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On the Problem of Jaśkowski and the Logics of Łukasiewicz.

by *J. KOTAS and N. C. A. da COSTA.*

ABSTRACT. In this paper we extend the results of I. M. L. D'Ottaviano and N. C. A. da Costa, *Sur un problème de Jaśkowski*, C. R. Acad. Sc. Paris, 270 A (1970), 1349-1353; by means of the *generalized logics of Łukasiewicz*, new solutions of the so called Jaśkowski's problem are presented and discussed.

Introduction.

Let $\mathcal{L}$ denote a given language containing negation. Any set of formulas of $\mathcal{L}$ is called a propositional system (or simply, a system) of $\mathcal{L}$. The elements of a system S are called theses of S.

A system S is said to be inconsistent if it contains at least two theses, such that one is the negation of the other; in the opposite case, S is called consistent. S is said to be trivial (or overcomplete) if any formula of $\mathcal{L}$ is also a thesis of S; otherwise, S is nontrivial (or not overcomplete).

A propositional system which has an underlying logic, i. e., that is based on a logic, is called a deductive system. If in such system the rule: "From α and not-α, infer β" is permissible, then it is inconsistent if and only if it is trivial. This is precisely what happens with deductive systems based on classical logic and on several other categories of logics, as, for example, the intuitionistic.

Of course, trivial systems have no practical importance. But the situation is completely different in connection with consistent systems, which are of fundamental relevance from the theoretical as well as from the practical points of view. Nonetheless, there are relatively few systems of which we really know that they are consistent. We cannot give absolute proofs of consistence even for certain rather elementary mathematical systems. The situation is still worse when we consider systems based on results of experimental research. These facts and other stronger reasons motivated Jaśkowski to formulate in [8] the problem: To construct logics satisfying the following conditions: 1) when they are employed as underlying logics of inconsistent proposi ionaltsystems, inconsistency does not necessarily imply triviality; 2) they are rich enough to make possible most common inferences.

In [8] and [9], Jaśkowski presented a solution for his problem at the level of the propositional calculus: in effect, he introduced the so called *discussive propositional calculus* $\mathbf{D}_2$, defined by means of an appropriate interpretation in the modal system **S5** of Lewis. Therefore, Jaśkowski's solution depends on standard modal logic. Results by Furmanowski (see [7]), Perzanowski (see [12] and Błaszczuk and Dziobiak (cf. [1], [2] and [3]) showed that we obtain very interesting solutions to Jaśkowski's problem using other modal systems instead of **S5** . In [5], a completely distinct solution to the problem is described, where a hierarchy of logical systems not restricted to the propositional calculus is studied and developed up to the construction of inconsistent but apparently nontrivial set theories. D'Ottaviano and da Costa in [6] studied still another solution, the calculus $\mathbb{J}_3$, which is founded on Łukasiewicz three-valued logic.

In every-day life the process of assertion of sentences is very complex. We may suppose that we consider any sentence as true when our conviction about its truth is strong enough or, is other words, when a "logical value" sufficiently large corresponds to the sentence. Clearly, the assertion of a sentence as probable is made analogously. If we restrict ourselves to $\mathbf{D}_2$ or $\mathbb{J}_3$, then we have to consider as probable all sentences which have (in our conviction, of couse) a "logical value" greater than 0, and moreover we are constrained to assume that "to be probable" and "to be true" have practically the same meaning. This attitude, although convenient and full of interesting consequences, is only a crude approximation to the actual procedure. Also, we do not look at sentences which we believe to have sufficiently small "logical values" as probable. To introduce finer distinction than it is possible with the help of $\mathbf{D}_2$ and $\mathbb{J}_3$, we study in this paper some new logical calculi.

In fact, the aim of our work is to show that if we take as bases the finite or infinite logics of Łukasiewicz, then it is possible to define a class of logical calculi, some of which are solutions of Jaškowski's problem. Therefore, this paper contains a generalization of D'Ottaviano – da Costa's results. It seems worthwhile to observe that the method of characterizing logical calculi employed here and applied to Lukasiewicz logics, can be extended to other logical calculi.

JAŠKOWSKI'S PROBLEM AND THE LOGICS OF ŁUKASIEWICZ.

Let M be a finite or infinite-valued matrix of Łukasiewicz, i.e., a matrix of the form:

$$\langle A, \{1\}, \Rightarrow, \neg \rangle,$$

where $A = \{0, 1/n, \ldots, n-1/n, 1\}$, $n \geqslant 1$, or A is the set of all real numbers x such that $0 \leqslant x \leqslant 1$, and the operations $\Rightarrow$ and $\neg$ are defined as usual. Let $M_{a,b}$, $0 < a \leqslant 1$, $0 < b \leqslant 1$, be the matrix obtained from M by the addition to its two operations, of the new operations $\cup$, $\cap$, and C_a defined as follows: for any $x,y \in A$, $x \cup y = \max(x,y)$, $x \cap y = \min(x,y)$, $C_a x = 0$ for $x < a$ and $C_a x = 1$ for $x \geqslant a$, and by replacing $\{1\}$ by the set of all $x \in A$ such that $x \geqslant b$. $M_{a,b}$ will be called a *generalized matrix of Łukasiewicz*, and may be a finite or an infinite-valued matrix.

We note that $\cup$ and $\cap$ are definable in terms of $\Rightarrow$ and $\neg$ in any matrix of Łukasiewicz, since for any $x,y \in A$, we have: $x \cup y = (x \Rightarrow y) \Rightarrow y$ and $x \cap y = \neg(\neg x \cup \neg y)$, but that C_a is definable only in finite-valued matrices.

Let p, q, r, ... be propositional variables and $\rightarrow$, $\vee$, $\wedge$, $\sim$, and $\diamond_a$, $0 < a \leqslant 1$, be the symbols of implication, disjunction, conjunction, negation and a-possibility, respectively. $\mathbb{F}$ will denote the set of formulas defined in the usual manner, and α, β, γ, ... will be variables whose values are formulas.

The set of all formulas valid in $M_{a,b}$, $0 < a \leqslant 1$, $0 < b \leqslant 1$, symbolized by $\mathfrak{L}_{a,b}$, will be called a *generalized logic of Łukasiewicz*. In the case that $M_{a,b}$ is a finite-valued matrix the notion just defined coincides with the concept of a generalized logic of Łukasiewicz, as introduced in [14] by Rosser and Turquete. If $M_{a,b}$ is a finite-valued matrix, then $\mathfrak{L}_{a,b}$ can also be defined as a *discussive* system, to wit, $\mathfrak{L}_{a,b}$ can be interpreted as the set of all formulas α such that $\Diamond_b\alpha$ is valid in M.

It is easy to verify that in those systems in which all sentences having "logical values" at least equal to b, $0 < b \leqslant 1$, are considered as true, that is, as having distinguished values, the implication of Łukasiewicz ($\rightarrow$) does not have some of the fundamental properties commonly associated with the notion of implication; for instance, the rule of modus ponens is not valid. This rule can be applied in connection with $\rightarrow$, only if the sentences considered true have the value 1. Intuitively, this means that assertion has an absolute character: we accept as true only sentences about which we are absolutely confident that they are true. Obviously, such condition is almost never satisfied in empirical systems. In particular, concerning the logic $\mathfrak{L}_{a,b}$, where $a < b$ and $M_{a,b}$ is finite-valued, it occurs that there are formulas α and β such that α and $\alpha\rightarrow\beta$ are valid, but β is not. To obtain the formulas α and β, we can make use of the criterion of the definability of functions in matrices of Łukasiewicz given by McNaughton in [11] or the criterion formulated by Prucnal in [13].

Since the rule of modus ponens plays an important role in deductive systems, the question arises, whether we can define in $\mathfrak{L}_{a,b}$ a binary operation which could be accepted as an implication. We want, especially, that the rule of detachement (modus ponens), relative to such operation, when applied to formulas of $\mathfrak{L}_{a,b}$, would always give formulas belonging to $\mathfrak{L}_{a,b}$. For this purpose, we proceed precisely as Jaśkowski in [8], where he defines the *discussive implication*; we extend the language of $\mathfrak{L}_{a,b}$ by the addition of the following operation, which it is natural to call a-discussive implication (or simply discussive implication):

DEFINITION 1. $\alpha \underset{a}{\rightarrow} \beta =_{\mathrm{def}} \Diamond_a\alpha \rightarrow \beta$.

This definition is analogous to Jaśkowski's definition of discussive implication in $\mathbf{D}_2$, and has a similar meaning. If $\Diamond_a$ is interpreted as possibility, then $\underset{a}{\rightarrow}$ coincides with discussive implication.

The logic obtained from $\mathfrak{L}_{a,b}$ by extending its language with the addition of $\underset{a}{\rightarrow}$, according the above definition, will still be denoted by the symbol $\mathfrak{L}_{a,b}$.

THEOREM 1. *If* $a \leqslant b$, *then the following rules of inference are permissible in* $\mathcal{L}_{a,b}$:

(1) *If* α *and* $\alpha \underset{a}{\rightarrow} \beta$, *then* β.

(2) *If* α, *then* $\beta \underset{a}{\rightarrow} \alpha$.

(3) *If* $\alpha \underset{a}{\rightarrow} \beta$ *and* $\beta \underset{a}{\rightarrow} \gamma$, *then* $\alpha \underset{a}{\rightarrow} \gamma$.

(4) *If* α *and* β, *then* $\alpha \wedge \beta$.

(5) *If* $\alpha \wedge \beta$, *then* α; *if* $\alpha \wedge \beta$, *then* β.

(6) *If* $\alpha \underset{a}{\rightarrow} \beta$ *and* $\alpha \underset{a}{\rightarrow} \gamma$, *then* $\alpha \underset{a}{\rightarrow} \beta \wedge \gamma$.

(7) *If* α, *then* $\alpha \vee \beta$; *if* β, *then* $\alpha \vee \beta$

(8) *If* $\alpha \underset{a}{\rightarrow} \gamma$ *and* $\beta \underset{a}{\rightarrow} \gamma$, *then* $\alpha \vee \beta \underset{a}{\rightarrow} \gamma$.

PROOF of (1): Let us suppose that $\alpha, \alpha \underset{a}{\rightarrow} \beta \in \mathcal{L}_{a,b}$, i.e., for every valuation ϑ, we have $\vartheta(\alpha) \geqslant b$ and $\vartheta(\alpha \underset{a}{\rightarrow} \beta) \geqslant b$. Then, we have also $\vartheta(\beta) \geqslant b$, because $\vartheta(\alpha \underset{a}{\rightarrow} \beta) = \vartheta(\Diamond_a \alpha \rightarrow \beta) = \vartheta(\Diamond_a \alpha) \Rightarrow \vartheta(\beta) = C_a \vartheta(\alpha) \Rightarrow \vartheta(\beta) = 1 \Rightarrow \vartheta(\beta) = \vartheta(\beta)$.

The proofs of (2)-(8) are similar. ■

It is rather interesting that, if $a < b$, then the formulas corresponding to the rules of inference of Theorem 1 do not belong to $\mathcal{L}_{a,b}$ (we assume, for example, that $(p \wedge (p \underset{a}{\rightarrow} q)) \underset{a}{\rightarrow} q$ corresponds to rule (1) and that $p \underset{a}{\rightarrow} (q \underset{a}{\rightarrow} p)$ corresponds to rule (2)). Usually those formulas are considered as characteristic of a good implication. Thus, the implication $\underset{a}{\rightarrow}$ apparently cannot find fundamental applications in deductive systems, because the set of formulas of $\mathcal{L}_{a,b}$ in which $\underset{a}{\rightarrow}$ occurs is "poor". However, it is known that in the application of logical calculi to propositional systems, the rules of the calculi are more important than their theses. Then, Theorem 1 says that the implication $\underset{a}{\rightarrow}$ is actually not so weak, and that $\mathcal{L}_{a,b}$ is apt to be used as underlying logic of deductive systems (in which the basic implication is $\underset{a}{\rightarrow}$ and not $\rightarrow$).

DEFINITION 2. *For any* $\alpha, \beta \in \mathbb{F}$,

$\alpha \approx \beta$ *if and only if* $\alpha \underset{a}{\rightarrow} \beta \in \mathfrak{L}_{a,b}$ *and* $\beta \underset{a}{\rightarrow} \alpha \in \mathfrak{L}_{a,b}$.

THEOREM 2. $\approx$ *is an equivalence relation if and only if* $a = b$. *Moreover, under this hypothesis, if* $\alpha_1 \approx \beta_1$ *and* $\alpha_2 \approx \beta_2$, *then* $\alpha_1 \vee \alpha_2 \approx \beta_1 \vee \beta_2$, $\alpha_1 \wedge \alpha_2 \approx \beta_1 \wedge \beta_2$ *and* $\alpha_1 \underset{a}{\rightarrow} \alpha_2 \approx \beta_1 \underset{a}{\rightarrow} \beta_2$, *but it is not true that* $\alpha_1 \rightarrow \alpha_2 \approx \beta_1 \rightarrow \beta_2$ *and* $\sim\alpha_1 \approx \sim\alpha_2$.

PROOF: If $a < b$, then $\approx$ is not reflexive, because $p \underset{a}{\rightarrow} p$ does not belong to $\mathfrak{L}_{a,b}$, and if $a > b$, then $\approx$ is not transitive, owing to the fact that it is not true that if $\alpha \underset{a}{\rightarrow} \beta \in \mathfrak{L}_{a,b}$ and $\beta \underset{a}{\rightarrow} \gamma \in \mathfrak{L}_{a,b}$, then $\alpha \underset{a}{\rightarrow} \gamma \in \mathfrak{L}_{a,b}$. (It follows from Prucnal's criterion that in the case of a finite-valued generalized Łukasiewicz logic we can find formulas α , β and γ such that $\alpha \approx \beta$ and $\beta \approx \gamma$, but not $\alpha \approx \gamma$.) Supposing $a = b$, $\alpha \approx \beta$ is true if and only if for any valuation ϑ one has $\vartheta(\alpha) < a$ and $\vartheta(\beta) < a$, or $\vartheta(\alpha) \geqslant a$ and $\vartheta(\beta) \geqslant a$. Therefore, it follows that in this case the relation $\approx$ has the properties required by the theorem. ■

We observe that $\alpha \underset{a}{\rightarrow} \beta \in \mathfrak{L}_{a,b}$, with $a \geqslant b$, if and only if, for any valuation ϑ, we have $\vartheta(\alpha) < a$ or $\vartheta(\beta) \geqslant b$. Consequently:

THEOREM 3. *If* $a \geqslant b$, *then the following formulas belong to* $\mathfrak{L}_{a,b}$:

(i) $p \underset{a}{\rightarrow} (q \underset{a}{\rightarrow} p)$,

(ii) $(p \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} ((q \underset{a}{\rightarrow} r) \underset{a}{\rightarrow} (p \underset{a}{\rightarrow} r))$,

(iii) $((p \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p$,

(iv) $p \vee q \underset{a}{\rightarrow} ((p \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} q)$,

(v) $((p \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} p \vee q$,

(vi) $p \wedge q \underset{a}{\rightarrow} p$,

(vii) $p \wedge q \underset{a}{\rightarrow} q$,

(viii) $p \underset{a}{\rightarrow} (q \underset{a}{\rightarrow} p \wedge q)$.

$\mathfrak{L}_a$, $0 < a \leqslant 1$, will designate the set of all formulas of $\mathfrak{L}_{a,a}$ in which, besides propositional variables and parentheses, only the symbols $\neg, \vee, \wedge$, and $\underset{a}{\rightarrow}$. occur. Let $\mathfrak{L}_a^+$, $0 < a \leqslant 1$, be the subset of $\mathfrak{L}_a$ containing all formulas in which negation does not appear. $\mathfrak{L}_a^+$ will be called the positive part of $\mathfrak{L}_a$.

It is a consequence of Theorem 1 that the rule of detachement for $\underset{a}{\rightarrow}$ is permissible in $\mathfrak{L}_a^+$. Evidently, $\mathfrak{L}_a^+$ is a set of formulas closed under substitution (limited to the set of formulas of the language $\mathfrak{L}_a^+$). According to a result of Sobociński [16], the formulas listed in Theorem 3 together with substitution and detachement constitute an axiomatization for the classical positive propositional calculus, which, by Theorems 1 and 3, is contained in $\mathfrak{L}_a^+$. But if $\alpha \in \mathfrak{L}_a^+$, then α is a thesis of the classical propositional calculus and, eo ipso, of the classical positive propositional calculus, since for 0 and 1 the operations of $M_{a,b}$, corresponding to the logical connectives $\vee, \wedge$ and $\underset{a}{\rightarrow}$, have the same values as the appropriate operations of the two-valued classical matrix. Hence, we have proved the following:

THEOREM 4. *$\mathfrak{L}_a^+$, $0 < a \leqslant 1$, is the classical positive propositional calculus.*

The algebraic version of the classical positive propositional calculus is constituted by the notion of classical implicative lattice (as well as by the concept of a Boolean ring; see [4] and [15]). Denoting by t_a^+, $0 < a \leqslant 1$, the *algebra of formulas* $\mathfrak{L}_a^+$, we deduce from Theorems 2 and 4 that:

COROLLARY 1. *The quotient algebra $t_a^+/\approx$, $0 < a \leqslant 1$, is a classical implicative lattice.*

It is not difficult to prove the following proposition:

THEOREM 5. *Suppose that $a < 1/2$; then, the following formulas are not theses of $\mathfrak{L}_a$* $(\alpha \underset{a}{\leftrightarrow} \beta =_{\mathrm{def}} (\alpha \underset{a}{\rightarrow} \beta) \wedge (\beta \underset{a}{\rightarrow} \alpha))$:

$p \underset{a}{\rightarrow} (\sim p \underset{a}{\rightarrow} q)$, $\quad (p \wedge \sim p) \underset{a}{\rightarrow} q$,

$(p \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} ((p \underset{a}{\rightarrow} \sim q) \underset{a}{\rightarrow} \sim p)$, $\quad (p \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} (\sim q \underset{a}{\rightarrow} \sim p)$,

$(p \underset{a}{\leftrightarrow} q) \underset{a}{\rightarrow} (\sim p \underset{a}{\leftrightarrow} \sim q)$, $(p \vee (q \wedge \sim q)) \underset{a}{\leftrightarrow} p$,

$(p \underset{a}{\rightarrow} q) \underset{a}{\leftrightarrow} \sim p \vee q$, $(p \underset{a}{\rightarrow} q) \underset{a}{\leftrightarrow} \sim (p \wedge \sim q)$.

Theorem 4 shows that $\mathfrak{L}_a$, $0 \leqslant a \leqslant 1$, is a rather rich calculus, since its positive part coincides with the classical positive propositional calculus. However, $\mathfrak{L}_a$, $0 < a < 1$, is not rich enough so as to make it impossible to found on it inconsistent nontrivial systems; on the contrary, by Theorem 5, $\mathfrak{L}_a$, $0 < a < 1/2$, can be used as underlying logic for those systems. Hence, we have:

COROLLARY 2. *Every $\mathfrak{L}_a$, $0 < a < 1/2$, constitutes a solution to Jaśkowski's problem.*

We shall denote by $M_{a,b}^{(2k)}$ the $2k$-valued generalized matrices of Łukasiewicz; it is clear that $1/2$ does not belong to the set of elements of $M_{a,b}^{(2k)}$. $\mathfrak{L}_{a,b}^{(2k)}$ symbolizes the set of valid formulas in $M_{a,b}^{(2k)}$.

LEMMA 1. (i) $p \underset{a}{\rightarrow} (\sim p \underset{a}{\rightarrow} q) \in \mathfrak{L}_{a,b}^{(2k)}$ *if and only if* $a \geqslant 1/2$;

(ii) $(\sim p \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p \in \mathfrak{L}_{a,b}^{(2k)}$ *if and only if* $b \leqslant a \leqslant 1/2$ *or* $a + b \leqslant 1$ *and* $a > 1/2$.

PROOF of (i): In this case $a \geqslant 1/2$. Hence, for any valuation ϑ, if $\vartheta(p) < a$, then $(p \underset{a}{\rightarrow} (\sim p \underset{a}{\rightarrow} q)) = 1$, and if $\vartheta(p) \geqslant a$, then also $\vartheta(p \underset{a}{\rightarrow} (\sim p \underset{a}{\rightarrow} q)) = 1$, because $\vartheta(\sim p) = 1 - \vartheta(p) < 1 - a \leqslant a$. Conversely, supposing that $a < 1/2$, it follows that for a valuation ϑ which satisfies the conditions $a \leqslant \vartheta(p) < 1/2$ and $\vartheta(q) = 0$, we have: $\vartheta(\sim p) = 1 - \vartheta(p) > 1/2 > a$ and $\vartheta(p \underset{a}{\rightarrow} (\sim p \underset{a}{\rightarrow} q)) = 0$; hence, $p \underset{a}{\rightarrow} (\sim p \underset{a}{\rightarrow} q)$ does not belong to $\mathfrak{L}_{a,b}^{(2k)}$.

PROOF of (ii): Admit that $b \leqslant a \leqslant 1/2$; then, we have for any valuation ϑ: $\vartheta((\sim p \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p) = 1$ for $\vartheta(p) < a$, and $\vartheta((\sim p \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p) \geqslant a \geqslant b$ for $\vartheta(p) \geqslant a$. If $a > 1/2$ and $a + b \leqslant 1$, then we have, for any valuation ϑ: $\vartheta((\sim p \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p) \geqslant b$ for $\vartheta(p) \leqslant 1 - a$, and $\vartheta((\sim p \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p) \geqslant 1 - a \geqslant a \geqslant b$ for $\vartheta(p) > 1 - a$. We can directly verify that $(\sim p \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p \notin \mathfrak{L}_{a,b}$ for values of a and b different from those especified in the lemma . ■

The proof of the following propositions offers no difficulty:

LEMMA 2. *The formulas*

(i) $p \vee q \underset{a}{\leftrightarrow} (\sim p \underset{a}{\rightarrow} q)$,

(ii) $p \wedge q \underset{a}{\leftrightarrow} \sim (p \underset{a}{\rightarrow} \sim q)$,

belong to $Ł^{(2k)}_{1/2,\,1/2}$.

It is known that $(p \underset{a}{\rightarrow} q) \underset{a}{\rightarrow} ((q \underset{a}{\rightarrow} r) \underset{a}{\rightarrow} (p \underset{a}{\rightarrow} r))$, $p \underset{a}{\rightarrow} (\sim p \underset{a}{\rightarrow} q)$ and $(\sim p \underset{a}{\rightarrow} p) \underset{a}{\rightarrow} p$, together with the rules of substitution and modus ponens, constitute an axiomatization for the classical propositional calculus with implication and negation as the sole primitive connectives. In order for the above formulas to belong to $Ł_{a,b}$ and for the rules of substitution and of modus ponens to be permissible in $Ł_{a,b}$, we must have: $a = b = 1/2$ and $M_{a,b}$ has $2k$ elements, as Theorems 1 and 3, and Lemma 1 show. Therefore, the classical propositional calculus with implication and negation as the sole primitive connectives is contained in $Ł^{(2k)}_{1/2,\,1/2}$. But from Lemma 2 we deduce that *all* classical propositional calculus is included in $Ł^{(2k)}_{1/2,\,1/2}$. Obviously, if $\alpha \in Ł^{(2k)}_{1/2,\,1/2}$, α is a thesis of the classical propositional calculus, since the operations of $M^{(2k)}_{1/2,\,1/2}$ which corresponds to $\vee, \wedge, \underset{a}{\rightarrow}$ and $\sim$ have the same values for 0 and 1 as the analogous operations of two-valued classical matrix. Consequently, $Ł^{(2k)}_{1/2,1/2}$ is contained in the classical propositional calculus. Thus, we proved the following:

THEOREM 6. $Ł^{(2k)}_{1/2,\,1/2}$ *is the classical propositional calculus.*

As we have already noted, the connective $\diamond_a$, $0 < a \leqslant 1$, is definable in any finite-valued logic $Ł^{(n)}$ of Łukasiewicz. Then, if we have an axiomatization for $Ł^{(n)}$, we have also an axiomatization for this logic enriched by the definition of $\diamond_a$: both are essentially the same. Rosser and

Turquette, in [14], were the first to give an axiomatization for the finite-valued generalized logics of Łukasiewicz. Now we present another, formally simpler, axiomatization of such logics.

A proof that the finite-valued logic of Łukasiewicz, $\mathfrak{L}$, which is the set of all formulas valid in the matrix $\langle [0,1], \{1\}, \Rightarrow, \neg \rangle$, was given by Wajsberg (see [10]).

The axioms of Wajsberg are:

A_1. $p \to (q \to p)$,

A_2. $(p \to q) \to ((q \to r) \to (p \to r))$,

A_3. $((p \to q) \to q) \to ((q \to p) \to p)$,

A_4. $(\sim p \to \sim q) \to (q \to p)$,

and the primitive rules are substitution and modus ponens.

Tokarz proved in [17] that for every natural number n, $n > 1$, there exists a formula α_n, which is called axiom of Tokarz, such that A_1-A_4, α_n and substitution and modus ponens form an axiomatics for $\mathfrak{L}^{(n)}$.

Let **A** be the following set of formulas:

(i) $\Diamond_1(p \to (q \to p))$,

(ii) $\Diamond_1((p \to q) \to ((q \to r) \to (p \to r)))$,

(iii) $\Diamond_1(((p \to q) \to q) \to ((q \to p) \to p)))$,

(iv) $\Diamond_1((\sim p \to \sim q) \to (q \to p))$,

(v) $\Diamond_1 \alpha_n$, where α_n is the axiom of Tokarz,

and let **R** be the set of rules (1)-(4) bellow:

(1) Substitution,

(2) If $\Diamond_1 \alpha$ and $\Diamond_1 (\alpha \to \beta)$, then $\Diamond_1 \beta$,

(3) If $\Diamond_1 \alpha$, then α,

(4) If $\Diamond_a \alpha$, then α $(0 < a \leqslant 1)$.

THEOREM 7. *$\mathfrak{L}^{(n)}_{a,b}$ can be axiomatized by taking* **A** *as the set of axioms and* **R** *as the set of rules.*

PROOF: Obviously, the calculus based on the axioms of **A** and on the rules of **R** is contained in $\mathfrak{L}^{(n)}_{a,b}$. We have to prove that the converse inclusion holds. Assume that $\alpha \in \mathfrak{L}^{(n)}_{a,b}$; hence, there exists a finite sequence of formulas

$$\beta_1, \beta_2, \beta_3, \ldots, \beta_n,$$

which is a proof of $\Diamond_a \alpha$ in the axiomatics of Tokarz, referred to above. It is easy to see that the sequence

$$\Diamond_1\beta_1, \Diamond_1\beta_2, \ldots, \Diamond_1\beta_n, \beta_n, \alpha$$

is a proof of α in the calculus whose axioms belong to **A** and whose rules of inference are members of **R**. Therefore, the theorem is proved.

OPEN QUESTIONS.

Concluding our paper, we present some open problems:

PROBLEM 1. Is $\mathfrak{L}_{a,b}$, when it is based on the infinite-valued Łukasiewicz logic, axiomatizable?

PROBLEM 2. Are there axiomatizations of $\mathfrak{L}_{a,b}$ in which the sole primitive connectives are $\vec{a}$, $\rightarrow$, $\wedge$, $\vee$ and $\sim$? (In this problem, $\mathfrak{L}_{a,b}$ may be a finite or infinite-valued logic.)

PROBLEM 3. What results of this paper can be extended to generalized logics of Łukasiewicz with quantification ? (Evidently, *some* of our results can easily be adapted to the level of the predicate calculus.)

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