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Approximations of Modal Logic K

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Abstract

Inspired by the recent work on approximating classical logic, we present a procedure that approximates the modal logic **K** by determining a maximum limit of introspection. This procedure has a clear semantics and a natural proof theory in the context of Massacci's modal Single Step Tableaux (SST).

Keywords: Approximated Inference, Modal Logic, Approximations of Modal Logic.

1 Introduction

George Boole defined logic as the science of "the fundamental laws of those operations of the mind by which reasoning is performed" [1]. Much of the later development emphasised the relations between logic and foundations of mathematics, leaving aside its use as a model for a thinking agent. With the development of Artificial Intelligence, however, the use of logic as the basis of

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the formalisation of problems of Knowledge Representation has been shown, in many cases, to be successful, returning in a certain way to Boole's ideal.

Ideal agents know all the consequences of their beliefs. However, real agents are limited in their capabilities. Following the work of Marco Cadoli and Marco Schaerf [9], Marcelo Finger and Renata Wassermann [3] developed a method for modelling limited and approximated reasoning for classical propositional logic. Their work considers a set of atoms — which is called *relevant propositions* — permitting a non-classical behaviour of the operator $\neg$ on sentences containing atoms outside this set. The set of relevant propositions is a *parameter* in the construction of a family of logics that "approximate" classical logic. Each logic in the family proves a subset of classical theorems; the approximation process happens when a logic cannot prove a specific theorem, so its parameter set is incremented, moving to a more powerful logic.

There are two sources of complexity in modal logics: *logical omniscience* and *unbounded logical introspection* [4]. Logical omniscience means that the thinking agent knows all the consequences of his beliefs. The work of Finger and Wassermann, among others, give a method to control this feature in classical logic, and this method is easily translated to modal logic. This paper is concerned with the other source of complexity. Unbounded logical introspection allows an agent to reason about his beliefs of his beliefs ... of his beliefs. Limitation is accomplished by selecting a *maximum limit of introspection*, inside which the agent can reason classically about its beliefs; outside, the behaviour is non-normal. By selecting arbitrarily large limits of introspection, we can recover classical normal modal logic; for the sake of this presentation, we restrict ourselves to modal logic **K**, but the ideas here are by no means restricted to **K**.

This modal logic of limited of introspection is initially presented in semantical terms. The semantics gives rise to a very simple proof theory in the context of the Massacci's SST-tableaux [8]. Massacci, in [7], has developed a different method of approximating modal logics. In the context of Hilbert-style derivation systems, this kind of approximations is known in the work of Ghilardi [5].

The rest of the paper develops as follows. Section 2 presents the notations used, some concepts of modal logics necessary for the following sections and what do labelled signed sentences (used in the tableau-based proof theory) mean. Section 3 presents the semantics of the approximating logics, examples and a theorem that states exactly what is needed to determine the truth of a sentence in an approximating logic. Section 4 presents a tableau-based proof

[4] We observe that the use of the word "introspection" is usually restricted to the context of epistemic logic. Following [7], we extend its use to any modal logic.

theory in the context of SST, examples and proves soundness and completeness of the tableaux rules with respect to the semantics of Section 3. In the conclusion, possible directions for future research are discussed.

2 Preliminaries

We assume familiarity with the basics of modal logics as presented, for instance, in [6], and Smullyan's tableaux [10]. Some notations will be explained. The language is based on a denumerable set A of propositional variables, or *atoms*, and contains the logical symbols $\neg$, $\wedge$, $\vee$, $\supset$, $\Box$ and $\Diamond$. Consider a model $\mathcal{M} = \langle W, R, v\rangle$ for the modal logic **K**. In this model, W is a set of *possible worlds*, $R \subseteq W \times W$ is the *accessibility relation* and $v : W \to (A \to \{0,1\})$ is a function that assigns to each world $w \in W$ a *propositional valuation*. The *reflexive transitive closure* of a world $w \in W$, denoted by w^*, is the set of all $t \in W$ such that there is a sequence wRw_1, w_1Rw_2, $\ldots$, $w_{n-1}Rt$ — this sequence is a *path* $wRw_1Rw_2R\cdots Rw_{m-1}Rt$ from w to t. The *distance* from w to t in the path $wRw_1Rw_2R\cdots Rw_{m-1}Rt$ is denoted by $|w,t|$ and equals to m. The *nesting of modal operators* in a sentence ϕ, $\mathcal{N}(\phi)$, is defined as follows: if ϕ is an atom, $\mathcal{N}(\phi) = 0$; $\mathcal{N}(\neg\phi) = \mathcal{N}(\phi)$; $\mathcal{N}(\phi \wedge \psi) = \mathcal{N}(\phi \vee \psi) = \mathcal{N}(\phi \supset \psi) = \max\{\mathcal{N}(\phi), \mathcal{N}(\psi)\}$; $\mathcal{N}(\Box\phi) = \mathcal{N}(\Diamond\phi) = \mathcal{N}(\phi) + 1$. As an example, $\mathcal{N}(\Diamond\neg\Box(p \wedge \Diamond q)) = 3$.

In Section 4, we use Massacci's Single Step Tableaux, SST, as presented in [8], and here we present only the main aspects needed. SST is a variant of Fitting's *prefixed tableaux* [4]. Tables 1 and 2 show unifying notations that simplify considerably the presentation of the tableaux rules which are applied to signed sentences. *Signed sentences* are sentences that start with a T or a F, as $T\phi$ and $F\phi$. Intuitively, $T\phi$ means "ϕ is true" and $F\phi$ means "ϕ is false"; so $T\phi$ has the same truth value as ϕ, and $F\phi$ has the same truth value as $\neg\phi$. The components of SST are *prefixed sentences*. In those sentences, before any signed sentence θ, there is a prefix σ, written as $\sigma : \theta$. A prefix, by definition, is a finite sequence of integers that begins with 1. Different elements of this sequence are separated by dots. So 1, 1.1, 1.2.1 are all examples of prefixes. Intuitively, the prefix σ in $\sigma : \theta$ "names" a world where the sentence θ is true [5]. The advantage of this representation is that it codifies all introspection steps needed in order to arrive from the "real" world denoted by 1 to the world denoted by σ (we "go" from one world w to another t when wRt by the accessibility relation R). For instance, the prefix 1.2.1.1 shows that, in order to arrive at the world 1.2.1.1 from the world 1, we need to go from 1 to 1.2, them from 1.2 to 1.2.1, and finally from 1.2.1 to 1.2.1.1. The *size* of a prefix

[5] We are not assuming that for a given world there is a single name.

$1.n_1.n_2.\ldots.n_m$ is denoted by $|1.n_1.n_2.\ldots.n_m|$ and equals to m.

α	α_1	α_2
$T\phi \wedge \psi$	$T\phi$	$T\psi$
$F\phi \vee \psi$	$F\phi$	$F\psi$
$F\phi \supset \psi$	$T\phi$	$F\psi$

neg	*pos*
$T\neg\phi$	$F\phi$
$F\neg\phi$	$T\phi$

β	β_1	β_2
$F\phi \wedge \psi$	$F\phi$	$F\psi$
$T\phi \vee \psi$	$T\phi$	$T\psi$
$T\phi \supset \psi$	$F\phi$	$T\psi$

Table 1
Smullyan's unifying notation

ν	ν_0
$T\Box\phi$	$T\phi$
$F\Diamond\phi$	$F\phi$

π	π_0
$F\Box\phi$	$F\phi$
$T\Diamond\phi$	$T\phi$

Table 2
Fitting's unifying notation

Suppose that we want to prove some unsigned sentence ϕ. First, we sign ϕ as $F\phi$ and prefix it with 1, obtaining $1 : F\phi$ in the root of the tableau. Then we add nodes by the tableaux rules. A *branch* R is a path from the root to the leaf. Intuitively, each branch is a tentative model, and the tableau is a systematic attempt to obtain a counter example — technically called *counter model* — to ϕ. A branch of the tableau *closes* when in it appears a pair of sentences $\sigma : T\phi$ and $\sigma : F\phi$, and the tableau *closes* when all its branches close. If the tableau closes, we succeeded in proving the original sentence. If some branch does not close even after we use every possible tableau rule, we say that it is *open* and *reduced* and we have the required counter model.

3 Semantics

The following semantics presents under what conditions a sentence ψ is to be considered true according to the truth of its subsentences. We have three parameters that determine the truth: the model $\mathcal{M} = \langle W, R, v\rangle$, the world $w \in W$ and an integer n (so $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$), that is to be called *maximum limit of introspection.*

Semantics of the n-approximations

(i) $\mathcal{M}, w, n \models p$ iff $v(w)(p) = 1$
(ii) $\mathcal{M}, w, n \models \neg\phi$ iff $\mathcal{M}, w, n \not\models \phi$
(iii) $\mathcal{M}, w, n \models \phi \wedge \psi$ iff $\mathcal{M}, w, n \models \phi$ and $\mathcal{M}, w, n \models \psi$
(iv) $\mathcal{M}, w, n \models \phi \vee \psi$ iff either $\mathcal{M}, w, n \models \phi$ or $\mathcal{M}, w, n \models \psi$
(v) $\mathcal{M}, w, n \models \phi \supset \psi$ iff either $\mathcal{M}, w, n \not\models \phi$ or $\mathcal{M}, w, n \models \psi$
(vi) if $n > 0$, $\mathcal{M}, w, n \models \Box\phi$ iff for every $t \in W$ such that wRt, $\mathcal{M}, t, n-1 \models \phi$

(vii) if $n > 0$, $\mathcal{M}, w, n \models \Diamond\phi$ iff there exists some $t \in W$ such that wRt and $\mathcal{M}, t, n-1 \models \phi$

(viii) if $n \leq 0$, either $\mathcal{M}, w, n \models \Box\phi$ or $\mathcal{M}, w, n \not\models \Box\phi$

(ix) if $n \leq 0$, either $\mathcal{M}, w, n \models \Diamond\phi$ or $\mathcal{M}, w, n \not\models \Diamond\phi$

For a fixed n, when $\mathcal{M}, w, n \models \psi$ we say that ψ is n-*true* in the world w on the model $\mathcal{M}$. Similarly, if for a fixed n we have $\mathcal{M}, w, n \not\models \psi$, then we say that ψ is n-*false* in the world w on the model $\mathcal{M}$. When ψ is n-true for every possible world w on every model $\mathcal{M}$, we say simply that ψ is n-valid, denoting this fact by $n \models \psi$. Note that the semantics above implies that every n-valid theorem is a **K**-valid theorem. The logic obtained by fixing some n is called n-logic.

When $n > 0$, clauses 6–7 can be applied and the semantics works in the same way as in classical modal logic. It is in clauses 8–9 that differences can be found. When $n \leq 0$, we only ask that one (and therefore, by clause 2, exactly one) of $\Box\phi$ and $\neg\Box\phi$ be true. This is a tautology, of course (we are indeed stating the absence of a condition), but the point in clauses 8–9 is that there is no other thing that must happen — therefore, we can choose the truth in any way we like, we have freedom to do that. The choice of truth could be made on the model itself, but because in our proof theory we shall be working with the construction of a model, this formulation is better suited. Clauses 8–9 could be absent from the definition, but we state it for the sake of clearness.

Now, some examples.

(i) The Rule of Necessitation, $n \models \phi \Rightarrow n \models \Box\phi$, does not hold in a n-approximation when $n = 0$. Let $\mathcal{M}$ be an arbitrary model and w some world in this model. Suppose that ϕ is n-valid (for instance, a tautology), so that $\mathcal{M}, t, n \models \phi$ for every t and n. Take $n = 0$. By Theorem 3.1, as $0 \leq 0$ and $w \in w^*$, it is possible to choose the 0-truth of $\Box\phi$ in w. So, assume that $\mathcal{M}, w, 0 \not\models \Box\phi$.

(ii) The principle K, $\Box(p \supset q) \supset (\Box p \supset \Box q)$, is not 0-valid. Let $\mathcal{M}$ be an arbitrary model and w some world in this model, and take $n = 0$. By Theorem 3.1 (in the same way as the last example), we can choose the 0-truth of $\Box p$, $\Box q$ and $\Box(p \supset q)$ in w; assuming that $\mathcal{M}, w, 0 \models \Box(p \supset q)$, $\mathcal{M}, w, 0 \models \Box p$ and $\mathcal{M}, w, 0 \not\models \Box q$, we obtain a counter model for K.

(iii) The equivalence $\Diamond\phi \equiv \neg\Box\neg\phi$ is not 0-valid. Let $\mathcal{M}$ be an arbitrary model and w some world in this model, and take $n = 0$. As in the previous examples, we can choose the truth of $\Diamond\phi$ and $\Box\neg\phi$. Assume that $\mathcal{M}, w, 0 \not\models \Box\neg\phi$ and that $\mathcal{M}, w, 0 \not\models \Diamond\phi$. By the semantics of $\neg$, condition 2, $\mathcal{M}, w, 0 \not\models \Box\neg\phi$ implies that $\mathcal{M}, w, 0 \models \neg\Box\neg\phi$. Similarly, the other equivalences $\neg\Box\phi \equiv \Diamond\neg\phi$, $\neg\Box\neg\phi \equiv \Diamond\phi$ are not 0-valid.

Examples 1 and 2 above show that the logic we are treating here is not normal, because neither the Rule of Necessitation nor K are n-valid when $n = 0$ (or when $n \leq 0$).

The following theorem looks at what determines the n-truth of a sentence at a world w on a model $\mathcal{M}$.

Theorem 3.1 *Let $\mathcal{M} = \langle W, R, v\rangle$ be a model. The n-truth of a sentence ϕ for a world $w \in W$ is determined by v and by the m-truth of every subsentence $\Box\phi'$ and $\Diamond\phi''$ of ϕ for $t \in w^*$, where $m \leq 0$ and $n - m = |w, t|$.*

Proof. The proof is by induction on the complexity of ϕ.

(i) If ϕ is some atom p, $\mathcal{M}, w, n \models p$ if $v(w)(p) = 1$, and $\mathcal{M}, w, n \not\models p$ if $v(w)(p) = 0$. So the n-truth of ϕ in w is determined by v.

(ii) If ϕ is $\neg\varphi$, the n-truth of ϕ in w is opposite to that of φ (condition 2 of the semantics), and by the inductive hypothesis the n-truth of φ in w is determined by v and the m-truth of every subsentence $\Box\varphi'$ and $\Diamond\varphi''$ of φ for $t \in w^*$ — subsentences of ϕ also —, where $m \leq 0$ and $n - m = |w, t|$.

(iii) If ϕ is $\varphi \wedge \psi$, the n-truth of ϕ in w is determined by the n-truth of φ and ψ (condition 3 of the semantics), and by the inductive hypothesis the n-truth of φ (ψ) in w is determined by v and the m-truth of every subsentence $\Box\varphi'$ ($\Box\psi'$) and $\Diamond\varphi''$ ($\Diamond\psi''$) of φ (ψ) for $t \in w^*$ — subsentences of ϕ also —, where $m \leq 0$ and $n-m = |w, t|$. Similarly for the connectives $\vee$ and $\supset$ (conditions 4–5 of the semantics).

(iv) Suppose that $n > 0$. If ϕ is $\Box\varphi$, the n-truth of ϕ for w is determined by the $(n-1)$-truth of φ for every t such that wRt, that is, $t \in w^*$ and $|w, t| = 1$ (condition 6 of the semantics). By the inductive hypothesis, the $(n-1)$-truth of φ for t is determined by v and by the m-truth of every subsentence $\Box\varphi'$ and $\Diamond\varphi''$ of φ for $t' \in t^*$ — subsentences of ϕ also —, where $m \leq 0$ and $n - 1 - m = |t, t'|$. Therefore, the n-truth of ϕ in w is determined by v and by the m-truth of every subsentence $\Box\varphi'$ and $\Diamond\varphi''$ of φ for $t' \in t^*$ for every t such that wRt, where $m \leq 0$ and $n-1-m = |t, t'|$. But wRt and $t' \in t^*$ implies that $t' \in w^*$, and $|t, t'| = n - 1 - m$ implies that $|w, t'| = (n - 1 - m) + 1 = n - m$. So the n-truth of ϕ in w is determined by v and by the m-truth of every subsentence $\Box\varphi'$ and $\Diamond\varphi''$ of ϕ for $t' \in w^*$, where $m \leq 0$ and $n - m = |w, t'|$. Similarly for the connective $\Diamond$ when $n > 0$ (condition 7 of the semantics).

(v) Suppose that $n \leq 0$. If ϕ is $\Box\varphi$, condition 8 of the semantics just says that either $\mathcal{M}, w, n \models \Box\varphi$ or $\mathcal{M}, w, n \not\models \Box\varphi$ — so we can choose which one. Therefore, the n-truth of ϕ for w is determined by the n-truth of $\Box\varphi$ for $t = w$, that obviously satisfies $t \in w^*$. Similarly for the connective $\Diamond$ when $n \leq 0$ (condition 9 of the semantics).

That completes the proof. Note that in usual modal logics, the truth value of a sentence at a world in a frame $\langle W, R\rangle$ is determined only by v. □

4 A Tableau-Based Proof Theory

Consider a fixed maximum limit of introspection n. The proof theory of the semantics presented in the previous section will be given by the so-called n-SST-rules, that characterise the n-Single Step Tableaux (n-SST). In the tables below, we use the notations explained on Tables 1 and 2.

$$\frac{\sigma : \alpha}{\begin{array}{l}\sigma : \alpha_1 \\ \sigma : \alpha_2\end{array}} \qquad \frac{\sigma : neg}{\sigma : pos} \qquad \frac{\sigma : \beta}{\sigma : \beta_1 \quad \sigma : \beta_2}$$

Table 3
Propositional n-SST-rules

Table 3 gives the n-SST rules for propositional connectives, and is exactly the same as the SST rules for propositional connectives, given in [8].

$$\frac{\sigma : \pi}{\sigma.m : \pi_0}\ (\pi') \quad \text{with } |\sigma| < n \text{ and } \sigma.m \text{ new in the branch}$$

$$\frac{\sigma : \nu}{\sigma.m : \nu_0}\ (K) \quad \text{with } \sigma.m \text{ already in the branch}$$

Table 4
Modal n-SST-rules

Table 4 gives the n-SST rules for modal connectives for the modal logic **K**. The only difference between these rules and those for SST in [8] is that rule (π') substitutes rule (π) there; rule (π') entitles us to accept $\sigma.m : \pi_0$ on the strength of $\sigma : \pi$ only if $|\sigma| < n$; this restriction does not exist for (π).

Below we have two examples of proofs by n-SST.

Figure 1 gives a proof of K by n-SST. Sentences (2) and (3) of both sides are obtained from (1) by application of the α-rule; in the same way we obtain (4) and (5). Now, in the left side, there is nothing more that can be done, because we need to apply (π') in (5) to obtain a new prefix and this is only possible for $|\sigma| < n$; when $n = 0$, as in the left side, this does not happen $(0 = |1| \geq 0)$. In the right side, we can apply (π'), obtaining (6). (7) and (8) are obtained from (2) and (4), respectively, by the (K)-rule. (9) and (10) come from (7) by the β-rule.

The left side of Figure 1 permits us to construct a counter model for K when $n = 0$. The counter model is seen in Example 2 of the previous section (using 1 in the place of w).

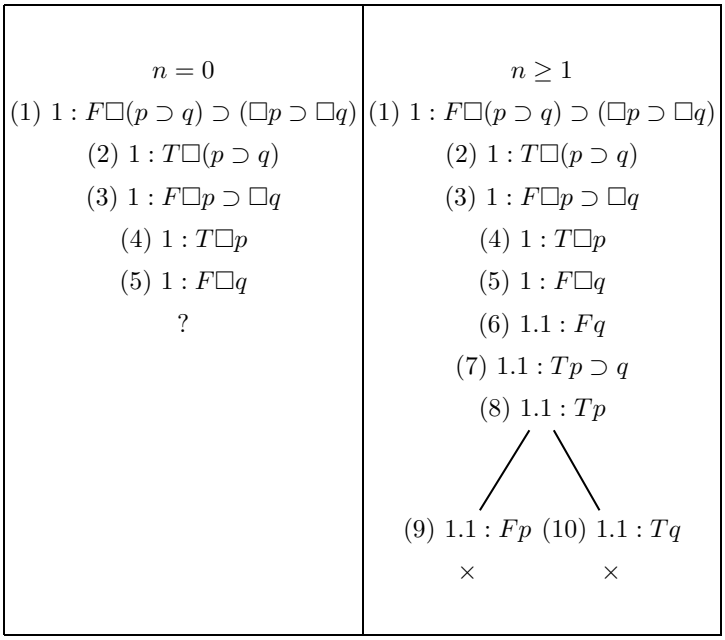

Fig. 1. Proof of K by n-SST

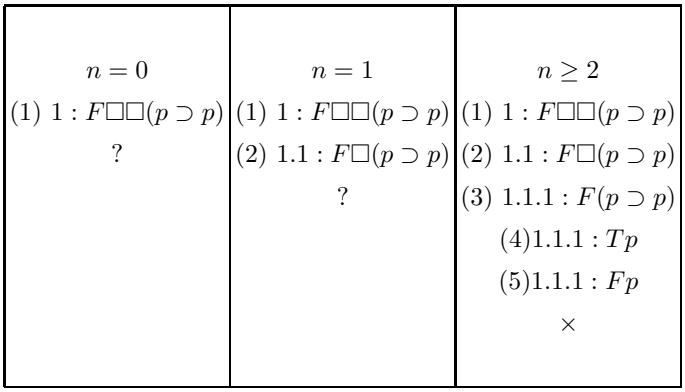

Fig. 2. Proof of $\Box\Box(p \supset p)$ by n-SST

Figure 2 gives a proof of $\Box\Box(p \supset p)$ by n-SST. Sentence (1) asks for application of (π')-rule, and this cannot be done in the left side because, there, $n = 0$, and $0 = |1| \geq 0$. Sentence (2) in the middle and right sides is obtained by (π')-rule, that can be applied because $|1| < 1$. We cannot proceed anymore in the middle, but because $1 = |1.1| \leq 2$ we can apply (π') again in the right side, obtaining (3). Finally, sentences (4) and (5) of the right side are obtained by application of the α-rule.

The left side of Figure 2 permits us to construct a counter model $\mathcal{M} = \langle W, R, v\rangle$ for $\Box\Box(p \supset p)$ when $n = 0$. We can take $W = \{1\}$ (the set of prefixes

in the branch), $R = \emptyset$ (because there is no prefix $1.m$ in the branch which is accessed by the world denoted by 1), and $v(1)$ is a fixed valuation of the atoms of the language (any will work). Because of Theorem 3.1, we can choose the 0-truth of $\Box\Box(p \supset p)$ in the world 1; we choose $\mathcal{M}, 1, 0 \not\models \Box\Box(p \supset p)$.

The middle of Figure 2 permits us to construct a counter model for $\Box\Box(p \supset p)$ when $n = 1$. Take $\mathcal{M} = \langle W, R, v\rangle$, where $W = \{1, 1.1\}$, $R = \{(1, 1.1)\}$ and $v(1)$, $v(1.1)$ fixed arbitrary valuations of the atoms of the language. As in the previous paragraph, we can choose the 0-truth of $\Box(p \supset p)$ in the world 1; we choose $\mathcal{M}, 1, 0 \not\models \Box(p \supset p)$. Then, because $1R1.1$, we must have $\mathcal{M}, 1, 1 \not\models \Box\Box(p \supset p)$.

We must show soundness and completeness of the n-SST with respect to the semantics of the n-approximations presented in the previous section.

Tableaux rules are *sound* with respect to some semantics if that every inference that the tableaux rules do can be done by the semantics — that is, the tableaux does not infer facts that are not permitted already by the semantics. To show this, we must first explain what it means for a model to satisfy sentences in a branch of a tableau. Two definitions are needed.

Definition 4.1 [[8]] Let $\mathcal{M} = \langle W, R, v\rangle$ a model and R a branch of some n-SST. An *SST-interpretation* is a mapping from the prefixes $\sigma \in R$ to worlds $i(\sigma) \in W$ such that, whenever σ, $\sigma.m \in R$, one has $i(\sigma)Ri(\sigma.m)$.

Definition 4.2 A tableau branch R is *satisfiable* if there is a model $\mathcal{M} = \langle W, R, v\rangle$ and an SST-interpretation i such that for every prefixed sentence $\sigma : T\phi \in R$, one has $\mathcal{M}, i(\sigma), n - |\sigma| \models \phi$, and for every prefixed sentence $\sigma : F\phi \in R$, one has $\mathcal{M}, i(\sigma), n - |\sigma| \not\models \phi$.

In Definition 4.2, $\sigma : T\phi$ is to be considered *satisfiable* in a model $\mathcal{M}$ with an *SST*-interpretation i if ϕ is $n - |\sigma|$-true in the world $i(\sigma)$, and $\sigma : F\phi$ is to be considered *satisfiable* in the same model and interpretation if ϕ is $n - |\sigma|$-false in the world $i(\sigma)$. So, to see whether in real world some sentence is false, we could have to analyse the $n - |\sigma|$-truth of the sentence ϕ in the world $i(\sigma)$, that requires $|\sigma|$ introspection steps to arrive at.

Now soundness follows. Its proof is pretty much the same as that of [8], with very simple modifications.

Theorem 4.3 (Soundness) *Suppose that the n-SST for ψ closes. Then $n \models \psi$.*

Proof. We shall show that, if a model $\mathcal{M}$ with an SST-interpretation i satisfies a branch R of the tableau prior to the application of some n-SST-rule, then one of the branches obtained by addition of the conclusions of the rule is also satisfied by $\mathcal{M}$ with some SST-interpretation j. This is done by con-

sidering separately the application of every rule, supposing that every kind of sentence is satisfied by some $\mathcal{M}$ and i.

(i) Suppose that $\sigma : F\phi \vee \psi$ is satisfiable by $\mathcal{M}$ and i. Then, by definition of satisfaction, $\mathcal{M}, i(\sigma), n - |\sigma| \not\models \phi \vee \psi$. The semantics forces us to accept that $\mathcal{M}, i(\sigma), n - |\sigma| \not\models \phi$ and $\mathcal{M}, i(\sigma), n - |\sigma| \not\models \psi$, that is, $\mathcal{M}$ and i satisfy both $\sigma : F\phi$ and $\sigma : F\psi$. The same happens for all other α-sentences.

(ii) Suppose that $\sigma : T\neg\phi$ is satisfiable by $\mathcal{M}$ and i. Then, by definition of satisfaction, $\mathcal{M}, i(\sigma), n - |\sigma| \models \neg\phi$. The semantics forces us to accept that $\mathcal{M}, i(\sigma), n - |\sigma| \not\models \phi$, that is, $\mathcal{M}$ and i satisfy $\sigma : F\phi$. The same happens for all other *neg*-sentences.

(iii) Suppose that $\sigma : T\phi \vee \psi$ is satisfiable by $\mathcal{M}$ and i. Then, by definition of satisfaction, $\mathcal{M}, i(\sigma), n - |\sigma| \models \phi \vee \psi$. The semantics forces us to accept that either $\mathcal{M}, i(\sigma), n - |\sigma| \models \phi$ or $\mathcal{M}, i(\sigma), n - |\sigma| \models \psi$, that is, $\mathcal{M}$ and i satisfy either $\sigma : T\phi$ or $\sigma : F\psi$. So some of the two branches that appear after the application of this β-rule is satisfied by $\mathcal{M}$ and i. The same happens for all other β-sentences.

(iv) Suppose that $\sigma : T\Box\phi$ is satisfiable by $\mathcal{M}$ and i and that there is some prefix $\sigma.m$ on the branch. For $\sigma.m$ to exist, the (π')-rule must have been used in a sentence with prefix σ, so $|\sigma| < n$, that is, $n - |\sigma| > 0$. By definition of satisfaction, $\mathcal{M}, i(\sigma), n - |\sigma| \models \Box\phi$. The semantics forces us to accept that for every world $i(\sigma')$ accessible by $i(\sigma)$, $\mathcal{M}, i(\sigma'), n - (|\sigma| + 1) \models \phi$. So this is the case when $\sigma' = \sigma.m$. Because $|\sigma.m| = |\sigma| + 1$, we have that $\mathcal{M}, i(\sigma.m), n - |\sigma.m| \models \phi$, that is, $\mathcal{M}$ and i satisfy $\sigma : \phi$. The same happens for all applications of the (K)-rule.

(v) Finally, suppose that $\sigma : F\Box\phi$ is satisfiable by $\mathcal{M}$ and i, and that the (π')-rule has been applied producing a new prefix $\sigma.m$ in a sentence $\sigma.m : F\phi$ on the branch. This means, first of all, that $|\sigma| < n$, so that $n - |\sigma| > 0$. By definition of satisfaction, $\mathcal{M}, i(\sigma), n - |\sigma| \not\models \Box\phi$. Because $n - |\sigma| > 0$, the semantics forces us to accept that, for some world t accessible by $i(\sigma)$, $\mathcal{M}, t, n - (|\sigma| + 1) \not\models \phi$. Extend the function i to another j, from the prefixes on the extended branch to the worlds, as follows:

$$j(s) = \begin{cases} t \text{ if } s = \sigma.m \\ i(s) \text{ otherwise} \end{cases}$$

First we shall prove that j is an SST-interpretation. We must show, therefore, that for all prefixes s and $s.l$ present on the branch, $j(s)Rj(s.l)$. This is obvious for every $s \neq \sigma$, $\sigma.m$ and every $s.l \neq \sigma.m$, because for such s the function j coincides with i that is an SST-interpretation, so $j(s)Rj(s.l)$

because already $i(s)Ri(s.l)$. Now consider $s = \sigma$. For every $\sigma.l \neq \sigma.m$, j coincides with i and we also have that $j(s)Rj(s.l)$. Moreover, by construction (because $j(\sigma.m) = t$ and $i(\sigma)Rt$), $j(\sigma)Rj(\sigma.m)$. Finally, because $\sigma.m$ is new on the branch, there is no prefix $\sigma.m.l$ on the branch that requires us to prove that $j(\sigma.m)Rj(\sigma.m.l)$. With the SST-interpretation j, $\mathcal{M}$ satisfies every sentence in the extended branch other than $\sigma.m : F\phi$, because for such sentences the prefixes involved are all different from $\sigma.m$ and therefore j coincides with i, so $\mathcal{M}$ and i already satisfied them. And, because the world t accessible by $i(\sigma) = j(\sigma)$, $\mathcal{M}, t, n - (|\sigma| + 1) \not\models \phi$ and $t = j(\sigma.m)$, we have that $\mathcal{M}, j(\sigma.m), n - |\sigma.m| \not\models \phi$ (remember that $|\sigma.m| = |\sigma| + 1$). Therefore the extended branch is satisfiable by $\mathcal{M}$ with the SST-interpretation j.

That completes the proof by induction. □

Tableaux rules are *complete* with respect to some semantics if every inference that the semantics does can also be done by the tableaux rules — that is, the tableau infers at least the facts permitted by the semantics. The strategy for proving this is as follows. Suppose that the tableau cannot prove some sentence. Then some reduced open branch appears. We use this branch to construct a counter model for the original sentence; this counter model shows that the original sentence cannot be inferred from the semantics. This means the same as saying that, if the original sentence can be inferred from the semantics, then there cannot be any reduced open branch in the tableau, that is, all branches must eventually close.

Theorem 4.4 (Completeness) *Suppose that $n \models \psi$. Then the n-SST for ψ closes.*

Proof. Let R be an reduced open branch of the tableau. Construct the model $\mathcal{M} = \langle W, R, v\rangle$ as follows:

- $W = \{\sigma : \sigma \in R\}$
- $\sigma R \sigma'$ iff $\sigma' = \sigma.m$
- $v(\sigma)(p) = 1$ if $\sigma : Tp \in R$; $v(\sigma)(p) = 0$ if $\sigma : Fp \in R$

There is no ambivalence in the definition of $v(\sigma)(p)$ because the branch does not close, that is, it does not happen simultaneously that $\sigma : Tp$ and $\sigma : Fp \in R$. Notice also that it is necessary, by Theorem 3.1, to define the $(n - |\sigma|)$-truth of the sentences $\Box\phi$ and $\Diamond\phi$ for σ such that $|\sigma| \geq n$ (we notice that every prefix in a branch is in 1^*, and for a prefix $1.n_1.n_2.\ldots.n_m$, we have $|1, 1.n_1.n_2.\ldots.n_m| = m$). Assume that $\mathcal{M}, \sigma, n - |\sigma| \models \Box\phi$ if $\sigma : T\Box\phi \in R$, $\mathcal{M}, \sigma, n - |\sigma| \not\models \Box\phi$ if $\sigma : F\Box\phi \in R$, and similarly for the sentences $\Diamond\phi$. For the atoms and sentences that do not appear in R, assume that some function

to worlds to valuations is fixed.

Now it is necessary to show that R is satisfiable by $\mathcal{M}$ so defined, with the SST-interpretation that is the identity function ($i(\sigma) = \sigma$ for every $\sigma \in R$). This is done by induction on the complexity of the sentences in the branch (the part on the right of the prefixes).

(i) $\mathcal{M}$ satisfies the atoms by definition.

(ii) If there is a sentence of the form $\sigma : F\phi \supset \psi$, then (because the branch is reduced) we have that $\sigma : T\phi$ and $\sigma : F\psi \in R$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma : T\phi$ and $\sigma : F\psi$. This means that $\mathcal{M}, \sigma, n - |\sigma| \models \phi$ and $\mathcal{M}, \sigma, n - |\sigma| \not\models \psi$. Therefore, by the semantics, $\mathcal{M}, \sigma, n - |\sigma| \not\models \phi \supset \psi$. In the same way we prove that $\mathcal{M}$ satisfies every α-sentence.

(iii) If there is a sentence of the form $\sigma : F\neg\phi$, then (because the branch is reduced) we have that $\sigma : T\phi \in R$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma : T\phi$. This means that $\mathcal{M}, \sigma, n-|\sigma| \models \phi$, so $\mathcal{M}, \sigma, n-|\sigma| \not\models \neg\phi$. In the same way we prove that $\mathcal{M}$ satisfies every neg-sentence.

(iv) If there is a sentence of the form $\sigma : T\phi \supset \psi$, then (because the branch is reduced) we have either $\sigma : F\phi$ or $\sigma : T\psi \in R$. By the induction hypothesis, $\mathcal{M}$ satisfies either $\sigma : F\phi$ or $\sigma : T\psi$. So either $\mathcal{M}, \sigma, n-|\sigma| \not\models \phi$ or $\mathcal{M}, \sigma, n-|\sigma| \models \psi$. Therefore, by the semantics, $\mathcal{M}, \sigma, n-|\sigma| \models \phi \supset \psi$. In the same way we prove that $\mathcal{M}$ satisfies every β-sentence.

(v) If there is a sentence of the form $\sigma : T\Box\phi$, then (because the branch is reduced) we have that $\sigma.m : T\phi$ for every $\sigma.m$ in the branch — that is, for every prefix accessible by σ. We observe that $|\sigma.m| = |\sigma| + 1$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m : T\phi$ for every $\sigma.m$ in the branch. This means that $\mathcal{M}, \sigma, n - (|\sigma| + 1) \models \phi$ for every prefix accessible by σ. Therefore, by the semantics, $\mathcal{M}, \sigma, n - |\sigma| \models \Box\phi$. In the same way we prove that $\mathcal{M}$ satisfies every ν-sentence.

(vi) If there is a sentence of the form $\sigma : F\Box\phi$, there are two cases to consider. First: if $|\sigma| \geq n$, the sentence is satisfied by the branch by definition. In the second case, $|\sigma| < n$, and the (π')-rule can be applied. Because the branch is reduced, we have that $\sigma.m : F\phi$ for some $\sigma.m$ in the branch — that is, for some prefix accessible by σ. We observe that $|\sigma.m| = |\sigma| + 1$. By the induction hypothesis, $\mathcal{M}$ satisfies $\sigma.m : F\phi$ for some $\sigma.m$ in the branch. This means that $\mathcal{M}, \sigma, n-(|\sigma|+1) \models \phi$ for some prefix accessible by σ. Therefore, by the semantics, $\mathcal{M}, \sigma, n - |\sigma| \not\models \Box\phi$. In the same way we prove that $\mathcal{M}$ satisfies every π-sentence.

That completes the proof by induction. □

We can be sure that if ϕ is a theorem of **K**, then there is and n such

that $n \models \phi$ as follows. Consider the nesting of modal operators in a sentence ϕ, $\mathcal{N}(\phi)$. We need at most $\mathcal{N}(\phi)$ introspection steps in order to prove ϕ, because this is the maximum number of times that rule (π'), that creates new prefixes, can be applied to ever increasing prefixes. This is the essence of the approximation process. If we can prove some sentence with some fixed n, then we are done; but if not, we can increase n and try to prove the same sentence. If the sentence ϕ can be proved classically, it will eventually be proved for some n-logic (it can be proved for the $\mathcal{N}(\phi)$-logic).

5 Conclusion

Although our procedure is described only for the modal logic **K**, the essence of the method applies for other basic modal logics. Essentially, what happens is that for other logics the accessibility relation is restricted to have some properties (for instance, reflexivity), and accordingly we have more SST-rules (SST are modular). But the only SST-rule that creates new prefixes is the (π)-rule, a rule required for all basic modal logics, and its change to (π') is the only one that has to be performed in the SST-rules. The arguments 5 in the proof of soundness and 6 in the proof of completeness, that depend only on this rule, apply to every basic modal logic, and all the other arguments for the soundness and completeness of those logics — found in [8] — can be retained without change. One example: in the modal logic **S4**, the accessibility relation between prefixes, that must be reflexive and transitive, is given by $\sigma R \sigma.\sigma'$. All the tableaux rules from Tables 3 and 4 are preserved, but there is also a new tableau rule, called (T), that allows us to add $\sigma : \nu_0$ on the branch of the tableau whenever $\sigma : \nu$ appears on it, and another tableau rule, called (4), that allows us to add $\sigma.m : \nu$ on the branch whenever the prefix $\sigma.m$ and the sentence $\sigma : \nu$ already appear on it.

One possible application of this procedure is the analysis of proofs in modal logics. For instance, "how many" proofs can we have with a given maximum limit of introspection, compared with another? As a possible direction for future research, there is the fundamental question of the complexity of the family of n-logics. The provability of sentences for most modal logics is a PSPACE-complete problem. In classical logic, the provability is a NP-complete problem. In [2], Marcelo Finger presents a family of logics that approximates classical inference, in such a way that each step in the approximation can be decided in polynomial time. We know that every NP-complete problem is PSPACE, but we don't know whether the reciprocal is true of false. Do the n-logics give rise to problems polynomially tractable? If so, as they approximate a PSPACE-complete problem, what light do they bring over the

question whether PSPACE $\subseteq$ NP?

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