Special Groups and Quadratic Forms over Rings with non Zero-Divisor Coefficients

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Abstract

In this paper we present an algebraic theory of diagonal quadratic forms with *non zero-divisor coefficients* over preordered (commutative, unitary) rings $\langle A, T \rangle$, where 2 is invertible and the preorder T satisfies a mild additional requirement (2.8). In § 5 we prove that several major results known to hold in the classical theory of quadratic forms over fields —e.g., the Arason-Pfister Hauptsatz, and Pfister's local-global principle— carry over to any class of preordered rings satisfying the property of $\mathfrak{N}T$-*quadratic faithfulness* (4.8), a notion central to our results. In § 7 we prove that this property holds —and hence the abovementioned results are valid— for many classes of rings frequently met in practice, such as: (i) the reduced f-rings and some of their extensions, for which Marshall's signature conjecture (7.8) and a vast generalization of Sylvester's inertia law (7.10) are also true; and (ii) the reduced partially ordered Noetherian rings and many of their quotients (a result of interest in real algebraic geometry). This paper provides a broad generalization of the theory developed in the monograph [DM5] and of the methods employed therein.

1 Introduction

The aim of this paper is to present and develop a theory of (diagonal) quadratic forms *with non zero-divisor coefficients* over several broad classes of preordered unitary commutative rings, in which 2 is invertible.

We shall consider preordered rings $\langle A, T \rangle$ with the additional property that the support $T \cap -T$ of T contains only zero-divisors of A; we call this the *zero-divisor property*, abridged **zdp** (see Definition 2.8). As we shall see along the paper, especially in §7, many classes of preordered rings frequently met in mathematical practice satisfy this property.

The theory presented here constitutes a vast generalization of that presented in the monograph [DM5] of the first two authors. The latter deals with quadratic forms *with invertible coefficients* over preordered rings where 2 is invertible. A few simple examples should convince

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the reader of the extent of the generalization presented here. Consider, for example, the ring $\mathbb{C}(X)$ of continuous real-valued functions on a (completely regular) topological space X, preordered by the squares (i.e., the everywhere non-negative functions). The invertible elements of $\mathbb{C}(X)$ are the functions without zeros, while the non zero-divisors of $\mathbb{C}(X)$ are the functions f whose zero-set $Z(f) = f^{-1}[0]$ has empty interior. Thus, diagonal quadratic forms having as coefficients polynomials or trigonometric functions on $X = \mathbb{R}$ were, in general, excluded from the theory developed in [DM5], but find their place in the theory presented here.

Many of the results presented in this paper are inspired from —or have ancestors in— [DM5], and their proofs generalize, often considerably, those of their ancestors. The converse, however, is far from true: the proofs of most of the results presented below *are not reducible to, nor consequences of* those of their ancestors (if any) in [DM5]. In spite of their affinity with results in [DM5], the proof of most of the results in this paper has to be done from scratch.

Section 2 contains a number of preliminaries needed in the rest of the paper. In sections 3 and 4 we set up the machinery required to establish a theory of quadratic forms in the present context; namely:

(i) In § 3 we introduce the representation relations for quadratic forms with coefficients in the multiplicative set $\mathfrak{N} = \mathfrak{N}_A$ of non zero-divisors of a ring A, and prove their basic properties. In Proposition 3.5 we show that the group $G_{\mathfrak{N},T}(A)$ associated to a preordered ring $\langle A,T\rangle$ and the set $\mathfrak{N}$ is a *proto-special group* (and a *pre-special group* if binary representation is 2-transversal). Note that the appropriate notion of morphism in the present context is that of *$\mathfrak{N}p$-ring morphism* (Definition 3.10), on account of the fact that ordinary ring morphisms may not preserve non zero-divisors.

(ii) In § 4 we define the intrinsic notion of *$\mathfrak{N}T$-isometry* in the p-ring $\langle A,T\rangle$ for forms with coefficients in $\mathfrak{N}$ (Definition 4.4) and lay down the axioms for the notion of quadratic faithfulness in the present context (Definition 4.8), called *$\mathfrak{N}T$-quadratic faithfulness.* Based on the notions of representation and isometry defined in § 3, these axioms give the appropriate adaptation of the concept of T- quadratic faithfulness, see [DM5], Definition 3.1, p. 27, to the present context. The main results in § 4 are:

(a) (Theorem 4.10) Under some of the axioms for $\mathfrak{N}T$-quadratic faithfulness, the proto special group $G_{\mathfrak{N},T}(A)$ associated to the p-ring $\langle A,T\rangle$ is, indeed, a special group.

(b) (Theorem 4.11) If $\langle A,T\rangle$ is $\mathfrak{N}T$-faithfully quadratic, then $\mathfrak{N}T$-isometry of forms of arbitrary dimension in the p-ring $\langle A,T\rangle$ is faithfully represented by isometry of their images in the special group $G_{\mathfrak{N},T}(A)$. A similar statement holds for representation by forms of arbitrary dimension in $\langle A,T\rangle$.

The theory presented above can be carried out, with obvious modifications, for saturated multiplicative subsets of non zero-divisors of A containing $A^\times$, instead of the set $\mathfrak{N}_A$ of *all* non zero-divisors of A. To simplify the exposition we have decided to carry it out in the latter case, omitting here this extra degree of generality.

The first structural consequences of the preceding theory for $\mathfrak{N}T$-faithfully quadratic p-rings come in § 5. They establish the validity in that context of the Arason-Pfister Hauptsatz (Theorem 5.2) and of Pfister's local-global principle for T-isometry (Proposition 5.8) for preorders T having the zdp.

In § 6 we discuss the relation between the notion of T-quadratic faithfulness —in the sense of [DM5], Definition 3.1, p. 27— and that of $\mathfrak{N}T$-quadratic faithfulness presented in § 4. The main result in this section is Theorem 6.5, which shows the equivalence of the $\mathfrak{N}T$-quadratic faithfulness of a p-ring $\langle A,T\rangle$, where the preorder T has the zdp, with that of T_*-quadratic faithfulness, in the sense of [DM5], of the total ring of fractions A_* of A, endowed with the natural preorder T_* induced by T on A_*, defined in 6.1 (b).

In §7 we have gathered a selection of the most significant applications of the previously developed theory. Here are some noteworthy examples of p-rings that we prove to be $\mathfrak{N}T$-faithfully quadratic:

- All partially ordered integral domains (Proposition 7.2). This includes the rings of polynomials in several variables and power series in an arbitrary number of variables over real integral domains, as well as valuation rings and the real holomorphy ring of formally real fields, all partially ordered by sums of squares (Corollary 7.3).
- All Boolean powers of zdp-rings with the induced order (Theorem 7.5).
- The preordered rings $\langle A, P \rangle$, where A is a reduced f-ring and the (proper) preorder P contains the natural partial order $T_\sharp$ of A and has the zdp. In particular, $\langle A, T_\sharp \rangle$ is $\mathfrak{N}T_\sharp$-faithfully quadratic (Theorem 7.7).

The results in §5 above and (the proofs of) those in §10 of [DM5] entail that the following classical results, (well) known to hold over formally real fields, extend to the class of preordered extensions of f-rings mentioned above:

- The Arason-Pfister Hauptsatz (cf. 5.2).
- Pfister's local-global principle (cf. 5.8).
- Marshall's signature conjecture (Proposition 7.8).
- A far reaching extension of Sylvester's "inertia law" (1853) (Theorem 7.10).
- Real closed rings endowed with a zdp-preorder (Corollary 7.11(a)). In particular, the ring $\mathbb{C}(X)$ of continuous real valued functions defined on a (completely regular) topological space endowed with any zdp preorder T is $\mathfrak{N}T$-faithfully quadratic, as well as its real holomorphy ring; cf. 7.11(b,c). These results hold, *mutatis mutandis*, for the ring $\mathbb{C}_{sa}(\mathbb{R}^n)$ of continuous semi-algebraic functions on $\mathbb{R}^n$, see Proposition 7.12.
- The ring $\mathbb{C}(X)_p$ of germs at a point $p \in X$ of continuous real valued functions defined on a perfectly normal topological space X, ordered by squares (Proposition 7.18).
- Any reduced partially ordered Noetherian ring $\langle A, T \rangle$ (Theorem 7.24 (c)).

In particular,

- If $\langle A, T \rangle$ is a partially ordered Noetherian ring and J is a T-convex ideal of A, then the quotient ring $\langle A/J, T/J \rangle$ is $\mathfrak{N}(T/J)$-faithfully quadratic (Corollary 7.25 (a)).
- All real Noetherian rings, A, and all quotients of them modulo real ideals, under the partial order Σ of sums of squares, are $\mathfrak{N}\Sigma$-faithfully quadratic (Corollary 7.25 (b)).

2 Preliminaries

2.1 Horn-geometric and Geometric Formulas and Theories. For the definitions of these concepts we refer the reader to Definition 1.4, p. 2, in [DM5], registering here the properties to be frequently employed below, namely:

a) A geometric theory is preserved by arbitrary directed colimits (or inductive limits; cf. [Mi2], section 4, chapter 17)

b) A Horn-geometric theory is preserved by arbitrary reduced products and inductive limits. In fact, by Theorem 6.2.5 (p. 412) of [CK], preservation by arbitrary reduced products characterizes Horn sentences. ■

2.2 General notational conventions. a) We shall employ standard set-theoretic notation. For real-valued functions on a set X, f, we write $[\![f \geq 0]\!] = \{x \in X : f(x) \geq 0\}$ and similarly for $>$, $\leq$ and $<$ in place of $\geq$.

b) Let A be a commutative, unitary ring, in which 2 is a unit. For $D \subseteq A$ and $x \in A$, we

$A^\times$ = groups of units of A; $D^\times = D \cap A^\times$;

$D^2 = \{d^2 \in A : d \in D\}$; $xD = \{xd : d \in D\}$ and $-D = \{-d : d \in D\}$;

$\Sigma D^2 = \{\sum_{i=1}^n d_i^2 : \{d_1, \ldots, d_n\} \subseteq D \text{ and } n \in \mathbb{N}, n \geq 1\}$.

c) Write $\mathbb{Z}_2 = \{\pm 1\}$ for the multiplicative group of units of the integers ($\mathbb{Z}$) and $\mathbb{F}_2$ for the two-element field. ■

Henceforth, the word **ring** *stands for a commutative unitary semi-real ring* (-1 *is not a sum of squares*)*, in which* 2 *is a unit.*

2.3 Reduced and Real Rings. Let A be a ring. We recall the following concepts:

a) A is **reduced** if its only nilpotent element is 0. It is straightforward to check that A is reduced iff it satisfies the Horn-geometric sentence

(*) $$\forall u\, (u^2 = 0 \ \Rightarrow \ u = 0).$$

b) An ideal I in A is **real** if for all integers $n \geq 1$ and $a_1, \ldots, a_n \in A$, $\sum_{i=1}^n a_i^2 \in I$ implies $a_i \in I$, $1 \leq i \leq n$. A ring is **real** if the zero ideal is real, i.e., for all integers $n \geq 1$ and $c_1, \ldots, c_n \in A$, $\sum_{i=1}^n c_i^2 = 0$ entails $c_i = 0$, $1 \leq i \leq n$. Note that the property of being real can be described by a countable set of Horn-geometric sentences. Clearly, every real ring is semireal and reduced (by (*) in item (a)). Moreover, the properties of being reduced or real are hereditary, i.e, are inherited by any subring of a ring possessing them. ■

If A is a ring and $Z = \{Z_k : k \in K\}$ is a non-empty set of variables, write $A[X_1, \ldots, X_n]$ and $A[[Z]]$ for the rings of polynomials in n variables and of formal power series in the variables Z, respectively, with coefficients in A.

Lemma 2.4 *Let A be a ring. With notation as above, if A is an integral domain, reduced or real, the same is true of both $A[X_1, \ldots, X_n]$ and $A[[Z]]$.*

Proof. It suffices to show that the stated properties are preserved by adding one variable, i.e., if A has any one of the properties in the statement, then so do $A[X]$ and $A[[X]]$. Once this is established, for polynomials one employs induction, recalling that

(I) $$A[X_1, \ldots, X_n, X_{n+1}] = A[X_1, \ldots, X_n][X_{n+1}],$$

while for formal power series over the set of variables Z, we use the fact that a formula analogous to (I) is true for finite subsets of Z and that $A[[Z]]$ is the inductive limit over the partially ordered set of the finite subsets of K of the formal power series in each finite subset of K (note: the sentence characterizing integral domains in the language of rings is *geometric*). We shall explicitly prove only that if A is real, then $A[[X]]$ is real; a straightforward modification of the argument yields the remaining statements.

Let $f_1, \ldots, f_n \in A[[X]]$, $n \geq 1$, and assume $\sum_{j=1}^n f_j^2 = 0$. We may suppose, to obtain a contradiction, that all of the $f_j \neq 0$. For each j $(1 \leq j \leq n)$, write

$$f_j(X) \ = \ a_{jk_j} X^{k_j} + \text{terms of higher order, with } a_{jk_j} \neq 0.$$

Let $p = \min \{k_j : 1 \leq j \leq n\}$ and set $\alpha = \{j \in \{1, \ldots, n\} : k_j = p\}$; clearly, α is not empty. Moreover, the coefficient of X^{2p} in $\sum_{j=1}^n f_j^2$ is $\sum_{j \in \alpha} a_{jp}^2 = 0$. Since R is real, we get $a_{jp} = 0$, for all $j \in \alpha$, contradicting the fact that they were assumed non-zero. ■

2.5 Preorders and Partial orders on a ring. Let A be a ring.

a) A subset $T \subseteq A$ is a **preorder** on A if it is closed under sums, products and contains A^2; we say T is **proper** if $T \neq A$. Since $2 \in A^{\times}$, we have:

(1) T is a proper preorder on A iff $-1 \notin T$;

(2) The set $\text{supp}(T) := T \cap -T$, called the **support** of T, is a proper ideal of A.

(3) The smallest preorder (under inclusion) on a ring is ΣA^2 (sometimes called the *weak preorder*); it is clearly proper because of our blanket assumption that all rings are semi-real.

b) A proper **p-ring**, $\langle A, T \rangle$ is a ring with a proper preorder on A; clearly, a proper p-ring is semi-real.

c) A **p-ring morphism**, $f : \langle A_1, T_1 \rangle \longrightarrow \langle A_2, T_2 \rangle$, is a morphism of unitary rings, such that $f[T_1] \subseteq T_2$.

d) A preorder, T, on A is a **partial order** if $\text{supp}(T) = \{0\}$; in this case $\langle A, T \rangle$ is said to be a partially ordered ring **(po-ring)**. Every preorder on a formally real field is a partial order. ■

In what follows, all p-rings are assumed to be proper.

If $\langle A, T \rangle$ is a p-ring, we refer the reader to 4.14.(a) in p. 48 of [DM5] for the notions of T-convex and T-radical ideal in A; for the convenience of the reader we state Fact 4.17 of the same reference as

Lemma 2.6 *Let A be a reduced ring and let T be a preorder on A.*

a) The following are equivalent:

(1) *T is a partial order on A;*

(2) *The zero ideal is T-convex, i.e. for all $s, t \in T$, $s + t = 0$ implies, $s = t = 0$;*

(3) *A is T-reduced, that is $\sqrt[T]{0} := \{a \in A : \exists\, m \in \mathbb{N}$ and $t \in T$ so that $a^{2m} + t = 0\} = \{0\}$* (cf. 4.14.4.(a), [DM5]).

In particular, ΣA^2 is a partial order on A $\Leftrightarrow$ A is real.

b) The following are equivalent:

(1) *A^2 is a partial order on A;* (2) *A is Pythagorean,* (i.e., $A^2 = \Sigma A^2$). ■

For the definition and basic properties of the real holomorphy ring of a formally real field see [L], Chapter 9, pp. 73-79, or [Be], pp. 21ff, or [M], § 3.5, p. 50 and § 5.2(7), pp. 87-88.

Since subrings of a real ring are real, we obtain

Corollary 2.7 *Let F be a formally real field.*

a) If V is a valuation ring of F, then ΣV^2 is a partial order on V.

b) If $H(F)$ is the real holomorphy ring of F, then $\Sigma H(F)^2$ is a partial order on $H(F)$. ■

In fact, it is shown in Proposition 9.8 of [DM5], that $\Sigma H(F)^2 = \Sigma F^2 \cap H(F)$. The same is true for any valuation ring of F whose residue field is formally real.

The preceding results furnish a varied and interesting number of integral domains, A, in which ΣA^2 is a partial order; the significance of this will became apparent shortly.

Let A be a ring and let $T = A^2$ or a (proper) preorder of A.

Let $$\mathfrak{N}_A = \{x \in A : \forall\, \alpha \in A\ (\alpha x = 0 \Rightarrow \alpha = 0)\}$$

be the multiplicative set of non zero-divisors in A. *Whenever A is clear from context, we omit the subscript A.* Note that the formula defining $\mathfrak{N}$ is Horn-geometric.

Note that for $a, b \in A$ and $s \in \mathfrak{N}$, $sa = sb$ implies $a = b$ ($s(a-b) = 0$ and $s \in \mathfrak{N}$).

Let $\mathfrak{t} = T \cap \mathfrak{N}$; note that if $T = A^2$, then $\mathfrak{t} = A^2 \cap \mathfrak{N} = (\mathfrak{N} \cap A)^2 = \mathfrak{N}^2$.

Definition 2.8 *Let $\langle R, P \rangle$ be a p-ring. The preorder P has the* **zero-divisor property (zdp)** *if* $\mathrm{supp}(P) \subseteq R \setminus \mathfrak{N}_R$, *i.e.,* $\mathrm{supp}(P)$ *contains only zero-divisors. Another, equivalent, formulation of this property* (to be henceforth employed) *is: if $\mathfrak{p} := P \cap \mathfrak{N}_R$, then $\mathfrak{p} \cap -\mathfrak{p} = \emptyset$. In case $\underline{T = A^2}$, we say that A^2 has the zdp if $\mathfrak{N}_R^2 \cap -\mathfrak{N}_R^2 = \emptyset$.*

Remarks 2.9 a) If A is a ring, the class of preorders on A with the zdp is closed under arbitrary intersections and upward directed unions.

b) If A is a ring such that $\mathfrak{N} = A^{\times}$, then **any** proper preorder on A has the zdp. An interesting class of examples with proper preorders is that of von Neumann regular rings. Indeed, if A is von Neumann regular and $a \in A$, then there is an idempotent $e \in A$ so that principal ideals (a) and (e) are the same, and so $a = ae$. If a is not a unit, then $e \neq 1$, entailing $a(1-e) = ae(1-e) = 0$ and a is a zero divisor. If T is a proper preorder on A, then $\mathrm{supp}(T)$ is a proper ideal in A and so cannot intersect $\mathfrak{N} = A^{\times}$ and $\langle A, T \rangle$ is a zdp-ring. ∎

Clearly, if $\langle R, P \rangle$ is a po-ring, then P has the zdp. For integral domains, we note:

Fact 2.10 *If $\langle R, P \rangle$ is a p-ring and R is an integral domain, then*

$$P \text{ has the zdp} \quad \Leftrightarrow \quad P \text{ is a partial order.}$$

Proof. It suffices to establish the implication ($\Rightarrow$); since R is an integral domain, $\mathfrak{N} = R \setminus \{0\}$ and so the zdp entails $\mathrm{supp}(P) = \{0\}$, as claimed. ∎

Henceforth, to simplify statements, a p-ring, $\langle A, T \rangle$, such that T has the zdp will be referred to as a **zdp-ring**.

Example 2.11 (a) Let X be a completely regular space and let $\mathbb{C}(X)$ be the ring of continuous real-valued functions on X. Recall that a subset of X is **rare** if it has empty interior. A closed set in X is **regular** if it is the closure of its interior. Moreover,

(*) $f \in \mathbb{C}(X)$ is a non zero-divisor iff $Z(f) = \{x \in X : f(x) = 0\}$ is a rare closed set.

If K is a non-rare closed set in X, then the preorder

$$P_K = \{f \in \mathbb{C}(X) : K \subseteq [\![f \geq 0]\!]\}$$

has the zdp. First note that $\mathrm{supp}(P_K) = \{f \in \mathbb{C}(X) : K \subseteq Z(f)\}$; since K has non-empty interior, the same is true of $Z(f)$, for $f \in \mathrm{supp}(P_K)$, whence the latter ideal contains only zero divisors, as needed.

b) Our reference for semialgebraic sets and maps (sa-set, sa-map) is [BCR]. Let $A := \mathbb{C}_{sa}(\mathbb{R}^n)$ be the ring of real-valued continuous semialgebraic maps defined on $\mathbb{R}^n$, a real closed ring (cf. second and third paragraphs, p. 535, [DST] and the references therein). Let K be a closed set in $\mathbb{R}^n$, with non-empty interior, and let, as above,

$$P_K = \{f \in A : K \subseteq [\![f \geq 0]\!]\}$$

be the preoder determined by K on A. We show that P_K has the zdp $\underline{\text{in } A}$. Assume, for $f \in A$, that $f \in \mathrm{supp}(P_K)$; then $K \subseteq Z(f)$ and $Z(f)$ is a sa-set. Moreover, if U is the interior of $Z(f)$, then $U \neq \emptyset$ is a sa-set (cf. Prop. 2.2.2, p. 27, [BCR]) and the same is true of its complement, F. Let $g(x) := \mathrm{dist}(x, F)$ be the distance function from $x \in \mathbb{R}^n$ to F. By Prop. 2.2.8, p. 29, [BCR], $g \in A$; note that $U = [\![g > 0]\!]$ (and $g \restriction F = 0$). Clearly, gf identically zero in $\mathbb{R}^n$, and so f is a zero-divisor in A, as required. ∎

2.12 First-order theories of certain classes of p-rings. a) Let $L(P, S, N) = \{ +, \cdot, P, N, S, 0, 1\}$ be the first-order language with equality, consisting of the language of rings augmented by three unary predicates, P, S, N and let $\mathcal{T}_{pS}$ be the first-order theory with equality in $L(P, S, N)$ consisting of the following finite set of sentences:

(*i*) The axioms for unitary commutative rings, in which 2 is a unit;

(*ii*) The axioms stating that P is a proper preorder (including $\neg P(-1)$);

(*iii*) The sentence $\forall x \forall u (N(x) \wedge xu = 0 \longrightarrow u = 0)$;

(*iv.a*) The axioms that S is a multiplicative set; (*iv.b*) $\forall x (S(x) \longrightarrow N(x))$;

(*v*) $\exists z (P(z) \wedge P(-z) \wedge S(z)) \longrightarrow P(-1)$.

Note that $\mathcal{T}_{pS}$ is a Horn-geometric theory; moreover, any model of this theory consists of a proper p-ring, $\langle A, T \rangle$, together with a multiplicative set S having empty intersection with the supp(T) (by (*v*)), while (*iii*), (*iv.a*)) and (*iv.b*) guarantee that any element of N is a non zero-divisor and $S \subseteq N$. If we wish P to be a partial order, it suffices to add the Horn-geometric sentence

(*vi*) $\forall t (P(t) \wedge P(-t) \longrightarrow t = 0)$ (in which case (*v*) may be omitted).

Write $\mathcal{T}_{poS}$ for the ensuing theory. The theories $\mathcal{T}_{pS}$ and $\mathcal{T}_{poS}$ are the **theories of p-rings and po-rings with a multiplicative set of non-zero divisors, S, having empty intersection with the support of the preorder**, respectively, both Horn-geometric in $L(P, S, N)$.

b) The theory of zdp-rings. Let $L(P, N)$ the language of rings together with two unary predicates, P and N. Let $\mathcal{T}_{zdp}$ be the theory in $L(P, N)$ consisting of the following finite set of sentences:

(*vii*) The sentences in (*i*), (*ii*) in item (a);

[zdp] : $\exists z (N(z) \wedge P(z) \wedge P(-z)) \longrightarrow P(-1)$.

[N] : $\forall x \forall u (N(x) \longleftrightarrow (ux = 0 \longrightarrow u = 0))$

Note that $\mathcal{T}_{zdp}$ is a consistent theory: if $\mathbb{R}$ is the real line, with P is interpreted as $\mathbb{R}^2$ and N as $(\mathbb{R} \setminus \{0\})$, we obtain a model for $\mathcal{T}_{zdp}$. Moreover, (*vii*) and [zdp] are Horn-geometric, as is the implication ($\longrightarrow$) in [N] (equivalent to (*iii*) in item (a)), but not its converse (whose matrix is of the form $(p_1 \longrightarrow p_2) \longrightarrow p_3$, with p_i atomic, $i = 1, 2, 3$).

It is straightforward that the models of $\mathcal{T}_{zdp}$ are exactly the zdp-rings. ■

For reduced products, we adopt the notation and terminology of [CK] (see p. 214 ff). With notation as in 2.12, we have:

Proposition 2.13 *a) The class of models of the theories $\mathcal{T}_{pS}$ and $\mathcal{T}_{poS}$ are closed under arbitrary reduced products and directed colimits* (or inductive limits).

b) The class of zdp-rings is closed under arbitrary non-empty reduced products. In particular, it is closed under products and $\mathcal{T}_{zdp}$ is Horn axiomatizable.

Proof. Item (a) follows from 2.1. For (b), let I be a non-empty set and let D be a (proper) filter on I. Let $\langle A_i, T_i \rangle$, $i \in I$, be a I-family of zdp-rings. For each $i \in I$, write $\mathcal{N}_i$ for the multiplicative set of non zero-divisors in A_i and

$$\langle A, T \rangle = \langle \textstyle\prod_{i \in I} A_i, \prod_{i \in I} T_i \rangle \quad \text{and} \quad \langle A_D, T_D \rangle = \langle \prod_D A_i, \prod_D T_i \rangle,$$

for the product and the reduced product mod D of the $\langle A_i, T_i \rangle$, respectively. Let $\mathcal{N}_D$ be the set of non zero-divisors in A_D. As in [CK], if $s \in A$, write s_D for its equivalence class in A_D.

We start with the following

Fact. *a) For all $z \in \prod_{i\in I} A_i$, we have* $z_D \in \mathcal{N}_D \quad \Leftrightarrow \quad \{i \in I : z(i) \in \mathcal{N}_i\} \in D$.

b) For all $x \in A$, $x_D \in \mathrm{supp}(T_D) \quad \Leftrightarrow \quad \{i \in I : x(i) \in \mathrm{supp}(T_i)\} \in D$.

Proof. a) $\underline{\Rightarrow:}$ Suppose $z_D \in \mathcal{N}_D$ and set $K = \{i \in I : z(i) \in \mathcal{N}_i\}$. For each $i \in K^c = I \setminus K$, $z(i) \notin \mathcal{N}_i$ and so there is $a_i \in A_i \setminus \{0\}$ such that $z(i)a_i = 0$. Define, for $i \in I$,

$$a(i) = \begin{cases} a_i & \text{if } i \in K^c \\ 0 & \text{if } i \in K. \end{cases}$$

Then, $az = 0$ and so, $a_D \cdot z_D = (az)_D = 0_D$. Since $z_D \in \mathcal{N}_D$, we get $a_D = 0_D$ whence, $\{i \in I : a(i) = 0\} = K \in D$, as needed.

$\underline{\Leftarrow:}$ Assume $a \in A$ satisfies $a_D \cdot z_D = (az)_D = 0_D$; the definition of reduced product entails that $J = \{i \in I : a(i)z(i) = 0\} \in D$; set $K = \{i \in I : z(i) \in \mathcal{N}_i\} \in D$. Hence, $G = K \cap J \in D$ and for all $i \in G$, $a(i) = 0$, whence $a_D = 0_D$ and $z_D \in \mathcal{N}_D$.

b) $\underline{\Rightarrow:}$ If $x_D, -x_D \in T_D$, then $J_1 = \{j \in I : x(j) \in T_j\} \in D$ and $J_2 = \{k \in I : -x(k) \in T_k\} \in D$. Hence, $J = J_1 \cap J_2 \in D$ and for all $i \in J$, $x(i), -x(i) \in T_i$, as needed. The converse can be treated similarly. □

For $c_D \in A_D$, if $c_D \in \mathrm{supp}(T_D) \cap \mathcal{N}_D$ the Fact entails $\{k \in I : c(i) \in \mathrm{supp}(T_i) \cap \mathcal{N}_i\} \in D$. In particular, this set is non-empty, contradicting the hypothesis that all coordinates are zdp-rings.

Clearly, the case of products is a special case of reduced products, while Horn axiomatizability follows from Theorem 6.5.2, p. 412 in [CK] (no need of the continuum hypothesis. cf. p. 414 ff, of the same reference.). ∎

Remark 2.14 a) We are not aware of an explicit Horn axiomatization for zdp-rings, although the multiplicative set of non zero-divisors in a ring is *defined by a Horn-geometric formula*. The question, of course, involves how to find a collection of Horn sentences equivalent to axiom [N] in 2.12.(b).

b) However, there is a significant subclass of zdp-rings for which one can give a Horn-geometric axiomatization: the po-rings; it is clear from 2.12.(a) that this theory has a Horn-geometric axiomatization in the language $L(P)$. Write $\mathcal{T}_{por}$ for the first-order theory with equality of partially ordered rings. Hence, the class of models of $\mathcal{T}_{por}$ is closed under arbitrary reduced products and directed colimits (or inductive limits). ∎

The final theme in this section is the description of the group of exponent 2 that will underlie all constructions to follow.

Lemma 2.15 *Let A be a ring and let $T = A^2$ or a proper preorder on A. We have:* [1]

a) (1) $1 \in \mathfrak{t}$, $-1 \notin \mathfrak{t}$, *and* $\mathfrak{t} \cdot \mathfrak{t} = \mathfrak{t}$. *Moreover, for all $x \in \mathfrak{N}$, $x^2 \in \mathfrak{t}$.*

(2) *If A is reduced and T is a partial order, then* $\mathfrak{t} + \mathfrak{t} \subseteq \mathfrak{t}$.

b) For $a, b \in \mathfrak{N}$, <u>define</u> $a \sim_{\mathfrak{t}} b$ iff $\exists\, s \in \mathfrak{t}$ so that $sab \in \mathfrak{t}$. Then:

(1) *$\sim_{\mathfrak{t}}$ is a congruence with respect to product in $\mathfrak{N}$.*

(2) *$a \sim_{\mathfrak{t}} b$ iff $\exists\, s, t \in \mathfrak{t}$ so that $sa = tb$. Hence, $a \sim_{\mathfrak{t}} 1$ iff $\exists\, s \in \mathfrak{t}$, so that $sa \in \mathfrak{t}$. In particular, $a \in \mathfrak{t}$ implies $a \sim_{\mathfrak{t}} 1$ and for all $a \in \mathfrak{N}$, $a^2 \sim_{\mathfrak{t}} 1$.*

(3) *$G_{\mathfrak{N},T}(A) = \mathfrak{N}/{\sim_{\mathfrak{t}}} := \{a^T = a/{\sim_{\mathfrak{t}}} : a \in \mathfrak{N}\}$ is a* group *of exponent two, with $1 = 1^T$ and $-1 = (-1)^T$.*

[1] Recall: for a preorder T, $\mathfrak{t} = T \cap \mathfrak{N}$; if $T = A^2$, then $\mathfrak{t} = \mathfrak{N}^2$.

(4) $1^T \neq (-1)^T$ *iff* $\mathfrak{t} \cap -\mathfrak{t} = \emptyset$ (i.e., T has the zdp). *In particular, this holds if T is a partial order.*

Proof. a) Item (1) is clear. For (2), assume A is reduced and T is a partial order and, for $s, t \in \mathfrak{t}$, that $s + t \notin \mathfrak{N}$ (it is clearly in T). Then, there is $\alpha \neq 0$ in A so that $\alpha(s + t) = 0$, entailing $\alpha^2 s + \alpha^2 t = 0$. Hence, since $\alpha^2 t \in T$, we conclude that $\alpha^2 s \in T \cap -T = \{0\}$; since $s \in \mathfrak{t} \subseteq \mathfrak{N}$, we have $\alpha^2 = 0$ and so, A being reduced, we get $\alpha = 0$, a contradiction.

b) (1) Clearly, $a \sim_\mathfrak{t} a$, and $a \sim_\mathfrak{t} b$ implies $b \sim_\mathfrak{t} a$. For transitivity, if $a \sim_\mathfrak{t} b$ and $b \sim_\mathfrak{t} c$, then there are $s, t \in \mathfrak{t}$ such that sab, $tbc \in \mathfrak{t}$ and so $(stb^2)ac \in \mathfrak{t}$, with $stb^2 \in \mathfrak{t}$, whence $a \sim_\mathfrak{t} c$. If $a \sim_\mathfrak{t} c$ and $b \sim_\mathfrak{t} d$, then for some $s, t \in \mathfrak{t}$, we have sac, $tbd \in \mathfrak{t}$, implying $(st)(abcd) \in \mathfrak{t}$, and $\sim_\mathfrak{t}$ is a congruence with respect to product in $\mathfrak{N}$.

(2) If, for $s \in \mathfrak{t}$, $sab = t \in \mathfrak{t}$, then $(sb^2)a = tb$; for the converse, if $s'a = t'b$, with $s', t' \in \mathfrak{t}$, then $s'ab = t'b^2 \in \mathfrak{t}$, as needed. The remaining statements in (2) are clear.

(3) With product induced by that in $\mathfrak{N}$ ($a^T b^T := (ab)^T$), the quotient $G_{\mathfrak{N},T}(A) : = \mathfrak{N}/\sim_\mathfrak{t}$ is a semigroup with $1 = 1^T$, which is of exponent 2 (by the last statement in (2)) and hence a <u>group</u> of exponent two.

(4) Note that $1^T = (-1)^T$ iff there is $s \in \mathfrak{t}$ so that $s \cdot 1 \cdot (-1) = -s \in \mathfrak{t}$ iff $\mathfrak{t} \cap -\mathfrak{t} \neq \emptyset$. ■

The results in Lemma 2.15 will be used below, oftentimes without explicit reference.

2.16 Notation. Let A be a ring and let $\varphi = \langle a_1, \ldots, a_n \rangle$ be a form over $\mathfrak{N}$.

- If T is a preorder on A, write $\varphi^T = \langle a_1^T, \ldots, a_n^T \rangle$ for the corresponding form in $G_{\mathfrak{N},T}(A)$;
- If $T = \Sigma A^2$, write $G_{\mathfrak{N},\Sigma}$ for $G_{\mathfrak{N},\Sigma A^2}(A)$, a^Σ for a^T and $\varphi^\Sigma = \langle a_1^\Sigma, \ldots, a_n^\Sigma \rangle$ for the corresponding form in $G_{\mathfrak{N},\Sigma}$;
- If $T = A^2$, write $G_\mathfrak{N}(A)$ for $G_{\mathfrak{N},A^2}(A)$, $\widehat{x}$ for x^T, $x \in \mathfrak{N}$, and $\widehat{\varphi} = \langle \widehat{a_1}, \ldots, \widehat{a_n} \rangle$ for the corresponding form in $G_\mathfrak{N}(A)$. ■

3 The Proto-Special Group Associated to $\mathfrak{N}$

Definition 3.1 *Let A be a ring, with $2 \in A^\times$, and let $T = A^2$ or a proper preorder on A. With notation as above, let $\varphi = \langle a_1, \ldots, a_n \rangle$ be an n-form ($n \geq 2$) over $\mathfrak{N}$.*

a) Define

$D_T^v(\varphi) = \{c \in \mathfrak{N} : \exists\, s \in \mathfrak{t}, \exists\, t_1, \ldots, t_n \in T \text{ so that } sc = \sum_{i=1}^n t_i a_i\}$ *and*

$D_T^t(\varphi) = \{x \in \mathfrak{N} : \exists s, \tau_1, \ldots, \tau_n \in \mathfrak{t} \text{ so that } sx = \sum_{i=1}^n \tau_i a_i\}$,

respectively the set of elements **value represented** *and* **transversally value represented** *by φ in $\mathfrak{N}$. Clearly, $\{a_1, \ldots, a_n\} \subseteq D_T^v(\varphi)$ and $D_T^t(\varphi) \subseteq D_T^v(\varphi)$. We also set $D_T^v(\langle a \rangle) = D_T^t(\langle a \rangle) = \{a\}$.*

b) Define $\mathfrak{D}_T(\varphi)$ (the inductive definition of value representation) *as follows:*

$*$ *If $n = 2$, $\mathfrak{D}_T(\varphi) = D_T^v(a_1, a_2)$;*

$*$ *If $n \geq 3$,*

$$\mathfrak{D}_T(\varphi) = \{x \in \mathfrak{N} : \forall\, 1 \leq k \leq n, \exists\, u \in \mathfrak{N} \text{ s. t. } u \in D_T^v(b_1, \ldots, \check{b}_k, \ldots, b_n) \text{ and } x \in D_T^v(b_k, u)\},$$

where $\left\langle a_1, \ldots, \check{a}_k, \ldots, a_n \right\rangle$ is the $(n-1)$-form obtained by removing the k^{th}-entry from φ.

At this stage it is important to describe the basic properties of D_T^v for 2-forms (compare with Lemma 2.6, p.11 ff in [DM5]).

Lemma 3.2 *Let A be a ring and let $T = A^2$ or a proper preorder on A. Let $x, y, u, v \in \mathfrak{N}$ and $w_1, w_2 \in \mathfrak{t} = T \cap \mathfrak{N}$. With notation as above:*

a) (1) $u \in D_T^v(1,-1)$; (2) *If T is a preorder on A, then* $u \in D_T^v(1,1) \Rightarrow u^T = 1^T$.

b) (1) $uD_T^v(x,y) \subseteq D_T^v(ux,uy)$, *and* $z \in D_T^v(ux,uy) \Rightarrow uz \in D_T^v(x,y)$.

(2) $D_T^v(x,y) = D_T^v(w_1x, w_2y)$. *In particular, this holds if* $w_1, w_2 \in T^\times = T \cap A^\times$.

c) $u \in D_T^v(x,y)$ *and* $u^T = v^T \Rightarrow v \in D_T^v(x,y)$.

d) $x^T = u^T$ *and* $y^T = v^T \Rightarrow D_T^v(x,y) = D_T^v(u,v)$.

e) (1) $D_T^v(1,x)$ *is a sub-semigroup of* $\mathfrak{N}$.

(2) *If T is a preorder on A,* $y \in D_T^v(1,x) \Rightarrow D_T^v(1,y) \subseteq D_T^v(1,x)$.

f) For all $z \in \mathfrak{N}$, $D_T^v(x,y) = D_T^v(u,v) \Rightarrow D_T^v(zx,zy) = D_T^v(zu,zv)$.

Proof. a) (1) Since $2 \in A^\times$, we have $u = \frac{1}{4}(1+u)^2 - \frac{1}{4}((1-u)^2$, as needed.

(2) If $u \in D_T^v(1, 1)$, then $su = t_1 + t_2$ ($s \in \mathfrak{t}$, $t_i \in T$); note that the left-hand side of this equality is in $\mathfrak{N}$, while its right-hand side is in T. Thus, $t_1 + t_2 \in \mathfrak{t}$ and so $su = (t_1 + t_2) \cdot 1$, whence $u \sim_{\mathfrak{t}} 1$, i.e., $u^T = 1^T$.

b) (1) If $w \in D_T^v(x, y)$, there are $s \in \mathfrak{t}$, $t_1, t_2 \in T$ so that $sw = t_1x + t_2y$ and so $s(uw) = t_1(ux) + t_2(uy)$, whence $uw \in D_T^v(ux, uy)$.

If $z \in D_T^v(ux, uy)$, then $sz = t_1ux + t_2uy$ ($s \in \mathfrak{t}$, $t_i \in T$), hence $suz = (t_1u^2)x + (t_2u^2)y$, entailing $uz \in D_T^v(x, y)$. Note: the latter relation yields $u^2z \in uD_T^v(x, y)$, with $(u^2z)^T = z^T$ (because $u^2z^2 \in \mathfrak{t}$).

(2) If $z \in D_T^v(x, y)$, i.e. $sz = t_1x + t_2y$ ($s \in \mathfrak{t}$, $t_i \in T$), then $(sw_1w_2t)z = (t_1w_2)(w_1x) + (t_2w_1)(w_2y)$. Since $sw_1w_2 \in \mathfrak{t}$ (recall: $w_1, w_2 \in \mathfrak{t}$) we get $z \in D_T^v(w_1x, w_2y)$. If $u \in D_T^v(w_1x, w_2y)$, then $s'u = t'_1w_1x + t'_2w_2y$ ($s' \in \mathfrak{t}$, $t'_i \in T$), hence $u \in D_T^v(x, y)$, establishing the desired equality.

c) By Lemma 2.15.(b.(2)), there are $s_1, s_2 \in \mathfrak{t}$ so that $s_1u = s_2v$. We also have $s \in \mathfrak{t}$, $t_1, t_2 \in T$ so that $su = t_1x + t_2y$; hence $(ss_2)\, v = s(s_1u) = (s_1t_1)x + (s_1t_2)y$ and $v \in D_T^v(x, y)$, as desired.

d) By 2.15.(b.(2)), there are $s_1, s_2, s_3, s_4 \in \mathfrak{t}$ with $s_1x = s_2u$ and $s_3y = s_4v$. If $w \in D_T^v(x, y)$, then $sw = t_1x + t_2y$ ($s \in \mathfrak{t}$, $t_i \in T$). Hence,

$$(ss_1s_3)w = (t_1s_3)s_1x + (t_2s_1)s_3y = (t_1s_3)s_2u + (t_2s_1)s_4v,$$

yielding $w \in D_T^v(u, v)$. By symmetry, we obtain the claimed equality.

e) If $u, v \in D_T^v(1, x)$, then $s_1u = t_1 + t_2x$ and $s_2v = t'_1 + t'_2x$ ($s_i \in \mathfrak{t}$, $t_i, t'_i \in T$). Thus,

$$s_1s_2uv = (t_1t'_1 + t_2t'_2x^2) + (t_1t'_2 + t'_1t_2)x,$$

with $s_1s_2 \in \mathfrak{t}$, and so $uv \in D_T^v(1, x)$. If $T = A^2$ and $a, b \in D_T^v(1, x)$, there are $w, z \in \mathfrak{N}$ and $p, q, u, v \in A$ so that $w^2a = p^2 + q^2x$ and $z^2b = u^2 + v^2x$. Hence,

$$\begin{aligned} w^2z^2ab &= (p^2u^2 + q^2v^2x^2) + (p^2v^2 + q^2u^2)x \\ &= (p^2u^2 + q^2v^2x^2) + 2puqvx - 2puqvx + (p^2v^2 + q^2u^2)x \\ &= (pu + qvx)^2 + (pv - qu)^2x \end{aligned}$$

and so $ab \in D_T^v(1, x)$.

Fix $y \in D_T^v(1, x)$, written

$$(*) \qquad sy = t_1 + t_2x. \qquad (s \in \mathfrak{t},\ t_i \in T)$$

If $z \in D_T^v(1, y)$, we may write $s'z = t'_1 + t'_2y$; hence, using (*), we obtain

$$ss'z = st'_1 + t'_2sy = st'_1 + t'_2(t_1 + t_2x) = (st'_1 + t'_2t_1) + t_2t'_2x,$$

yielding $z \in D^v_T(1, x)$, as claimed.

f) Suppose $w \in D^v_T(zx, zy)$, i.e.,

$$(+) \qquad sw = t_1 zx + t_2 zy. \qquad (s \in \mathfrak{t},\ t_i \in T)$$

Since $x, y \in D^v_T(x, y) = D^v_T(u, v)$, we have $s'x = t'_1 u + t'_2 v$ and $s''y = t''_1 u + t''_2 v$; these equalities and (+) yield

$$\begin{aligned}(s's''s)w &= t_1 z s''(s'x) + t_2 z s'(s''y) = t_1 z s''(t'_1 u + t'_2 v) + t_2 z s'(t''_1 u + t''_2 v) \\ &= [t_1 t'_1 s'' + t_2 t''_1 s'](zu) + [t_1 t'_2 s'' + t_2 t''_2 s'](zv),\end{aligned}$$

whence $w \in D^v_T(zu, zv)$. Similarly, one shows $D^v_T(zu, zv) \subseteq D^v_T(zx, zy)$, ending the proof. ∎

The next Lemma is the analog in the present setting of items (f) and (g) in Lemma 2.6 (p. 12) of [DM5].

Lemma 3.3 *Let $x, y, u, v \in \mathfrak{N}$. With notation as above:*

a) $u \in D^v_T(x,y) \Leftrightarrow D^v_T(u, uxy) = D^v_T(x,y)$.

b) The following are equivalent:

(1) $(xy)^T = (uv)^T$ *and* $D^v_T(x,y) = D^v_T(u,v)$;

(2) $(xy)^T = (uv)^T$ *and* $D^v_T(x,y) \cap D^v_T(u,v) \neq \emptyset$.

Proof. a) Since $a \in D^v_T(a, b)$, only ($\Rightarrow$) needs proof. For the argument that follows, one should keep in mind 3.2.(c). Assume that $u \in D^v_T(x, y)$; then,

$$(*) \qquad ux \in D^v_T(1, xy).$$

Now, if $z \in D^v_T(u, uxy)$, then $zu \in D^v_T(1, xy)$, hence (*) and 3.3.(e.1) yield $zxu^2 \in D^v_T(1, xy)$ and so $zu^2 \in D^v_T(x, y)$, entailing $z \in D^v_T(x, y)$. Conversely, if $z \in D^v_T(x, y)$, then $zx \in D^v_T(1, xy)$, and another application of 3.2.(e.1) and (*) yields $zux^2 \in D^v_T(1, xy)$; hence, just as above, $z \in D^v_T(u, uxy)$, as claimed.

b) It suffices to show (2) $\Rightarrow$ (1). Let $z \in D^v_T(x, y) \cap D^v_T(u, v)$; by item (a) we have

$$(\sharp) \qquad D^v_T(z, zxy) = D^v_T(x, y) \quad \text{and} \quad D^v_T(z, zuv) = D^v_T(u, v).$$

Since $(zxy)^T = (zuv)^T$ (because $(xy)^T = (uv)^T$), 3.2.(d) and ($\sharp$) entail $D^v_T(x, y) = D^v_T(u, v)$, ending the proof. ∎

Definition 3.4 *Define a binary relation, $\equiv_{\mathfrak{t}}$, on $G_{\mathfrak{N},T}(A) \times G_{\mathfrak{N},T}(A)$, called* **binary isometry mod t**, *as follows: for $a, b, c, d \in \mathfrak{N}$*

$$(\equiv_{\mathfrak{t}}) \qquad \langle a^T, b^T \rangle \equiv_{\mathfrak{t}} \langle c^T, d^T \rangle \quad \Leftrightarrow \quad a^T b^T = c^T d^T \ \text{ and } \ D^v_T(a,b) = D^v_T(c,d).$$

For the definitions of *proto-special* (π-SG), *pre-special* (p-SG) *and special* (SG) *groups, reducibility and of morphisms of these structures*, we refer the reader to Definitions 1.9 (pp. 5, 6) and 1.12 (p. 7) of [DM5].

Proposition 3.5 *Let A be a ring and let $T = A^2$ or a proper preorder on A. Let $\mathfrak{N}$ be the multiplicative set of non zero-divisors in A.*

a) (1) The structure $\langle G_{\mathfrak{N},T}(A), \equiv_{\mathfrak{t}}, -1 \rangle$ is a proto-special group. As usual, write $D_{G_{\mathfrak{N},T}(A)}(*,*)$ for binary representation in $G_{\mathfrak{N},T}(A)$.

(2) If T is a preorder on A, $G_{\mathfrak{N},T}(A)$ is reduced iff $\mathfrak{t} \cap -\mathfrak{t} = \emptyset$, i.e., iff $\langle A, T \rangle$ is a zdp-ring. In particular, $G_{\mathfrak{N},T}(A)$ is reduced if T is a partial order.

b) (1) For $x, y, z \in \mathfrak{N}$, $z \in D^v_T(x,y) \Leftrightarrow z^T \in D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$.

(2) *For* $u,\ v,\ x,\ y \in \mathfrak{N}$, $D^v_T(u,v) \subseteq D^v_T(x,y) \ \Leftrightarrow\ D_{G_{\mathfrak{N},T}(A)}(u^T, v^T) \subseteq D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$.

(3) *For all* $x \in \mathfrak{N}$, $D_{G_{\mathfrak{N},T}(A)}(1, x^T)$ *is a saturated subgroup of* $G_{\mathfrak{N},T}(A)$. [2]

c) For all $z,\ x,\ y \in \mathfrak{N}$, $z^T D_{G_{\mathfrak{N},T}(A)}(x^T, y^T) = D_{G_{\mathfrak{N},T}(A)}(z^T x^T, z^T y^T)$.

d) If binary value representation is **2-transversal**, *i.e., for all* $x,\ y \in \mathfrak{N}$, $D^t_T(x,y) = D^v_T(x,y)$, *then* $G_{\mathfrak{N},T}(A)$ *is a* **pre-special group**, *which is reduced if* T *satisfies the zdp; in particular, this holds if* T *is a partial order.*

e) If $T = A^2$, *the following are equivalent:*

(1) $G_{\mathfrak{N}}(A)$ *is a reduced proto-special group;* (2) $-\mathfrak{N}^2 \cap \mathfrak{N}^2 = \emptyset$ *and* $(A^2 + A^2) \cap \mathfrak{N} \subseteq \mathfrak{N}^2$.

In particular, if A *is Pythagorean* ($A^2 = \Sigma A^2$) *and the weak preorder is a partial order, then* $G_{\mathfrak{N}}(A)$ *is a reduced special group.*

Proof. a) (1) By 2.15.(b.3), $G_{\mathfrak{N},T}(A)$ is a group of exponent two. Clearly, $\equiv_\mathfrak{t}$ is an equivalence relation on $G_{\mathfrak{N},T}(A) \times G_{\mathfrak{N},T}(A)$ and $\langle a^T, b^T \rangle \equiv_\mathfrak{t} \langle b^T, a^T \rangle$, and so $G_{\mathfrak{N},T}(A)$ satisfies [SG 0] and [SG 1] in Def. 1.9, p. 5, [DM5]. Axiom [SG 2] is a consequence of 3.2.(a) and 3.3.(a): since $a \in D^v_T(1, -1)$, we get $D^v_T(a, -a) = D^v_T(1, -1)$, with $(a \cdot (-a))^T = -(a^2)^T = (-1)^T$, because $-1 \cdot -a^2 = a^2 \in \mathfrak{t}$. Axiom [SG 3] is immediate from the definition of $\equiv_\mathfrak{t}$. Fix $x \in \mathfrak{N}$; if $\langle a^T, b^T \rangle \equiv_\mathfrak{t} \langle c^T, d^T \rangle$, then $(ab)^T = (cd)^T$ and $D^v_T(a, b) = D^v_T(c, d)$. Hence, $(xab)^T = (xcd)^T$, while 3.2.(f) yields $D^v_T(xa, xb) = D^v_T(xc, xd)$, entailing, by 3.4, $\langle x^T a^T, x^T b^T \rangle \equiv_\mathfrak{t} \langle x^T c^T, x^T d^T \rangle$, establishing $G_{\mathfrak{N},T}(A)$ to be a proto-special group.

(2) If T is a preorder on A, item (b.4) in 2.15 and $\mathfrak{t} \cap -\mathfrak{t} = \emptyset$ guarantee $1^T \neq -1^T$. Now, the reducibility of $G_{\mathfrak{N},T}(A)$ follows straightforwardly from 3.2.(a.2).

Proof of (b.1) : If $z \in D^v_T(x, y)$, then 3.3.(a) entails $D^v_T(z, zxy) = D^v_T(x, y)$, while it is clear that $z^T \cdot (z^T x^T y^T) = x^T y^T$. Thus, $\langle z^T, z^T x^T y^T \rangle \equiv_\mathfrak{t} \langle x^T, y^T \rangle$ and $z^T \in D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$. Conversely, if $z^T \in D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$, then, $\langle z^T, z^T x^T y^T \rangle \equiv_\mathfrak{t} \langle x^T, y^T \rangle$, entailing $z \in D^v_T(z, zxy) = D^v_T(x, y)$, as needed.

Item (b.2) follows from (b.1), while (b.3) is an immediate consequence of 3.2.(e), (b.1) and (b.2).

c) If $v^T \in z^T D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$, then $z^T v^T \in D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$ and (b.1) yields $zv \in D^v_T(x, y)$. Hence, for some $s \in \mathfrak{t}$, $t_1, t_2 \in T$, we have $szv = t_1 x + t_2 y$, entailing $(sz^2)v = t_1(zx) + t_2(zy)$, i.e., $v \in D^v_T(zx, zy)$. Another application of (b.1) yields $v^T \in D_{G_{\mathfrak{N},T}(A)}(z^T x^T, z^T y^T)$, and $z^T D_{G_{\mathfrak{N},T}(A)}(x^T, y^T) \subseteq D_{G_{\mathfrak{N},T}(A)}(z^T x^T, z^T y^T)$. For the reverse inclusion, suppose w^T is in $D_{G_{\mathfrak{N},T}(A)}(z^T x^T, z^T y^T)$; then 3.2.(b.1) yields $zw \in D^v_T(x, y)$. By (b.1), $z^T w^T \in D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$ and so $(z^T)^2 w^T = w^T \in z^T D_{G_{\mathfrak{N},T}(A)}(x^T, y^T)$, completing the proof of (c).

d) To verify [SG 4] in Def. 1.9, p. 5, [DM5], suppose $\langle u^T, v^T \rangle \equiv_\mathfrak{t} \langle x^T, y^T \rangle$. Then, $u^T v^T = x^T y^T$ and so, since $G_{\mathfrak{N},T}(A)$ is a group of exponent two, we obtain $-x^T u^T = -v^T y^T$ or, equivalently, $-x^T = -u^T v^T y^T$. By items (a) and (b) in Lemma 3.3 and the definition of $\equiv_\mathfrak{t}$, the desired conclusion is equivalent to $u \in D^v_T(-v, y)$: indeed, once this established, we obtain $D^v_T(u, -uvy) = D^v_T(-v, y)$, with $u^T \cdot -(u^T v^T y^T) = -v^T y^T$, whence $\langle -v^T, y^T \rangle \equiv_\mathfrak{t} \langle u^T, -u^T v^T y^T \rangle = \langle u^T, -x^T \rangle$. Now, since $y^T \in D_{G_{\mathfrak{N},T}(A)}(u^T, v^T)$, by item (b.1), $y \in D^v_T(u, v) = D^t_T(u, v)$. Hence, there are $s, s_1, s_2 \in \mathfrak{t}$, so that $sy = s_1 u + s_2 v$, whence $s_1 u = s_2(-v) + sy$. Thus, $u \in D^t_T(-v, y) = D^v_T(-v, y)$, as needed.

e) (1) $\Rightarrow$ (2) : By 2.15.(b.4), the first condition in (2) is satisfied. If $a \in (A^2 + A^2) \cap \mathfrak{N}$, then, $a \in D^v_T(1, 1)$ and so $\widehat{a} \in D_{\mathfrak{N}}(1, 1)$; the reducibility of $G_{\mathfrak{N}}(A)$ entails $\widehat{a} = \widehat{1}$, that is, $a \in \mathfrak{N}^2$.

(2) $\Rightarrow$ (1) : Again, the first condition in (2) and 2.15.(b.4) imply $1^T \neq -1^T$. Now, assume

[2] Recall: a subgroup Δ of a π-SG, G, is *saturated* if for all $x \in G$, $x \in \Delta \Leftrightarrow D_G(1, x) \subseteq \Delta$.

$\widehat{a} \in D_{G_{\mathfrak{N}}(A)}(1, 1)$; but then, $a \in D^v_T(1, 1)$, and there are $u \in \mathfrak{N}$ and $x, y \in A$ so that $u^2a = x^2 + y^2$; the left-hand side of this last equality is in $\mathfrak{N}$, while it right-hand side is in $A^2 + A^2$; thus, for some $z \in \mathfrak{N}$, $u^2a = z^2$, and so $\widehat{a} = \widehat{1}$, as needed. The last assertion in (e) follows straightforwardly from item (a.2). ■

Remark 3.6 a) Employing Definition 1.7.(d) (p. 4) in [DM5], binary isometry in $G_{\mathfrak{N},T}(A)$, $\equiv_\mathfrak{t}$, can be extended to forms of all dimensions $n \geq 1$, still indicated by the same symbol, as follows:

(1) $n = 1 : \langle a \rangle \equiv_\mathfrak{t} \langle b \rangle$ iff $a = b$; (2) $n = 2 : \equiv_\mathfrak{t}$ is the binary isometry defined in $G_{\mathfrak{N},T}(A)$;

(3) For $n \geq 3$, proceed by induction: $\langle a_1, \ldots, a_n \rangle \equiv_\mathfrak{t} \langle b_1, \ldots, b_n \rangle$ iff there are $x, y, z_3, \ldots, z_n$ in $G_{\mathfrak{N},T}(A)$ so that

(*i*) $\langle a_1, x \rangle \equiv_\mathfrak{t} \langle b_1, y \rangle$; (*ii*) $\langle a_2, \ldots, a_n \rangle \equiv_\mathfrak{t} \langle x, z_3, \ldots, z_n \rangle$;

(*iii*) $\langle b_2, \ldots, b_n \rangle \equiv_\mathfrak{t} \langle y, z_3, \ldots, z_n \rangle$.

b) As in Definition 1.3, p. 3, [DM1], if $n \geq 1$ and φ is n-form over $G_{\mathfrak{N},T}(A)$, let

$$D_{G_{\mathfrak{N},T}(A)}(\varphi) = \{u \in G_{\mathfrak{N},T}(A) : \exists\, z_2, \ldots z_n \in G_{\mathfrak{N},T}(A) \text{ so that } \langle u, z_2, \ldots, z_n \rangle \equiv_\mathfrak{t} \varphi\},$$

be the **set of elements represented by** φ in $G_{\mathfrak{N},T}(A)$. ■

Definition 3.7 *For a form $\varphi = \langle a_1, \ldots, a_n \rangle$ over $\mathfrak{N}$, define:*

$$D_{\mathfrak{N},T}(\varphi) = \{z \in \mathfrak{N} : \exists\, z_2, \ldots, z_n \in \mathfrak{N} \text{ so that } \langle z^T, z_2^T, \ldots, z_n^T \rangle \equiv_\mathfrak{t} \varphi^T\},$$

the set of elements in $\mathfrak{N}$ **isometry-represented mod T** *by φ^T in $G_{\mathfrak{N},T}(A)$. In case $T = A^2$ or $T = \Sigma A^2$, write $D_{\mathfrak{N}}(\varphi)$ and $D_{\mathfrak{N},\Sigma}(\varphi)$ for the set of elements of $\mathfrak{N}$ isometry represented by φ in $G_{\mathfrak{N}}(A)$ and $G_{\mathfrak{N},\Sigma}(A)$, respectively.*

Remark 3.8 Clearly, $D_{\mathfrak{N},T}(\varphi)$ is a subset of $\mathfrak{N}$. However, it is straightforward from 3.6.(b) and 3.7 that if φ is a form over $\mathfrak{N}$ and $x \in \mathfrak{N}$, then $x \in D_{\mathfrak{N},T}(\varphi)$ <u>iff</u> $x^T \in D_{G_{\mathfrak{N},T}(A)}(\varphi^T)$. ■

Lemma 3.9 *With notation as above, let $\varphi = \langle a_1, \ldots, a_n \rangle$ be a form over $\mathfrak{N}$.*

a) $\mathfrak{D}_T(\varphi) \cup D^t_T(\varphi) \subseteq D^v_T(\varphi)$.

b) For all $x \in \mathfrak{N}$, $x \in D_{\mathfrak{N},T}(\varphi) \Rightarrow x \in D^v_T(\varphi)$, i.e., $D_{\mathfrak{N},T}(\varphi) \subseteq D^v_T(\varphi)$.

c) If $\dim(\varphi) \leq 3$, *then* $\mathfrak{D}_T(\varphi) \subseteq D_{\mathfrak{N},T}(\varphi)$.

Proof. a) It is clear that $D^t_T(\varphi) \subseteq D^v_T(\varphi)$; for $\dim(\varphi) = 2$, $\mathfrak{D}_T(\varphi) = D^v_T(\varphi)$ by definition. Assume the result holds for forms of dimension $n \geq 2$ and let $\varphi = \langle a \rangle \oplus \psi$, with $\psi = \langle b_1, \ldots, b_n \rangle$. If $x \in \mathfrak{D}_T(\varphi)$, there is $u \in \mathfrak{N}$ with $x \in D^v_T(a, u)$ and $u \in D^v_T(\psi)$. Hence, there are $s_1, s_2 \in \mathfrak{t}$ and $t_1, t_2 \in T$ so that $s_1x = t_1a + t_2u$; this equation together with $s_2u = \sum_{i=1}^n t'_ib_i$ ($t'_i \in T$), imply $s_1s_2x = s_2t_1a + t_2s_2u = s_2t_1a + t_2(\sum_{i=1}^n t'_ib_i)$ and so $u \in D^v_T(\varphi)$.

b) By 3.5.(b.1), the result holds for 2-forms. We proceed by induction on the dimension of φ; if $\varphi = \langle a \rangle \oplus \psi$ and $x^T \in D_{G_{\mathfrak{N},T}(A)}(\varphi^T)$, the definition of representation in $G_{\mathfrak{N},T}(A)$ (cf. 3.6.(b)) yields $u, z_1, \ldots, z_n \in \mathfrak{N}$ so that $\langle x^T, z_1^T, \ldots, z_n^T \rangle \equiv_\mathfrak{t} \langle a^T \rangle \oplus \psi^T$. Now the definition of isometry in $G_{\mathfrak{N},T}(A)$ furnishes a n-form θ over $\mathfrak{N}$ and $u, v, w_2, \ldots, w_n \in \mathfrak{N}$ so that $\langle x^T, u^T \rangle \equiv_\mathfrak{t} \langle a^T, v^T \rangle$, $\theta^T \equiv_\mathfrak{t} \langle u^T, w_2^T, \ldots, w_n^T \rangle$ and $\psi^T \equiv_\mathfrak{t} \langle v^T, w_2^T, \ldots, w_n^T \rangle$. The isometry $\langle x^T, u^T \rangle \equiv_\mathfrak{t} \langle a^T, v^T \rangle$ entails $x \in D^v_T(a, v)$, while $\psi^T \equiv_\mathfrak{t} \langle v^T, w_2^T, \ldots, w_n^T \rangle$ yields, by the induction hypothesis, $v \in D^v_T(\psi)$. Proceeding as in item (a), it is straightforward to show that $x \in D^v_T(\varphi)$, as desired. The second statement in (b) follows from the first, since $x \in D_{\mathfrak{N},T}(\varphi)$ iff $x^T \in D_{G_{\mathfrak{N},T}(A)}(\varphi^T)$ (cf. 3.8).

c) If $\dim(\varphi) = 2$, by items (a) and (b) of Lemma 3.3, we have then $D_{\mathfrak{N},T}(\varphi) = D^v_T(\varphi) = \mathfrak{D}_T(\varphi)$. Let $\dim(\varphi) = 3$. In *any* π-*SG*, G (presently, $G_{\mathfrak{N},T}(A)$), the condition

(†) $$D_G(b_1, b_2, b_3) = \bigcup \{D_G(b_1, u) : u \in D_G(b_2, b_3)\}$$

tells, for $a \in D_G(b_1, b_2, b_3)$, how to complete $\langle a, \cdot, \cdot \rangle$ so that it becomes G-isometric to $\langle b_1, b_2, b_3 \rangle$: picking $u \in D_G(b_2, b_3)$ so that $a \in D_G(b_1, u)$, then $\langle a, z \rangle \equiv_G \langle b_1, u \rangle$, where $z = ab_1u$, and $\langle b_2, b_3 \rangle \equiv_G \langle u, c \rangle$, where $c = b_2b_3u$ ([SG 3]); the third condition required to get $\langle a, z, c \rangle \equiv_G \langle b_1, b_2, b_3 \rangle$, namely $\langle z, c \rangle \equiv_G \langle z, c \rangle$, holds automatically. ■

Not every morphism of p-rings induces a morphism of the π-special groups associated to non zero-divisors because a ring morphism might not take a non zero-divisor to a non zero-divisor. To establish the analog in the present setting of Lemma 2.8 (p. 14) in [DM5] (or Lemma 8.14 in [DM2]), and the Remarks in 2.27.(c) in [DM5], we introduce the following

Definition 3.10 *Let $\langle A, T \rangle$ and $\langle R, P \rangle$ be p-rings. Let $\mathfrak{t} = T \cap \mathfrak{N}_A$ and $\mathfrak{p} = P \cap \mathfrak{N}_R$. A* **$\mathfrak{N}$p-ring morphism** *from $\langle A, T \rangle$ to $\langle R, P \rangle$ is a a ring morphism, $h : A \longrightarrow R$, such that* $h[T] \subseteq P$ and $h[\mathfrak{N}_A] \subseteq \mathfrak{N}_R$, *that is, h is a p-ring morphism taking $\mathfrak{N}_A$ into $\mathfrak{N}_R$. Clearly, we have $h[\mathfrak{t}] \subseteq \mathfrak{p}$.*

Lemma 3.11 *With notation as in 3.10, let $h : \langle A, T \rangle \longrightarrow \langle R, P \rangle$ be a $\mathfrak{N}$p-ring morphism. The morphism h induces a morphism of proto-special groups*

(∗) $$h^\pi : G_{\mathfrak{N},T}(A) \longrightarrow G_{\mathfrak{N},P}(R),$$

given by $h^\pi(a^T) = h(a)^P$. Moreover, $Id^\pi_A = Id_{G_{\mathfrak{N},T}(A)}$ and if $g : \langle R, P \rangle \longrightarrow \langle R', P' \rangle$ is a $\mathfrak{N}$p-ring morphism, then $(g \circ h)^\pi = g^\pi \circ h^\pi$.

Proof. Since h is a $\mathfrak{N}$p-ring morphism, $h^* = h \restriction \mathfrak{N}_A : \mathfrak{N}_A \longrightarrow \mathfrak{N}_R$ preserves multiplication, with $h^*(1) = 1$ and $h^*(-1) = -1$. To see that h^π is well-defined, assume, for $a, b \in \mathfrak{N}_A$, that $a^T = b^T$; then, there are $s, t \in \mathfrak{t}$ so that $sa = tb$ and so $h(s)h(a) = h(t)h(b)$. Since h takes $\mathfrak{t}$ to $\mathfrak{p}$, we obtain $h(a)^P = h(b)^P$, as needed. It is straightforward that h^π is a morphism of groups of exponent two, taking -1 to -1. To show that it is a morphism of π-SGs, it must be shown that for a, b, $c \in \mathfrak{N}_A$,

(I) $$a^T \in D_{G_{\mathfrak{N},T}(A)}(b^T, c^T) \;\Rightarrow\; h^\pi(a^T) = h(a)^P \in D_{G_{\mathfrak{N},P}(R)}(h(b)^P, h(c)^P).$$

By Proposition 3.5.(b.1), (I) is equivalent to

(II) $$a \in D^v_T(b, c) \;\Rightarrow\; h(a) \in D^v_P(h(b), h(c)).$$

The antecedent in (II) means there are $s \in \mathfrak{t}$, $t_1, t_2 \in T$ so that $sa = t_1b + t_2c$, whence, $h(s)h(a) = h(t_1)h(b) + h(t_2)h(c)$ and (II) follows immediately, recalling that h is a p-ring morphism, taking $\mathfrak{t} \subseteq A$ to $\mathfrak{p} \subseteq R$. The remaining statements are clear. ■

Remark 3.12 By Lemma 1.13.(d.2) (p. 7) of [DM5], any π-SG morphism preserves isometry of forms of arbitrary dimension. ■

4 $\mathfrak{N}T$ - Quadratic Faithfulness

Besides the already adopted conceptual framework and notational conventions, we set down:

4.1 Notation. Let A be a ring and let $\mathfrak{N}$ be the multiplicative set of non zero-divisors in A. Let $T = A^2$ or a proper preorder on A. Recall that $\mathfrak{t} = T \cap \mathfrak{N}$.

a) If $a_1, \ldots, a_n$ are elements of $\mathfrak{N}$, let $\mathscr{M}(a_1, \ldots, a_n)$ be the diagonal matrix whose non-zero entries are $a_1, \ldots, a_n$.

b) Let $M_n(A)$ be the ring of $n \times n$ matrices with coefficients in A. As usual, I_n is the identity matrix in $M_n(A)$. ■

Definition 4.2 *Let A be a ring and let $T = A^2$ or a proper preorder on A. Let $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ be forms over $\mathfrak{N}$. We say φ* **is simply isometric to** *ψ, written $\varphi \approx_{sT} \psi$, if there are $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathfrak{t}$ and a matrix $M \in M_n(A)$, so that $\det(M) \in \mathfrak{N}$ and $M\mathcal{M}(u_1a_1, \ldots, u_na_n)M^t = \mathcal{M}(v_1b_1, \ldots, v_nb_n)$.*

If T is clear from context, write $\approx_s$ for $\approx_{sT}$. Moreover, if $T = A^2$, $\approx$ will stand for $\approx_{sA^2}$.

Lemma 4.3 *Let A be a ring and let $T = A^2$ or a proper preorder on A.*

a) The relation of simple isometry between $\mathfrak{N}$-forms of the same dimension is reflexive and symmetric.

b) If $T = A^2$, then, for each integer $n \geq 1$, $\approx$ is an equivalence relation on the set of n-forms over $\mathfrak{N}$.

Proof. a) Clearly, $\approx_s$ is reflexive; for symmetry, assume $\varphi = \langle a_1, \ldots, a_n \rangle \approx_s \psi = \langle b_1, \ldots, b_n \rangle$, i.e.,

(I) $$M\mathcal{M}(u_1a_1, \ldots, u_na_n)M^t = \mathcal{M}(v_1b_1, \ldots, v_nb_n),$$

with $\det(M) \in \mathfrak{N}$, $u_i, v_i \in \mathfrak{t}$, $1 \leq i \leq n$. Let $c = \det(M)$. By Proposition 8, p. 334, in [La], there is a matrix $\widetilde{M}$ in $M_n(A)$ so that $M\widetilde{M} = \widetilde{M}M = c(I_n)$. Hence, $\det(\widetilde{M})c = \det(cI_n) = c^n$; since $c \in \mathfrak{N}$, the latter equality entails $\det(\widetilde{M}) = c^{n-1} \in \mathfrak{N}$. From (I) we then obtain,

$$\begin{aligned}\widetilde{M}\mathcal{M}(v_1b_1, \ldots, v_nb_n)\widetilde{M}^t &= \widetilde{M}M\mathcal{M}(u_1a_1, \ldots, u_na_n)M^t\widetilde{M}^t = cI_n\mathcal{M}(u_1a_1, \ldots, u_na_n)cI_n \\ &= \mathcal{M}(c^2u_1a_1, \ldots, c^2u_na_n),\end{aligned}$$

whence $\psi \approx_s \varphi$, as needed.

b) If $T = A^2$, equation (I) takes the form

(II) $$M\mathcal{M}(x_1^2a_1, \ldots, x_n^2a_n)M^t = \mathcal{M}(y_1^2b_1, \ldots, y_n^2b_n) = \mathcal{M}(y_1, \ldots, y_n)\mathcal{M}(\psi)\mathcal{M}(y_1, \ldots, y_n),$$

where $\mathcal{M}(\psi) = \mathcal{M}(b_1, \ldots, b_n)$, with $x_i, y_i \in \mathfrak{N}$, $1 \leq i \leq n$.

Claim : *With notation as above, if D is a diagonal matrix in $M_n(A)$, then $\widetilde{D} = \widetilde{D^t} = \widetilde{D}^t$.*

Proof of Claim. Follows straightforwardly from the definition of $\widetilde{D}$ and the proof of Proposition 8 (pp. 334-335) in [La]. □

To simplify exposition, write $\mathcal{M}(\overline{y})$ for $\mathcal{M}(y_1, \ldots, y_n)$. Multiplying (II) on both sides by $\widetilde{\mathcal{M}(\overline{y})} = \widetilde{\mathcal{M}(\overline{y})}^t$ obtains

(III) $$\begin{aligned}\widetilde{\mathcal{M}(\overline{y})}M\mathcal{M}(x_1^2a_1, \ldots, x_n^2a_n)M^t\widetilde{\mathcal{M}(\overline{y})}^t &= \widetilde{\mathcal{M}(\overline{y})}\mathcal{M}(\overline{y})\mathcal{M}(\psi)\mathcal{M}(\overline{y})\widetilde{\mathcal{M}(\overline{y})} = dI_n\mathcal{M}(\psi)dI_n \\ &= d^2I_n\mathcal{M}(\psi),\end{aligned}$$

where $d = \det(\mathcal{M}(\overline{y})) = y_1 \cdot \ldots \cdot y_n \in \mathfrak{N}$. Let $N = \widetilde{\mathcal{M}(\overline{y})}M$; recalling (from the proof of (a)), that $\det(\widetilde{\mathcal{M}(\overline{y})}) = d^{n-1}$, it is clear that $\det(N) = \det(\widetilde{\mathcal{M}(\overline{y})})\det(M) \in \mathfrak{N}$. Moreover, (III) may be written

(IV) $$N\mathcal{M}(x_1^2a_1, \ldots, x_n^2a_n)N^t = d^2I_n\mathcal{M}(\psi).$$

To prove transitivity of $\approx$, assume, in addition, that $\psi \approx \theta$, with $\theta = \langle c_1, \ldots, c_n \rangle$. As in the proof of (IV) we get $w_i, z_i \in \mathfrak{N}$ and $K \in M_n(A)$ so that

(V) $$K\mathcal{M}(\overline{w})\mathcal{M}(\psi)\mathcal{M}(\overline{w})K^t = \mathcal{M}(z_1^2c_1, \ldots, z_n^2c_n).$$

Since constant multiples of the identity matrix are in the center of the ring $M_n(A)$, (V) yields

(VI) $$K\mathcal{M}(\overline{w})\big[d^2I_n\mathcal{M}(\psi)\big]\mathcal{M}(\overline{w})K^t = d^2I_n\mathcal{M}(z_1^2c_1, \ldots, z_n^2c_n) = \mathcal{M}(d^2z_1^2c_1, \ldots, d^2z_n^2c_n).$$

Now (IV) and (VI) imply $\varphi \approx \theta$, establishing the transitivity of $\approx$. ∎

Definition 4.4 *Let A be a ring and let $T = A^2$ or a proper preorder of A. Let φ, ψ be n-forms over $\mathfrak{N}$. We say that φ is* **$\mathfrak{N}T$-isometric** *to ψ, written $\varphi \approx_{\mathfrak{N}T} \psi$, if there is a sequence φ_0, ..., φ_m of n-forms over $\mathfrak{N}$, such that $\varphi_0 = \varphi$, $\varphi_m = \psi$, and for each $1 \leq i \leq m-1$, $\varphi_i \approx_s \varphi_{i+1}$. The integer m is called the* **length** *of the sequence of simple isometries connecting φ to ψ.*

Lemma 4.5 *With notation as above, let $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ be $\mathfrak{N}$-forms, with $n \geq 1$. Write $\approx_s$ for $\approx_{sT}$.*

a) For each $n \geq 1$, $\approx_{\mathfrak{N}T}$ is an equivalence relation on n-forms over $\mathfrak{N}$. If $T = A^2$, then $\approx_{\mathfrak{N}T}$ coincides with $\approx$.

b) For $c \in \mathfrak{N}$, $\varphi \approx_{\mathfrak{N}T} \psi \;\Rightarrow\; c\varphi \approx_{\mathfrak{N}T} c\psi$.

c) If θ is a $\mathfrak{N}$-form and $\varphi \approx_{\mathfrak{N}T} \psi$, then

(1) $\theta \oplus \varphi \approx_{\mathfrak{N}T} \theta \oplus \psi$; (2) $\theta \otimes \varphi \approx_{\mathfrak{N}T} \theta \otimes \psi$.

(3) *The operations $\oplus$ and $\otimes$ are associative with respect to $\approx_{\mathfrak{N}T}$.*

d) If φ_i, ψ_i $(i = 1, 2)$ are $\mathfrak{N}$-forms of the same dimension, then

$$\varphi_1 \approx_{\mathfrak{N}T} \psi_1 \;\text{ and }\; \varphi_2 \approx_{\mathfrak{N}T} \psi_2, \;\Rightarrow\; \begin{cases} \varphi_1 \oplus \varphi_2 \approx_{\mathfrak{N}T} \psi_1 \oplus \psi_2; \\ \varphi_1 \otimes \varphi_2 \approx_{\mathfrak{N}T} \psi_1 \otimes \psi_2. \end{cases}$$

e) $\varphi \approx_{\mathfrak{N}T} \psi \;\Rightarrow\; d(\varphi)^T = d(\psi)^T$, where $d(\varphi)$ is the discriminant of φ.

f) If σ is a permutation of $\{1, \ldots, n\}$, then $\varphi \approx_{\mathfrak{N}T} \psi$ implies $\varphi^\sigma \approx_{\mathfrak{N}T} \psi^\sigma$, where $\varphi^\sigma = \langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} \rangle$. Hence, the operations $\oplus$ and $\otimes$ are commutative with respect to $\approx_{\mathfrak{N}T}$.

g) (1) $\varphi \approx_{\mathfrak{N}T} \psi \;\Rightarrow\; D^v_T(\varphi) = D^v_T(\psi)$;

(2) $\varphi = \langle a_1, \ldots, a_n \rangle \approx_{\mathfrak{N}T} \psi \;\Rightarrow\; \{a_1, \ldots, a_n\} \in D^v_T(\psi)$.

h) (1) If $t = (t_1, \ldots, t_n) \in \mathfrak{t}^n$, then $D^v_T(t\varphi) = D^v_T(t_1a_1, \ldots, t_na_n) = D^v_T(\varphi)$.

(2) For forms $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ over $\mathfrak{N}$, $a_k^T = b_k^T$, for $1 \leq k \leq n$, implies $\varphi \approx_{\mathfrak{N}T} \psi$.

(3) If x, $y \in \mathfrak{N}$ and φ is a n-form over $\mathfrak{N}$, then $x \in D^v_T(\varphi)$ and $x^T = y^T$ implies $y \in D^v_T(\varphi)$

i) Let $1 \leq k \leq m$ be integers, $\varphi_1, \ldots, \varphi_m$ be $\mathfrak{N}$-forms and let $x_j \in D^v_T(\varphi_j)$, $1 \leq j \leq k$. Then,

$$D^v_T(x_1, \ldots, x_k) \subseteq D^v_T\left(\oplus_{j=1}^m \varphi_j\right).$$

In particular, if ψ is a m-form over $\mathfrak{N}$, then $D^v_T(\varphi) \subseteq D^v_T(\varphi \oplus \psi)$.

j) For a, $b \in \mathfrak{N}$, $\langle a \rangle \approx_{\mathfrak{N}T} \langle b \rangle$ iff $a^T = b^T$.

k) If $f : \langle A, T \rangle \longrightarrow \langle R, P \rangle$ is a $\mathfrak{N}p$-ring morphism and φ, ψ are n-forms over A, then $\varphi \approx_{\mathfrak{N}T} \psi$ implies $f \star \varphi \approx_{\mathfrak{N}P} f \star \psi$. [3]

Proof. Throughout the proof, $\varphi \approx_s \psi$ shall be explicitly written as

$$\text{(I)} \qquad M\mathcal{M}(u_1a_1, \ldots, u_na_n)M^t = \mathcal{M}(v_1b_1, \ldots, v_nb_n),$$

with $\det(M) \in \mathfrak{N}$ and u_i, $v_i \in \mathfrak{t}$, an equation to be used repeatedly below.

a) Clearly, $\approx_{\mathfrak{N}T}$ is the transitive closure of the reflexive and symmetric relation of simple isometry and it is well-known that this closure yields an equivalence relation on the n-forms over $\mathfrak{N}$. The second assertion in (a) is immediate from 4.3.(b).

b) It suffices to show that multiplication by $c \in \mathfrak{N}$ preserves simple isometry. From equation (I) above, we obtain

[3] Recall: if $f : A \longrightarrow R$ is a map and $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over A, $f \star \varphi = \langle f(a_1), \ldots, f(a_n) \rangle$ is the image form over R.

$$\begin{aligned} M\mathscr{M}(c^2u_1(ca_1), \ldots, c^2u_n(ca_n))M^t &= M\mathscr{M}(c^3u_1a_1, \ldots, c^3u_na_n)M^t \\ &= \mathscr{M}(c^3v_1b_1, \ldots, c^3v_nb_n) \\ &= \mathscr{M}(c^2v_1(cb_1), \ldots, c^2v_n(cb_n)), \end{aligned}$$

as needed.

c) (1) Again it suffices to check that the statement holds for simple isometry and $\theta = \langle a \rangle$, $a \in \mathfrak{N}$. If M is the matrix in equation (I), consider $K \in M_{n+1}(A)$ given by

$$K = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$$

i.e., K has as its first row and column $\langle 1, 0, 0, \ldots, 0 \rangle$ and $M \in M_n(A)$ as its (1,1) minor (usually denoted by K_{11}). It is straightforward that

$$K\mathscr{M}(a, u_1a_1, \ldots, u_na_n)K^t = \mathscr{M}(a, v_1b_1, \ldots, v_nb_n),$$

as desired. Item (2) follows easily from (1) and item (b), while (3) is clear.

d) Adding φ_1 to both sides $\varphi_2 \approx_{\mathfrak{N}T} \psi_2$, yields $\varphi_1 \oplus \varphi_2 \approx_{\mathfrak{N}T} \varphi_1 \oplus \psi_2$, while adding ψ_2 to both sides of $\varphi_1 \approx_{\mathfrak{N}T} \psi_1$ obtains $\varphi_1 \oplus \psi_2 \approx_{\mathfrak{N}T} \psi_1 \oplus \psi_2$, and the transitivity of $\approx_{\mathfrak{N}T}$ yields the desired conclusion. Similarly, one proves that $\approx_{\mathfrak{N}T}$ preserves tensor products.

e) If M is the matrix in (I), then, since $\det(M) := c \in \mathfrak{N}$, we obtain, computing determinants on both sides of (I), $c^2u_1 \cdots u_na_1 \cdots a_n = v_1 \cdots v_nb_1 \cdots b_n$ and so $d(\varphi)^T = d(\psi)^T$; thus, the same will be true of $\approx_{\mathfrak{N}T}$.

f) It is well-known that $\varphi \approx_s \varphi^\sigma$, where the matrices involved are invertible. Hence, the conclusion follows from the transitivity of $\approx_{\mathfrak{N}T}$. It is then clear that $\oplus$ is commutative with respect to $\approx_{\mathfrak{N}T}$, whence the same is true for $\otimes$.

g) (1) It suffices to verify the statement for simple isometry and use induction. Suppose $a \in D_T^v(\psi)$, i.e.,

(*) $$sa = \textstyle\sum_{j=1}^n t_jb_j. \qquad (s \in \mathfrak{t},\ t_j \in T,\ 1 \le j \le n)$$

It follows straightforwardly from (I) above that for each $1 \le j \le n$,

(**) $$v_jb_j = \textstyle\sum_{i=1}^n x_{ji}^2u_ia_i,$$

where $(x_{j1}, \ldots, x_{jn})$ is the j^{th}-row of M, Thus, (*) and (**) yield

$$\begin{aligned} sv_1 \cdots v_na &= \textstyle\sum_{j=1}^n t_jv_1 \cdots v_{j-1}v_{j+1} \cdots v_n(v_jb_j) \\ &= \textstyle\sum_{j=1}^n t_jv_1 \cdots v_{j-1}v_{j+1} \cdots v_n(\sum_{i=1}^n x_{ji}^2u_ia_i), \end{aligned}$$

wherefrom, upon collecting terms in each a_i, $1 \le i \le n$, yields $a \in D_T^v(\varphi)$. Since simple isometry is symmetric, we also have $D_T^v(\varphi) \subseteq D_T^v(\psi)$, entailing the equality of these sets. Item (2) follows directly from (1), because $\{a_1, \ldots, a_n\} \subseteq D_T^v(\varphi)$.

h) (1) Clearly, $D_T^v(t_1a_1, \ldots, t_na_n) \subseteq D_T^v(\varphi)$. For the reverse inclusion, assume $a \in D_T^v(\varphi)$, that is, $sa = \sum_{i=1}^n x_ia_i$, with $s \in \mathfrak{t}$ and $x_i \in T$, $1 \le i \le n$. Then,

$$s(t_1 \cdots t_n)a = \textstyle\sum_{i=1}^n (t_1 \cdots t_{i-1} \cdot x_i \cdot t_{i+1} \cdots t_n)(t_ia_i),$$

and so $a \in D_T^v(t_1a_1, \ldots, t_na_n)$, as desired.

(2) The hypothesis means that there are $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathfrak{t}$ so that $s_ka_k = t_kb_k$, $1 \le k \le n$, and the conclusion follows immediately from item (h.1).

(3) Let $\varphi = \langle a_1, \ldots, a_n \rangle$; there are $s, t \in \mathfrak{t}$ so that $sx = ty$, together with $w \in \mathfrak{t}$ and $u_1, \ldots, u_n \in T$ so that $wx = \sum_{k=1}^n u_ka_k$. Hence, $wty = wsx = \sum_{k=1}^n (su_k)a_k$, with $wt \in \mathfrak{t}$ and $su_k \in T$, $1 \le k \le n$; thus, $y \in D_T^v(\varphi)$, as needed.

Item (i) is straightforward, while (j) is an immediate consequence of the definitions.

k) It suffices to establish the result for simple isometry steps. Let $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$. With the relation $\varphi \approx_{sT} \psi$ written as in the matrix equality (I), by taking images under h on both sides and recalling items (b.1) and (b.2) in Remarks 2.27 (p. 24) in [DM5], we get

$$(+) \qquad h(M)\, \mathscr{M}(h(s_1)h(a_1), \ldots, h(s_n)h(a_n))\, h(M)^t \;=\; \mathscr{M}(h(t_1)h(b_1), \ldots, h(t_n)h(b_n)).$$

Since for all $1 \le k \le n$, $h(s_k)$, $h(t_k) \in \mathfrak{p} = P \cap \mathfrak{N}_R$ and $\det h(M) = h(\det M) \in \mathfrak{N}_R$, equality (+) implies that $h \star \varphi \approx_{sP} h \star \psi$ in R, as needed. ∎

Remark 4.6 The proof of item (c.1) in 4.5 shows that if θ is a k-form over $\mathfrak{N}$ and $\varphi_0, \varphi_1, \ldots, \varphi_m$ is a sequence of n-forms over $\mathfrak{N}$, witnessing $\varphi_0 \approx_{\mathfrak{N}T} \varphi_m$, then the sequence $\theta \oplus \varphi_0, \ldots, \theta \oplus \varphi_m$ of $(k+n)$-forms over $\mathfrak{N}$, witnesses $\theta \oplus \varphi_0 \approx_{\mathfrak{N}T} \theta \oplus \varphi_m$. ∎

Lemma 4.7 *Let $\langle A, T \rangle$ be a p-ring.*

a) If binary value representation is 2-transversal, i.e., for all x, $y \in \mathfrak{N}$, $D^v_T(x, y) = D^t_T(x, y)$, *then for all a, b, c, $d \in \mathfrak{N}$, the following are equivalent:*

(1) $\langle a^T, b^T \rangle \equiv_{\mathfrak{t}} \langle c^T, d^T \rangle$; (2) $\langle a, b \rangle \approx_{sT} \langle c, d \rangle$;

(3) $\langle a, b \rangle \approx_{\mathfrak{N}T} \langle c, d \rangle$.

b) In case $T = A^2$, we can dispense with the transversality assumption and have, for all a, b, c, $d \in \mathfrak{N}$, $\left\langle \widehat{a}, \widehat{b} \right\rangle \equiv \left\langle \widehat{c}, \widehat{d} \right\rangle \;\Leftrightarrow\; \langle a, b \rangle \approx \langle c, d \rangle$.

c) Let φ, ψ be $\mathfrak{N}$-forms of the same dimension ≥ 2 over $\mathfrak{N}$. Then

$$\varphi^T \equiv_{\mathfrak{t}} \psi^T \text{ (in } G_{\mathfrak{N},T}(A)) \;\Rightarrow\; \varphi \approx_{\mathfrak{N}T} \psi.$$

Proof. a) (1) $\Rightarrow$ (2): The definition of isometry in $G_{\mathfrak{N},T}(A)$ (see 3.4) yields $a^T b^T = c^T d^T$ and $D^v_T(a, b) = D^v_T(c, d)$. These equalities and the transversality assumption furnish s_1, s_2, t, t_1, $t_2 \in \mathfrak{t}$ so that $s_1 d = abct$ and $s_2 c = t_1 a + t_2 b$. Hence,

$$\text{(I)} \qquad s_1 s_2 d = (tab)s_2 c = tab(t_1 a + t_2 b) = (tt_2 b^2)a + (tt_1 a^2)\, b.$$

Set $M = \begin{pmatrix} t_1 & t_2 \\ -bt & at \end{pmatrix}$; then $\det(M) = tt_1 a + tt_2 b = t(t_1 a + t_2 b) = s_2 t\, c \in \mathfrak{N}$. Moreover, taking into account the equations in (I), obtains

$$\begin{aligned} M\begin{pmatrix} t_2 a & 0 \\ 0 & t_1 b \end{pmatrix} M^t &= \begin{pmatrix} t_1 t_2 a & t_2 t_1 b \\ -abtt_2 & abtt_1 \end{pmatrix}\begin{pmatrix} t_1 & -bt \\ t_2 & at \end{pmatrix} \\ &= \begin{pmatrix} t_1^2 t_2 a + t_2^2 t_1 b & -tt_1 t_2 ab + tt_1 t_2 ab \\ -abtt_1 t_2 + abtt_1 t_2 & b^2 t^2 t_2 a + a^2 t^2 t_1 b \end{pmatrix} \\ &= \begin{pmatrix} t_1 t_2 (t_1 a + t_2 b) & 0 \\ 0 & t[(b^2 tt_2)a + (a^2 tt_1)b] \end{pmatrix} = \begin{pmatrix} s_2 t_1 t_2 c & 0 \\ 0 & ts_1 s_2 d \end{pmatrix}. \end{aligned}$$

By 4.2, this establishes $\langle a, b \rangle \approx_{sT} \langle c, d \rangle$.

(2) $\Rightarrow$ (1): By 4.2, the relation $\langle a, b \rangle \approx_{sT} \langle c, d \rangle$ is equivalent to the existence of a matrix $M \in M_2(A)$ and elements u, v, x, $y \in \mathfrak{t} = T \cap \mathfrak{N}$ such that $\det(M) \in \mathfrak{N}$ and

$$M \begin{pmatrix} ua & 0 \\ 0 & vb \end{pmatrix} M^t = \begin{pmatrix} xc & 0 \\ 0 & yd \end{pmatrix}.$$

Taking determinants on both sides of this equality yields $xycd = \det(M)^2 uvab$ and so $a^T b^T = c^T d^T$. Also, $xc = s^2 ua + t^2 vb$, where (s, t) is the first row of M. Since x, u, $v \in \mathfrak{t}$, we obtain $c \in D^v_T(a, b) \cap D^v_T(c, d)$ and 3.3.(b) entails $\langle a^T, b^T \rangle \equiv_{\mathfrak{t}} \langle c^T, d^T \rangle$.

b) $(\Rightarrow)$: The hypothesis entails $\widehat{ab} = \widehat{cd}$ and $D^v_T(a, b) = D^v_T(c, d)$. Thus, $c \in D^v_T(a, b)$ and $\widehat{d} = \widehat{abc}$. Thus, there are α β, $e \in \mathfrak{N}$, s, $t \in A$ so that

(I) $$\alpha^2 d = cabe^2 \quad \text{and} \quad \beta^2 c = s^2 a + t^2 b.$$

Then,

(II) $$\alpha^2\beta^2 d \;=\; (\beta^2 c)(abe^2) \;=\; abe^2(s^2a + t^2b) \;=\; (t^2b^2e^2)a + (s^2a^2e^2)b.$$

Consider the matrix

$$M \;=\; \begin{pmatrix} s & t \\ -bte & ase \end{pmatrix}$$

Then, $\det(M) = s^2ae + t^2be = e(s^2a + t^2b) \;=\; \beta^2 ce \in \mathfrak{N}$. Now, taking into account (I) and (II) above, obtains

$$M\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}M^t \;=\; \begin{pmatrix} \beta^2 c & 0 \\ 0 & \beta^2\alpha^2 d \end{pmatrix},$$

whence $\langle a, b \rangle \approx \langle c, d \rangle$.

The implication $(\Leftarrow)$ can be established employing the same reasoning and results as in the proof of (2) $\Rightarrow$ (1) in item (a). Since clearly (2) implies (3), to finish the proof of (a) it suffices to show:

(3) $\Rightarrow$ (1) : Let $\langle x_1, y_1 \rangle, \ldots, \langle x_k, y_k \rangle$ be a witnessing sequence for the relation in (3). Hence, $\langle x_i, y_i \rangle \approx_{sT} \langle x_{i+1}, y_{i+1} \rangle$ for $1 \le i < k$ (and $\langle a, b \rangle = \langle x_1, y_1 \rangle$, $\langle x_k, y_k \rangle = \langle c, d \rangle$). The equivalence of (1) and (2) yields $\langle x_i^T, y_i^T \rangle \equiv_t \langle x_{i+1}^T, y_{i+1}^T \rangle$, for $1 \le i < k$. The transitivity of the relation $\equiv_t$ on binary forms, cf. 3.5.(a.1), entails $\langle a^T, b^T \rangle \equiv_t \langle c^T, d^T \rangle$.

c) By items (a) and (b), the asserted implication holds for $\dim(\varphi) = 2$. We proceed by induction, assuming it holds for forms of dimension n. Let $\varphi = \langle a \rangle \oplus \varphi_1$ and $\psi = \langle b \rangle \oplus \psi_1$, where φ_1 and ψ_1 are n-forms over $\mathfrak{N}$. By the definition of isometry in $G_{\mathfrak{N},T}(A)$, the hypothesis $\varphi^T \equiv_t \psi^T$ yields elements u, v, $z_2, \ldots, z_3 \in \mathfrak{N}$ so that

(IV) $$\langle a^T, u^T \rangle \equiv_t \langle b^T, v^T \rangle, \quad \varphi_1^T \equiv_t \langle u^T, z_2^T, \ldots, z_n^T \rangle \quad \text{and} \quad \psi_1^T \equiv_t \langle v^T, z_2^T, \ldots, z_n^T \rangle.$$

The induction hypothesis and item (a) entail

$$\langle a, u \rangle \approx_{\mathfrak{N}T} \langle b, v \rangle, \quad \varphi_1 \approx_{\mathfrak{N}T} \langle u, z_2, \ldots, z_n \rangle \quad \text{and} \quad \psi_1 \approx_{\mathfrak{N}T} \langle v, z_2, \ldots, z_n \rangle.$$

Now items (a), (c) and (d) in Lemma 4.5 imply

$$\varphi \approx_{\mathfrak{N}T} \langle a \rangle \oplus \langle u, z_2, \ldots, z_n \rangle \;=\; \langle a, u \rangle \oplus \langle z_2, \ldots, z_n \rangle \approx_{\mathfrak{N}T} \langle b, v \rangle \oplus \langle z_2, \ldots, z_n \rangle \;=\;$$
$$=\; \langle b \rangle \oplus \langle v, z_2, \ldots, z_n \rangle \approx_{\mathfrak{N}T} \langle b \rangle \oplus \psi_1 = \psi,$$

completing the induction step and the proof. ■

Following the lead of [DM5], we set down

Definition 4.8 *Let A be a ring and let $T = A^2$ or a proper preorder of A. The ring A is said to be* **$\mathfrak{N}T$- faithfully quadratic** *if it satisfies the following requirements:*

[$\mathfrak{N}T$-FQ 1] : *For all a, $b \in \mathfrak{N}$, $D^v_T(a, b) \;=\; D^t_T(a, b)$.*

[$\mathfrak{N}T$-FQ 2] : *For all $n \ge 2$ and all n-forms over $\mathfrak{N}$, $\mathfrak{D}_T(\varphi) = D^v_T(\varphi)$.*

[$\mathfrak{N}T$-FQ 3] : *For all $a \in \mathfrak{N}$ and all forms φ, ψ over $\mathfrak{N}$ of the same dimension,*
$\langle a \rangle \oplus \varphi \approx_{\mathfrak{N}T} \langle a \rangle \oplus \psi \;\Rightarrow\; \varphi \approx_{\mathfrak{N}T} \psi$.

As in [DM5], we write [$\mathfrak{N}T$-FQ 2]$_m$ or [$\mathfrak{N}T$-FQ 3]$_m$, if the statements [$\mathfrak{N}T$-FQ 2] or [$\mathfrak{N}T$-FQ 3] are only required to hold for forms of dimension $2 \le n \le m$.

If $T = A^2$, we say A is **$\mathfrak{N}$-faithfully quadratic** and indicate the corresponding axioms by omitting the mention of T: [$\mathfrak{N}$-FQ i], $i = 1, 2, 3$. Similarly, if $T = \Sigma A^2$, we say A is **$\mathfrak{N}\Sigma$-faithfully quadratic** and write the axioms as [$\mathfrak{N}\Sigma$-FQ i], $i = 1, 2, 3$,

Theorem 4.9 *Let A be a ring and let $T = A^2$ or a proper preorder of A. Let $k, n \geq 2$ be integers. Assume A verifies [$\mathfrak{N}T$-FQ 1] and [$\mathfrak{N}$T-FQ 2]$_n$.*

a) For all m-forms φ over $\mathfrak{N}$, with $2 \leq m \leq n$, $D^v_T(\varphi) = D^t_T(\varphi)$.

b) If $\varphi_1, \ldots, \varphi_k$ are forms over $\mathfrak{N}$ and $\varphi = \bigoplus_{i=1}^k \varphi_i$ is such that $\dim \varphi \leq n$, then

$$\begin{aligned} D^v_T(\varphi) &= \bigcup \{D^v_T(u_1, \ldots, u_k) : u_i \in D^v_T(\varphi_i),\ 1 \leq i \leq k\} \\ &= \bigcup \{D^t_T(v_1, \ldots, v_k) : v_i \in D^t_T(\varphi_i),\ 1 \leq i \leq k\}. \end{aligned}$$

Proof. a) By [$\mathfrak{N}T$-FQ 1], the result is true for $m = 2$. We proceed by induction on m, recalling that, by 3.9.(a), it suffices to verify that $D^v_T(\varphi) \subseteq D^t_T(\varphi)$. Let $\varphi = \langle a_1 \rangle \oplus \psi$, with $m = \dim \psi < n$ and let $x \in D^v_T(\varphi)$. By [$\mathfrak{N}T$-FQ 1], [$\mathfrak{N}$T-FQ 2]$_n$ and the induction hypothesis, there is $u \in D^v_T(\psi) = D^t_T(\psi)$ such that $x \in D^v_T(a_1, u) = D^t_T(a_1, u)$. It is now straightforward that $x \in D^t_T(\varphi)$: if $\psi = \langle c_1, \ldots, c_m \rangle$, there are $\alpha, \beta, s, t, z_1, \ldots, z_m \in$ t, such that $\alpha x = sa_1 + tu$, with $\beta u = \sum_{i=1}^m z_i c_i$, and so $x \in D^t_T(\langle a_1 \rangle \oplus \psi)$, as needed.

b) It suffices to verify the first equality for $k = 2$; a straightforward induction will complete its proof, while the second follows from (a) and the fact that $k \leq n$ and $\max_{1 \leq i \leq k} (\dim \varphi_i) \leq n$. Moreover, we may assume that $n \geq 3$, otherwise there is nothing to prove. Suppose, then, that $\varphi = \varphi_1 \oplus \varphi_2$; by [$\mathfrak{N}$T-FQ 2]$_n$ the result is true if $\dim \varphi_1 = 1$. We proceed by induction on $m = \dim \varphi_1 < n$, letting $\varphi_1 = \langle a_1 \rangle \oplus \psi$, with $\dim \psi = m - 1$. Fix $a \in D^v_T(\varphi)$; by [$\mathfrak{N}$T-FQ 2]$_n$, there is $x \in D^v_T(\psi \oplus \varphi_2)$ such that

(I) $$a \in D^v_T(a_1, x).$$

Since $\dim \psi = m - 1$, the induction hypothesis yields $u \in D^v_T(\psi)$ and $v \in D^v_T(\varphi_2)$ such that

(II) $$x \in D^v_T(u, v).$$

It follows from (I) and (II) that $a \in D^v_T(a_1, u, v)$. Because $n \geq 3$, we have

$$D^v_T(a_1, u, v) = \mathfrak{D}_T(a_1, u, v) \subseteq \bigcup\{D^v_T(z, v) : z \in D^v_T(a_1, u)\},$$

thus there is $z \in D^v_T(a_1, u)$ so that $a \in D^v_T(z, v)$. Since $u \in D^v_T(\psi)$, 4.5.(i) entails $z \in D^v_T(\varphi_1) = D^v_T(\langle a_1 \rangle \oplus \psi)$ and so $a \in D^v_T(z, v)$, with and $v \in D^v_T(\varphi_2)$, completing the induction step. ∎

Theorem 4.10 *Let A be a ring and let $T = A^2$ or a proper preorder of A.*

a) Assume that value representation of $\mathfrak{N}$-forms verifies [$\mathfrak{N}$T-FQ 2]$_3$. If φ is a form of dimension ≤ 3 over $\mathfrak{N}$, then $D^v_T(\varphi) = D_{\mathfrak{N},T}(\varphi)$, that is, an element of $\mathfrak{N}$ is value represented modulo T iff it is isometry represented by φ^T in $G_{\mathfrak{N},T}(A)$.

b) Suppose value representation of $\mathfrak{N}$-forms verifies [$\mathfrak{N}T$-FQ 1], [$\mathfrak{N}$T-FQ 2]$_3$ and [$\mathfrak{N}$T-FQ 3]$_3$. Then,

(1) For all 3-forms φ, ψ over $\mathfrak{N}$, $\varphi \approx_{\mathfrak{N}T} \psi \Leftrightarrow \varphi^T \equiv_t \psi^T$;

(2) $G_{\mathfrak{N},T}(A)$ is a special group, which is reduced iff $\langle A, T \rangle$ is a zdp-ring. In particular, $G_{\mathfrak{N},T}(A)$ is a RSG if T is a partial order.

Proof. a) Lemma 3.9.(b), (c) gives $\mathfrak{D}_T(\varphi) \subseteq D_{\mathfrak{N},T}(\varphi) \subseteq D^v_T(\varphi)$ and so, [$\mathfrak{N}$T-FQ 2]$_3$ implies the equality of these sets.

b) (1) By Lemma 4.7.(c), it suffices to prove ($\Rightarrow$). Let $\varphi = \langle a, x, y \rangle$ and $\psi = \langle b_1, b_2, b_3 \rangle$. By Lemma 4.5.(g.1), $\varphi \approx_{\mathfrak{N}T} \psi$ entails $a \in D^v_T(\psi) = \mathfrak{D}_T(\psi)$. Following the recipe in the proof 3.9.(c) (see (†), therein), we set $z = ab_1u$ and $c = b_2b_3$, where $u \in D^v_T(b_2, b_3)$, to obtain

(I) $\langle a^T, c^T, z^T \rangle \equiv_t \langle b_1^T, b_2^T, b_3^T \rangle$ and

(II) $\langle a^T, z^T \rangle \equiv_t \langle b_1^T, u^T \rangle$ and $\langle b_2^T, b_3^T \rangle \equiv_t \langle u^T, c^T \rangle$.

By Lemma 4.7.(c), (I) entails $\langle a, c, z \rangle \approx_{\mathfrak{N}T} \langle b_1, b_2, b_3 \rangle$. Since $\approx_{\mathfrak{N}T}$ is transitive, this relation implies $\langle a, c, z \rangle \approx_{\mathfrak{N}T} \langle a, x, y \rangle$, and $[\mathfrak{N}\text{T-FQ } 3]_3$ entails $\langle c, z \rangle \approx_{\mathfrak{N}T} \langle x, y \rangle$; now, item (a) in 4.7 yields $\langle x^T, y^T \rangle \equiv_t \langle c^T, z^T \rangle$. This isometry, together with those in (II) above yield

$$\langle a^T, z^T \rangle \equiv_t \langle b_1^T, u^T \rangle, \quad \langle b_2^T, b_3^T \rangle \equiv_t \langle u^T, c^T \rangle \quad \text{and} \quad \langle x^T, y^T \rangle \equiv_t \langle c^T, z^T \rangle,$$

that is equivalent to $\langle a^T, x^T, y^T \rangle \equiv \langle b_1^T, b_2^T, b_3^T \rangle$, as needed.

(2) From the equivalence in (1), we conclude that $\equiv_t$ is 3-transitive, and so $G_{\mathfrak{N},T}(A)$ is a special group. If T is a preorder, then 3.5.(a) implies $G_{\mathfrak{N},T}(A)$ is reduced; in particular, this holds if T is a partial order. ■

Theorem 4.11 *Let A be a ring and let T be a preoder of A or $T = A^2$. Suppose A satisfies* $[\mathfrak{N}T\text{-FQ } 1]$, $[\mathfrak{N}T\text{-FQ } 2]$ and $[\mathfrak{N}\text{T-FQ } 3]_3$. *Then,*

a) For all $\mathfrak{N}$-forms φ, $D^v_T(\varphi) = D_{\mathfrak{N},T}(\varphi)$, i.e., an element of $\mathfrak{N}$ is value represented by φ iff it is isometry represented by φ^T in $G_{\mathfrak{N},T}(A)$.

If in addition A is $\mathfrak{N}T$-faithfully quadratic, then

b) For all $\mathfrak{N}$-forms, φ, ψ, of the same dimension, $\varphi \approx_{\mathfrak{N}T} \psi \Leftrightarrow \varphi^T \equiv_t \psi^T$.

c) $G_{\mathfrak{N},T}(A)$ is a special group, which is reduced if T is a preorder with the zdp; in particular, $G_{\mathfrak{N},T}(A)$ is reduced if T is a partial order.

Proof. By Theorem 4.10.(b.2), we know that $G_{\mathfrak{N},T}(A)$ is a special group.

a) By Lemma 3.9.(a) it suffices verify that $D^v_T(\varphi) \subseteq D_{\mathfrak{N},T}(\varphi)$, which will be achieved by induction on dim $\varphi \geq 2$. It follows from 4.10.(a) that the result holds true for dim $\varphi \leq 3$. Assume it valid for forms of dimension n and let $\varphi = \langle b \rangle \oplus \psi$, where dim $\psi = n$. If $a \in \mathfrak{N}$ is value-represented by $\langle b \rangle \oplus \psi$, then $[\mathfrak{N}\text{T-FQ } 2]_n$ implies that there is $u \in \mathfrak{N}$ such that

(I) $$a \in D^v_T(b, u) \quad \text{and} \quad u \in D^v_T(\psi).$$

The induction hypothesis yields $z_2, \ldots, z_n \in \mathfrak{N}$ such that

(II) $$\langle u^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \psi^T,$$

while the first representation relation in (I) implies $\langle a^T, (abu)^T \rangle \equiv_t \langle b^T, u^T \rangle$. Adding $\langle b^T \rangle$ to both sides of (II), Lemma 1.13.(c), p. 7, in [DM5] yields

(III) $$\langle b^T, u^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \langle b^T \rangle \oplus \psi^T.$$

Since $\langle a^T, (abu)^T \rangle \equiv_t \langle b^T, u^T \rangle$, another application of Lemma 1.13.(c) in [DM5] yields

(IV) $$\langle a^T, (abu)^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \langle b^T, u^T, z_2^T, \ldots, z_n^T \rangle.$$

Now, (III), (IV) and the transitivity of $\equiv_t$ entail

$$\langle a^T, (abu)^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \langle b^T \rangle \oplus \psi^T,$$

wherefrom we conclude $a \in D_{\mathfrak{N},T}(b \oplus \psi)$, as needed.

b) By Lemma 4.7.(c), it is enough to prove the implication ($\Rightarrow$), that we know, by 4.10.(b.1), to hold for forms of dimension ≤ 3. We proceed by induction on dimension; assume the result holds for forms of dimension n and suppose $\varphi = \langle a \rangle \oplus \theta_1$, $\psi = \langle b \rangle \oplus \theta_2$, where dim $\theta_i = n$

$(i = 1, 2)$. If $\varphi \approx_{\mathfrak{N}T} \psi$, then 4.5.(g.2) entails $a \in D^v_T(\psi)$ and so, as above, $[\mathfrak{N}\text{T-FQ } 2]_n$ and item (a) yield $u, z_2, \ldots, z_n \in \mathfrak{N}$ such that

(V) $$\langle a^T, (abu)^T \rangle \equiv_t \langle b^T, u^T \rangle \quad \text{and} \quad \langle u^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \theta_2^T.$$

Adding $\langle b^T \rangle$ to both sides of the second isometry in (V) gives

(VI) $$\langle b^T, u^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \psi^T.$$

On the other hand, by adding $\langle z_2^T, \ldots, z_n^T \rangle$ to both sides, the first isometry in (V) entails,

$$\langle a^T, (abu)^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \langle b^T, u^T, z_2^T, \ldots, z_n^T \rangle,$$

that together with (VI) and the transitivity of $\equiv_t$ implies

(VII) $$\langle a^T, (abu)^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \psi^T.$$

Lemma 4.7.(b) and the assumption $\varphi \approx_{\mathfrak{N}T} \psi$ give:

$$\langle a, abu, z_2, \ldots, z_n \rangle \approx_{\mathfrak{N}T} \psi \approx_{\mathfrak{N}T} \varphi = \langle a \rangle \oplus \theta_1.$$

Now, using $[\mathfrak{N}T\text{-FQ } 3]$ we can cancel out $\langle a \rangle$ to get $\langle abu, z_2, \ldots, z_n \rangle \approx_{\mathfrak{N}T} \theta_1$. Since $\dim \theta_1 = n$, the induction hypothesis applies, to yield $\langle (abu)^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \theta_1^T$, wherefrom, adding $\langle a^T \rangle$ to both sides and using (VII) we get

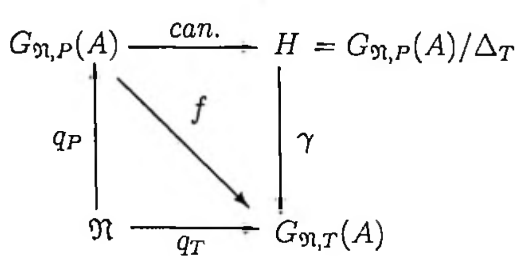
$$\psi^T \equiv_t \langle a^T, (abu)^T, z_2^T, \ldots, z_n^T \rangle \equiv_t \langle a^T \rangle \oplus \theta_1^T = \varphi^T,$$

as required. The reducibility of $G_{\mathfrak{N},T}(A)$ follows from 3.5.(a.2), and, in particular, it holds if T is a partial order, completing the proof. ■

There is a analog of Theorem 3.9, p. 33, [DM5] in our present context. We state the pertinent result, omitting the (rather lengthy) proof.

Theorem 4.12 *Let $\langle A, P \rangle$ be a p-ring. Let T be a proper preorder of A, containing P, verifying* $[\mathfrak{N}T\text{-FQ } 1]$ *and with the zdp. As above, let $\mathfrak{p} = P \cap \mathfrak{N} \subseteq \mathfrak{t} = T \cap \mathfrak{N}$. Let $\iota : \langle A, P \rangle \longrightarrow \langle A, T \rangle$ be the natural inclusion, a $\mathfrak{N}p$-morphism. Let $f := \iota^\pi : G_{\mathfrak{N},P}(A) \longrightarrow G_{\mathfrak{N},T}(A)$, be given, for $a \in A$, by $f(a^P) = a^T$ (cf. 3.10 and 3.11).*

a) Assume $G_{\mathfrak{N},P}(A)$ is a RSG. Then, $\Delta_T = \ker f = \{a^P : a^T = 1^T\}$ is a proper saturated subgroup of $G_{\mathfrak{N},P}(A)$. Moreover, the map γ given by $\gamma(a^P/\Delta_T) = a^T$ is the unique isomorphism *of reduced proto-special groups, making the following diagram commute:*



where q_P, q_T and can. are the natural quotient maps. Hence, $G_{\mathfrak{N},T}(A)$ is a reduced special group.

b) Assume $\langle A, P \rangle$ is a $\mathfrak{N}P$-faithfully quadratic zdp-ring (hence, $G_{\mathfrak{N},P}(A)$ is a RSG, cf. 4.10.(b.2)). *The following are equivalent:*

(1) *A is $\mathfrak{N}T$-faithfully quadratic;*

(2) *For all $x, a_1, \ldots, a_n \in \mathfrak{N}$, if $x \in D^v_T(a_1, \ldots, a_n)$, then there are $x_2, \ldots, x_n \in \mathfrak{N}$ and $t_1, \ldots, t_n \in \mathfrak{t}$ such that $\langle x, x_2, \ldots, x_n \rangle \approx_{\mathfrak{N}P} \langle t_1 a_1, \ldots, t_n a_n \rangle$;*

(3) *For all $\mathfrak{N}$-forms φ, $D^v_T(\varphi) = D^t_T(\varphi)$, i.e., representation of $\mathfrak{N}$-forms of arbitrary dimension is T-transversal.* ■

Proposition 4.13 *Let $\langle A, T \rangle$ be $\mathfrak{N}T$-faithfully quadratic p-ring. If φ, ψ are forms of dimension $n \geq 2$ over $\mathfrak{N}$ and $\varphi \approx_{\mathfrak{N}T} \psi$, then there is a witnessing sequence for this T-isometry of length at most $\ell(n) = 2^{n-1} - 1$.*

Proof. The result is clear for $n = 2$ in view of 4.7.(a), showing that $\ell(n) \leq 1$. We proceed by induction, assuming the result true for $n \geq 2$. Let $\varphi = \langle x \rangle \oplus \varphi_1$, $\psi = \langle y \rangle \oplus \psi_1$ satisfy $\varphi \approx_{\mathfrak{N}T} \psi$. If $G_{\mathfrak{N},T}(A)$ is the special group associated to $\langle A, T \rangle$ as in Theorem 4.11, its item (b) yields $\varphi^T \equiv_{\mathfrak{t}} \psi^T$. The definition of isometry in $G_{\mathfrak{N},T}(A)$ furnishes $u, v \in A$ and a $(n-1)$-form θ over $\mathfrak{N}$ so that

(I) $\quad\langle x^T, u^T \rangle \equiv_{\mathfrak{t}} \langle y^T, v^T \rangle,\quad \varphi_1^T \equiv_{\mathfrak{t}} \langle u^T \rangle \oplus \theta^T \quad\text{and}\quad \psi_1^T \equiv_{\mathfrak{t}} \langle v^T \rangle \oplus \theta^T.$

But then, (I) and 4.11.(b) entail

(II) $\quad\langle x, u \rangle \approx_{\mathfrak{N}T} \langle y, v \rangle,\quad \varphi_1 \approx_{\mathfrak{N}T} \langle u \rangle \oplus \theta \quad\text{and}\quad \psi_1 \approx_{\mathfrak{N}T} \langle v \rangle \oplus \theta.$

By Remark 4.6, since the first isometry in (II) has a witnessing sequence of length ≤ 1, it follows that $\langle x, u \rangle \oplus \theta \approx_{\mathfrak{N}T} \langle y, v \rangle \oplus \theta$ also has a witnessing sequence of length ≤ 1. By the induction hypothesis, the last two isometries in (II) have witnessing sequences of length at most $\ell(n)$; concatenating these sequences yields a witnessing sequence for $\varphi \approx_{\mathfrak{N}T} \psi$ of length at most $2\ell(n) + 1 = 2^n - 1 = \ell(n+1)$, as needed. ■

Remark 4.14 a) Let $\langle R, P \rangle$ be a p-ring. If $\mathfrak{N}_R = R^{\times}$ (and so $\mathfrak{p} = P^{\times}$), our definition of $\mathfrak{N}P$-isometry and that in [DM5], though equivalent, are distinct. The advantage of the notion of $\mathfrak{N}P$-isometry we are here employing is that it allows for a direct Horn-geometric axiomatization of the notion of P-quadratic faithfulness. (cf. [HR]).

b) For $\mathfrak{N}P$-quadratic faithfulness, the situation is subtler and distinct: since we are interested in *reduced* special groups, we require the rings involved to be zdp-rings, i.e., models of the theory $\mathcal{T}_{zdp}$ (cf. 2.12), which know to be Horn (by 2.13.(b)), but whose *explicit* axiomatization by Horn-geometric (or even geometric) sentences is an open question.

Let $L(P, N)$ be the first-order language of equality set down in 2.12.(b). Then, the axioms $[\mathfrak{N}P\text{-FQ } i]$, $i = 1, 2, 3$, can be given Horn-geometric descriptions in $L(P, N)$. The recipe for this follows that in the proof of Theorem 5.2, p. 54 of [DM5], replacing the role of units by N and that of units in the preorder by $P \cap N$ (i.e., $\mathfrak{t}$); the most important change is: the underlined disjunction appearing in the expression of $\psi^{nk}(\overline{x}, \overline{y})$ in page 56, [DM5], may be replaced by a conjunction of atomic formulas, expressing a simple isometry step. Hence, $\psi^{nk}(\overline{x}, \overline{y})$ becomes a Horn-geometric formula in the language $L(P, N)$. We omit further details.

Let $\mathcal{T}_{FQ}$ be the set of Horn-geometric sentences of $L(P, N)$ that axiomatize $[\mathfrak{N}P\text{-FQ } i]$, $i =$ 1, 2, 3. The **theory of faithfully quadratic zdp-rings**, $\mathcal{T}_{FQzdp}$, is given by $\mathcal{T}_{zdp} \cup \mathcal{T}_{FQ}$. ■

The preceding observations yield:

Proposition 4.15 *The class of faithfully quadratic zdp-rings is closed under arbitrary reduced products. In particular, it has a Horn axiomatization.*

Proof. The theory $\mathcal{T}_{FQzdp}$ is a Horn-geometric extension of $\mathcal{T}_{zdp}$ and the latter is closed under arbitrary reduced products (by 2.13.(b)), and hence so must be the former. ■

5 The Hauptsatz and the Local-Global Principle

5.1 The Witt ring of $\mathfrak{N}$T-faithfully quadratic p-rings. Let $\langle A, T \rangle$ be a $\mathfrak{N}T$-faithfully quadratic p-ring. If φ, ψ are forms over A, we say that they are **Witt equivalent mod T,**

written $\varphi \sim_T \psi$, if there are integers n, m such that $\varphi \oplus m\langle 1, -1 \rangle \approx_{\mathfrak{N}T} \psi \oplus n\langle 1, 1 \rangle$. Since $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic, T-isometry is faithfully reflected by isometry in the special group $G_{\mathfrak{N},T}(A)$, and so, by Proposition 1.6.(b), p. 4, of [DM1] Witt-cancellation holds for $\approx_{\mathfrak{N}T}$. Hence, the set of equivalence classes $\overline{\varphi}$, of forms φ over $\mathfrak{N}$ with respect to $\sim_T$,

$$W_{\mathfrak{N},T}(A) = \{ \overline{\varphi} = \varphi/\sim_T : \varphi \text{ is a form over } \mathfrak{N}\},$$

with the operations of sum and product of classes induced by $\oplus$ and $\otimes$, is a commutative ring with identity $\overline{\langle 1 \rangle}$, whose zero is the class of any hyperbolic form. In fact, it is straightforward that $W_T(A)$ is naturally isomorphic to $W(G_{\mathfrak{N},T}(A))$, the Witt ring of the special group $G_{\mathfrak{N},T}(A)$, via the map induced on Witt rings by $\varphi \longmapsto \varphi^T$. Hence, $W_T(A)$ has all the properties described in paragraph 1.25 and Fact 1.26 of [DM1], pp. 19, 20. In particular,

- $I_T(A) = I(G_{\mathfrak{N},T}(A))$ is the fundamental ideal of $W_T(A)$, consisting of the classes of even dimensional forms;
- For $n \geq 1$, $I^n_T(A) = I^n(G_{\mathfrak{N},T}(A))$, the n^{th} -power of $I_T(A)$, consists of all linear combinations, with coefficients in A, of Pfister forms of degree n over A. ∎

As a consequence of these observations and the fact that RSGs satisfy the Arason-Pfister Hauptsatz (Theorem 7.31, p. 171, [DM1]), we obtain:

Theorem 5.2 (The Arason-Pfister Hauptsatz) *If $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic zdp-ring, then* $\bigcap_{n\geq 1} I^n_T(A) = \{0\}$. ∎

Our next goal is to prove a version of the Pfister's local-global principle in the present setting (Proposition 5.8, below), establishing that if $\langle A, T \rangle$ is a $\mathfrak{N}T$-faithfully quadratic zdp-ring, then an analog of that classical principle holds for $\langle A, T \rangle$.

We shall follow the path set in section 2 of chapter 3 in [DM5]. However, some care must be exercised; in particular, that we shall not be dealing with *groups* (as in the case of [DM5]), but with *semigroups*.

Blanket assumption : *$\langle A, T \rangle$ is a zdp-ring.*

Recall that $\mathbb{Z}_2 = \{1, -1\}$ is the two-element reduced special group. Following the ideas in [DM5] and [KRW], we set down the following

Definition 5.3 *Let $\langle A, T \rangle$ be a proper zdp-ring.*

(1) *A* **T-signature** *on $\mathfrak{N}$ is a a map $\tau : \mathfrak{N} \longrightarrow \mathbb{Z}_2$, preserving product, 1, -1 and for all $a_1, a_2 \in \mathfrak{N}$,*

$$a_1, a_2 \in \ker \tau := \{x \in \mathfrak{N} : \tau(x) = 1\} \quad \Rightarrow \quad D^v_T(a_1, a_2) \subseteq \ker \tau.$$

Write $Z_{\mathfrak{N},T}$ for the set of T-signatures on $\mathfrak{N}$. Clearly, for each $\tau \in Z_{\mathfrak{N},T}$, the set $\ker \tau$ *is closed under product,* $1 \in \ker \tau$ *and* $-1 \notin \ker \tau$.

(2) *If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over $\mathfrak{N}$ and $\tau \in Z_{\mathfrak{N},T}$, the* integer $sgn_\tau(\varphi) = \sum_{i=1}^n \tau(a_i)$ *is the* **signature of φ at τ**. ∎

Fact 5.4 *If $\tau \in Z_{\mathfrak{N},T}$, then:*

a) $\mathfrak{t} \subseteq \ker \tau$.

b) $a \sim_{\mathfrak{t}} b$ implies $\tau(a) = \tau(b)$. In particular, $a \sim_{\mathfrak{t}} b$ and $a \in \ker \tau$ implies $b \in \ker \tau$.

Proof. For (a), note that since $\mathfrak{t} \subseteq D^v_T(1, 1)$, we get $\mathfrak{t} \subseteq \ker \tau$. For (b), let s, $t \in \mathfrak{t}$ be such that $sa = tb$; then (a) entails $\tau(a) = \tau(sa) = \tau(tb) = \tau(b)$, as needed. The last assertion is clear. ∎

5.5 Example and Remarks. Let $\langle A, T \rangle$ be a proper p-ring and let $Y_T = \text{Sper}(A,T)$ be the real spectrum of $\langle A, T \rangle$.

a) Each $\alpha \in Y_T$ satisfying supp $\alpha \cap \mathfrak{N} = \emptyset$, gives rise to a signature, $\tau_\alpha : \mathfrak{N} \longrightarrow \mathbb{Z}_2$, given by

$$\tau_\alpha(x) = \begin{cases} 1 & \text{if } x \in \alpha \setminus (-\alpha); \\ -1 & \text{if } x \in -\alpha \setminus \alpha. \end{cases}$$

Clearly, τ_α preserves product, 1 and -1. Note that for all $s \in \mathfrak{t}$, $\tau_\alpha(s) = 1$. For $a, b, c \in \mathfrak{N}$, if $a, b \in \ker \tau_\alpha$ and $c \in D^v_T(a, b)$, then there is $s \in \mathfrak{t}$ so that

(I) $$sc = t_1 a + t_2 b,$$

with $t_1, t_2 \in T$. Clearly, the right-hand side of (I) is in α and since $sc \in \mathfrak{N}$, it cannot be in the support of α. Hence, $1 = \tau_\alpha(sc) = \tau_\alpha(s)\tau_\alpha(c) = \tau_\alpha(c)$ and τ_α is indeed a signature on $\mathfrak{N}$. Note that if $\alpha \subseteq \beta$ in Y_T and supp $\beta \cap \mathfrak{N} = \emptyset$, then $\tau_\alpha = \tau_\beta$.

b) Let $A_* := A\mathfrak{N}^{-1}$ be the ring of fractions of A by $\mathfrak{N}$, called the **total ring of fractions** of A. Let

$$T_* = \left\{\frac{t}{b^2} : t \in T \text{ and } b \in \mathfrak{N}\right\}.$$

Clearly, T_* is a preorder on A_*; in fact, it is proper: if $-1 = \frac{t}{b^2}$, then $-b^2 = t \in T$ and so $b^2, -b^2 \in \mathfrak{t}$, contradicting the assumption that T has zdp. It is well-known (cf. Theorem 13.3.7, p. 508, [DST]) that $\text{Sper}(A_*, T_*)$ is naturally homeomorphic to the proconstructible (in fact, inversely closed) subspace of $\text{Sper}(A,T)$:

$$\mathfrak{T} = \{\beta \in \text{Sper}(A,T) : \text{supp } \beta \cap \mathfrak{N} = \emptyset\}.$$

We shall identify $\text{Sper}(A_*, T_*)$ with the subspace $\mathfrak{T}$ of $\text{Sper}(A,T)$, often writing $\mathfrak{T} = \text{Sper}(A_*, T_*) \subseteq \text{Sper}(A,T)$.

c) In general, sub-semigroups do not classify quotients in semigroups. In the case at hand, we show this to be the case for signatures.

c.1) Let $\tau : \mathfrak{N} \longrightarrow \mathbb{Z}_2$ be a signature. The category-theoretic kernel of τ, namely, Cker $\tau = \{\langle a,b \rangle \in \mathfrak{N}^2 : \tau(a) = \tau(b)\}$ is an equivalence relation on $\mathfrak{N}$. Since the only possible values for τ are ± 1, we obtain $\langle a,b \rangle \in \text{Cker } \tau \Leftrightarrow \tau(ab) = \tau(a)\tau(b) = \tau(a)^2 = 1 \Leftrightarrow ab \in \ker \tau$. Thus, for $a, b \in \mathfrak{N}$, the relation $a \sim_\tau b$ iff $ab \in \ker \tau$ is an equivalence relation, equal to that determined by Cker τ.

c.2) Fact 5.4.(b) yields

(*) For each $\tau \in Z_{\mathfrak{N},T}$, the equivalence relation $\sim_\mathfrak{t}$ defining $G_{\mathfrak{N},T}(A)$ from $\mathfrak{N}$ is contained in that determined by $\ker \tau$. ■

If G is a proto-SG, then the set of SG-characters of G is closed in $\mathbb{Z}_2{}^G$ (product topology; discrete topology on $\mathbb{Z}_2$), and so is a Boolean space in the induced topology, called the **space of orders** of G and denoted X_G. Similarly, $Z_{\mathfrak{N},T}$ is a closed subset of $\mathbb{Z}_2{}^{\mathfrak{N}}$, and a Boolean space if endowed with the induced topology.

Let $\mathfrak{N} \xrightarrow{q_\mathfrak{t}} G_{\mathfrak{N},T}(A)$ be the canonical quotient map (cf. 2.15.(b.3)). Then:

Lemma 5.6 *With notation as above, the map*

$$\sigma \in X_{G_{\mathfrak{N},T}(A)} \longmapsto \tau_\sigma = \sigma \circ q_\mathfrak{t} \in Z_{\mathfrak{N},T}$$

is a natural homeomorphism between the space of proto-SG-characters of $G_{\mathfrak{N},T}(A)$ and the space of T-signatures on $\mathfrak{N}$. Moreover, for all forms φ over $\mathfrak{N}$, $sgn_{\tau_\sigma}(\varphi) = sgn_\sigma(\varphi^T)$.

Proof. Since $\mathfrak{N}$, T, will remain fixed throughout the proof, write G for $G_{\mathfrak{N},T}(A)$, X_G for $X_{G_{\mathfrak{N},T}(A)}$, Z for $Z_{\mathfrak{N},T}$ and q for $q_\mathfrak{t}$.

Fix $\sigma \in X_G$; we first show that $\tau = \sigma \circ q$ is a T-signature on $\mathfrak{N}$. Since q preserves product, 1 and -1, while σ is a group morphism taking -1 to -1, τ will preserve product, 1 and -1. For $a, b, c \in \mathfrak{N}$, assume $c \in D_T^v(a, b)$ and $\tau(a) = \tau(b) = 1$. Proposition 3.5.(b.1) yields $q(c) = c^T \in D_G(q(a), q(b)) = D_G(a^T, b^T)$. Since $\sigma(a^T) = \sigma(q(a)) = \sigma(b^T) = \sigma(q(b)) = 1$, and σ is a character (and thus a proto-SG morphism), we obtain

$$\sigma(c^T) \in D_{\mathbb{Z}_2}(\sigma(a^T), \sigma(b^T)) \;=\; D_{\mathbb{Z}_2}(1, 1),$$

entailing, because $\mathbb{Z}_2$ is a RSG, that $\sigma(c^T) = \tau(c) = 1$, and that τ is a signature on $\mathfrak{N}$. Since q is onto, we immediately obtain the injectivity of $\sigma \longmapsto \tau_\sigma$. For its surjectivity, fix $\tau \in Z$. By (*) in 5.5.(c.2), τ factors uniquely through q, to yield a semigroup morphism, $\sigma : G \longrightarrow 2$, preserving 1, -1, and such that $\tau = \sigma \circ q = \tau_\sigma$. Since G is of exponent two and σ preserves product it is clearly a *group* morphism. It remains to check that σ is a morphism of proto-SGs. Let $a, b \in \mathfrak{N}$ be such that $a^T \in D_G(1, b^T)$; then, 3.5.(b.1) entails $a \in D_T^v(1, b)$; in order to show that σ is a SG morphism it suffices to prove that $\sigma(b^T) = 1$ implies $\sigma(a^T) = 1$. If $\sigma(b^T) = \sigma(q(b)) = \tau(b) = 1$, then $b \in \ker \tau$ and so $a \in \ker \tau$, which in turn yields $1 = \sigma(q(a)) = \sigma(a^T)$, as needed.

Note that for every $\sigma \in X_G$ and $x \in \mathfrak{N}$, we have

(I) $$\sigma(x^T) \;=\; \sigma(q(x)) \;=\; \tau_\sigma(x),$$

which immediately implies that for all forms φ over $\mathfrak{N}$, $sgn_\sigma(\varphi^T) = sgn_{\tau_\sigma}(\varphi)$.

That the map $\sigma \longmapsto \tau_\sigma$ is a homeomorphism is proved exactly as in the end of the proof of Lemma 3.16, p. 38, of [DM5]. ∎

If $\langle A, T \rangle$ is a p-ring it is a natural question whether all T-signatures on $\mathfrak{N}$ come from orderings in Sper(A, T) in the way indicated in Example 5.5.(a). If the p-ring $\langle A, T \rangle$ satisfies [$\mathfrak{N}$T-FQ 2] (in particular, if it is $\mathfrak{N}T$-faithfully quadratic), we have the following:

Lemma 5.7. *Assume $\langle A, T \rangle$ satisfies* [$\mathfrak{N}$T-FQ 2], *let τ be a T-signature on $\mathfrak{N}$ and let $n \geq 2$ be an integer.*

a) If $a_1, \ldots, a_n \in \ker \tau$, then $D_T^v(a_1, \ldots, a_n) \subseteq \ker \tau$.

b) With notation as in 5.5.(a), there is $\alpha \in$ Sper(A, T) so that supp $\alpha \cap \mathfrak{N} = \emptyset$ *and $\tau = \tau_\alpha$.*

Proof. a) If $n = 2$, this follows from the definition of signature on $\mathfrak{N}$. We proceed by induction on $n \geq 2$; if $a_1, \ldots, a_n, a_{n+1}$ are in $\ker \tau$ and $a \in D_T^v(a_1, \ldots, a_n, a_{n+1})$ $(a \in \mathfrak{N})$, [$\mathfrak{N}$T-FQ 2] yields $u \in D_T^v(a_2, \ldots, a_{n+1})$ such that $a \in D_T^v(a_1, u)$. The induction hypothesis and the case $n = 2$ entail $a \in \ker \tau$, completing the induction.

b) Let $Q = \{\sum_{i=1}^n a_i t_i : n \in \mathbb{N},\ a_i \in \ker \tau \text{ and } t_i \in T\}$ be the preorder on A generated by $\ker \tau$ and T. We claim that Q is proper; otherwise, $-1 \in Q$, and there would be $a_1, \ldots, a_n \in \ker \tau$ and $t_1, \ldots, t_n \in T$ such that $-1 = \sum_{i=1}^n a_i t_i$, i.e., $-1 \in D_T^v(a_1, \ldots, a_n)$, which is impossible by item (a). Let A_* be the total ring of fractions of A (cf. 5.5.(b)) and set

$$Q_* \;=\; \left\{\frac{q}{b^2} : q \in Q \text{ and } b \in \mathfrak{N}\right\}.$$

Then, Q_* is a proper preorder of A_*: if $-1 = \frac{q}{b^2}$, with $q \in Q$ and $b \in \mathfrak{N}$, then $-b^2 = q$ in A. Hence there are $c_1, \ldots, c_m \in \ker \sigma$ and $t_1, \ldots, t_m \in T$ so that $q = \sum_{i=1}^m t_i c_i$, whence $q \in D_T^v(c_1, \ldots, c_m)$ $\subseteq \ker \tau$ (by item (a)).Thus, $\tau(q) = 1$, while, recalling that $b^2 \in \mathfrak{t} \subseteq \ker \tau$, obtains $\tau(-b^2) = -1$, a contradiction.

Keeping in mind the identification made in 5.5.(b), there is, by Propositions 4.3.8, p. 90 and, 4.2.7, p. 87, of [BCR], $\alpha \in$ Sper$(A_*, T_*) \subseteq$ Sper(A, T) so that $Q \subseteq Q_* \subseteq \alpha$ and supp $\alpha \cap \mathfrak{N} = \emptyset$. With notation as in 5.5.(b), by construction we have $\ker \tau \subseteq \ker \tau_\alpha$. Now, note that for $a \in \mathfrak{N}$,

$\tau(a) = 1$ iff $a \in \ker \tau$ iff $\tau_\alpha(a) = 1$ and so, since both τ and τ_α take values in $\mathbb{Z}_2 = \{1, -1\}$, we obtain $\tau = \tau_\alpha$, as needed. ■

We now state, with notation as above:

Proposition 5.8 (Pfister's local-global principle for $\mathfrak{N}T$-isometry) *Let $\langle A, T \rangle$ be a $\mathfrak{N}T$-faithfully quadratic zdp-ring. For all forms φ, ψ of the same dimension over $\mathfrak{N}$, the following are equivalent:*

(1) $\varphi \approx_{\mathfrak{N}T} \psi$; (2) *For all* $\tau \in Z_{\mathfrak{N},T}$, $sgn_\tau(\varphi) = sgn_\tau(\psi)$.

(3) *For all* $\alpha \in \mathrm{Sper}(A, T)$ *such that* $\mathrm{supp}\ \alpha \cap \mathfrak{N} = \emptyset$, $sgn_{\tau_\alpha}(\varphi) = sgn_{\tau_\alpha}(\psi)$.

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows immediately from Example 5.5 and Lemma 5.7.(b). It remains to prove (1) $\Leftrightarrow$ (2).

<u>(1) $\Leftrightarrow$ (2) :</u> We recall Proposition 3.7, p. 51, in [DM1]:

(I) *If G is a reduced special group, then for all forms φ, ψ of the same dimension over G,* $\varphi \equiv_G \psi \;\Leftrightarrow\;$ *For all* $\sigma \in X_G$, $sgn_\sigma(\varphi) = sgn_\sigma(\psi)$.

Let φ, ψ be forms of the same dimension over $\mathfrak{N}$. Since $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic, Theorem 4.11.(b) yields

(II) $$\varphi \approx_{\mathfrak{N}T} \psi \quad \Leftrightarrow \quad \varphi^T \equiv_{\mathfrak{t}} \psi^T.$$

Recalling that $\equiv_{\mathfrak{t}}$ is isometry in the *reduced special group* $G_{\mathfrak{N},T}(A)$, (I) and Lemma 5.6 entail

(III) $$\begin{aligned} \varphi^T \equiv_{\mathfrak{t}} \psi^T \quad &\Leftrightarrow \quad \text{For all } \sigma \in X_{G_{\mathfrak{N},T}(A)},\ sgn_\sigma(\varphi^T) = sgn_\sigma(\psi^T) \\ &\Leftrightarrow \quad \text{For all } \tau \in Z_{\mathfrak{N},T},\ sgn_\tau(\varphi) = sgn_\tau(\psi). \end{aligned}$$

From (II) and (III) we immediately conclude the equivalence of (1) and (2), ending the proof. ■

6 A relation between $\mathfrak{N}T$- and T-quadratic faithfulness

6.1 Notation and Remarks. Let $\langle A, T \rangle$ be a zdp-ring. Our notational conventions remain in force; in particular, those in 2.2 and 5.5.(b).

a) Write $G := G_{\mathfrak{N},T}(A) = \{a^T : a \in \mathfrak{N}\}$ for the proto-SG associated to $\mathfrak{N}$ (cf. 3.5).

b) Let $\langle A_*, T_* \rangle$ be the total ring of fractions of $\langle A, T \rangle$ (cf. 5.5.(b)).

b.1) Note that for $a \in A \setminus \{0\}$ and $s \in \mathfrak{N}$,

(I) $$\frac{a}{s} \text{ is a non zero-divisor in } A_* \;\Leftrightarrow\; a \in \mathfrak{N} \;\Leftrightarrow\; \frac{a}{s} \in A_*^\times.$$

It is enough to check the implication ($\Rightarrow$) in the first equivalence (the second is clear). If $a \notin \mathfrak{N}$, then there is $b \in A \setminus \{0\}$ so that $ab = 0$, yielding $\frac{a}{s}\frac{b}{1} = 0$ in A_*, an impossibility. Hence

$$\mathfrak{N}_{A_*} = A_*^\times \text{ and } T_* \cap \mathfrak{N}_{A_*} = T_*^\times.$$

In fact, (I) above entails

(II) $$T_*^\times = \left\{\frac{t}{s^2} : \underline{t \in \mathfrak{t}} \text{ and } s \in \mathfrak{N}\right\}.$$

Note that if $x \in \mathfrak{t}$, then $\frac{1}{x} = \frac{x}{x^2} \in T_*^\times \subseteq T_*$, a fact constantly used below.

b.2) Write $\iota_A : A \longrightarrow A_*$ for the natural injective ring morphism $a \in A \longmapsto \frac{a}{1} \in A_*$. By the remarks in (b.1), ι_A is a <u>$\mathfrak{N}p$-*ring morphism*</u> (cf. 3.10). We register:

Fact 6.2 *Let $\langle A, T\rangle$ be a zdp-ring.*

(i) If T is a partial order on A, then T_* is partial order on A_*.

(ii) Let $\langle R, P\rangle$ be a zdp-ring and let $\langle A, T\rangle \xrightarrow{f} \langle R, P\rangle$ be a $\mathfrak{N}$p-ring morphism. Then, there is a unique p-ring morphism, $f_* : \langle A_*, T_*\rangle \longrightarrow \langle R_*, P_*\rangle$, making the following diagram commute:

$$\begin{array}{ccc} \langle A, T\rangle & \xrightarrow{\quad f \quad} & \langle R, P\rangle \\ \Big\downarrow \iota_A & & \Big\downarrow \iota_R \\ \langle A_*, T_*\rangle & \xrightarrow[\quad f_* \quad]{} & \langle R_*, P_*\rangle \end{array}$$

Proof. (*i*) If u, $-u \in T_*$, there are p, $q \in T$ and s, $t \in \mathfrak{N}_A$, so that $u = \frac{p}{s^2}$ and $-u = \frac{q}{t^2} = \frac{-p}{s^2}$; thus, $qs^2 = -pt^2$, whence pt^2, $-pt^2 \in T$, implying $pt^2 = 0$. Now, from $t^2 \in \mathfrak{N}_A$ comes $p = 0$, and so $u = 0$, as needed.

Item (*ii*) follows from the universal property of rings of fractions (cf. Prop, 3.1, p. 37, [AM]): since f and ι_R are $\mathfrak{N}$p-ring morphisms, for all $s \in \mathfrak{N}_A$, $\iota_R \circ f(s) \in \mathfrak{N}_R$ and is thus a unit in R_*, yielding the existence and uniqueness of f_* (in fact, given by $f_*(a/s) := f(a)/f(s)$). □

c) To simplify exposition, write $G_* = A_*^{\times}/T_*^{\times}$ for the proto-SG associated to $\langle A_*, T_*\rangle$, write the elements of G_* as u^* (instead of u^{T_*}), write ι for the injective ring morphism ι_A. By 6.1.(b.2) and Lemma 3.11, the $\mathfrak{N}$p-ring embedding ι induces a morphism of proto-special groups,

[ι] $\quad \iota^{\pi} : G \longrightarrow G_*$, given by $a^T \in G \longmapsto \left(\frac{a}{1}\right)^* \in G_*$. ■

Proposition 6.3 *The proto-SG morphism ι^{π} of* 6.1.(*c*) *is an* <u>isomorphism</u> *of reduced proto-special groups.*

Proof. We first show that ι^{π} is bijective.

- Suppose a, $b \in \mathfrak{N}$ and $\left(\frac{a}{1}\right)^* = \left(\frac{b}{1}\right)^*$; then (cf. equivalence (*) in 2.5 (p. 11) in [DM5]), there is $\frac{t}{s^2} \in T_*^{\times}$ so that $\frac{b}{1} = \frac{t}{s^2}\frac{a}{1}$ and so $s^2 b = ta$. Since $s^2 \in \mathfrak{t}$ and $t \in \mathfrak{t}$ (cf. 6.1(b.1)), Lemma 2.15.(b.2) yields $a^T = b^T$, establishing the injectivity of ι^{π};
- For a, $t \in \mathfrak{N}$, note that $\frac{t^2}{1}\frac{a}{t} = \frac{at}{1}$, with $\frac{t^2}{1} \in T_*^{\times}$; hence, $\left(\frac{a}{t}\right)^* = \left(\frac{at}{1}\right)^*$, showing ι^{π} to be surjective.

Since ι^{π} is a π-SG morphism, to show it to be an isomorphism it remains to check that for a, b, $c \in \mathfrak{N}$,

$$\left(\frac{a}{1}\right)^* \in D_{G_*}\left(\left(\frac{b}{1}\right)^*, \left(\frac{c}{1}\right)^*\right) \;\Rightarrow\; a^T \in D_G(b^T, c^T),$$

where D_{G_*} is (as usual) representation in G_*. By Proposition 3.5.(b.1), the preceding implication is equivalent to

$$(+) \qquad \frac{a}{1} \in D^v_{T_*}\left(\frac{b}{1}, \frac{c}{1}\right) \;\Rightarrow\; a \in D^v_T(b, c).$$

The antecedent in (+) means there are $\frac{t}{s^2}$, $\frac{u}{v^2} \in T_*$ so that $\frac{a}{1} = \frac{t}{s^2}\frac{b}{1} + \frac{u}{v^2}\frac{c}{1}$, hence, in A, we obtain $(s^2v^2)a = (tv^2)b + (us^2)c$; since $s^2v^2 \in \mathfrak{t}$ and tv^2, $us^2 \in T$, the desired conclusion follows immediately, completing the proof. ■

Lemma 6.4 *For x, y, $c_1, \ldots, c_n \in \mathfrak{N}$ and a* unit $\frac{u}{s} \in A_*^{\times}$,

(1) $\frac{x}{1} \in D^v_{T_*}\left(\frac{y}{1}, \frac{u}{s}\right) \quad \Leftrightarrow \quad x \in D^v_T(y, us)$.

(2) $\frac{u}{s} \in D^v_{T_*}\left(\frac{c_1}{1}, \ldots, \frac{c_n}{1}\right) \quad \Leftrightarrow \quad us \in D^v_T(c_1, \ldots, c_n)$.

Proof. Note that by (I) in 6.1.(b.1), $u \in \mathfrak{N}$, because $\frac{u}{s} \in A_*^{\times}$.

(1) (⇒) : There are $\frac{t}{v^2}, \frac{w}{z^2} \in T_*$ so that $\frac{x}{1} = \frac{t}{v^2}\frac{y}{1} + \frac{w}{z^2}\frac{u}{s} = \frac{t}{v^2}\frac{y}{1} + \frac{w}{z^2 s^2}\frac{us}{1}$, whence $s^2v^2z^2x = (ts^2z^2)y + (wv^2)us$, with $(svz)^2 \in \mathfrak{t}$ and (ts^2z^2), $(wv^2) \in T$, as needed.

(⇐) : There are $v \in \mathfrak{t}$, t_1, $t_2 \in T$ so that $vx = t_1y + t_2(us)$, whence, $\frac{xv}{1} = \frac{t_1}{1}\frac{y}{1} + \frac{t_2}{1}\frac{us}{1}$, which entails $\frac{x}{1} = \frac{t_1}{v}\frac{y}{1} + \frac{t_2s^2}{v}\frac{u}{s}$, with $\frac{t_1}{v}, \frac{t_2s^2}{v} \in T$ (recall: $v \in \mathfrak{t}$), establishing (⇐) and item (1).

(2) (⇒) : There are $\frac{t_k}{v_k^2} \in T_*$, $1 \leq k \leq n$ so that

$$\text{(I)} \qquad \frac{u}{s} = \frac{us}{s^2} = \sum_{k=1}^{n} \frac{t_k}{v_k^2}\frac{c_k}{1}.$$

Set $d_0 = \prod_{k=1}^{n} v_k^2$, and for $1 \leq j \leq n$, $d_j = t_js^2 \prod_{i \neq j} v_i^2$. Elementary computations yield

$$d_0(us) = \sum_{j=1}^{n} d_jc_j,$$

with $d_0 \in \mathfrak{t}$ and $d_j \in T$, whence $us \in D^v_T(c_1, \ldots, c_n)$, as desired.

(⇐) : There is $v \in \mathfrak{t}$ and $t_k \in T$, $1 \leq k \leq n$, such that $vus = \sum_{i=1}^{n} t_kc_k$. Hence, in A_* we get

$$\frac{vus}{1} = \frac{vs^2}{1}\frac{u}{s} = \sum_{i=1}^{n} \frac{t_kc_k}{1}$$

entailing $\frac{u}{s} = \sum_{i=1}^{n} \frac{t_k}{vs^2}\frac{c_k}{1}$, with $\frac{t_k}{vs^2} \in T_*$, because $\frac{vs^2}{1}$ is a unit in A_*, ending the proof. ∎

The next result is crucial in what follows

Theorem 6.5 *With notation as above, let $\langle A, T \rangle$ be a zdp-ring, The following are equivalent:*

(1) *$\langle A_*, T_* \rangle$ is T_*-faithfully quadratic* (in the sense of [DM5], Chapter 3, §1; see also 4.14.(a));

(2) *$\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic.*

Proof. (1) ⇒ (2):

• $\langle A, T \rangle$ satisfies [$\mathfrak{N}T$-FQ1] : By Lemma 3.9.(a), it suffices to show, for b, $c \in \mathfrak{N}$, that $D^v_T(b, c) \subseteq D^{\mathfrak{t}}_T(b, c)$. Assume, for $a \in \mathfrak{N}$, that $a \in D^v_T(b, c)$; then $\frac{a}{1} \in D^v_{T_*}\left(\frac{b}{1}, \frac{c}{1}\right)$ in A_*. Since value representation in A_* is 2-transversal, there are $\frac{u}{v^2}, \frac{x}{y^2} \in T_*^{\times}$ (thus, u, $x \in \mathfrak{t}$) so that

$$\frac{a}{1} = \frac{u}{v^2}\frac{b}{1} + \frac{x}{y^2}\frac{c}{1}.$$

Hence, in A we obtain $v^2y^2a = (uy^2)b + (xv^2)c$, with v^2y^2, uy^2, $xv^2 \in \mathfrak{t}$, and $a \in D^{\mathfrak{t}}_T(b, c)$, showing that binary value representation in $\langle A, T \rangle$ is also 2-transversal, as needed.

• $\langle A, T \rangle$ satisfies [$\mathfrak{N}T$-FQ2] : Again, by 3.9.(a), it is enough to prove that if $a_1, \ldots, a_n \in \mathfrak{N}$, then $D^v_T(a_1, \ldots, a_n) \subseteq \mathfrak{D}_T(a_1, \ldots, a_n)$. Suppose $b \in D^v_T(a_1, \ldots, a_n)$, $b \in \mathfrak{N}$. Then, in A_*

$$\frac{b}{1} \in D^v_{T_*}\left(\frac{a_1}{1}, \ldots, \frac{a_n}{1}\right).$$

Fix k, $1 \leq k \leq n$; since $\langle A_*, T_* \rangle$ is T_*- faithfully quadratic, there is a *unit* $\frac{u}{s}$ in A_* so that:

$$(1) \quad \frac{b}{1} \in D^v_{T_*}\left(\frac{a_k}{1}, \frac{u}{s}\right) \quad \text{and} \quad (2) \quad \frac{u}{s} \in D^v_{T_*}\left(\frac{a_1}{1}, \frac{a_2}{1}, \ldots, \overset{\vee}{\frac{a_k}{1}}, \ldots, \frac{a_n}{1}\right).$$

Now, items (1) and (2) of Lemma 6.4 immediately entail

$$b \in D^v_T(a_k, us) \quad \text{and} \quad us \in D^v_T(a_1, \ldots, \overset{\vee}{a_k}, \ldots, a_n),$$

as needed.

• $\langle A, T \rangle$ satisfies [$\mathfrak{N}T$-FQ3] : Let $a \in \mathfrak{N}$ and let φ, ψ be n-forms over $\mathfrak{N}$, $n \geq 2$, such that $a \oplus \varphi \approx_{\mathfrak{N}T} a \oplus \psi$. Since $\iota : A \longrightarrow A_*$ is a $\mathfrak{N}$p-ring morphism, 4.5.(k) yields $\iota(a) \oplus (\iota \star \varphi) \approx_{T_*} \iota(a) \oplus (\iota \star \psi)$. The quadratic faithfulness of $\langle A_*, T_* \rangle$ entails $\iota \star \varphi \approx_{T_*} \iota \star \psi$, which in turn yields $(\iota \star \varphi)^{T_*} \equiv_{T_*} (\iota \star \psi)^{T_*}$ in G_* (cf. 3.12). Now note that for $x \in \mathfrak{N}$,

$$\iota(x)^{T_*} = \iota^{\pi}(x^T).$$

Hence, $(\iota \star \varphi)^{T_*} = \iota^{\pi}(\varphi^T)$ and $(\iota \star \psi)^{T_*} = \iota^{\pi}(\psi^T)$, and so the isomorphism of Proposition 6.3 yields $\varphi^T \equiv_t \psi^T$ in G. But then Lemma 4.7.(c) entails $\varphi \approx_{\mathfrak{N}T} \psi$, completing the proof of the $\mathfrak{N}T$-quadratic faithfulness of $\langle A, T \rangle$.

Before presenting the proof of (2) $\Rightarrow$ (1), we need some preliminaries. Recall that if R is a ring, $M_n(R)$ is the ring of $n \times n$ matrices with entries in R.

Remarks 6.6 As noted in the proof of Proposition 6.3, if a, $t \in \mathfrak{N}$ (i.e., $\frac{a}{t} \in A^{\times}_*$), then $\left(\frac{a}{t}\right)^*$ $= \left(\frac{at}{1}\right)^*$, because $\frac{t^2}{1}\frac{a}{t} = \frac{at}{1}$, with $\frac{t^2}{1} \in T^{\times}_*$. But then, items (h.2) and (h.3) of 4.5, together with Lemma 4.5.(g.1) entail, for $a_1, \ldots, a_n$, $t_1, \ldots, t_n \in \mathfrak{N}$:

(1) $\left\langle \frac{a_1}{t_1}, \ldots, \frac{a_n}{t_n} \right\rangle \approx_{T_*} \left\langle \frac{a_1 t_1}{1}, \ldots, \frac{a_n t_n}{1} \right\rangle$;

(2) $D^v_{T_*}\left(\left\langle \frac{a_1}{t_1}, \ldots, \frac{a_n}{t_n} \right\rangle\right) = D^v_{T_*}\left(\left\langle \frac{a_1 t_1}{1}, \ldots, \frac{a_n t_n}{1} \right\rangle\right)$

(3) By (1), if φ, ψ are forms over $A^{\times}_*$ and $\varphi \equiv_{T_*} \psi$, we may assume that every step of a witnessing sequence for this isometry is of the form $\left\langle \frac{u_1}{1}, \ldots, \frac{u_n}{1} \right\rangle$, with $u_1, \ldots, u_n \in \mathfrak{N}$. ■

Lemma 6.7 *For forms* $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ *over* $\mathfrak{N}$,

$$\left\langle \frac{a_1}{1}, \ldots, \frac{a_n}{1} \right\rangle \approx_{T_*} \left\langle \frac{b_1}{1}, \ldots, \frac{b_n}{1} \right\rangle \quad \Leftrightarrow \quad \varphi \approx_{\mathfrak{N}T} \psi.$$

Proof. It suffices to prove the implication ($\Rightarrow$), since the converse is an immediate consequence of Lemma 4.5.(k) (recall: for $x \in A$, $\frac{x}{1} = \iota(x)$). By Remark 6.6.(3), we may assume that all steps in a witnessing sequence of the isometry in A_* are of the form $\left\langle \frac{u_1}{1}, \ldots, \frac{u_n}{1} \right\rangle$ and so it suffices to establish ($\Rightarrow$) for simple isometry.

Suppose $\left\langle \frac{u_1}{1}, \ldots, \frac{u_n}{1} \right\rangle \approx_{sT_*} \left\langle \frac{v_1}{1}, \ldots, \frac{v_n}{1} \right\rangle$; then there are $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $w_1, \ldots, w_n$, $z_1, \ldots, z_n \in \mathfrak{N}$ and $M \in M_n(A_*)$, with $\det M \in A^{\times}_*$ (cf. (I) in 6.1.(b)), so that

$$\text{(I)} \qquad M\mathscr{M}\left(\frac{x_1}{y_1^2}\frac{u_1}{1}, \ldots, \frac{x_n}{y_n^2}\frac{u_n}{1}\right) M^t = \mathscr{M}\left(\frac{w_1}{z_1^2}\frac{v_1}{1}, \ldots, \frac{w_n}{z_n^2}\frac{v_n}{1}\right).$$

Let $M = \left(\frac{a_{ij}}{t_{ij}}\right)$, $1 \leq i, j \leq n$ and set $t = \prod_{1 \leq i,j \leq n} t_{ij}$; clearly, $t \in \mathrm{t}$.

Fact 6.8 *With notation as above,*

a) $\det M = \frac{d}{t}$, *with* $d \in \mathfrak{N}$.

b) Set $N = \mathcal{M}\left(\frac{t}{1},\dots,\frac{t}{1}\right)M = \left(\frac{a_{ij}t}{t_{ij}}\right)$; *then* N *is a matrix with entries of the form* $\frac{c_{ij}}{1}$, *whose determinant is* $\frac{dt^{n-1}}{1} \in A_*^\times$.

Proof. a) By Proposition 4 (p. 331) in [La], if S_n is the group of permutations on $\{1, \dots, n\}$, then

$$\det M = \textstyle\sum_{\sigma\in S_n} \varepsilon(\sigma)\frac{a_{\sigma(1),1}}{t_{\sigma(1),1}} \cdots \frac{a_{\sigma(n),n}}{t_{\sigma(n),n}},$$

with $\varepsilon(\sigma)$ being the sign of the permutation σ; it is then clear that t is a common denominator for this sum and so $\det M$ is of the form $\frac{d}{t}$; since it is a unit in A_*, by (I) in 6.1.(b), we obtain $d \in \mathfrak{N}$, as claimed.

b) Since t is the product of all t_{ij}, it is clear that the coefficients of N are of the form $\frac{c_{ij}}{1}$; now (a) yields

$$\det N = \det\left(\mathcal{M}\left(\tfrac{t}{1},\dots,\tfrac{t}{1}\right)M\right) = \det\left(\mathcal{M}\left(\tfrac{t}{1},\dots,\tfrac{t}{1}\right)\right)\cdot \det M = \frac{t^n}{1}\cdot\frac{d}{t} = \frac{dt^{n-1}}{1},$$

with $\frac{dt^{n-1}}{1} \in A_*^\times$. □

Let $y^2 := \prod_{i=1}^n y_i^2$ and $z^2 := \prod_{i=1}^n z_i^2$, that clearly are in $\mathfrak{t}$. Multiplying both sides of (I) by $\mathcal{M}\left(\frac{t^2}{1},\dots,\frac{t^2}{1}\right)$, $\mathcal{M}\left(\frac{y^2}{1},\dots,\frac{y^2}{1}\right)$ and $\mathcal{M}\left(\frac{z^2}{1},\dots,\frac{z^2}{1}\right)$ and recalling that scalar multiples of the identity matrix are in the center of the ring $M_n(R)$, we obtain, with notation as in 6.8.(b)

$$\text{(II)}\quad \left[\mathcal{M}\left(\tfrac{t}{1},\dots,\tfrac{t}{1}\right)M\right]\cdot\mathcal{M}\left(\frac{z^2x_1u_1}{1},\dots,\frac{z^2x_nu_n}{1}\right)\cdot\left[\mathcal{M}\left(\tfrac{t}{1},\dots,\tfrac{t}{1}\right)M\right]^t =$$

$$= N\,\mathcal{M}\left(\frac{z^2x_1u_1}{1},\dots,\frac{z^2x_nu_n}{1}\right)N^t = \mathcal{M}\left(\frac{y^2w_1v_1}{1},\dots,\frac{y^2w_nv_n}{1}\right),$$

with $N = \mathcal{M}\left(\frac{c_1}{1},\dots,\frac{c_n}{1}\right)$ and $\det N = \frac{dt^{n-1}}{1} \in A_*^\times$. Let $Q = \mathcal{M}(c_1,\dots,c_n)$; (II) entails that in A we have $Q\,\mathcal{M}(z^2x_1u_1, \dots, z^2x_nu_n)\,Q^t = \mathcal{M}(y^2w_1v_1, \dots, y^2w_nv_n)$, with $\det Q = dt^{n-1} \in \mathfrak{N}$ and $z^2x_k, y^2w_k \in \mathfrak{t}$, $1 \le k \le n$, establishing the simple T-isometry between $\langle u_1,\dots,u_n\rangle$ and $\langle v_1,\dots,v_n\rangle$ in $\langle A,T\rangle$, as desired. ■

Proof of (2) ⇒ (1) in Theorem 6.5.

• $\langle A_*,T_*\rangle$ satisfies [T_*-FQ 1]: Let $\varphi = \left\langle\frac{a_1}{t_1},\frac{a_2}{t_2}\right\rangle$ be a form over $A_*^\times$ and let $\frac{u}{s} \in A_*^\times$ be such that $\frac{u}{s} \in D_{T_*}^v(\varphi)$. By 6.6.(2), this is equivalent to $\frac{u}{s} \in D_{T_*}^v\left(\frac{a_1t_1}{1},\frac{a_2t_2}{1}\right)$ and so 6.4.(2) yields $us \in D_T^v(a_1t_1, a_2t_2)$. One should keep in mind :

(*) Since $\frac{u}{s}$ and the coefficients of φ are in $A_*^\times$, u, s, a_i, t_i, $i = 1, 2$ are in $\mathfrak{N}$ (cf. (I), 6.1.(b.1));

(**) If $t \in \mathfrak{t}$, then $t \in T_*^\times$ $(1/t = t/t^2)$.

The $\mathfrak{N}T$-quadratic faithfulness of $\langle A,T\rangle$ yields τ, α, β in $\mathfrak{t}$ so that $\tau us = \alpha a_1t_1 + \beta a_2t_2$, that in turns yields in A_*

$$\text{(I)}\qquad \frac{\tau us}{1} = \frac{\alpha a_1t_1}{1} + \frac{\beta a_2t_2}{1}.$$

Dividing (I) by $\tau s^2 \in T_*^{\times}$ obtains $\frac{u}{s} = \left(\frac{\alpha t_1^2}{\tau s^2}\right)\frac{a_1}{t_1} + \left(\frac{\beta t_2^2}{\tau s^2}\right)\frac{a_2}{t_2}$, with $\frac{\alpha t_1^2}{\tau s^2}$ and $\frac{\beta t_2^2}{\tau s^2}$ in $T_*^{\times}$, as needed to show $D^v_{T_*}(\varphi) \subseteq D^t_{T_*}(\varphi)$.

• $\langle A_*, T_* \rangle$ satisfies [T_*-FQ2]: Let $x \in A_*^{\times}$ and let φ be a n-form over $A_*^{\times}$, so that $x \in D^v_{T_*}(\varphi)$. As registered in Remark 6.6, we may assume that $x = \frac{u}{1}$ and $\varphi = \left\langle \frac{a_1}{1}, \ldots, \frac{a_n}{1} \right\rangle$. Then, 6.4.(2) yields $u \in D^v_T(a_1, \ldots, a_n)$. Fix k, $1 \le k \le n$. The $\mathfrak{N}T$-quadratic faithfulness of $\langle A, T \rangle$ furnishes $v \in \mathfrak{N}$ so that

$$u \in D^v_T(a_k, v) \quad \text{and} \quad v \in D^v_T(a_1, \ldots, \check{a}_k, \ldots, a_n),$$

and another application of 6.4.(2) guarantees that

$$x = \frac{u}{1} \in D^v_{T_*}\left(\left\langle \frac{a_k}{1}, \frac{v}{1} \right\rangle\right) \quad \text{and} \quad \frac{v}{1} \in D^v_{T_*}\left(\left\langle \frac{a_1}{1}, \ldots, \frac{\check{a}_k}{1}, \ldots, \frac{a_n}{1} \right\rangle\right),$$

as required.

• $\langle A_*, T_* \rangle$ satisfies [T-FQ 3]: We may assume that for $a \in \mathfrak{N}$, $\varphi = \left\langle \frac{x_1}{1}, \ldots, \frac{x_n}{1} \right\rangle$ and $\psi = \left\langle \frac{y_n}{1}, \ldots, \frac{y_n}{1} \right\rangle$ (forms over $A_*^{\times}$), we have $\left\langle \frac{a}{1} \right\rangle \oplus \varphi \approx_{T_*} \left\langle \frac{a}{1} \right\rangle \oplus \psi$. Lemma 6.7 then yields $\langle a \rangle \oplus \langle x_1, \ldots, x_n \rangle \approx_{\mathfrak{N}T} \langle a \rangle \oplus \langle y_1, \ldots, y_n \rangle$, and the $\mathfrak{N}T$-quadratic faithfulness of $\langle A, T \rangle$ entails $\langle x_1, \ldots, x_n \rangle \approx_{\mathfrak{N}T} \langle y_1, \ldots, y_n \rangle$. Hence, the equivalence in 6.7 implies $\varphi \approx_{T_*} \psi$, completing the proof of Theorem 6.5. ∎

7 Applications

A. von Neumann Regular Rings, Integral Domains and Boolean Powers

Since a preordered von Neumann regular ring is its own total ring of fractions, Theorem 6.5 above together with Theorem 6.5, p. 63 (in view of Remark 6.2(a), p. 61) of [DM5]), yields:

Proposition 7.1 *If $\langle A, T \rangle$ is a preordered von Neumann regular ring such that all residue fields of A have cardinality at least 7, then $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic.* ∎

Proposition 7.2 *If $\langle A, T \rangle$ is a partially ordered integral domain, then $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic.*

Proof. The total ring of fractions of $\langle A, T \rangle$ is a partially ordered formally real field, $\langle A_*, T_* \rangle$, and so T_*-faithfully quadratic (A_* is a ring with many units, that are all completely faithfully quadratic by Thm 6.5, p. 63, [DM5]); then, Theorem 6.5 above entails the desired conclusion. ∎

Proposition 7.2 yields a number of interesting examples of $\mathfrak{N}\Sigma$-faithfully quadratic rings. With notation as in 2.4 and 2.7, we have:

Corollary 7.3 *a) If A is a real integral domain, then the ring of polynomials, $A[X_1, \ldots, X_n]$, and the ring of power series, $A[[Z]]$, in any number of variables, are $\mathfrak{N}\Sigma$-faithfully quadratic.*

b) If V is a valuation ring of a formally real field, then $\langle V, \Sigma V^2 \rangle$ is $\mathfrak{N}\Sigma$-faithfully quadratic.

c) If F is a formally real field, its real holomorphy ring, $H(F)$, is $\mathfrak{N}\Sigma$-faithfully quadratic. ∎

Remark 7.4 Boolean Powers. a) Let L be a first-order language with equality, let M be a L-structure and let X be a Boolean space. Endow M with the discrete topology (all points are open) and let $M(X) := \mathbb{C}(X, M)$ be the set of all continuous maps M-valued maps defined on X. Note that every f is locally constant and its image, Im f, is a *finite* subset of M (recall: X is Hausdorff and compact). $M(X)$ is made into an L-structure by endowing it with the structure induced by the power M^X; hence, relations and functions are pointwise defined, while constants correspond to constant functions with the appropriate value, and the map $\gamma^X : M(X) \longrightarrow M^X$ is an L-embedding.

If R is a n-ary relation in L, write $R(X)$ for its interpretation in $M(X)$, i.e. the set of n-tuples of locally constant functions on X, whose n-tuple of values at each $x \in X$ are in R. The structure $M(X)$ is the **Boolean power of M by X.**

b) If $f \in M(X)$, as noted in (a), Im f is a *finite* subset of M and $\mathcal{P}(f) = \{f^{-1}[m] : m \in \text{Im } f\}$ is a finite set of pairwise disjoint non-empty clopens in X, whose union is X, that is, a **partition** of X.

Recall that if $\mathcal{P}$, $\mathcal{Q}$ are partitions of X, $\mathcal{Q}$ is a **refinement** of $\mathcal{P}$, written $\mathcal{P} \preceq \mathcal{Q}$, if every $V \in \mathcal{Q}$ is contained in a (necessarily unique) element of $\mathcal{P}$. Clearly, if $W \in \mathcal{P}$, then W is the disjoint union of the clopens in $\mathcal{Q}$ contained in W. The relation $\preceq$ is a partial order, endowing the set of partitions of X with a join-semilattice structure: if $\mathcal{P}_1$, $\mathcal{P}_2$ are partitions of X, then $\mathcal{P}_1 \vee \mathcal{P}_2 = \{U \cap V : U \in \mathcal{P}_1, V \in \mathcal{P}_2 \text{ and } U \cap V \neq \emptyset\}$ is the smallest (in the po $\preceq$) common refinement of $\mathcal{P}_i$, $i = 1, 2$. Hence, if $f_1, \ldots, f_n \in M(X)$ (with notation as above), $\mathcal{Q} := \bigvee_{i=1}^{n} \mathcal{P}(f_i)$ is a partition of X, such that, on each $V \in \mathcal{Q}$, *all* f_i are constant, $1 \leq i \leq n$. We shall employ these observations below, without further comment. ■

Theorem 7.5 *With notation as in 7.4, let $\langle A, T \rangle$ be a zdp-ring and write $\mathfrak{N}$ for $\mathfrak{N}_A$. Let X be a Boolean space and let $A(X)$ be the Boolean power of A by X.*

a) $A(X)$ is a unitary commutative ring, in which 2 (the constant function with value 2) *is a unit.*

b) $\mathfrak{N}_{A(X)} = \mathfrak{N}(X)$ and $T(X)$ is a zdp preorder on $A(X)$.

c) There is a natural p-ring isomorphism, $\mu : \langle A(X)_*, T(X)_* \rangle \longrightarrow \langle A_*(X), T_*(X) \rangle$ (the Boolean power of the total rings of fractions of $\langle A, T \rangle$ by X).

d) If $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic, then $\langle A(X), T(X) \rangle$ is $\mathfrak{N}T(X)$-faithfully quadratic.

Proof. Item (a) is straightforward.

b) Clearly, $\mathfrak{N}(X) \subseteq \mathfrak{N}_{A(X)}$. For the reverse inclusion, let $f \in \mathfrak{N}_{A(X)}$, and suppose that for some $x \in X$, $f(x) \notin \mathfrak{N}$. Then, there would be a non-empty clopen, V, in X, such that the constant value of f on V is a zero-divisor, $f(x)$. Select $u \neq 0$ in A so that $uf(x) = 0$ and define $g \in A(X)$ by $g(z) = \begin{cases} u & \text{if } z \in V; \\ 0 & \text{otherwise.} \end{cases}$ Then, $g \neq 0$, but $fg = 0$, contradicting the fact that $f \in \mathfrak{N}_{A(X)}$.
Thus, $\mathfrak{N}_{A(X)} = \mathfrak{N}(X)$, as desired. It is straightforward that $T(X)$ is a proper preorder on $A(X)$; moreover, for $f \in A(X)$, we have

$$(*) \begin{cases} f \in \text{supp}(T(X)) \text{ iff } f, -f \in T(X) \text{ iff } \forall\, x \in X,\, f(x), -f(x) \in T \text{ iff} \\ \forall\, x \in X,\, f(x) \in \text{supp}(T) \text{ iff } f \in \text{supp}(T)(X). \end{cases}$$

Hence (*) and the first statement in (b) entail $\text{supp}(T(X)) \cap \mathfrak{N}(X) = \emptyset$, and $\langle A(X), T(X) \rangle$ is a zdp-ring.

c) Define $\mu : A(X)_* \longrightarrow A_*(X)$ as follows: for $f \in A(X)$ and $g \in \mathfrak{N}(X)$, set

$$(\mu) \qquad \mu\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

To see that $\mu\left(\frac{f}{g}\right) \in \mathbb{C}(X, A_*)$, select a partition $\mathcal{P}$ of X so that both f and g are constant in each $W \in \mathcal{P}$, with values a and s, respectively. Then, $\mu\left(\frac{f}{g}\right)$ will have constant value $\frac{a}{s}$ on W, showing that it is continuous (A_* has the discrete topology). Note that

$$\text{(I)} \qquad f \in T(X) \text{ and } h \in \mathfrak{N}(X) \;\Rightarrow\; \mu\left(\frac{f}{h^2}\right) \in \mathbb{C}(X, T_*),$$

and so μ takes $T(X)_*$ into $\mathbb{C}(X, T_*) = T_*(X)$.

Since all operations in Boolean powers are pointwise defined, it is now straightforward that μ is p-ring morphism; by its very construction, $\mu\left(\frac{f}{g}\right) = 0$ entails $f = 0$ and hence μ is injective.

Now, suppose $\zeta \in \mathbb{C}(X, A_*)$; then there is a partition $\mathcal{Q}$ of X, such that on each $V \in \mathcal{Q}$, ζ has constant value on V, say $\frac{a_V}{s_V} \in A_*$, with $a_V \in A$ and $s_V \in \mathfrak{N}$. Define $f, g : X \longrightarrow A$ so that on each $V \in \mathcal{Q}$, f has constant value a_V and g has constant value s_V. By item (b), $g \in \mathfrak{N}(X)$ and the definition of μ yields $\mu\left(\frac{f}{g}\right) = \zeta$, establishing surjectivity. A similar argument proves μ to be a p-ring isomorphism, i.e., the converse of (I) is also verified, completing the proof of (c).

d) Since $\langle A, T \rangle$ is $\mathfrak{N}$T-faithfully quadratic, by Theorem 6.5, its total ring of fractions, $\langle A_*, T_* \rangle$ is T_*-faithfully quadratic (in the sense of [DM5]). In view of item (c), to establish (d) it suffices to prove the Boolean power of $\langle A_*, T_* \rangle$ by X, $\mathbb{C}(X, \langle A_*, T_* \rangle)$, is $T_*(X)$-faithfully quadratic.

By Theorem 4.6, p. 44, [DM5], $\langle A_*, T_* \rangle^X$ is T_*-faithfully quadratic. Moreover, Theorem 5.2, p. 54, [DM5], shows that the theory of T-faithfully quadratic rings is geometric (it is even Horn-geometric). We now invoke Proposition 2.1, p. 950, [DM2a], guaranteeing that the natural embedding $\gamma^X : \mathbb{C}(X, \langle A_*, T_* \rangle) \longrightarrow \langle A_*, T_* \rangle^X$ is existentially closed (and hence reflects geometric sentences), to conclude that $\mathbb{C}(X, \langle A_*, T_* \rangle) = \langle A_*(X), T_*(X) \rangle$ is $T_*(X)$-faithfully quadratic. Once again, Theorem 6.5 yields that $\langle A(X), T(X) \rangle$ is $\mathfrak{N}$T(X)-faithfully quadratic, ending the proof of Theorem 7.5. ∎

B. The $\mathfrak{N}T$-quadratic faithfulness of f-rings

A reduced f-ring, A, is a subdirect product of linearly ordered domains. The ring A is endowed with a natural partial order, pointwise defined and written $T_\sharp$, with which it is a lattice-ordered ring. As references on f-rings we mention [BKW] and [Da]. A good amount of information concerning f-rings appears in chapter 8 of [DM5].

Definition 7.6 (Definition 3, p.80, [KZ]) *Let $\langle A, T_\sharp \rangle$ be a f-ring. A* **f-ring extension** *of A is a f-ring, R, containing A, such that A is a subring and a sublattice of R, and $T_\sharp$ is the restriction to A of the natural partial order of R.*

Theorem 7.7 *Let $\langle A, T_\sharp \rangle$ be a reduced f-ring and let P be a proper preorder of A, containing $T_\sharp$ and having the zdp. Then, $\langle A, P \rangle$ is $\mathfrak{N}P$-faithfully quadratic. In particular, $\langle A, T_\sharp \rangle$ is $\mathfrak{N}T_\sharp$-faithfully quadratic. Moreover, the reduced special group associated to $\langle A, P \rangle$ is a Boolean algebra. Further, $\langle A, P \rangle$ satisfies the Arason-Pfister Hauptsatz* (cf. 5.2) *and Pfister's local-global principle* (cf. 5.8).

Proof. With notation as above, let $\langle A_*, T_{\sharp_*} \rangle$ be the total ring of fractions of $\langle A, T_\sharp \rangle$; as before, write $\mathfrak{N}$ for the multiplicative set of non zero-divisors in A. Moreover, just as in 6.1.(b),

$$P_* \;=\; \left\{\frac{p}{s^2} : p \in P \text{ and } s \in \mathfrak{N}\right\}$$

is a proper preorder on A_*, clearly containing $T_{\sharp_*}$. By Corollary 10.13 in [KZ], $\langle A_*, T_{\sharp_*} \rangle$ is a reduced f-ring extension of $\langle A, T_\sharp \rangle$. Since $T_{\sharp_*} \subseteq P_*$, Theorem 8.21 (p. 90) in [DM5] guarantees the P_*-quadratic faithfulness of $\langle A_*, P_* \rangle$ and that the special group associated to it is a Boolean algebra . But then Theorem 6.5 entails the $\mathfrak{N}$P-quadratic faithfulness of $\langle A, P \rangle$, while Proposition 6.3 yields an isomorphism between the reduced special groups associated to $\langle A_*, P_* \rangle$ and $\langle A, P \rangle$, as required. ■

With notation as in 5.3 and 5.5, we also have, with a proof following the same pattern as that of Theorem 10.7.(b), p. 114, of [DM5]:

Proposition 7.8 (Marshall's signature conjecture) *Let A be an f-ring and let $T_\sharp$ be its natural partial order. Let T be a preorder on A, such that $T_\sharp \subseteq T$ and having the zdp. Let φ be a form over $\mathfrak{N}_A$. If for all $\alpha \in$ Sper(A,T) satisfying* supp$(\alpha) \cap \mathfrak{N}_A = \emptyset$, *we have $sgn_{\tau_\alpha}(\varphi) \equiv 0 \bmod 2^n$, then $\varphi \in I^n_T(A)$.*

Proof. Write $\mathfrak{N}$ for $\mathfrak{N}_A$. By 5.5.(a) and Lemma 5.7.(b), the hypothesis on φ is equivalent to

$$\text{For all } \tau \in Z_{\mathfrak{N},T},\ sgn_\tau(\varphi) \equiv 0 \bmod 2^n,$$

whence Lemma 5.6 entails

(*) For all SG-characters σ on $G_{\mathfrak{N},T}(A)$, $sgn_\sigma(\varphi^T) \equiv 0 \bmod 2^n$.

But then:

- By Theorem 7.7, $G_{\mathfrak{N},T}(A)$ is a Boolean algebra, and
- Boolean algebras are [SMC]-special groups (Theorem 2.9, p. 203, of [DM3]), and hence by Proposition 4.4, p. 168 of [DM2], verify Marshall's signature conjecture.

From (*) and these observations we obtain $\varphi^T \in I^n(G_{\mathfrak{N},T}(A))$ and the quadratic faithfulness of $\langle A, T \rangle$ entails $\varphi \in I^n_T(A)$, as claimed. ■

Before our next result, we register:

Remarks 7.9 a) Let A be a ring and let $B := B(A)$ be the Boolean algebra (BA) of idempotents of A (cf. 4.1. p. 41 ff in [DM5]). Let $e_1, \ldots, e_m$ be an orthogonal decomposition of B into non-zero idempotents. We can consider B as a RSG and every RSG-character of B is also a BA-morphism (cf. Corollary 4.4, p. 62, and Proposition 4.5, p. 63, [DM1], respectively). We hence have, recalling that 0 and 1 are the bottom and top elements in B,

(1) $$1 = \bigvee_{j=1}^m e_j \quad \text{and} \quad e_j e_k = e_j \wedge e_k = 0, \text{ if } 1 \leq k \neq j \leq m,$$

where $\vee$ and $\wedge$ denote join and meet in B. Fix $\tau \in X_B = S(B)$ (the Stone space of B); then (1) and the fact that τ is a BA-morphism entail

$$1 = \tau(1) = \bigvee_{j=1}^m \tau(e_j) = 1 \quad \text{and} \quad \tau(e_j) \wedge \tau(e_k) = 0, \text{ for } 1 \leq j \neq k \leq m.$$

It is then clear that there is a unique k between 1 and m so that $\tau(e_k) = 1$, while $\tau(e_j) = 0$ for all $j \neq k$ $(1 \leq j \neq k \leq m)$. Therefore, X_B is the disjoint union of the clopens $[\![e_j = 1]\!]$,

(#) $$X_B = \coprod_{j=1}^m [\![e_j = 1]\!].$$

The same reasoning will show that if $e \in B$, then the space of orders of the Boolean algebra Be (whose bottom is 0 and top is e) is naturally homeomorphic to the clopen $[\![e = 1]\!]$ in X_B, via the Stone dual of the natural BA-morphism $f \in B \longmapsto fe \in Be$.

b) Let $\langle R, P \rangle$ be a proper *partially ordered* ring and let e be a *non-zero idempotent* in R. Clearly, $Pe = \{pe : p \in P\}$ is a preorder of the ring Re. In fact, it is a proper partial order of Re: indeed, since $Pe \subseteq P$ (e is a square in R), if $x, -x \in Pe$, then $x \in$ supp $P = \{0\}$, whence $x = 0$. In particular, $-e$ cannot be in Pe (recall: $e = 1$ in Re), whence $\langle Re, Pe \rangle$ is a proper partially ordered ring. Note that this result might be false if P is not a *partial order* of R. ■

The next Theorem is an analog in our present setting of Theorem 10.10, p. 116, [DM5].

Theorem 7.10 (Local-global Sylvester's inertia law) *Let A be an f-ring and let $T_\sharp$ be its natural partial order. Write X for the space of orders of the reduced special group $G = G_{\mathfrak{N},T_\sharp}(A)$ and write $a^\sharp$ for the elements of G* (cf. proof of Theorem 8.13.(d), p. 81, [DM5]). *Write $\mathfrak{t}_\sharp$ for $T_\sharp \cap \mathfrak{N}$. Let $\varphi = \langle a_1, \ldots, a_n \rangle$ and $\psi = \langle b_1, \ldots, b_n \rangle$*[4] *be n-forms over $\mathfrak{N} := \mathfrak{N}_A$, $n \geq 1$. The following are equivalent:*

(1) $\varphi \approx_{\mathfrak{N}T_\sharp} \psi$;

(2) *There is a disjoint finite covering of X by non-empty clopens, $U_1, \ldots, U_m$, such that for every $1 \leq j \leq m$, the following conditions are satisfied:*

(2.i) *No entry in $\varphi^\sharp$ and $\psi^\sharp$ changes sign in U_j, i.e., if σ, $\sigma' \in U_j$, then for each $1 \leq \ell \leq n$, $\sigma(a_\ell^\sharp) = \sigma'(a_\ell^\sharp)$ and $\sigma(b_\ell^\sharp) = \sigma'(b_\ell^\sharp)$;*

(2.ii) *The number of entries of $\varphi^\sharp$ and $\psi^\sharp$ that are strictly negative in U_j is the same.*

Proof. Implication (2) $\Rightarrow$ (1) is clear: (2) guarantees that for each $\sigma \in X$, the signature of $\varphi^\sharp$ and $\psi^\sharp$ are the same at σ and so Pfister's local-global principle for reduced special groups (Proposition 3.7, p. 51, [DM1]) entails $\varphi^\sharp \equiv_{\mathfrak{t}_\sharp} \psi^\sharp$. Hence, the $\mathfrak{N}T_\sharp$-quadratic faithfulness of $\langle A, T_\sharp \rangle$ yields $\varphi \approx_{\mathfrak{N}T_\sharp} \psi$, as needed.

(1) $\Rightarrow$ (2): We shall adopt here the conventions and notation set down in 6.1. Let $\langle A_*, T_{\sharp_*} \rangle$ be the total ring of fractions of $\langle A, T_\sharp \rangle$ (cf. beginning of the proof of Theorem 7.7), which we know to be an f-ring extension of $\langle A, T_\sharp \rangle$. To simplify exposition, write $\equiv_*$ for isometry in the RSG G_* associated to $\langle A_*, T_{\sharp_*} \rangle$ and for $d \in \mathfrak{N}$, write $\mathfrak{d}$ for $\dfrac{d}{1} \in A_*$.

Assume (1) holds; then, the $\mathfrak{N}T_\sharp$-quadratic faithfulness of $\langle A, T_\sharp \rangle$ entails $\varphi^\sharp \equiv_{\mathfrak{t}_\sharp} \psi^\sharp$ in G and the isomorphism ι^π of Proposition 6.3 yields $\langle \mathfrak{a}_1^*, \ldots, \mathfrak{a}_n^* \rangle \equiv_* \langle \mathfrak{b}_1^*, \ldots, \mathfrak{b}_n^* \rangle$ in G_*. Thus, the $T_{\sharp_*}$-quadratic faithfulness of $\langle A_*, T_{\sharp_*} \rangle$ (Theorem 6.5) yields $\mathfrak{f} := \langle \mathfrak{a}_1, \ldots, \mathfrak{a}_n \rangle \approx_{T_{\sharp_*}} \mathfrak{g} := \langle \mathfrak{b}_1, \ldots, \mathfrak{b}_n \rangle$ in $\langle A_*, T_{\sharp_*} \rangle$. By item (3) of the equivalence in Theorem 10.10, p. 166, [DM5], there is an orthogonal decomposition of A_* into idempotents, $e_1, \ldots, e_m$, (that we may assume to be all non-zero), such that for every $1 \leq j \leq m$ (numbering is as in the just mentioned Theorem 10.10):

(3.i) Each entry in $\mathfrak{f}$ and $\mathfrak{g}$ is either in $T_{\sharp_*}^{\times} e_j$ (strictly positive in Ae_j) or in $-T_{\sharp_*}^{\times} e_j$ (strictly negative in Ae_j);

(3.ii) The number of entries of $\mathfrak{f}$ and $\mathfrak{g}$ that are strictly negative in Ae_j is the same.

Fix j between 1 and m; by 7.9.(b), $\langle A_* e_j, T_{\sharp_*} e_j \rangle$ is a proper partially ordered ring and in fact an f-ring (cf. Lemma 8.10, p. 78, [DM5]). Let G_{*j} be RSG associated to $\langle A_* e_j, T_{\sharp_*} e_j \rangle$ (which is $(T_{\sharp_*} e_j)$-faithfully quadratic by Corollary 4.7, p. 46, [DM5] and so G_{*j} is a BA).

If $\mathfrak{a}_\ell$ is an entry in $\mathfrak{f}$, $1 \leq \ell \leq n$, such that $\mathfrak{a}_\ell e_j$ is strictly positive in $A_* e_j$, then the same will be true for all orders on A_* containing $T_{\sharp_*}$. Now, Example 3.15, together with Lemmas 3.16 and 3.17, pp. 36-38 of [DM5], entail that $\tau \in X_{G_{*j}}$ implies $\tau(\mathfrak{a}_\ell^*) = 1$, or equivalently, employing the identification registered in 7.9.(a), $\tau \in [\![e_j = 1]\!] \subseteq X_{G_*}$ implies $\tau(\mathfrak{a}_\ell^*) = 1$. Similarly, we treat the case in which $\mathfrak{a}_\ell$ is strictly negative in $A_* e_j$ (to obtain $\tau(\mathfrak{a}_\ell^*) = -1$, for each $\tau \in [\![e_j = 1]\!]$). This same reasoning applies, of course, to every entry in $\mathfrak{g} = \langle \mathfrak{b}_1, \ldots, \mathfrak{b}_n \rangle$. Thus, (3.$i$) and (3.$ii$) above yield :

[4] Whence, $\varphi^\sharp = \left\langle a_1^\sharp, \ldots, a_n^\sharp \right\rangle$; similarly for ψ.

(I) No entry in $\langle \mathfrak{a}_1^*, \ldots, \mathfrak{a}_n^* \rangle$ and $\langle \mathfrak{b}_1^*, \ldots, \mathfrak{b}_n^* \rangle$ changes sign in $[\![e_j = 1]\!] \subseteq X_{G_*}$.

(II) The number of negative values of each entry of $\langle \mathfrak{a}_1^*, \ldots, \mathfrak{a}_n^* \rangle$ and $\langle \mathfrak{b}_1^*, \ldots, \mathfrak{b}_n^* \rangle$ on $[\![e_j = 1]\!]$ are the same.

By (#) in 7.9.(a), we have $X_{G_*} = \coprod_{j=1}^m [\![e_j = 1]\!]$, a disjoint union of non-empty clopens. The RSG-isomorphism $\iota^\pi : G \longrightarrow G_*$, induces, by composition, a homeomorphism, $h : X_{G_*} \longrightarrow X_G$, given by $h(\tau) = \tau \circ \iota^\pi$. Set $U_j = f\left[[\![e_j = 1]\!]\right]$, $1 \leq j \leq m$; then, X_G is a disjoint union of the non-empty clopens $U_1, \ldots, U_m$. We claim that this covering satisfies conditions (2.(i)) and (2.(ii)) of the statement; in fact, it suffices to check the former, since the latter is an immediate consequence of (2.(i)) and (II) above.

The reader should keep in mind that in our current notation $\iota^\pi : G \longrightarrow G_*$ is given by $d^\sharp \in G \longmapsto \mathfrak{d}^* \in G_*$.

Fix j between 1 and m; if a_ℓ is an entry in φ, $1 \leq \ell \leq n$, and $\sigma \in U_j$, there is $\tau \in [\![e_j = 1]\!]$ so that $f(\tau) = \tau \circ \iota^\pi = \sigma$. Hence, $\sigma(a_\ell^\sharp) = \tau(\iota^\pi(a_\ell^\sharp)) = \tau(\mathfrak{a}_\ell^*)$, and since the value of $\mathfrak{a}_\ell^*$ does not change on $[\![e_j = 1]\!]$, the value of $a_\ell^\sharp$ cannot change on U_j. This same reasoning applies to the entries in ψ, completing the proof. ■

Theorems 7.7, 7.10 and Proposition 7.8 apply, in particular, to rings of continuous real valued functions on any completely regular space as well as to rings of continuous real valued semi-algebraic functions defined on $\mathbb{R}^n$. In what follows, X stands for a completely regular topological space:

- Write $\mathbb{C}_b(X)$ for the ring of real-valued bounded continuous functions on X, preordered by squares.
- Write $\mathbb{C}(X)$ for the ring of real valued continuous maps on X, preordered by squares.
- Write $\mathbb{C}_{sa}(\mathbb{R}^n)$ for the ring real valued continuous semi-algebraic maps defined on $\mathbb{R}^n$.

Corollary 7.11 *a) If A is a real closed ring and T is a preorder on A with the zdp, then $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic and $G_{\mathfrak{N},T}(A)$ is a Boolean algebra (BA).*

b) If T is a preorder on $\mathbb{C}(X)$ with the zdp, then $\langle \mathbb{C}(X), T \rangle$ is $\mathfrak{N}T$-faithfully quadratic and $G_{\mathfrak{N},T}(\mathbb{C}(X))$ is a BA. In particular, this holds for the zdp preorder

$$P_K = \{f \in \mathbb{C}(X) : K \subseteq [\![f \geq 0]\!]\},$$

where K is a closet set in X with non-empty interior. An analogous result holds for $\mathbb{C}_b(X)$.

c) If T is a preorder on $\mathbb{C}_{sa}(\mathbb{R}^n)$ with the zdp, then $\langle \mathbb{C}_{sa}(\mathbb{R}^n), T \rangle$ is $\mathfrak{N}T$-faithfully quadratic. In particular, this holds for the zdp preorder

$$P_K = \{f \in \mathbb{C}_{sa}(\mathbb{R}^n) : K \subseteq [\![f \geq 0]\!]\},$$

where K is a closet set in $\mathbb{R}^n$ with non-empty interior.

Proof. For the statements concerning P_K in (b) and (c), just recall items (a) and (b) in 2.11. ■

For the notion of real holomorphy ring of a ring, we refer the reader to section 2 in [KZ].

Proposition 7.12 *The real holomorphy ring of $\mathbb{C}(X)$, is $\mathfrak{N}P$- faithfully quadratic, where P is any zdp preorder on $\mathbb{C}_b(X)$.*

Proof. By Example 4.13, pp. 37-38 in [KZ], the real holomorphy ring of $\mathbb{C}(X)$ is isomorphic to $A := \mathbb{C}_b(X)$, and so (by 7.11), $\mathfrak{N}$P-faithfully quadratic for any zdp preorder P on A. ■

C. Germs of real-valued functions at a point of a perfectly normal space

We assume the reader is acquainted with the properties of inductive systems and limits of structures in a first-order language with equality over a right-directed poset. References can be found [Mi1] or [Mi2], chapter 17, §4 or [DM4], §2.3, p. 1701 and §6.5, p. 1728. To establish notation, we register:

Remark 7.13 a) Let $\langle I, \leq \rangle$ be an right directed partially ordered set (rd-poset), i.e. for all i, j, there is $k \in I$ so that i, $j \leq k$. Let L is a first-order language with equality. Let

(#) $$\mathcal{M} = \langle M_i; \{\mu_{ij}\colon i \leq j\} \rangle,$$

inductive system of L-structures over I, where $\mu_{ij} : M_i \longrightarrow M_j$ are L-morphism such that $\mu_{ii} = Id_{M_i}$ and for all $i \leq j \leq k$, $\mu_{jk} \circ \mu_{ij} = \mu_{ik}$. Let $\varinjlim \mathcal{M} = \langle M; \{\mu_i \ : i \in I\} \rangle$ be the inductive limit of $\mathcal{M}$, where, for $i \in I$, $\mu_i : M_i \longrightarrow M$ is a L-morphism, such that $\mu_j \circ \mu_{ij} = \mu_i$. Recall that $M = \bigcup_{i \in I} \mu_i[M_i]$.

b) If $\mathcal{N} = \langle N_i; \{\nu_{ij}\colon i \leq j\} \rangle$ is an inductive system over I, a **morphism** from $\mathcal{M}$ to $\mathcal{N}$ is a family of L-morphisms, $\eta = \{M_i \xrightarrow{\eta_i} N_i\}$ such that for all $i \leq j$, we have $\eta_j \circ \mu_{ij} = \nu_{ij} \circ \eta_i$. ■

The fundamental properties of inductive limits we need, all consequence of Fact 2.4 in [DM4] (or Corollary 17.8, [Mi1] or [Mi2]), are summarized in the following

Fact 7.14 *Let $\langle I, \leq \rangle$ be rd-poset and let $\mathcal{M}$ be an inductive system of L-structures over I* (as in (#) in item 7.13.(a)). *Let i, $j \in I$.*

a) For $x \in M_i$, $y \in M_j$, there is $k \geq i$, j and w, $z \in M_k$, so that $w = \mu_{ik}(x)$, $z = \mu_{jk}(y)$ and $\mu_k(w) = \mu_i(x)$ and $\mu_k(z) = \mu_j(y)$.

b) If $\varphi(\overline{v})$ is a conjunction of atomic formulas in L and $x_\ell \in M_{i_\ell}$, $1 \leq \ell \leq n$, then

$$M \models \varphi[\mu_{i_1}(x_1), \ldots, \mu_{i_n}(x_n)] \ \Leftrightarrow \ \begin{cases} \text{There is } k \geq i_1, \ldots, i_n, \text{ so that} \\ M_k \models \varphi[\mu_{i_n k}(x_1), \ldots, \mu_{i_n k}(x_n)]. \end{cases}$$

In particular

(1) For $x \in M_i$ and $y \in M_j$, if $\mu_i(x) = \mu_j(y)$, there is $k \geq i$, j, so that $\mu_{ik}(x) = \mu_{jk}(y)$.

(2) If c is a constant in L and $\mu_i(x) = c$ ($x \in M_i$), there is $j \geq i$ so that $\mu_{ij}(x) = c$ in M_j.

c) Let $\mathcal{N} = \langle N_i; \{\nu_{ij} : i \leq j\} \rangle$ be an inductive system of L-structures and let $\eta = \{\eta_i : i \in I\}$ be a morphism from $\mathcal{N}$ to $\mathcal{M}$. With notation as in Fact 2.4.(c), [DM4], *let $\varinjlim \eta : \varinjlim \mathcal{N} \longrightarrow \varinjlim \mathcal{M}$ be the morphism of inductive limits induced by η. If for each $i \in I$, η_i is injective, the same is true of $\varinjlim \eta$.* ■

Recall (2.12.(b)) that $L(P, N)$ is the first-order language with equality of zdp-rings.

Lemma 7.15 *Let I be a rd-poset and let $\mathcal{A} = \langle \langle A_i, P_i \rangle; \{f_{ij} : i \leq j\} \rangle$ be an inductive system over I, consisting of zdp-rings and $\mathfrak{N}$p-ring morphisms. For $i \in I$, write $\mathfrak{N}_i$ for $\mathfrak{N}_{A_i}$. Let $\langle \langle A, P \rangle; \{f_i : i \in I\} \rangle = \varinjlim \mathcal{A}$. Then,*

a) $A = \bigcup_{i \in I} f_i[A_i]$ is a unitary commutative ring, in which 2 is a unit.

b) $P = \bigcup_{i \in I} f_i[P_i]$ is a proper preorder on A and for all $i \in i$, $f_i : \langle A_i, P_i \rangle \longrightarrow \langle A, P \rangle$ is a p-ring morphism.

c) Let $N = \bigcup_{i \in I} f_i[\mathfrak{N}_i]$. Then

(1) N is multiplicative set in A, contained in $\mathfrak{N}_A$; (2) $\text{supp}(P) \cap N = \emptyset$.

Proof. Item (a) is clear, while (b) follows from Fact 6.5.(a), p. 1718, [DM4].

c) (1) : Note that N is the union of a I-directed set of proper multiplicative subsets of A and hence a multiplicative subset of A. That it is proper ($0 \notin N$), will follow immediately once it is verified that $N \subseteq \mathfrak{N}_A$.

For $u \in N$, suppose there is $v \in A$ so that $uv = 0$; hence, there are $i, j \in I$, together with $x \in \mathfrak{N}_i$ and $y \in A_j$ so that $f_i(x) = u$, $f_j(y) = v$ and $f_i(x)f_j(y) = 0$. By 7.14.(b), there is $k \geq i, j$, so that in A_k we have $f_{ik}(x)f_{jk}(y) = 0$ whence, $f_{jk}(y) = 0$, because $f_{ik}(x) \in \mathfrak{N}_k$. But then, $f_k(f_{jk}(y)) = f_j(y) = v = 0$, and so $N \subseteq \mathfrak{N}_A$ [5].

(2) : Assume there is $u \in A$ so that $u, -u \in P$ and $u \in N$. Hence, there are $i_1, i_2, i_3 \in I$ and $x_\ell \in A_{i_\ell}$, $\ell = 1, 2, 3$, so that $x_1 \in P_{i_1}$, $x_2 \in P_{i_2}$, $x_3 \in \mathfrak{N}_{i_3}$ and $f_{i_1}(x_1) = u$, $f_{i_2}(x_2) = -u = f_{i_1}(-x_1)$ and $f_{i_3}(x_3) = f_{i_1}(x_1)$. By 7.14.(b) (the formulas involved are atomic), there is $k \geq i_1, i_2, i_3$, so that in A_k we have $f_{i_1k}(x_1) = f_{i_2k}(-x_2)$ and $f_{i_1k}(x_1) = f_{i_3k}(x_3) \in \mathfrak{N}_k$. Thus, $f_{i_1k}(x_1)$, $-f_{i_1k}(x_1) \in P_k$ with $f_{i_1k}(x_1) \in \mathfrak{N}_k$, contradicting the assumption that $\langle A_k, P_k \rangle$ is a zdp-ring. ■

Remark 7.16 Write $A_N := AN^{-1}$ for the ring of fractions of A with respect to N and $P_N = \{p/s^2 : p \in P \text{ and } s \in N\}$ for the preorder on A_N induced by P; by 7.15.(c.2), P_N is a *proper* preorder. Write ι_N for the natural injective p-ring morphism from $\langle A, P \rangle$ to $\langle A_N, P_N \rangle$. ■

With notation as in 6.1, for $k \in I$ write ι_k for the natural injective p-ring morphism from $\langle A_k, P_k \rangle$ to its total ring of fractions, $\langle A_{k*}, P_{k*} \rangle$. The inductive system $\mathcal{A}$ in the statement of Lemma 7.15 gives rise to:

(*) An inductive system over I, $\mathcal{A}_* = \langle \langle A_{i*}, P_{i*} \rangle ; f_{ij*} : i \leq j\} \rangle$; if $i \leq j$ in I, we have $f_{ij*}(x/z) = f_{ij}(x)/f_{ij}(z)$ in A_{j*} (cf. 6.2.(b)). Let $\langle \langle \mathcal{Q}, \mathcal{P} \rangle ; \{f_{i*} : i \in I\} \rangle = \varinjlim \mathcal{A}_*$;

(**) A morphism of inductive systems, $\iota = \{\iota_k : k \in I\} : \mathcal{A} \longrightarrow \mathcal{A}_*$; since all ι_k are injective p-ring morphisms, the same is true of $j := \varinjlim \iota : \langle A, P \rangle \longrightarrow \langle \mathcal{Q}, \mathcal{P} \rangle$ (cf. Fact 7.14.(c)). For each $k \in I$, we have a commutative diagram (ι) (below left):

$$\begin{array}{ccc} \langle A_k, P_k \rangle & \xrightarrow{f_k} & \langle A, P \rangle \\ \iota_k \downarrow & (\iota) & \downarrow j \\ \langle A_{k*}, P_{k*} \rangle & \xrightarrow[f_{k*}]{} & \langle \mathcal{Q}, \mathcal{P} \rangle \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\iota_N} & A_N \\ & j \searrow \ (\mathrm{Q}) \ \swarrow j_* & \\ & \mathcal{Q} & \end{array}$$

Hence, for $a \in A$, select $k \in I$ and $x \in A_k$ so that $f_k(x) = a$ and then $j(a) = f_{k*}(x/1)$; clearly, $j(a)$ is independent of the chosen $k \in I$ and the witness $x \in A_k$.

Theorem 7.17 *With notation as above:*

a) If $u \in N$, then $j(u)$ is a unit in $\mathcal{Q}$. Let $j_* : \langle A_N, P_N \rangle \longrightarrow \langle \mathcal{Q}, \mathcal{P} \rangle$, given by $j_*(a/u) = j(a)j(u)^{-1}$, be the unique p-ring morphism so that $j_* \circ \iota_N = j$, i.e., the diagram (Q) *above right is commutative*

b) The map j_* is a p-ring isomorphism.

c) Assume $\langle A_i, P_i \rangle$ is $\mathfrak{N}P_i$-faithfully quadratic, for all $i \in I$. Moreover, suppose *the colimit of the inductive system $\mathcal{A}$ satisfies the following condition:*

(*) $$N = \mathfrak{N}_A.$$

Then, $\langle A, P \rangle$ is $\mathfrak{N}P$-faithfully quadratic.

[5] Since axiom [N] in 2.12.(b) is not geometric, we cannot guarantee $N = \mathfrak{N}_A$; however, since $f_i[\mathfrak{N}_i] \subseteq N \subseteq \mathfrak{N}_A$ (by 7.15.(c.1)), the f_i are $\mathfrak{N}$p-ring morphisms.

Proof. a) If $u \in N$, there are $k \in I$ and $x \in \mathfrak{N}_k$ so that $f_k(x) = u$. Then, $\iota_k(u)$ is a unit in A_{k*} and so $f_{k*}(u)$ is a unit in $\mathcal{Q}$. The commutative diagram (ι) above then yields $j(u) = j(f_k(u)) = f_{k*}(\iota_k(u)) \in \mathcal{Q}^\times$. The universal property of ring of fractions (Prop. 3.1, p.37, [AM] and its proof) yields the desired conclusion. The same argument employed above shows that if $v \in \mathcal{P}$, then $j_*(v) \in \mathcal{P}$. Therefore if $w = v/u^2 \in P_N$, then $j_*(w) = j(v)j(u)^{-2} \in \mathcal{P}$ and j_* is a p-ring morphism.

b) For $a/b \in A_N$, if $j_*(a/b) = j(a)j(b)^{-1} = 0$, then $j(a) = 0$ and the injectivity of j entails $j(a) = 0$, whence j_* is an injection.

To lighten notation, we identify A_k with its image by ι_k in A_{k*}. Let $\zeta \in \mathcal{Q}$; then there are $k \in I$, $x \in A_k$, $y \in \mathfrak{N}_k$ so that $f_{k*}(x/y) = f_{k*}(x)f_{k*}(y)^{-1} = \zeta$. Let $c = f_k(x)$ and $d = f_k(y) \in N$. The commutativity of diagram (ι) above yields $j(c) = f_{k*}(x)$ and $j(d) = f_{k*}(y)$. Hence, $j_*(c/d) = j(c)j(d)^{-1} = f_{k*}(x)f_{k*}(y)^{-1} = \zeta$, establishing the surjectivity of j_*. An analogous argument will show that $j_*[P_N] = \mathcal{P}$ and j_* is a p-ring isomorphism, as desired.

c) In the presence of (*), item (b) yields a p-ring isomorphism between $\langle \mathcal{Q}, \mathcal{P} \rangle$ and $\langle A_*, P_* \rangle$, the total ring of fractions of $\langle A, P \rangle$. If $\langle A_i, P_i \rangle$ are $\mathfrak{N}P_i$-faithfully quadratic for each $i \in I$, $\mathcal{A}_*$ is an inductive system of P_i-faithfully quadratic rings (by 6.5); by Corollary 5.4, p. 57, [DM5], $\langle \mathcal{Q}, \mathcal{P} \rangle$ is $\mathcal{P}$-faithfully quadratic and so, by item (b), $\langle A_*, P_* \rangle$ is P_*-faithfully quadratic. Another application of Theorem 6.5 guarantees the $\mathfrak{N}$P-quadratic faithfulness of $\langle A, P \rangle$, ending the proof. ■

We now apply Theorem 7.17 to establish the following

Proposition 7.18 *Let X be a perfectly normal space and let $A = \mathbb{C}(X)$ be the ring of real-valued functions defined on X, partially ordered by squares ($= \Sigma A^2$). Let p be a point in X and let A_p be the ring of germs of functions of A at p. Then, A_p is $\mathfrak{N}$-faithfully quadratic.*

Remark 7.19 A topological space, X, is **perfectly normal** if it is a (Hausdorff) normal space in which all closed sets are G_δ (countable intersection of opens); cf. paragraphs before Theorem 1.5.19 in [En]. The following are some of the basic properties of a perfectly normal space X :

(1) If F, K are disjoint closed sets in X, then there is $f \in \mathbb{C}(X, [0,1])$ so that $K = Z(f)$ and $F = \{x \in X : f(x) = 1\}$ (cf. Theorem 1.5.19, p. 45, [En]). In particular, X is completely regular;

(2) By Theorem 1.5.19, [En], every open set U in X is a cozero set, that is, there is a continuous map $g : X \longrightarrow [0, 1]$, so that $U = \{x \in X : g(x) > 0\}$ (in [En], U is called functionally open);

(3) By Theorem 2.1.6, p. 68, [En], perfect normality is a hereditary property, i.e., it is inherited by every subspace of X. ■

Proof of Proposition 7.18. Notation will be as in 8.5 and 8.6, p. 1728-1729 in [DM4].

For an open U in X, write $A(U)$ for $\mathbb{C}(U)$, the real-valued functions defined on U. If q is a point in X, let ν_q be the (filter) of open neighborhoods of q in X. We endow ν_q with an rd-poset structure where the partial order, $\leq$, is the opposite of the inclusion. Thus, for $U \leq V$ (i.e. $V \subseteq U$), the restriction $\alpha_{UV} : A(U) \longrightarrow A(V)$, taking $s \in A(U)$ to $s{\restriction}V \in A(V)$ is a natural $\mathfrak{N}$p-morphism. Indeed, for $U \in \nu_q$ and $s \in A(U)$, suppose s is not a zero-divisor in $A(U)$. By 2.11.(a), the zero set of s has empty interior in U; hence, if $q \in U$ and $g \in A(V)$ ($q \in V \subseteq U$) is such that gs is identically zero in $W \in \nu_q$, $W \subseteq V$, then $g{\restriction}W = 0$ and its germ at q is zero, whence the germ of s at q is a non zero-divisor, as needed.

It is well-known that $A_p = \varinjlim \langle \langle A(U), A(U)^2 \rangle; \{\alpha_{UV} : U \leq V\} \rangle$. For $U \in \nu_p$ and $t \in A(U)$, write t_p for the germ of t at p.

Since every open V in X is perfectly normal (7.19.(3)) and $A(V)^2$ is a partial order, each $\langle A(V), A(V)^2 \rangle$ is a $\mathfrak{N}$-faithfully quadratic zdp-ring (cf. Theorem 7.7). By 7.15 and Theorem 7.17.(c), the desired conclusion will follow once it is shown that the inductive system giving rise to A_p satisfies condition (*) in 7.17. In the present setting, this amounts to proving that if x is a non zero-divisor in A_p, there is $W \in \nu_p$ and $s \in A(W)$, so that $s_p = x$ and $s \in \mathfrak{N}_{A(W)}$.

Let x be a non zero-divisor in A_p, let $U \in \nu_p$ and let $s \in A(U)$ be such that $s_p = x$. Then:

- If $s(p) \neq 0$, then $s_p = x$ is a unit in A_p and we are done;
- Let $Z_U(s) = \{x \in U : s(x) = 0\}$ (a closed set in U). If the interior of $Z_U(s)$ in U is empty, then s is a non zero-divisor in $A(U)$ (by (*) in 2.11), as needed.

We are left to treat the case in which $s(p) = 0$ (i.e., x is in the maximal ideal of the local ring A_p) and $K :=$ interior $Z_U(s)$, is a non-empty open set in U.

Claim: $p \notin \overline{K}$.

Proof. Since U is perfectly normal, there is $g \in \mathbb{C}(U, [0,1])$ so that $\{x \in U : g(x) > 0\} = K$. Note that $gs = 0$ on U. Moreover, if $p \in \overline{K}$, every $V \in \nu_p$ has non-empty intersection with K and so $g_p \neq 0$ in A_p (g is not identically zero in any neighborhood of p) . But then, $x \cdot g_p = s_p \cdot g_p = 0$, with $g_p \neq 0$, and x would be a zero-divisor in A_p, a contradiction that establishes the Claim.

By the Claim there is a $W \in \nu_p$ so that $W \subseteq U$ and $W \cap \overline{K} = \emptyset$. Hence, the set of zeros of $s \upharpoonright W$ in W is contained in the frontier of $Z_U(s)$ and so has empty interior in W (the frontier of any closed has empty interior). Therefore, $s \upharpoonright W$ is a non zero-divisor in $A(W)$, whose germ at p is x, completing the verification of (*) in 7.17.(c) and the proof of Proposition 7.18. ■

D. Noetherian Rings

Herein, all rings will be commutative and unitary. If R is a ring, $\mathfrak{N}_R$ is the multiplicative set of non zero-divisors in R; when R is clear from context, write $\mathfrak{N}$ for $\mathfrak{N}_R$.

Remark 7.20 If R is a ring, we recall:

a) Let T be a preorder on R. An ideal I in R is

- ***T*-convex** if for all $s, t \in T$, $s + t \in I \;\Rightarrow\; s, t \in I$;
- ***T*-radical** if for all $a \in A$ and $t \in T$ $a^2 + t \in I \;\Rightarrow\; a \in I$.

By Prop. 4.2.5 in [BCR], an ideal of R is T-radical iff it is T-convex and radical.

b) Every prime ideal in R contains a minimal prime. Thus, the nilradical of R = the intersection of all primes in R = intersection of all minimal primes in R. ■

Lemma 7.21 *Let A be a reduced ring.*

a) $A \setminus \mathfrak{N}$ = union of the minimal prime ideals in A.

b) Let T be a partial order on A. If $p \in T \cap \mathfrak{N}$, then for all $q \in T$, $p + q \in \mathfrak{N}$. In particular, $1 + T \subseteq \mathfrak{N}$ and so is disjoint from all minimal prime ideals in A.

Proof. a) (1) Let $\mathfrak{p}$ be a minimal prime in A; then $\mathfrak{p}^c = A \setminus \mathfrak{p}$ is a saturated multiplicative set in A. Let $S = \{xz \in A : x \in \mathfrak{p}^c, z \in \mathfrak{N}\}$ be the multiplicative set generated by $\mathfrak{p}^c$ and $\mathfrak{N}$; note that S is proper: if $xz = 0$, since $z \in \mathfrak{N}$, we must have $x = 0$, that is impossible since x is the complement of a prime ideal. By a well-known result, there is a prime ideal, Q, so that $Q \cap S = \emptyset$. Thus, $Q \cap \mathfrak{p}^c = \emptyset$ and so $Q \subseteq \mathfrak{p}$. By the minimality of $\mathfrak{p}$, we must have $\mathfrak{p} = Q$, whence $\mathfrak{p} \cap \mathfrak{N} = \emptyset$ and $\mathfrak{p} \subseteq A \setminus \mathfrak{N}$, i.e., $\mathfrak{p}$ consists of zero divisors.

(2) Let u be a zero divisor in A; then, there is $v \neq 0$ such that $uv = 0$. Since A is reduced, there must be a minimal prime, $\mathfrak{q}$, so that $v \notin \mathfrak{q}$. But $uv = 0 \in \mathfrak{q}$ and so $u \in \mathfrak{q}$.

Item (a) follows immediately from (1) and (2).

b) Suppose $x(p + q) = 0$; then $x^2(p + q) = x^2p + x^2\ q = 0$ and so $x^2p \in \operatorname{supp}(T) = 0$. Since $p \in \mathfrak{N}$, we conclude that $x^2 = 0$ and reducibility entails $x = 0$, as needed. ∎

Remark 7.22 a) If A is a Noetherian ring, by Exercise 9, p. 79 in [AM], A has only a finite number, $\mathfrak{p}_k$, $1 \le k \le n$, of minimal primes.

b) Write $\widehat{a}$ for the elements of the quotient ring $A/\mathfrak{p}_k$ (any k). ∎

Lemma 7.23 *Let $\langle A, T \rangle$ be reduced Noetherian po-ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the set of minimal primes in A and let $\mathfrak{p}$ be any of them.*

a) If $y \notin \mathfrak{p}$, then there are $u_y \notin \mathfrak{p}$ and $z_y \in \mathfrak{N}$, so that $z_y - yu_y \in \mathfrak{p}$.

b) $\mathfrak{p}$ is T-convex.

c) With notation as in 7.22.(b), $\langle A/\mathfrak{p}, T/\mathfrak{p} \rangle$ is a proper po-ring and its field of fractions, $k(\mathfrak{p})$, partially ordered by the $T_\mathfrak{p} = \{\widehat{t}/\widehat{y}^2 \ : \widehat{x} \in T/\mathfrak{p}$ and $\widehat{y} \in A/\mathfrak{p}$, $\widehat{y} \neq 0.\ \}$, is formally real.

Proof. Without loss of generality, we may assume $\mathfrak{p} = \mathfrak{p}_1$.

a) Let $I = \{m + \lambda y \in A\colon\ m \in \mathfrak{p}_1$ and $\lambda \in A\}$ be the ideal generated by $\mathfrak{p}_1$ and y. Clearly, I is not contained in $\mathfrak{p}_1$; moreover, I is not contained in $\bigcup_{k=2}^n \mathfrak{p}_k$, otherwise, by Prop. 1.11.(i) in [AM], $\mathfrak{p}_1 = \mathfrak{p}_k$, for some $k \ge 2$, which is impossible. Therefore I is not contained in $\bigcup_{k=1}^n \mathfrak{p}_k$ and so, by 7.21.(a), $I \cap \mathfrak{N} \neq 0$. The desired conclusion follows immediately. Note that u_y cannot be in $\mathfrak{p}_1$, otherwise, $\mathfrak{p}_1 \cap \mathfrak{N} \neq \emptyset$, an impossibility.

b) Let p, $q \in T$ and assume $p + q \in \mathfrak{p}_1$. If $p \notin \mathfrak{p}_1$, item (a) yields $u_p \notin \mathfrak{p}_1$ and $z_p \in \mathfrak{N}$ so that $pu_p \equiv z_p \bmod \mathfrak{p}_1$. But then, $p^2u_p^2 \equiv z_p^2 \bmod \mathfrak{p}_1$ and so $z_p^2 - p^2u_p^2 \in \mathfrak{p}_1$. But then,

$$pu_p^2(p + q) = p^2u_p^2 + pqu_p^2 \in \mathfrak{p}_1 \quad \text{and} \quad z_p^2 - p^2u_p^2 \in \mathfrak{p}_1$$

and so $x = z_p^2 + pqu_p^2 \in \mathfrak{p}_1$; since $z_p^2 \in T \cap \mathfrak{N}$ and $pqu_p^2 \in T$, by 7.21.(b), we would have $x \in \mathfrak{p}_1 \cap \mathfrak{N}$, a contradiction. Hence, $p \in \mathfrak{p}_1$, whence the same is true of q and $\mathfrak{p}_1$ is T convex.

c) By (b) and 7.21.(b), $T/\mathfrak{p}_1$ is a proper partial order on $A/\mathfrak{p}_1$; the remaining statement is clear. ∎

Theorem 7.24 *Let $\langle A, T \rangle$ be a reduced partially ordered Noetherian ring and let $\langle A_*, T_* \rangle$ be its total ring of quotients. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes in A. Then:*

a) $\langle A_*, T_* \rangle$ is a partially ordered reduced Noetherian ring.

b) The only prime ideals in A_* are $\mathfrak{p}_k\mathfrak{N}^{-1}$, $1 \le k \le n$; in particular, these ideal being pairwise distinct, are minimal and maximal in A_*. Moreover,

(1) $\bigcap_{k=1}^n \mathfrak{p}_k\mathfrak{N}^{-1} = \{0\}$; (2) *The $\mathfrak{p}_k\mathfrak{N}^{-1}$ are pairwise coprime.*

c) With notation as in 7.23.(c), $\langle A_*, T_* \rangle$ is naturally isomorphic to a product of partially order formally real fields, in fact, $\langle A_*, T_* \rangle \approx \prod_{k=1}^n \langle k(\mathfrak{p}_k), T_{\mathfrak{p}_k} \rangle$, and so $\langle A_*, T_* \rangle$ is T_*-faithfully quadratic. Hence, $\langle A, T \rangle$ is $\mathfrak{N}T$-faithfully quadratic.

Proof. a) By Prop. 7.3, p. 80, [AM], A_* is Noetherian; its reducibility is straightforward.

b) By Prop. 3.11, p. 41, [AM], the prime ideals in A_* are in bijective correspondence with the prime ideals of A disjoint from $\mathfrak{N}$ and so 7.21 yields the first statement in (b) and (b.1). Since the $\mathfrak{p}_k\mathfrak{N}^{-1}$ are all distinct and maximal, they must be pairwise coprime, establishing (b.2).

c) By the Chinese Remainder Theorem (Prop. 1.10, p. 7, [AM]), there is a natural ring isomorphism

$$\gamma : A_* \longrightarrow \prod_{k=1}^n A_*/\mathfrak{p}_k\mathfrak{N}^{-1} \;=\; \prod_{k=1}^n A\mathfrak{N}^{-1}/\mathfrak{p}_k\mathfrak{N}^{-1}$$

Since T_* is a partial order on A_* and $\mathfrak{p}_k\mathfrak{N}^{-1}$ are minimal primes in A_*, by 7.23.(c), $T_*/\mathfrak{p}_k\mathfrak{N}^{-1}$ is a proper partial order on $A_*/\mathfrak{p}_k\mathfrak{N}^{-1}$, and so a partially ordered field ($\mathfrak{p}_k\mathfrak{N}^{-1}$ is also maximal). It is straightforward that $T_* = \prod_{k=1}^n T_*/\mathfrak{p}_k\mathfrak{N}^{-1}$. Thus, each component of the product, $\langle A_*/\mathfrak{p}_k\mathfrak{N}^{-1}, T_*/\mathfrak{p}_k\mathfrak{N}^{-1} \rangle$ is a partially ordered field, (hence, formally real). Since preordered fields are completely faithfully quadratic (cf. Theorem 6.5, p. 63, [DM5]) and quadratic faithfulness is preserved by products (cf. Theorem 4.6, p. 44, [DM5] or employ the fact that P-quadratic faithfulness is Horn-geometric, cf. [HR]), we conclude that $\langle A_*, T_* \rangle$ is T_*-faithfully quadratic.

It only remains to establish that $A_*/\mathfrak{p}_k\mathfrak{N}^{-1}$ is isomorphic to the field of fractions, $k(\mathfrak{p}_k)$, $1 \leq k \leq n$.

Employing the notation set down in 7.22.(b), define $f : A_* \longrightarrow k(\mathfrak{p}_k)$, by $f(a/s) = \widehat{a}/\widehat{s}$; it is straightforward that f is a ring morphism, whose kernel is precisely $\mathfrak{p}_k\mathfrak{N}^{-1}$. Hence, f factors through the quotient morphism from A_* to $A_*/\mathfrak{p}_k\mathfrak{N}^{-1}$ to yield an injective ring morphism, $g : A_*/\mathfrak{p}_k\mathfrak{N}^{-1} \longrightarrow k(\mathfrak{p}_k)$. Given an element $\widehat{a}/\widehat{y} \in k(\mathfrak{p}_k)$, $\widehat{y} \neq 0$, then $y \notin \mathfrak{p}_k$. By 7.23.(a), there are $u_y \notin \mathfrak{p}_k$ and $z_y \in \mathfrak{N}$ so that $yu_y - z_y \in \mathfrak{p}_k$. But then, $\widehat{a}/\widehat{y} = \widehat{u_y}\,\widehat{a}/\widehat{z_y}$ in $k(\mathfrak{p}_k)$, showing that g is surjective and so a ring isomorphism. It is clear from the definitions that $g(T_*/\mathfrak{p}_k\mathfrak{N}^{-1})$ $= T_{\mathfrak{p}_k}$, completing the proof of (c). The last statement in (c) is an immediate consequence of Theorem 6.5, ending the proof. ■

Corollary 7.25 *a) If $\langle A, T \rangle$ is a partially ordered Noetherian ring and J is a proper T-convex radical ideal in A, $\langle A/J, T/J \rangle$ is $\mathfrak{N}(T/J)$-faithfully quadratic.*

b) If A is a real Noetherian ring and I is a real ideal in A, then A/I is $\Sigma\mathfrak{N}$-faithfully quadratic. In particular, A is $\Sigma\mathfrak{N}$-faithfully quadratic. Moreover, these statements apply to rings of polynomials and of formal power series with coefficients in A, in any finite number of variables.

Proof. We comment briefly on item (a). Since J is radical, A/J is a reduced Noetherian ring. Moreover, J being T-radical (cf. 7.20) and proper, we must have $(1 + T) \cap J = \emptyset$; since it is T-convex, T/J is a proper partial order on A/J and the desired conclusion is immediate from Theorem 7.24. ■

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