular example g is semi-conjugate to $P_{\underline{a}}$ for $0 < d \leq d^*$. This is because we are not interested in the behavior of $P_{\underline{a}}$ for $x > 1$. Figure 2 illustrates these results.

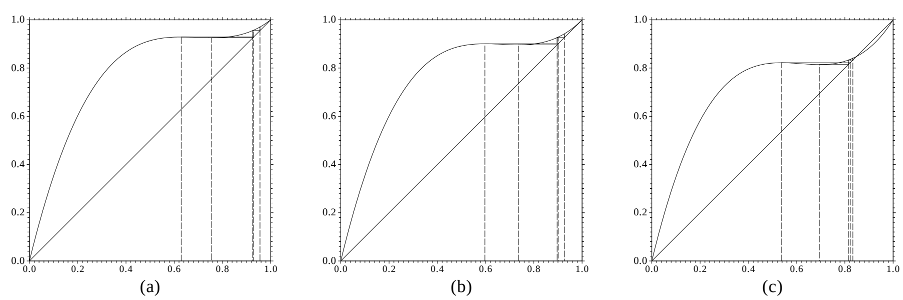

Figure 2. An example of cases A1 and A2a. The dashed lines correspond to the fixed point of the Thurston operator. In (a), for $d = 0.6$, we find $P(x) \approx 4.09x - 5.95x^2 + 2.87x^3$ and the fixed point $x = 1$ is a two-sided attractor. In (b), $d = 0.712$, $P(x) \approx 4.14x - 6.27x^2 + 3.14x^3$ and $x = 1$ is approximately a one-sided attracting fixed point. In (c), $d = 0.9$, $P(x) \approx 4.13x - 6.81x^2 + 3.69x^3$ and $x = 1$ is a repelling fixed point.

Now we treat Case A2b. Let $P_{\underline{a}}$ corresponds to the fixed point $y = (y_1, \ldots, y_k)$ of the Thurston operator and consider the same notations as before.

First note that y_s may not be a periodic point for $P_{\underline{a}}$ since its construction is not based on a pull-back argument but on the affine map S. This implies that if a turning point $\tilde{c}$ of g is eventually periodic and $g^{n_1}(\tilde{c}) = z_s$ for some $n_1 > 0$ then g and $P_{\underline{a}}$ may not even be combinatorially equivalent. We now show how to overcome this problem by choosing the proportions appropriately.

Let $I_1 = [y_r, y_m]$, $I_2 = [y_i, y_j]$, $J = [y_r, y_j]$ and $J' = [y_m, y_i]$ where the points are defined in Section 6.1. Note that y_s is contained in the interior of J'. Let $d_1 = \frac{|y_i - y_m|}{|y_i - y_r|} = \frac{|z_i - z_m|}{|z_i - z_r|}$ and $d_2 = \frac{|y_i - y_s|}{|J'|} = \frac{|z_i - z_s|}{|z_i - z_m|}$. We will show how to find $0 < d_1, d_2 < 1$ such that y_s is an attracting periodic orbit with prime period n and c is in the connected component of $B_0(y_s)$ which contains y_s. First we find d_1.

Lemma 6.4. (Case A2b). *There exists $d^* > 0$ which depends only on l (the modality of g) and n (the period of y_s) such that if $0 < d_1 < d^*$ then there exists a point $\tilde{y}$ in the interior of J' which is an essential periodic attractor of $P_{\underline{a}}$, has prime period n and $[c, y_j] \subset B_0(\tilde{y})$.*

Proof. The proof is identical to case (i) of Lemma 6.1. Let $h = P_{\underline{a}}^n$. Note that since $h(J) \subset J'$, $h(y_r) = y_m$ and $h(y_j) = y_i$ there must exist an attracting fixed point $\tilde{y}$ in the interior of J'. □

For arbitrary $0 < d_2 < 1$, $\tilde{y} \neq y_s$. Next Lemma shows how to choose d_2 in order that they coincide. We now do the same dichotomy procedure of Lemma 6.3 to choose d_2 appropriately. We have

Lemma 6.5. (Case A2b). *For $d_1 > 0$ sufficiently small as in Lemma 6.4, there exists $0 < d^* < 1$ such that if $d_2 = d^*$ then $y_s = \tilde{y}$, i.e., y_s is an attracting periodic orbit with prime period n attracting all the interval $[c, y_j] \subset B_0(y_s)$.*

Proof. We know that $\tilde{y}$ is the only fixed point of h in J'. Moreover, if $d_1 > 0$ is fixed sufficiently small, then there are constants $C_1, C_2 > 0$ such that $0 < C_1 < h'(\tilde{y}) < C_2 < 1$ for any $0 < d_2 < 1$. This implies that if d_2 is sufficiently small then $y_s > \tilde{y}$ otherwise the interval J should collapse for a subsequence of $d_2 \to 0$ forcing the critical point to approach $\tilde{y}$ contradicting the lower bound of $h'(\tilde{y})$. A similar argument implies that if d_2 is chosen sufficiently close to 1, $y_s < \tilde{y}$. Again, as in Lemma 6.3 we find the desired d^*. □

Remark 6.1. For $d_1 \to 1$ and $|I_2| \to 0$, h has 3 fixed points in J', one of them repelling and the others attracting.

Let us illustrate Case A2b with the following example. Let g be a 4-modal map of $[0, 1]$ satisfying $g(0) = 0$, $g(1) = 1$ and there is only one fixed point $\tilde{z} \in (0, 1)$ which attracts all the turning points $0 < c_1 < c_2 < \tilde{z} < c_3 < c_4 < 1$ of g. Assume also that

1. $g(x) > x$ if $x \in (0, \tilde{z})$ and $g(x) < x$ if $x \in (\tilde{z}, 1)$.
2. $I_1 = [c_2, g(c_2)]$ and $I_2 = [g(c_3), c_3]$ are fundamental domains in the left and right side of $\tilde{g}$ respectively. In this case $J = [c_2, c_3]$, $J' = [g(c_2), g(c_3)]$, $d_1 = \frac{|J'|}{|g(c_3) - c_2|}$ and $d_2 = \frac{|g(c_3) - \tilde{z}|}{|J'|}$.
3. $g(c_1) = c_3$ and $g(c_4) = \tilde{z}$.

In this situation there are 7 points considered $0 < z_1 < \ldots < z_7 < 1$ where $z_1 = c_1$, $z_2 = c_2$, $z_3 = g(c_2)$, $z_4 = \tilde{z}$, $z_5 = g(c_3)$, $z_6 = c_3$ and $z_7 = c_4$. The function $\pi : \{1, 2, 6, 7\} \to \{1, \ldots, 7\}$ is defined by $\pi(1) = 6$, $\pi(2) = 4$, $\pi(6) = 5$ and $\pi(7) = 4$. Let $t_1 = 1$, $t_2 = 2$, $t_3 = 6$ and $t_4 = 7$. The Thurston operator T acts on

$W = \{0 < x_1 < \ldots < x_7 < 1\}$ as follows: let $\underline{a} = (a_1, a_2, a_3, a_4)$ be such that the polynomial $P_{\underline{a}} = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + (1 - a_1 - a_2 - a_3 - a_4)x^5$ has 4 critical points $0 < \tilde{c}_1 < \ldots < \tilde{c}_4 < 1$ and $P_{\underline{a}}(\tilde{c}_i) = x_{\pi(t_i)}$ for $i = 1, \ldots, 4$. Let $y_{t_i} = \tilde{c}_i$, $i = 1, \ldots, 4$. To define y_3, y_4 and y_5 consider the affine map S satisfying $S(z_2) = y_2$ and $S(z_6) = y_6$. Let $y_i := S(z_i)$, for $i = 3, 4, 5$. Iterations of the Thurston map converge to a fixed point of T which corresponds to a polynomial also denoted by $P_{\underline{a}}$. We can choose d_1 small enough so that $P_{\underline{a}}$ has only one fixed point $\tilde{x} \in (0, 1)$ attracting all of its turning points also denoted by $0 < \tilde{c}_1 < \ldots < \tilde{c}_4 < 1$. Also, by a dichotomy procedure, we find $0 < d_2 < 1$ such that $P_{\underline{a}}(\tilde{c}_4) = \tilde{x}$. The results are depicted in Figure 3.

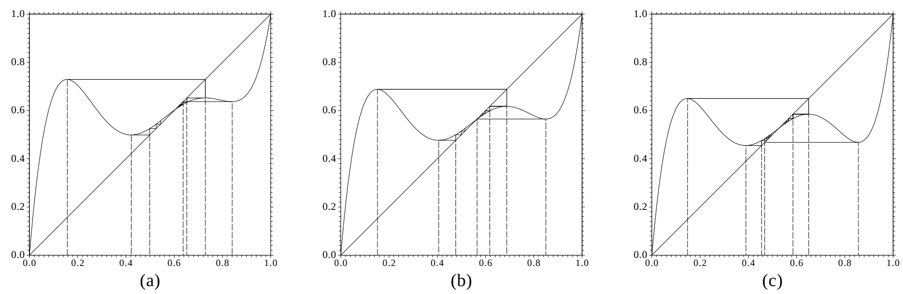

Figure 3. An example of case A2b. The dashed lines correspond to the fixed point of the Thurston operator. Fixing $d_1 = 0.67$, we have an attracting fixed point $\tilde{y}$ in $(0, 1)$ with all turning points in its immediate basin. In (a), $d_2 = 0.1$ and $y_s > \tilde{y}$. In (b), $d_2 = 0.375$ and $y_s \approx \tilde{y}$. In (c), $d_2 = 0.9$ and $y_s < \tilde{y}$.

In the case A3 since the corresponding attracting periodic points of g and $P_{\underline{a}}$ are turning points, they must be of the same local type and there is nothing else to do.

Remark 6.2. Choices of the proportions in the cases A1 and A2b must be done after all others cases. A dichotomy-like procedure in the choices of the corresponding proportions is needed in order to find $\underline{a}$ such that $P_{\underline{a}}$ satisfies simultaneously all the adjustments.

7. Proof of Theorem 1. First consider the finite and semi-finite combinatorics. To construct the topological conjugacy h it is necessary that g and $P_{\underline{a}}$ have the same combinatorics. Moreover, it is necessary that the corresponding attracting periodic orbits of g and $P_{\underline{a}}$ have the same type. Choosing appropriately the proportions for the semi-finite combinatorics, as explained in Lemmas 6.1-6.5, we can assume it is true. Let h_0 the the conjugation between $C^*(g)$ and $C^*(P_{\underline{a}})$, the sets of forward orbits of the turning points of g and $P_{\underline{a}}$, respectively. Extend h_0 to a conjugation h_1 defined in the corresponding basin of all attracting periodic orbits and the set of pre-images of all turning points. Since g and $P_{\underline{a}}$ satisfy [H1]-[H4], h_1 is defined in a dense set of the interval $[0, 1]$ and its image is also dense in $[0, 1]$. The conjugacy h is the continuous extension of h_1 in the interval $[0, 1]$.

In the infinite combinatorial case, it is also possible to find $P_{\underline{a}}$ which is combinatorially equivalent to g. In fact, there is a sequence $g_n \to g$ in the C^0 topology as $n \to \infty$ satisfying for all $n \geq 0$:

1. The first n iterates of the turning points of g_n and g coincide.

2. the maps g_n have semi-finite combinatoric for all $n \geq 0$.

For each g_n we find the corresponding $P_{\underline{a}_n}$. Note that if necessary we can choose appropriately the proportions so that the attractors of $P_{\underline{a}_n}$ and g with period less than n have the same local type.

In [4], it is shown that $P_{\underline{a}_n} \to P_{\underline{a}}$ as $n \to \infty$, where $P_{\underline{a}}$ is combinatorially equivalent to g. Also, by construction, all attracting periodic orbits have the same local type as those of g. The same steps done before define the conjugacy h between g and $P_{\underline{a}}$. This completes the proof of Theorem 1. □

8. Interpolating critical values. Thurston operator defined in Sections 5 and 6 is a contraction in the finite combinatorial case and it seems that the same holds for semi-finite combinatorics. Accordingly, in order to find a good approximation for its fixed point, which realizes the desired combinatoric, we only need to iterate an initial condition. In each iteration we face the following problem: given $(x_1, \dots, x_k)$, find $\underline{a} = (a_1, \dots, a_l)$ such that $P_{\underline{a}}(c_i(\underline{a})) = x_{\pi(t_i)}$ for each $i = 1, \dots, l$, where $c_i(\underline{a})$ are the critical points of $P_{\underline{a}}$. The values of $P_{\underline{a}}$ in the end points of the interval $[0, 1]$ are also determined by $g(0)$ and $g(1)$. As we have seen, there is only one such $\underline{a}$.

The following method gives an approach to it. Let $v_i = x_{\pi(t_i)}$, $i = 1, \dots, l$. Choose points $0 < x_1(0) < \dots < x_l(0) < 1$ and let P_1 be the polynomial with degree $l + 1$ interpolating $(0, g(0))$, $(x_i(0), v_i)$ and $(1, g(1))$ for $i = 1, \dots, l$. Note that P_1 can be found explicitly by Lagrange formula. It follows that P_1 has l turning points $0 < c_1(1) < \dots < c_l(1) < 1$ which can also be easily found. Now let $x_i(1) = c_i(1)$, $i = 1, \dots, l$. Repeat the interpolation with these new x_i's to find P_2 and so on.

Let S be the map $x_i(n) \mapsto x_i(n + 1) = c_i(n + 1)$, $i = 1, \dots, l$, defined in the simplex $W = \{x = (x_1, \dots, x_l) \in [0, 1]^l : 0 < x_1 < \dots < x_l < 1\}$. A fixed point $\tilde{x}$ of S corresponds to a solution of the interpolation problem. The existence of $\tilde{x}$ is given by the surjectiveness of $F : \Delta \to V$ defined in Section 4.1. A direct computation shows that the Jacobian matrix of S at $\tilde{x}$ is identically 0 and this implies a fast convergence to the fixed point $\tilde{x}$.

Remark 8.1. All figures presented in this paper were obtained by this method.

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