Topology and its Applications 359 (2025) 109086

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

Fixed point index bounds for self-maps on surfacelike complexes

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A R T I C L E   I N F O

Article history:
Received 2 November 2023
Received in revised form 24 March 2024
Accepted 26 March 2024
Available online 16 October 2024

MSC:
55M20
54H25
57M99
37B30

Keywords:
Fixed point index
Surface
Two-complex
Minimal map
Nielsen class

A B S T R A C T

For a certain family of aspherical 2-complexes it is shown that a pair of inequalities, known as hyperbolic index bounds, involving fixed point indices are satisfied for all fixed point minimal self-maps. As a corollary we verify the hyperbolic index bounds for the Nielsen fixed point classes of self-maps $f : X \to X$, when X is a finite wedge of compact surfaces each having non-positive Euler characteristic.

0. Introduction

The classical Poincaré-Birkhoff theory has lead to an interest in the fixed point behavior of mappings on 2-dimensional manifolds (see for example [2] or [15]). Consequently, many results dealing with fixed point indices for surfaces mappings have appeared in the literature, including the papers [1], [4], [9], [10], [11], [12], [13], [16], [18], [19], [20] and [24]. In particular, the papers [5], [9], [10], [11], [12], [13] and [24] present a number of results which establish global bounds for the indices of fixed points, and also for Nielsen classes. Some of these results have been generalized to the setting of higher dimensional manifolds. See [21], [22], [23] and references therein for related work.

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https://doi.org/10.1016/j.topol.2024.109086

In another direction one may consider 2-complexes as a generalization of surfaces. The paper [6] presents results regarding index bounds for a wedge of surfaces with a common wedge point. This paper continues the study of index bounds for a wedge of surfaces.

Let X be a compact topological space. In particular, a polyhedron, CW-complex, or ANR so that the space admits a fixed point index. See [3] for definition and properties of the fixed point index.

Let K denote a family of self-maps on X, together with a partition of the fixed points of each member. We say that K has the *index bounds property*, denoted by IBK, if there exist integers ℓ and u such that $\ell \leq \text{index}(f, P) \leq u$ holds for all maps $f \in K$ and each P a member of the partition on $\text{Fix}(f)$.

In the study of fixed point indices there are two natural families for consideration:

- The class of maps consists of all *fixed point minimal* maps. Those maps having the least number of fixed points possible among all maps in a given homotopy class. The partition classes are single fixed points. We denote by $MF[f]$ this minimal number for the homotopy class of f, and write IBMF for the bounds for this class.
- The class consists of all continuous maps and the Nielsen fixed point classes provide a partition. Given a map f a pair of fixed points are Nielsen related if there exists a path α joining the two points with $f(\alpha)$ homotopic to α rel endpoints. We write IBN for the bounds for these Nielsen classes. See the references [8], [14] for more information regarding Nielsen fixed point classes.

It is a result due to B. Jiang [7] that, for many polyhedra, property IBN is the same as property IBMF, simply because each homotopy class of maps has a representative where each essential Nielsen class is a single point. The exceptions to this occur either when X has local separating points or when X is a hyperbolic surface (i.e. a surface with negative Euler characteristic). The results given in this paper will make extensive use of this result of Jiang.

In the case of a hyperbolic surface index bounds were studied in [5], [9], [12], [13] and, in particular, the following global bounds were shown to be relevant:

$$\sum_{\text{index}(x)>1} (\text{index}(x) - 1) \leq 0 \ , \quad \text{and} \quad \sum_{\text{index}(x)<-1} (\text{index}(x) + 1) \geq 2\chi(F) \ .$$

These bounds were shown to hold in the two natural settings; (i) when x represents a Nielsen class, and (ii) when x is an isolated fixed point of a fixed point minimal map. We remark that these bounds hold trivially for the two closed surfaces with zero Euler characteristic; the torus and the Klein bottle. Motivated by these results we say that X has the *hyperbolic index bounds property*, denoted HIBK, for the class K if it satisfies the above inequalities for the given class K. It is proved in [9] that the HIBN property is also satisfied for graphs.

The main purpose of this paper is to generalize the methods used in [5] to a more general setting of 2-complexes. As a consequence we extend known results about hyperbolic index bounds for fixed point minimal maps, HIBMF. In particular, as a consequence of our HIBMF result we obtain a result regarding HIBN which is stated in Corollary 4.

We remark that in [11] the authors define the notion of *characteristic* of a Nielsen fixed point class and use this to give an improvement on the hyperbolic index bounds. Moreover, in doing so they show a connection between hyperbolic index bounds and the rank of the fixed subgroup of the induced endomorphism on fundamental group, which is either a free group or a surface group. This was further developed in [24]. It is natural to ask if the improved index bounds hold in the setting presented here, where the fundamental group is now a free product of surface groups. Work related to this can be found in [17].

The structure of the remainder of the paper is as follows. In Section 1 we present the variation on the method given in [5], where a large class K of maps is identified and shown to satisfy HIBK. This is stated

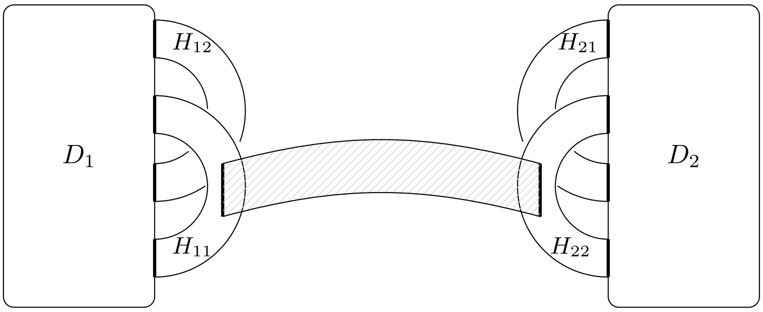

Fig. 1. A complex $X(B)$.

in Proposition 1. Due to the connection with the work in [5] it would be helpful for the reader to be familiar with methods developed in that paper. The proof of Proposition 1 is given in Section 2. It also follows from this variation that the HIBMF property holds for surfacelike complexes, obtained by joining surfaces with non-empty boundary using handles attached along the interior. See Corollary 1. Moreover, as these complexes do not have any local separating points, this corollary provides a new proof of the known result that the HIBN hold. See Corollary 2 and remark following its proof. Taking advantage of this variation, in Section 3 we adapt the relative version of the method given in [5] to the setting of 2-complexes and establish the HIBMF property for a family of 2-complexes, now involving closed surfaces. The result is stated in Theorem 1, which leads to the main result regarding HIBN stated in Corollary 4.

1. Family of (A,B)-transverse maps

The paper [5] studies self-maps of surfaces with non-empty boundary and defines class of surface maps, called *A-transverse* which satisfy hyperbolic index bounds when fixed points are grouped using a critical region graph. In this section we present a generalization of this notion for certain 2-complexes. We will then show that the hyperbolic index bounds can be recovered in a similar manner.

Let $F_1, \ldots, F_\ell$ be compact, connected surfaces with non-empty boundary ∂F_i and non-positive Euler characteristic. Exactly as in [5] we fix a handle structure on each F_i consisting of one 0-handle, which is a disk denoted by D_i, and $1 - \chi(F_i)$ 1-handles. These 1-handles are disks that are glued to the border of D_i along a pair of disjoint arcs, called attaching arcs. Let A denote the union of all the attaching arcs for the 1-handles, a total of $2(\ell - \sum \chi(F_i))$ pairwise disjoint arcs on the boundary of $\cup D_i$.

We now attach slightly different 1-handles. A *b-handle* attached to a space X is a disk with two preferred arcs on its boundary, with these arcs being joined by a homeomorphism to two given arcs in X.

Let B denote the union of $2\ell - 2$ pairwise disjoint arcs contained in the interiors of the handles for $\cup F_i$ satisfying the following conditions: (1) each handle contains at most one component of B, (2) two surfaces, say F_1, F_ℓ contain exactly one component of B and all others contain two components.

Attach $\ell - 1$ of these b-handles to B so that the resulting complex, denoted by $X(B)$, is connected. It follows from the construction that $X(B)$ does not contain any local separating points. Note also that the complex obtained by attaching a b-handle is no longer a surface. On the other hand, just as $\cup F_i$ is homotopy equivalent to a graph so is the complex $X(B)$. In fact, any finite collection of $b - handles$ attached to $\cup F_i$ is homotopy equivalent to a graph. The purpose for defining $X(B)$ is to help simplify the proof of Proposition 1 and to be general enough for the main results obtained in the last section of the paper.

The notion of an A-transverse map used in [5] is almost the same in this setting. We highlight here the distinction. A given map $f : X(B) \to X(B)$ will be $A - transverse$ if (1) the preimage curves $f^{-1}(A)$ are proper 1-manifolds when restricted to $\cup F_i$ and to each b-handle and (2) $f^{-1}(A) \cap B_0 = \emptyset$, where B_0 denotes the set of endpoints for the arcs in B. Thus globally the components of $f^{-1}(A)$ are one manifolds except at any point of intersection with B. At such points we have a vertex of valence 3 formed by an arc in $\cup F_i$ joining a half open interval in the b-handle.

We now consider the situation corresponding to B. By the same arguments as in [5] we can deform an A-transverse map f, without changing the fixed point set, so that (1) the result, which we still denote by f remains A-transverse, (2) $f^{-1}(B)$ is a 1-manifold inside of $\cup F_i$ and in each b-handle and (3) $f^{-1}(B) \cap B_0 = \emptyset$, and there are no fixed points on B.

But now, since components of B are not proper arcs, in the F_i (or a b-handle), a component of $f^{-1}(B) \cap F_i$ is either a simple closed curve or it is an arc, quiet likely not proper. As in the situation for $f^{-1}(A)$, at a point of intersection of $f^{-1}(B)$ with B we have a valence 3 vertex. We also assume so that if p is an endpoint of an arc in $f^{-1}(B)$, then $p \notin B$.

In summary, an A-transverse map f is said to be *(A,B)-transverse* if it satisfies the conditions above regarding $f^{-1}(B)$. The following analog of [5, Lemma 1] is proved in the same manner.

Lemma 1. *Given a fixed point minimal map $f : X(B) \to X(B)$ there is an (A, B)-transverse map homotopic to f with the same number of fixed points and the same fixed point indices.*

We now discuss the important notions of region and segment, leading up to the critical region graph $CRG(f)$. Let $\mathcal{R}$ denote the union of the set of components of $\cup F_i \setminus A$ together with the set of b-handles with B removed.

Given an (A, B)-transverse map f, a *region* is a component R of either $(\cup F_i) \setminus \big(A \cup f^{-1}(A \cup B)\big)$ or $J \setminus (f^{-1}(A \cup B))$, where J denotes a b-handle attached to B. Note that the role of B is not the same as that of A for this notion. This is done so that the methods given here align with those in [5].

By definition a region R is contained inside an element of $\mathcal{R}$. We observe that $f(R)$ is also contained inside an element of $\mathcal{R}$. If R and $f(R)$ are contained in the same element of $\mathcal{R}$ the region R is said to be *critical*.

Let $\bar{R}$ denote the closure of the region R. A component of $f^{-1}(B) \cap \bar{R}$ that has an endpoint contained in the interior of R is called a *null-segment*. These correspond to non-proper arcs in $f^{-1}(B)$. A *segment* is either a component of $f^{-1}(A) \cap \bar{R}$ that is an arc or a component of $f^{-1}(B) \cap \bar{R}$ that is an arc, but is not a null-segment. By a *cc-segment* we mean a component of $(f^{-1}(A) \cup f^{-1}(B)) \cap \bar{R}$ that is homeomorphic to S^1. We remark that a cc-segment from $f^{-1}(A)$ was considered in [5] and was called a segment in that paper. It was an observation that they had no impact on the index results (see proof of Lemma 3). In what follows we will see that the cc-segments coming from $f^{-1}(B)$ will play a different role.

We define a graph G_f associated to f called the *critical region graph*. The vertices of G_f are the critical regions, and two vertices are joined by an edge if and only if there is a point $p \in (A \cup B)$ common to both boundaries such that p and $f(p)$ lie in the same component A_0 of $A \cup B$. Let δ denote the union of segments with the point p being an endpoint. If $A_0 \subset A$ the two vertices lie on opposite sides of A_0 and on opposite sides of δ and the edge simply crosses A_0. If $A_0 \subset B$ the situation is similar. In D_i one side of δ is a critical region, and the other side maps into the b-handle attached to A_0. In this b-handle one side of δ is a critical region and the other side maps into D_i. So the critical region graph may be embedded in $X(B)$. Since f has no fixed points on $A \cup B$, critical regions have a well-defined fixed point index. The index of a component of G_f is the sum of the indices of its vertices.

Both Lemma 2 and Proposition 1 in [5] adapt readily to this setting. We recall that the version of Lemma 2 in [5] was proved by deforming the given map so that each component of the critical region graph has at most one vertex of nonzero index. As the corresponding vertices were (implicitly) disks there is a representative map which has exactly one fixed point in each such vertex, and no fixed points in the others. In the current setting there is more to check. See the proof of Lemma 3.

Lemma 2. *Let $f : X(B) \to X(B)$ be an (A, B)-transverse map and let M be the number of components of G_f that have a non-zero index. Then there is an (A, B)-transverse map g homotopic to f with G_g isomorphic to G_f and g has exactly M fixed points. In addition, the graph isomorphism preserves indices of components.*

Given a homotopy class of maps γ let $\mathcal{I}$ denote the collection of (A, B)-transverse maps f in γ such that G_f has exactly $\mathrm{MF}[\gamma]$ components with non-zero index. The class CRG consists of maps in $\mathcal{I}$ with the fixed points placed into collections determined by the components of the critical region graph.

Proposition 1. *For any 2-complex $X(B)$ as constructed above with each surface F_i having non-positive Euler characteristic, and for any $f \in \mathcal{I}$ the HIBCRG hold. That is*

$$\sum_{\mathrm{index}(C)\geq 1} (\mathrm{index}(C) - 1) \;\leq\; 0 \tag{1}$$

and

$$\sum_{\mathrm{index}(C)\leq -1} (\mathrm{index}(C) + 1) \;\geq\; 2\chi(X(B)) \;, \tag{2}$$

where C denotes a component of G_f.

Immediate from this proposition and Lemma 2 we have

Corollary 1. *The HIBMF property holds for all 2-complexes $X(B)$ where the surfaces each have non-positive Euler characteristic.*

Also from Proposition 1 we have the following corollary.

Corollary 2. *The HIBN property holds for the 2-complex $X(B)$.*

Proof. If $B = \emptyset$, then $X(B)$ is a surface with boundary. In this case the HIBN result was proved independently in [9] and [13]. When $B \neq \emptyset$, by the theorem of B. Jiang [7] we have that $MF = N$ for the complex $X(B)$, as this complex does not contain any local separating points. The result follows from Corollary 1. □

Remark. The result given in Corollary 2 was already known. The complex $X(B)$ and, more generally, any connected complex obtained by attaching b-handles to surfaces with nonempty boundary is homotopy equivalent to a graph, and hence a surface with boundary. So HIBN follows from [9], [13] and the homotopy type invariance of the Nielsen number.

The main reason for the choice of $X(B)$ was to take care of some technical difficulties in proving Proposition 1. Later in this paper we extend the class $X(B)$ to a larger class for which the proposition applies, but the reasoning given in the previous paragraph does not work.

The following terminology will be used in the proof of Proposition 1. Let ∂A denote the set consisting of the endpoints of the arcs in A. Consider a handle H (either a $0-$, 1-handle), which is bordered by the simple closed curve ∂H. Let η be a segment that is not null and which lies in H and let U_1, U_2 be the two components of ∂H with the endpoints of η removed. Define the *length of* η to be the cardinality of the smaller of the sets $U_1 \cap \partial A$ and $U_2 \cap \partial A$. Length for segments in a b-handle is defined in the same way as length in a 1-handle. We will not need the notion of length for cc-segments.

We divide regions into three types; either large, small or exceptional. A region is *large* if either it has a segment on its border with length at least 2 or the closure of the region meets all components of $(\partial H \setminus A)$, where H is the handle containing the region. Otherwise, the region is *small* or *exceptional*, where those that are exceptional is related to the arcs B and cc-segments. These will be given in context and studied in the sub-section at the end of this section. Outside of the exceptional regions the notions of large and small are the same as defined in [5] except that we have large (and small) regions in a b-handle. Two large critical regions are said to be *adjacent* if they meet a common component A_0 of $A \cup B$, and are in the same component of G_f, joined by a sequence of edges and vertices where each vertex corresponds to a small (or exceptional) critical region.

The notion of large and small segments does not change from [5]. Note that a segment with length one meets exactly one component of A. The segment η is *large* if either its length is greater than 1 or its length is exactly 1 and the following holds: $f(\eta)$ lies in the same component of A as the one that meets η, and η is on the border of a large critical region that is adjacent to another large critical region. A segment is *small* if it is not large.

We also define large and small for cc-segments. A cc-segment ω bounds a disk D_ω in the handle H. By definition of $X(B)$ we have that H is a 1-handle. We say ω is *large* if D_ω contains a component of B and *small* otherwise. If ω is large we say it is *contracting* if the image of ω is in a component of B that is contained in D_ω.

With these notions we have the analog of [5, Lemma 3] to be used for computing fixed point indices for critical regions. Given region R let H (as above) denote the handle for which $R \subset H$. We say that R *is enclosed by a cc-segment* if there is a cc-segment that separates R from ∂H in H.

Lemma 3. *Let f be (A, B)-transverse and let R be a non-exceptional critical region for f that is not enclosed by a cc-segment. Let $\alpha_1, \dots, \alpha_\ell$ denote the segments on the boundary of R. Suppose that for each i, $f(\alpha_i)$ is a single point and that each small cc-segment maps to a point outside of D_ω. Set $x_i = 0$ if $f(\alpha_i)$ and $R \setminus \alpha_i$ lie in the same component of $H \setminus \alpha_i$, otherwise set $x_i = 1$. Then*

$$\text{index}(R) = 1 - w - \sum_{i=1}^{\ell} x_i \ ,$$

where w denotes the number of large cc-segments on the border of R which have image in D_ω.

Proof. First note that f maps the null-segments to B without any fixed points so we may assume that these segments have been absorbed into the interior of R. Because of the assumptions in the first sentence of the statement the proof in the case when there are no large cc-segments on the border of R is exactly the same as in [5, Lemma 3].

Similarly, if a large cc-segment is mapped outside of D_ω the index can be computed by extending the map to include D_ω by mapping the disk outside, and so no fixed points are added. Finally if a large cc-segment maps into D_ω then the extension to D_ω adds a fixed point of index $+1$. Thus the index of R should see a corresponding decrease.

Here is an alternative proof following [5] that also shows that f can be deformed in R to have one fixed point, which can be removed in the case that $\text{index}(R) = 0$.

Consider a tree T in H which has one vertex v_0 in the interior of R, an edge joining v_0 to a vertex on each α_i, and an edge in each component of $H \setminus R$ ending at vertex $f(\alpha_i)$ in the case $x_i = 1$. In case $x_i = 0$ add edges as needed to $f(\alpha_i)$ and denote the union by T_1. Similarly, for each cc-segment ω that maps into D_ω add an edge from v_0 to ω and then to $f(\omega)$. Deform f so that it maps T into T_1. Let $r : R \to T$ be a retraction that sends each segment to a point. Then the map $f \circ r$ has only one fixed point v_0 and the index at v_0 is determined by the restriction to T_1, a graph self-map. This index is determined by the invariant expanding germs at v_0 and gives the desired formula. □

Remarks. (1) The statement of Lemma 3 is made to highlight the distinction between A and B. Alternately, one can define x_i for each cc-segment ($x_i = 1$ when ω maps into D_ω) and reach the conclusion that $\text{index}(R) = 1 - \sum x_i$. (2) In our setting w is either 0 or 1. It is certainly possible to have a complex with more than two components of B in a given handle and we state the lemma for the more general case. (3) Index for exceptional critical regions and those enclosed by a cc-segment will be treated later.

In order to verify the index bounds we will first need to make some adjustments to a given (A,B)-transverse map.

Suppose that $p \in A$ is such that p and $f(p)$ lie on the same component of A. If we deform f with support on a prescribed neighborhood of A so that $f^{-1}(A)$ remains unchanged as a set, and also all critical regions remain unchanged, then we get a new map f' which is in $\mathcal{I}$ as long as $f'(p) \neq p$. Moreover, $G(f') = G(f)$ and the index of each component is unchanged during the homotopy. As an illustration consider a small segment β which meets a point p as above and the arc I_β which joins the endpoints of β along $A \cup \partial F$. We arrange that β is mapped to a single point on $A \setminus I_\beta$ by a homotopy with support on a neighborhood of β. More generally, consider a connected finite union of small segments as above. Assuming that there is a point q on the component of A that is not contained in the union of the I_β we adjust so that the entire union of the small segments is mapped to q.

The second adjustment used is the notion of *coalescing* of segments. For this consider a pair of points p, q in $A \cap f^{-1}(A)$ that are adjacent along A_i and both are mapped by f to A_j. Let β_p, β_q be segments ending respectively at p, q, contained in the same component of $F \setminus A$, and where we assume these segments are not equal. This determines three regions; R_0 between the segments, and R_1, R_2 outside. Let δ be an arc in R_0 parallel to the subarc of A_i joining p, q with endpoints in β_p, β_q.

For our application we will also require that if R_0 is critical and $i = j$, then in G_f the vertex R_0 only joins across A_i a vertex of valence 1 corresponding to a small region.

Now apply a homotopy with support on a small neighborhood of δ which has the effect of joining the two segments. That is, viewing δ as vertical and β_p, β_q as horizontal, remove small arcs from each of β_p, β_q and replace with vertical arcs parallel to δ, and contained in R_0.

A variation on the coalescing move is to have δ as above, but now joining a segment to the boundary of F.

Let g denote the end of the coalescing homotopy. We now consider the difference between the graphs G_f and G_g. When R_0 is critical, then the assumption made above will imply that G_f and G_g differ by at most a single component which corresponds to small critical regions. By the adjustment above this component will have index zero. If R_0 is not critical, then this move will join R_1 and R_2 into a single region R'. If these regions are critical, then the index of R' is the sum of the two indices, and moreover, the index of the component of G_g containing R' is the sum of the indices of the components containing R_1 and R_2. Hence, by Lemma 2 if f is fixed point minimal, then one of these two components must have zero index. Thus, the coalescing move may change the graph, but assuming fixed point minimal the number of fixed points and their indices are unchanged.

Two applications of these adjustments are the following lemmas. The first lemma concerns small critical regions and gives control on the location of fixed point index. The second lemma concerns large critical regions and will be used to obtain the bounds on the index in the components of G_f.

For the following, since exceptional has not been defined, we take small to be regions that are not enclosed by a cc-segment and are not large. Small regions have the following feature. Let R be a small critical region in handle H (contained in $\cup F_i$). It meets at most one component A_0 of A and also at most one component ∂_0 of $\partial H \setminus A$. Let I_R be the subarc of $A_0 \cup \partial_0$ determined by R and let $\alpha(R)$ be the segment on the border of R such that $I_R = I_{\alpha(R)}$. Hence $\alpha(R)$ separates the interior of R from all other attaching arcs in H. A similar statement holds for small regions in a b-handle.

Consider a small critical region R and the component $K(R)$ of G_f that contains R. We say that R (or $K(R)$) is *small isolated* if each vertex of $K(R)$ corresponds to a small critical region.

Lemma 4. *Given a fixed point minimal (A, B)-transverse map there is another (A, B)-transverse map f in its homotopy class, which has the same fixed point data and such that for each small critical region R (1) if R is not small isolated, then* $\text{index}(R) = 0$ *and has a segment on its border that contributes $x_i = 1$ to its index calculation, or (2) if R is small isolated, then* $\text{index}(K(R))$ *is in* $\{-1, 0, 1\}$.

Proof. Following the methods from [12], which are used in the proof of Lemma 2 to show that each component of G_f has at most one fixed point, if R is not small isolated the fixed point will be in a large region. Thus each small region has zero index and a segment with $x_i = 1$.

Now suppose that R is small isolated.

Given R, $K(R)$ meets at most one component A_0 of A and possibly one component B_0 of B with small regions in the corresponding b-handle.

We first consider the case when $K(R)$ contains segments that map to A_0. Consider the following reduction move. Let η denote a small segment meeting A_0, is mapped by f to A_0, is contained in a 0-handle and is an outermost such segment.

Since η is in a 0-handle all critical regions inside of $\eta \cup I_\eta$ that do not have a segment mapping to A_0 on the border have zero index. Apply the first adjustment to arrange that the others also have zero index. Now successively apply a coalescing move along A_0 to the segments inside $\eta \cup I_\eta$ to form a new collection $K(R')$, where we ignore the regions inside $\eta \cup I_\eta$ enclosed by cc-segments. Clearly, $\text{index}(K(R')) = \text{index}(K(R))$.

The reason for η being outermost is now used. Consider the two regions, one just outside of η and one just inside of η. Exactly one is critical. If the one inside of η is critical then the same coalescing move can be used to reduce η. On the other hand if the index of the region outside R_0 is non-zero the reduction move will split the component of G_f into two components. Hence changing index. Now, using the hypothesis that $K(R)$ is isolated we have that R_0 is small and bordered by a small segment γ. Using the first adjustment we arrange that R_0 has zero index and that η can be reduced without changing index.

Apply the above reduction to each such outermost η to obtain a new configuration which preserves index and has the property that $K(R)$ does not contain a segment mapping to A_0. We remark that the reduction process may convert small segments to large. Consider for example a proper arc in a neighborhood of A_0 meeting A_0 in exactly 2 points. A reduction of the middle third converts 3 small segments to 1 large segment. This idea was also used in [5].

So we assume that $K(R)$ remains small isolated without segments mapping to A_0 and verify statement (2) of the lemma. We assume that R is contained in the 1-handle that also contains B_0. In this case $K(R)$ consists of a unique small critical region R, possibly meeting A_0, joined by $f^{-1}(B_0)$ to small critical regions in the b-handle containing B_0. Let A_1 denote the other attaching arc, where $A_1 \cap R = \emptyset$. We also take into consideration cc-segments on the border of R as there relation to B_0 impacts index calculation.

The computation of $\text{index}(R)$ will use Lemma 3 and is divided into two cases. Recall that $\alpha(R)$ separates the interior of R from A_1 in the 1-handle.

Case 1: A segment or cc-segment from $f^{-1}(B_0)$ meets B_0.

Thus there exist critical regions in the b-handle. These regions form a finite pairwise disjoint collection along B_0 that does not cover all of B_0. Use the first adjustment to map each segment from $f^{-1}(B_0)$ to a point b_0 that is not covered, and as a result each region in the b-handle has zero index. Choosing b_0 to be in the interior of R as well it follows that outside of α_R all other segments / cc-segments contribute $x_i = 0$ to the index calculation. Hence, $\text{index}(K(R)) = \text{index}(R) \in \{0, 1\}$.

Case 2: $f^{-1}(B_0) \cap B_0 \cap R = \emptyset$. So no critical regions in the b-handle.

If B_0 is on the same side of α_R as A_1, then α_R contributes $x_i = 1$ and all others $x_i = 0$. Thus index$(K(R)) = 0$. Otherwise B_0 is enclosed by a cc-segment or one of the segments on the border of R. Denote by β. Any segment / cc-segment on the border of R other than β or $\alpha(R)$ contributes $x_i = 0$ to the index calculation and so index$(K(R)) \in \{-1, 0, 1\}$. □

Lemma 5. *Let g be a fixed point minimal (A, B)-transverse map. Then g is homotopic to an (A, B)-transverse map f that has the same fixed point data as g such that associated to each pair of adjacent large critical regions there are two large segments, one contributes $x_i = 0$ to the index calculation of the region it borders and the other contributes $x_i = 1$ to its corresponding region.*

Proof. Large critical regions and segments are the same as in [5], except that we now include large regions in a b-handle. So the proof of Lemma 5 given in [5] adapts without change. □

1.1. Exceptional critical regions

Here we list and discuss the exceptional critical regions, most of which involve cc-segments. Recall that a cc-segment ω is large if D_ω contains a component of B, otherwise it is small. Let B_ω denote the unique component of B contained in D_ω. We often assume that small cc-segments have non-empty intersection with B.

(i) critical region bounded by a small cc-segment ω. By applying the first adjustment, as needed, we arrange that ω maps outside of D_ω. Thus, such critical regions, as in Lemma 4, have index zero. The purpose of such regions is to allow the CRG to bridge across b-handles.
(ii) critical region bounded by a large cc-segment ω. This region is D_ω and has index either 0 or 1 depending on the image of ω. As these regions are isolated in the CRG they will be ignored for computing index bounds.
(iii) critical region bounded by a large cc-segment ω and small cc-segments. This important case is treated below in Lemma 6.
(iv) critical region bounded by two large cc-segments. These have index in $\{-1, 0, 1\}$ and are isolated in the CRG. This is very similar to the last case in the proof of Lemma 4. As in (ii) these regions will be ignored for computing index bounds.
(v) large critical region with a contracting large cc-segment ω on its boundary. The index calculation appears in Lemma 3 and we note that if ω was not contracting we can arrange for image outside of D_ω, thus not contributing to the index calculation.

By definition of B distinct components of B are in distinct handles. Thus large cc-segments are naturally nested about components of B. So associated to a component B_0 of B there are m_0 large cc-segments, with an *innermost* and an *outermost*. In case (iii) above ω is the innermost while in case (v) it is outermost.

We now make a closer analysis related to case (iii). This will lead to an adjustment on a fixed point minimal (A, B)-transverse map f that will be exploited in the proof of Proposition 1.

Consider an innermost ω with D_ω containing component B_0 of B with B_0 contained in the b-handle H_0. Let R be the region in D_ω which has ω on its border.

First consider the case when R is not critical. So no edge of G_f will traverse ω. Consider a sequence of coalescing moves along arcs in B_0 that removes all the small cc-segments that intersect B_0. In general such a homotopy can impact indices in the CRG, but in the present setting if two (large) critical regions in H_0 are joined by a critical region in D_ω, then after the homotopy the two regions merge into a single (large) critical region. Thus a component of G_f meeting H_0 is sent to a component of $G_{f'}$ with the same index.

Any critical region in D_ω becomes isolated inside a cc-segment that does not meet B_0 and has zero index. Hence, after a sequence of these adjustments there are no segments or cc-segments meeting B_0.

We remark that the previous case did not require the curve ω. If R was a large region that enclosed B_0 the above proof applies as well.

We now assume that R is critical. If R does not meet a small cc-segment mapping to B_0 then the argument in the previous case can be applied as R is isolated from the vertices of G_f inside H_0. Its index is either 0 or +1. So we assume that there exists such a small cc-segment.

We can assume that the image of ω is either in A or B_0. If $\omega \mapsto A$, then by the first adjustment we assume that the small cc-segments map to a point outside of each. Then R has zero index and the argument in the first case can be applied. Finally, if $\omega \mapsto B_0$ choose an arc in R from ω to one of the small cc-segments and apply a coalescing move along the arc. This converts the two cc-segments into a single small cc-segment with the region outside being non-critical. Thus reducing to the first case above.

We summarize the above with the following lemma

Lemma 6. *Let g be a fixed point minimal (A, B)-transverse map. Then g is homotopic to an (A, B)-transverse map f that has the same fixed point data as g such that a component B_0 of B that is enclosed by a cc-segment satisfies $f^{-1}(A \cup B) \cap B_0 = \emptyset$.*

2. Proof of Proposition 1

The purpose of this section is to verify the two inequalities in the statement of Proposition 1.

Proof of Proposition 1. Let $f \in \mathcal{I}$ and let G_f be the CRG as defined in the previous section. Without loss we may assume that $f \in \mathcal{I}$ satisfies the conclusion of Lemmas 4 and 5. Let C be a component of G_f with v vertices, v_l of which correspond to large critical regions.

Equation (1) of the proposition is equivalent to showing that the index of a component of G_f is at most one. By Lemma 4 each small region contributes zero to the index. By Lemma 5 there are at least $v_l - 1$ segments with $x_i = 1$ corresponding to the large regions in C.

Note that all exceptional critical regions, where Lemma 3 does not apply, either have index zero or are isolated with index in $\{-1, 0, 1\}$ and can be removed from consideration.

Thus, following Lemma 3 (or remark afterwords)

$$\text{index}(C) = \sum \left(1 - \sum x_i\right) \ ,$$

where the outer sum is over the large vertices in C and the inner sum is over the segments on the border of a critical region. The right hand side is equal to $v_l - \sum x_i$, with the sum now over all x_i corresponding to large regions C. But this is less than or equal to $v_l - (v_l - 1)$, which establishes equation (1).

We now verify equation (2). If R is a small region in C, then by Lemma 4 its index is zero and so does not contribute to the index calculation. We ignore these regions. In this case $\text{index}(C) = \sum(1 - x_i)$, where the sum is taken over the large critical regions in C. Also, if R is large, then each small segment on its border contributes $x_i = 0$ to the index calculation.

So the conclusion of Lemma 3 can be stated as

$$\text{index}(R) = 1 - l_1(R) - w(R) \ ,$$

where $l_1(R)$ counts the number of large segments with $x_i = 1$ and $w(R)$ is as defined in Lemma 3.

Let $l_0(R)$ denote the number of large segments with $x_i = 0$ and $l(R)$ the total number of large segments on the border of R.

Summing over the large regions corresponding to C we have

$$\begin{aligned}\text{index}(C) &= \sum_R (\ 1 - l_1(R) - w(R)\ ) = \sum_R (\ 1 + l_0(R) - l(R) - w(R)\ ) \\ &= v_l + \sum_R l_0(R) - \sum_R l(R) - \sum_R w(R) \\ &\geq v_l + (v_l - 1) - \sum_R l(R) - \sum_R w(R)\ ,\end{aligned}$$

where the inequality above is a result of Lemma 5. Hence,

$$\text{index}(C) + 1 + \sum_R w(R) \geq \sum_R (\ 2 - l(R)\ )\ .$$

Now let $\mathcal{C}$ be any collection of components of G_f. Then,

$$\sum_{C\in\mathcal{C}} (\text{index}(C) + 1) + \sum_R w(R) \ \geq\ \sum_R (2 - l(R))\ , \tag{3}$$

where the summation on the right is over all large regions corresponding to $\mathcal{C}$.

In [5] one obtains the topological inequality $\sum_R (l(R) - 2) \leq -2\chi(F)$.

We now generalize to the setting of the complex $X(B)$ where $(\ell - 1)$ b-handles are attached to ℓ surfaces. The proof of the equation above given in [5] reduces the problem to the case where all large critical regions meeting the boundary are critical. Here it is shown that, after moving all length 1 large segments into the 1-handles, that $\sum(l(R) - 2)$ is at most -2 in the 0-handle and at most 2 in each of the 1-handles. Note that to get equality in a 1-handle all four length 1 segments must appear. Either (i) there is just one critical region with $l(R) = 4$ or (ii) there are two critical regions each with $l(R) = 3$ and all others have at most two large segments on the border. Now consider a b-handle. Such a handle meets exactly two components of B just as a 1-handle meets exactly two components of A. So the structure of large segments and regions is the same. Thus, $\sum(l(R) - 2) \leq 2$ in each b-handle. Moreover, whenever a component of B is disjoint from $f^{-1}(A \cup B)$ this summation drops by 1. Let N denote the number of such components of B. Then summing over all R we get

$$\begin{aligned}\sum (l(R) - 2) &\leq -2\ell + 2\sum(1 - \chi_i) + 2(\ell - 1) - N \\ &= 2\sum(1 - \chi_i) - 2 - N = 2(\ell - 1 - \sum \chi_i) - N\end{aligned}$$

Hence

$$\sum_R (l(R) - 2) \leq -2\chi(X(B)) - N\ . \tag{4}$$

Putting together equations (3) and (4) yields

$$\sum_{C\in\mathcal{C}} (\text{index}(C) + 1) + \sum_R w(R) \ \geq\ 2\chi(X(B)) + N \tag{5}$$

From Lemma 6 it follows that $N \geq \sum w(R)$, which establishes equation (2). □

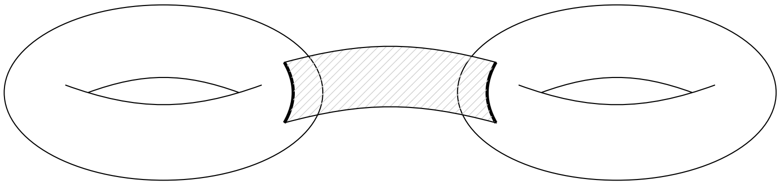

Fig. 2. A complex $Y(B)$, the result of attaching disks to the two boundary curves to X(B) in Fig. 1.

In [12, Proposition 6.1] the additional hypothesis that for a given self-map $f : F \to F$ there is a boundary component of the surface F with null homotopic image leads to a difference of 2 between (index$(C)+1$) and $2\chi(F)$. This adapts directly to each of the surfaces $F_1, \dots, F_\ell$ forming $X(B)$.

Suppose that $g : X(B) \to X(B)$ has the property that K of the ℓ surfaces each contain a boundary component with null homotopic image. We obtain the following improvement on equation (2) of Proposition 1.

Proposition 2. *With $f \in \mathcal{I}$ and K as above*

$$\sum_{\text{index}(C)\leq -1} (\text{index}(C)+1) \ \geq \ 2\chi(B) + 2K \ ,$$

where C denotes a component of G_f.

3. Extension to closed surfaces

In this section we revisit the relative version of the method as introduced in [5] and apply it to the natural extension of the complexes $X(B)$ to the setting where the surfaces F_i may be closed. The main application of this will be to verify the HIBMF property for these complexes. The result appears at the end of this section in Theorem 1.

As a consequence we obtain the main result of this paper given in Corollary 4, which verifies the hyperbolic index bounds for the Nielsen classes of self-maps on a wedge of surfaces.

Construct a family $Y(B)$ of 2-complexes as follows. Let $F_1, \dots, F_\ell$ be compact, connected surfaces each with non-positive Euler characteristic. A given F_i may or may not have boundary. Let B denote the union of $2\ell - 2$ pairwise disjoint arcs contained in the interiors of the F_i so that each F_i contains exactly two arcs in B except F_1 and F_ℓ, which contain one arc each. Attach $\ell - 1$ b-handles to B so that the resulting complex, denoted by $Y(B)$, is connected. Just as with $X(B)$ this complex has no local separating points (Fig. 2). Note that we have not yet prescribed a handle structure for the F_i. This will be done soon.

For a given F_i that is a closed surface consider a disk d_i in F_i. We assume that $d_i \cap B = \emptyset$. Let F_0 denote the union of all such d_i, which is bordered by a collection of inessential simple closed curves which we denote by λ. Let $f : F \to F$ be given and suppose that there are no fixed points on F_0 and that $f^{-1}(\lambda)$ is a 1-manifold that does not intersect F_0. Throughout this section we assume that all maps have this property. In particular a given fixed point minimal map can be deformed to have this property for a suitable choice of F_0.

We also assume all homotopies are *relative to* F_0, that is, constant on both F_0 and its preimage. For a suitable choice of F_0 this assumption, as in [5], is consistent with the lemmas used in proving Proposition 1. As the homotopies in these lemmas have support in a neighborhood of a 1-dimensional set (an arc in either $A \cup B$ or $f^{-1}(A \cup B)$ choosing F_0 small, i.e. $(F_0 \cup f(F_0)) \cap (A \cup B) = \emptyset$ will ensure that homotopies used in the proof are relative to the set F_0.

Let S denote the closure of $Y(B) \setminus F_0$. Consider a handle structure for S consisting of a single 0-handle in each F_i and a total of $\sum(1 - \chi(F_i))$ 1-handles in the various F_i. For now we assume that each component of B lies in the interior of a handle so that b-handles are consistent with given handles. We use the symbol D to denote a 0-handle, and A will denote the union of all the attaching arcs for the various 1-handles.

Given this setting we have the analogous notion of an (A, B)-transverse map. The following lemma is the analog of Lemma 1 for this relative setting. The proof is exactly the same.

Lemma 7. *Let F_0 be as above. Given a fixed point minimal map $f : Y(B) \to Y(B)$, relative to F_0, there is an (A, B)-transverse map homotopic to f rel F_0 with the same number of fixed points and the same fixed point indices.*

Given the handle structure above and an (A, B)-transverse map as above the notions *region*, *critical region* and *segment* are as in [5] (closed surface case). We also have *cc-segments* as given in this paper. These lead to the corresponding critical region graph for f in S, denoted G_f. Assuming $\lambda \cap B = \emptyset$ the notion of *singular segment* and *null* critical region is exactly as given in [5] and the issue of index computation is done in the same way and given in the following lemma, which is the adaptation of [5, Lemma 7].

Lemma 8. *Given a self-map of $Y(B)$ there is a fixed point minimal representative f and a union of disks F_0, with one disk in each F_i that is closed, such that $F_0 \cap B = \emptyset$ and $f^{-1}(F_0)$ is a finite set of disks, each disjoint from F_0. Moreover, if $f^{-1}(F_0)$ is non-empty there is an associated handle structure of S such that*

(1) *singular segments have length zero and all have endpoints contained in a single component of A,*
(2) *if an edge in G_f corresponds to a point that is an endpoint of a singular segment, then the singular segment borders a null critical region,*
(3) *singular segments never intersect small segments,*
(4) *components of B lie in distinct handles.*

Proof. As the disks in F_0 are small neighborhoods of points it is easy to arrange $F_0 \cap B = \emptyset$. The proof is now the same as in [5], with the exception of statement (4). We recall that the 1-handles are constructed in the proof as neighborhoods of certain proper arcs with endpoints in λ. As a component of B is an arc in the interior of surface F_i it is possible to choose these proper arcs so that one or two of them, as needed, will contain the component(s) of B that lie in a given F_i. □

As noted in [5] this lemma is consistent with the three lemmas used to prove index bounds for closed surfaces [5, Proposition 3]. This applies just as well in the current setting of the complex $Y(B)$. In addition, Lemma 6 applies in this case. In the same way we obtain the following relative version of the index bounds for certain critical region graphs.

Proposition 3. *For any $g : Y(B) \to Y(B)$ that satisfies the conclusion of Lemma 8 above, the HIBCRG hold. That is*

$$\sum_{\text{index}(C)\geq 1} (\text{index}(C) - 1) \ \leq \ 0 \tag{6}$$

and

$$\sum_{\text{index}(C)\leq -1} (\text{index}(C) + 1) \ \geq \ 2\chi(S) \ , \tag{7}$$

where C denotes a component of G_g.

We now return to fixed point minimal maps on the complex $Y(B)$.

As a consequence of Proposition 3 we obtain the following generalization.

Theorem 1. *The HIBMF property holds for the complex* $Y(B)$.

Proof. Let $f : Y(B) \to Y(B)$ be a fixed point minimal self-map. Without loss we may assume that f satisfies the conclusion of Lemma 8 for some collection of disks F_0. Since f is minimal, just as in Lemma 2, each component of G_f has one fixed point if its index is nonzero and no fixed points if its index is zero. Equation (6) of Proposition 3 automatically gives the desired upper bound. Equation (7) of the proposition gives a lower bound of $2\chi(S)$, where S was obtained by removing the interior of the disks in F_0. Since the image of the border of each disk is clearly null homotopic, by Proposition 2 we obtain a lower bound of $2\chi(S) + 2K$, where K is the number of closed surfaces in the construction of $Y(B)$. But this is equal to $2\chi(Y(B))$. □

This theorem when combined with the result of B. Jiang [7] that $MF = N$ for the complex $Y(B)$ yields the following corollary.

Corollary 3. *The HIBN property holds for* $Y(B)$.

Finally, as a consequence of this corollary we obtain the main result of the paper.

Corollary 4. *The HIBN property holds for a finite wedge of surfaces, each with non-positive Euler characteristic.*

Proof. Each such wedge of surfaces is homotopy equivalent to some $Y(B)$. The result follows from the homotopy type invariance property in Nielsen theory. □

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