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Non-Linear Equation of Motion for Page–Wootters Mechanism with Interaction and Quasi-Ideal Clocks

Leandro R. S. Mendes, Frederico Brito and Diogo O. Soares-Pinto *

Instituto de Física de São Carlos, Universidade de São Paulo, CP 369, São Carlos 13560-970, SP, Brazil; leandrorsm@gmail.com (L.R.S.M.); fbb@ifsc.usp.br (F.B.)
* Correspondence: dosp@usp.br

Abstract

We explore a timeless approach to quantum theory, in the form of the Page–Wootters mechanism, in which a gravitational interaction is introduced between the system and a finite-dimensional clock. The clock model used is the recently proposed quasi-ideal clock, a construction that can approximate the time–energy canonical commutation relation. We derive equations of motion for the case in which the system is in a pure and mixed state, obtaining a Schrödinger-like equation that leads to a non-linear equation exhibiting decoherence due to the non-ideal nature of the clock and gravitational coupling. A distinctive feature of this equation is that it exhibits dependence on the system's initial conditions.

Keywords: Page–Wootters mechanism; quantum clocks; equation of motion

1. Introduction

The goal of combining Einstein's theory of gravitation with quantum theory has been demonstrated to be an arduous task. At present, a theory pertaining to quantum gravity has not been discovered, and much effort is being put into trying to understand scenarios in which we can study the effects of relativity on quantum mechanical systems without the commitment to high energies [1–4]. One way to achieve this is to adopt the approach of general relativity and investigate quantum theory constructions that do not depend on background structures, such as time. In quantum mechanics, time has a special place; it is a parameter with respect to which all other events happen, a condition that is certainly not exhibited by general relativity, as a result of the principle of general covariance [5].

Time-covariant quantum mechanics, in which the dependence of a state on an external reference timeframe is excluded and therefore time is treated like any other quantity, can exhibit a Hamiltonian constraint that takes the form of a Wheeler–DeWitt equation, leading to a timeless condition. The dynamics in this formalism can be recovered by assigning a clock as the state of one of the subsystems, which is used as a reference for time. Then, in this construction, time is what is read from a clock in the same manner as it is viewed in general relativity. This way of restoring a kind of time evolution in a static global state is recognized as the Page–Wootters mechanism [6,7], also known as the timeless approach to quantum mechanics or the relational Schrödinger picture, an approach that has garnered much attention [8–20].

Among the many advances made regarding the Page–Wootters mechanism, criticism of those propagating the formulation was addressed [21], an indefinite causal order in the gravitational system was identified [22], and limits on precision when constructing an observable time were established [2]. A point in common with all the advances mentioned

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above is the use of idealized quantum clocks. Even though such clocks are usually effective when determining the physical constraints of the problem, we wish to explore a more realistic scenario where we rely on an imperfect clock, i.e., a clock with a finite dimension and states which are not necessarily orthogonal. Although several models have been proposed for the construction of quantum clocks, we are going to focus on a clock model that mimics several properties of the ideal clock while possessing a finite dimension. This model is known as the *quasi-ideal clock* [23], which has already been shown to satisfy a covariant condition akin to the Page–Wootters condition [24].

Here, we further expand the timeless formulation of quantum mechanics in a manner which is twofold: we employ the quasi-ideal clock model as the "timekeeper" in the timeless formulation, obtaining a Schrödinger-like equation for the case where the clock and system interact gravitationally. In doing that, we extend a previous result in which a temporally non-local Schrödinger equation was obtained when the system was in a pure state, and there was an interaction between the two [25,26]. Additionally, we consider the system to be in a mixed state, and we derive an equation of motion that describes gravitational decoherence. This equation is unique in that it depends on the system's initial conditions; therefore, it is non-linear in nature. The fact that the clock is not ideal is reflected in a dependence upon its dimension, where the dimension alleviates these non-linear and non-unitary effects.

2. Temporally Non-Local Schrödinger Equation

We start with a brief review of previous work [6,25] describing the effects of interaction in the Page–Wootters model. A physical Hilbert space $\mathcal{H}_{phys}$ composed of all global states $|\Psi\rangle$ is considered, which satisfy a Wheeler–DeWitt-like constraint [27,28]

$$H\,|\Psi\rangle = 0, \tag{1}$$

where H is the total Hamiltonian. The global state is considered to comprise a clock with Hilbert space $\mathcal{H}_C$ and a system (the system of interest) with Hilbert space $\mathcal{H}_S$ and Hamiltonians H_C and H_S. Then, the global state is said to contain the full history of the system of interest, with the clock being the timekeeper of the system. The clock states $|\tau\rangle \in \mathcal{H}_C$ are such that a state $|\tau\rangle$ is associated with time τ. The initial clock state can be defined as $|0\rangle \in \mathcal{H}_C$ and state evolution between its states is given by

$$|\tau\rangle = e^{-iH_C\tau}\,|0\rangle\,. \tag{2}$$

Any state of the system can be obtained by conditioning the global state $|\Psi\rangle$ to a clock state associated with a time τ as

$$|\psi_S(\tau)\rangle = (\langle\tau| \otimes \mathbb{1}_s)\,|\Psi\rangle\,, \tag{3}$$

where $|\psi_S(\tau)\rangle \in \mathcal{H}_S$ is what we will call a system state. This action can be interpreted as selecting a slice of the global state, representing the system state at the desired time. Then, an equation of motion for the system state in terms of the clock can be obtained using Equations (1) and (3):

$$i\frac{d}{d\tau}\,|\psi_S(\tau)\rangle = H_S\,|\psi_S(\tau)\rangle\,. \tag{4}$$

The constraint in Equation (1) can be modified to include a general interaction Hamiltonian H_I; then, the constraint can be written as

$$(H_S + H_C + H_I)\,|\Psi\rangle = 0. \tag{5}$$

This new constraint can also be shown to lead to a Schrödinger-like evolution; more specifically, the system state will obey a temporally non-local Schrödinger equation:

$$i\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle = H_S\left|\psi_S(\tau)\right\rangle + \int d\tau' K(\tau,\tau')\left|\psi_S(\tau')\right\rangle, \tag{6}$$

where the new term is a self-adjoint integral operator and $K(\tau,\tau') = \langle\tau| H_I |\tau'\rangle$ is seen as its kernel. Given that this equation is non-local in time, it is implied that to verify its solution, knowledge of the system state is required at all times [25].

3. Ideal and Quasi-Ideal Clocks

The result in Equation (6) considers ideal clocks, usually defined to be a quantum system associated with a time observable $\mathcal{T}$, which acts on an infinite dimensional Hilbert space possessing a distinguishable basis of time states. The use of an ideal clock can be traced back to the requirement that the time observable obeys the canonical commutation relation with the clock Hamiltonian H_C

$$[H_C, \mathcal{T}] = -i, \tag{7}$$

which is equivalent to requiring that the expectation value of $\mathcal{T}$ varies linearly with time. This condition is supposed to be satisfied only by the ideal momentum clock [29], with the clock Hamiltonian being the momentum operator $H_C = P_C$ and the position operator as the time observable $X = \mathcal{T}$.

Respecting this condition is not straightforward, and well-known clock models fail to adhere to it. Notably, even the prominent Salecker–Wigner–Peres (SWP) clock cannot approximately adhere to this condition [30,31]. The discussion surrounding the construction of a time observable is extensive, and we will not delve further into it. What is of interest in the present work is that there are constructions that can approximate the canonical commutation relation in Equation (7), meaning that there are clock models that can mimic the evolution of the idealized clock model while possessing finite dimensions. This is the case for the quasi-ideal clock (QIC).

Let $|\psi_{QI}(k_0)\rangle$ be a QIC state (which will be introduced shortly) is conditioned on the parameter $k_0 \in \mathbb{R}$, which is taken to be the initial time position of the clock. Then, we can write the commutation relation as

$$[H_C, \mathcal{T}]\left|\psi_{QI}(k_0)\right\rangle = -i\left|\psi_{QI}(k_0)\right\rangle + \left|\varepsilon_{comm}\right\rangle, \tag{8}$$

where $|\varepsilon_{comm}\rangle$ is a non-unity state vector that quantifies the error in the commutation relation of the QIC when compared to the ideal case, being exponentially small in dimension [23].

The QIC is based on the SWP clock, which can be seen as a quantum rotor [32] possessing a Hamiltonian with evenly spaced energy levels:

$$H_C = \sum_{j=0}^{d-1} \omega j \left|E_j\right\rangle\left\langle E_j\right|, \tag{9}$$

where d is the dimension of its Hilbert space, i.e., the number of energy eigenstates of the clock, which may be determined through some spectroscopic technique, with the period $T = 2\pi/\omega$. The SWP clock states form an orthonormal set of states $\{|\theta_k\rangle\}$ for

$k = 0, 1, \ldots, d-1$, which is dubbed the time basis. With this, it is possible to construct a time operator:

$$\mathcal{T} = \sum_k t_k |\theta_k\rangle \langle\theta_k| , \tag{10}$$

where $t_k = (T/d)k$. These states are connected to energy eigenstates through a discrete Fourier transform:

$$|\theta_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{-i2\pi jk/d} |E_j\rangle , \tag{11}$$

and evolve according to $e^{-iH_C T/d} |\theta_k\rangle = |\theta_{k+1}\rangle$, thus establishing regular, integer time intervals t_k, such that the periodic condition $e^{-iH_C T} |\theta_k\rangle = |\theta_{k \bmod (d)}\rangle$ is satisfied.

The QIC states that approximately obey the canonical commutation relation above are defined as a coherent superposition of time states [23]:

$$|\psi_{QI}(k_0)\rangle = \sum_{k \in S_d(k_0)} A e^{-\pi(k-k_0)^2/\sigma^2} e^{i2\pi j_0 (k-k_0)/d} |\theta_k\rangle , \tag{12}$$

where $A \in \mathbb{R}^+$ is a normalization constant, $\sigma \in (0, d)$ is the width of the clock in the time basis, ωj_0 is the average energy of the clock for which $j_0 \in (0, d-1)$, and $S_d(k_0) = \{k : k \in \mathbb{Z} \text{ and } -d/2 \leq k_0 - k < d/2\}$ is a set of d consecutive integers centered about k_0. When applicable, we will write $\psi(k_0; k) = A e^{-\pi(k-k_0)^2/\sigma^2} e^{i2\pi j_0(k-k_0)/d}$ so that

$$|\psi_{QI}(k_0)\rangle = \sum_{k \in S_d(k_0)} \psi(k_0; k) |\theta_k\rangle . \tag{13}$$

The evolution of the clock states is given by a relation equivalent to Equation (2):

$$e^{-iH_C t} |\psi_{QI}(k_0)\rangle = |\psi_{QI}(k_0 + td/T)\rangle + |\varepsilon\rangle , \tag{14}$$

where $|\varepsilon\rangle$ is an error term composed of two distinct errors, an error in changing the mean position of the clock, i.e., passing from k_0 to $k_0 + 1$, and an error in changing the consecutive integers, i.e., $S(k_0)$ to $S(k_0 + 1)$, whose norm becomes exponentially smaller as the number of dimensions decreases. Unlike the SWP clock, the quasi-ideal states are continuous, in the sense that their evolution holds, according to the idealized case, for arbitrarily small time intervals, where $t \in \mathbb{R}$.

4. Relative State for the Quasi-Ideal Clock

Without loss of generality, we choose the initial state of the clock to be centered around $k_0 = 0$ and rewrite the evolution, so that Equation (14) becomes

$$e^{-iH_C t} |\psi_{QI}(0)\rangle = |\psi_{QI}(td/T)\rangle + |\varepsilon\rangle . \tag{15}$$

As in Equation (3), we wish to define the relative state of the system as a slice of the global state with respect to the QIC at time t:

$$|\psi_S(\tau d/T)\rangle_e = \langle\psi_{QI}(\tau d/T)|\Psi\rangle . \tag{16}$$

Although we are defining the system state with the parameter $\tau d/T$, we will alter $td/T \to \tau$, so that all results are written in terms of τ.

The subscript in Equation (16) indicates that the state is an effective state for the system, as can be seen in the equation below (see Appendix A):

$$\begin{aligned} |\psi_S(\tau)\rangle_e &= \frac{1}{T}\int_0^T d\tau' \langle \psi_{QI}(\tau)|\psi_{QI}(\tau')\rangle |\psi_S(\tau')\rangle \\ &= \frac{1}{T}\int_0^T d\tau' F_{QI}(\tau-\tau') |\psi_S(\tau')\rangle . \end{aligned} \tag{17}$$

The state we obtain is not exactly $|\psi_S(\tau)\rangle$, the one we would like to prepare, but rather an approximation that depends on how well the QIC can distinguish the clock states at different times. This effect is due to the non-ideal condition of the clock: time will be determined with a certain accuracy that depends on the function $F_{QI}(\tau-\tau')$. In order for this function to peak sharply around τ, and hence increase the accuracy of the clock, we need to time-squeeze the QIC, which is accomplished for a regime where $\sigma < \sqrt{d}$. Thereby, we reduce the uncertainty in time readings in exchange for an increased uncertainty in energy, making the QIC closer to a time state. This also implies that if we aim to use this clock to implement a controlled unitary, it will be subject to larger errors, since for squeezed states the QIC becomes more vulnerable to a back-reaction.

Another aspect of Equation (16) worth noting is that if we consider the dimension of the clock to be large enough that we can ignore the error in the evolution given by Equation (14), we obtain $\langle \psi_{QI}(\tau)|\Psi\rangle \approx \langle \psi_{QI}(0)| e^{i\tau H_C} |\Psi\rangle$. Using the definition of a relative state, we can obtain a slightly different form for the effective system state, which will be useful; this is carried out as follows:

$$\begin{aligned} |\psi_S(\tau)\rangle_e &= \sum_{k\in S_d(\tau)} \psi^*(\tau;k) \langle \theta_k|\Psi\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{k\in S_d(\tau)} \psi^*(\tau;k) |\psi_S(k)\rangle , \end{aligned} \tag{18}$$

where we used $|\psi_S(k)\rangle = \sqrt{d}\,\langle \theta_k|\Psi\rangle$. We can see that, written this way, the effective system state is a superposition of the system state conditioned to the time basis $|\psi_S(k)\rangle$. The relative state $|\psi_S(k)\rangle$ is not an effective version, as in Equation (16), due to the fact that the time basis forms a complete distinguishable basis. Hereafter, we refer to the effective system state only as the system state, for simplicity.

5. Equation of Motion

To obtain the equation of motion for the mixed system state, we first derive an equation of motion for the pure system state. We introduce a lemma (see Appendix B) that will be very useful in describing the time evolution of the QIC with respect to τ.

Lemma 1. *Given a QIC state evolved to time τ, its derivative will be given by*

$$\frac{d}{d\tau}|\psi_{QI}(\tau)\rangle = -i\frac{T}{d}H_C|\psi_{QI}(\tau)\rangle - |\varepsilon'\rangle , \tag{19}$$

with the error decreasing exponentially with the dimension

$$\| \,|\varepsilon'\rangle\, \| \leq \mathcal{O}(poly(d)e^{\frac{-\pi d}{4}}). \tag{20}$$

The process of obtaining a Schrödinger-like equation for the system state conditioned on the QIC states follows the same steps given by the Page–Wootters method.

$$\begin{aligned}
i\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e &= i\frac{d}{d\tau}\left\langle\psi_{QI}(\tau)|\Psi\right\rangle \\
&= \left(-\frac{T}{d}\left\langle\psi_{QI}(\tau)\right| H_C + i\left\langle\varepsilon'\right|\right)\left|\Psi\right\rangle \\
&= -\frac{T}{d}\left\langle\psi_{QI}(\tau)\right| H_C\left|\Psi\right\rangle + i\left\langle\varepsilon'|\Psi\right\rangle \\
&= \frac{T}{d}\left\langle\psi_{QI}(\tau)\right|(H_S + H_I - H)\left|\Psi\right\rangle + \left|\varepsilon_s\right\rangle \\
&= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \frac{T}{d}\left\langle\psi_{QI}(\tau)\right| H_I\left|\Psi\right\rangle + \left|\varepsilon_s\right\rangle.
\end{aligned}$$

For the derivation, we also defined $|\varepsilon_s\rangle := i\langle\varepsilon'|\Psi\rangle$, which can be interpreted as the error given by the quasi-ideal clock in a certain slice of time associated with the system state. From this result, we need to address the term regarding the interaction H_I. Many choices could be made; here, we choose the one the seems the most interesting—we consider what happens when the QIC interacts gravitationally with the system of interest. For the case of weak gravitational fields and slowly moving particles, we can consider the Newtonian formula for the gravitational potential $\Phi(x) = -Gm_S m_C/x$; using the mass–energy equivalence principle, we obtain the interaction Hamiltonian:

$$H_I = -\mathbb{G}H_S \otimes H_C, \tag{21}$$

where we define $\mathbb{G} := G/c^4x$. G as the gravitational constant, c as the speed of light, and x as the coordinate distance between the clock and the system. The interaction term can be computed as follows:

$$\begin{aligned}
\left\langle\psi_{QI}(\tau)\right| H_I\left|\Psi\right\rangle &= -\mathbb{G}H_S\left\langle\psi_{QI}(\tau)\right| H_C\left|\Psi\right\rangle \\
&= \mathbb{G}H_S\left[i\frac{d}{T}\frac{d}{d\tau}\left\langle\psi_{QI}(\tau)\right| + \frac{d}{T}i\left\langle\varepsilon'\right|\right]\left|\Psi\right\rangle \\
&= i\mathbb{G}H_S\left[\frac{d}{T}\sum_k\frac{d}{d\tau}\psi^*(\tau;k)\left\langle\theta_k|\Psi\right\rangle + \frac{d}{T}i\left\langle\varepsilon'|\Psi\right\rangle\right] \\
&= i\mathbb{G}H_S\left[\frac{d}{T\sqrt{d}}\sum_k\frac{d}{d\tau}\psi^*(\tau;k)\left|\psi_S(k)\right\rangle + \frac{d}{T}\left|\varepsilon_s\right\rangle\right].
\end{aligned}$$

Hence,

$$\left\langle\psi_{QI}(\tau)\right| H_I\left|\Psi\right\rangle = i\mathbb{G}H_S\frac{d}{T}\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle. \tag{22}$$

With this result, we return to Equation (21), obtaining

$$\begin{aligned}
i\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle &= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + i\mathbb{G}H_S\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle + \left|\varepsilon_s\right\rangle \\
&= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\left[H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + i\mathbb{G}H_S\frac{d}{d\tau}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle + \left|\varepsilon_s\right\rangle\right] \\
&+ \mathbb{G}H_S\frac{d}{T}\left|\varepsilon_s\right\rangle + \left|\varepsilon_s\right\rangle \\
&= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S^2\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \left(\mathbb{G}H_S + \mathbb{G}H_S\frac{d}{T} + 1\right)\left|\varepsilon_s\right\rangle + \mathcal{O}(\mathbb{G}^2) \\
&= H_S\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \mathbb{G}H_S^2\frac{T}{d}\left|\psi_S(\tau)\right\rangle_e + \left|\varepsilon_{sg}\right\rangle + \mathcal{O}(\mathbb{G}^2),
\end{aligned} \tag{23}$$

The new error $|\varepsilon_{sg}\rangle$ will be the same error as the one in Lemma 1, with the addition of terms related to the gravitational constant and the system Hamiltonian; it will be exponentially small in terms of dimension if the Hamiltonian of the system H_S is bounded and the system state in relation to the time basis is normalized, i.e., $\langle\psi_S(k)|\psi_S(k)\rangle = 1$. With these considerations in mind, it is very easy to see that considering a large enough clock dimension allows us to neglect it and approximate the equation by disregarding terms of $\mathbb{G}$ that are second-order and above, through which we recover the expected approximate Schrödinger-like equation:

$$i\frac{d}{d\tau}|\psi_S(\tau)\rangle_e = H_S\frac{T}{d}|\psi_S(\tau)\rangle_e + \mathbb{G}H_S^2\frac{T}{d}|\psi_S(\tau)\rangle_e. \tag{24}$$

We now consider the system state to be a mixture of the pure system states in Equation (17):

$$\rho_S(\tau) = \sum_\iota p_\iota |\psi_S^\iota(\tau)\rangle_e \langle\psi_S^\iota(\tau)|, \tag{25}$$

where $\rho_S(\tau) = U(\tau)\rho_S(0)U^\dagger(\tau) = \sum_\iota p_\iota |\psi_S^\iota(\tau)\rangle_e \langle\psi_S^\iota(\tau)|$. Taking the derivative of this state and using Equation (24), we find

$$\frac{d\rho_S(\tau)}{d\tau} = -i\frac{T}{d}[H_S, \rho_S(\tau)] - i\mathbb{G}\frac{T}{d}[H_S^2, \rho_S(\tau)]. \tag{26}$$

We readily see that the effect of the gravitational interaction appears in the form of a second term, with the system Hamiltonian squared, analogously to the Schrödinger-like equation. This suggests that we can consider the system state as evolving according to the Hamiltonian $H_d = \frac{T}{d}H_S(\mathbb{1} + \mathbb{G}H_S)$. In [22], when considering how the equations of motion will appear according to different subsystems, it was proposed that terms of the form $1 + 2\Phi(x)/c^2$ express time dilation. This interpretation comes from the weak field approximation to the metric $ds^2 = -(1 + 2\Phi(x)/c^2)c^2dt^2 + dx^2$, which will have a dominant term $g_{00} = -(1 + 2\Phi(x)/c^2)$, for the Newtonian potential $\Phi(x)$. If we observe the Hamiltonian in the equation above, it seems to be analogous to a time dilation effect or a "blue-shifted" Hamiltonian, guiding the evolution of the system state. We note that we only considered the interaction mediated by gravity without explicitly introducing gravitational time dilation. We will see shortly that, when accounting for the clock error, the interaction between the clock and the system state induces a non-unitary behavior in the evolution of the system state, provided that Equation (24) holds, which is in contrast to the ideal clock case (see Appendix C).

To obtain Equation (26), the evolution is considered with negligible clock error. At this point, we continue by considering the effect of the small contribution given by the error of the QIC. Let us use the following transformations:

$$\tilde{\rho}(\tau) = e^{iH_d\tau}\rho_S(\tau)e^{-iH_d\tau} \tag{27}$$

and

$$\tilde{V}(\tau) = e^{iH_d\tau}V(\tau)e^{-iH_d\tau}, \tag{28}$$

where we define the potential $V(\tau) = \frac{\mathbb{G}_H\varepsilon'}{\sqrt{d}}V_{k,k'}(\tau)$ with $\mathbb{G}_H = \left(1 + \mathbb{G}H_S + \mathbb{G}H_S\frac{d}{T}\right)$ and ε' is the sum of errors in Lemma 1 (more details can be found in Appendix D). With these definitions, the initial state of the system will coincide with the transformed initial state $\tilde{\rho}(0) = \rho_S(0)$, and we obtain

$$\frac{d\tilde{\rho}(\tau)}{d\tau} = -i[\tilde{V}(\tau), \tilde{\rho}(\tau)]. \tag{29}$$

Integrating the above equation and iterating it, we obtain

$$\frac{d\tilde{\rho}(\tau)}{d\tau} = -i[\tilde{V}(\tau), \tilde{\rho}(0)] - \int_0^\tau [\tilde{V}(\tau), [\tilde{V}(s), \tilde{\rho}(s)]]ds, \tag{30}$$

an equation that is only dependent on the density operator of the system of interest. We assume a weak coupling between the clock and the system due to the nature of the assumed potential, i.e., gravitational, and expand the transformed density operator around τ, i.e.,

$$\tilde{\rho}(s) = \tilde{\rho}(\tau) + (s-\tau)\frac{d\tilde{\rho}(\tau)}{d\tau} + \mathcal{O}((s-\tau)^2). \tag{31}$$

It is easy to see that the derivatives will contribute with higher-order terms in $\mathbb{G}$; retaining only those up to second-order terms, we reach

$$\frac{d\tilde{\rho}(\tau)}{d\tau} = -i[\tilde{V}(\tau), \rho_S(0)] - \int_0^\tau [\tilde{V}(\tau), [\tilde{V}(s), \tilde{\rho}(\tau)]]ds. \tag{32}$$

This leads us to

$$\begin{aligned} \frac{d\rho_S(\tau)}{d\tau} &= -i[H_d, \rho_S(\tau)] - i[V(\tau), U^\dagger \rho_S(0) U] \\ &\quad - \int_0^\tau [V(\tau), [V(s), \rho(\tau)]_{\tau-s}]ds, \end{aligned} \tag{33}$$

where

$$[V(s), \rho(\tau)]_{t-s} = e^{-iH_d(\tau-s)}[V(s), \rho(\tau)]e^{iH_d(\tau-s)}.$$

Finally, we use the Baker–Campbell–Hausdorff formula [33],

$$U^\dagger \rho_S(0) U = \rho_S(0) + i\tau[H_d, \rho_S(0)] - \frac{\tau^2}{2}[H_d, [H_d, \rho_S(0)]] + \mathcal{O}(\tau^3),$$

to expand the second term of the equation above. For a sufficiently small time step τ, we may truncate this series to the first order in τ, finding

$$\frac{d\rho_S(\tau)}{d\tau} = -i[H_d, \rho_S(\tau)] - i[V(\tau), \rho_S(0)] + \tau[V(\tau), [H_d, \rho_S(0)]] - \int_0^\tau [V(\tau), [V(s), \rho(\tau)]_{\tau-s}]ds. \tag{34}$$

There are several interesting aspects in the above equation. First, since we do not assume that the initial state of the system is a product state with the environment (which is not included in the present work), the resulting equations contain terms that depend on the initial conditions of the system. This feature can be attributed to the timeless prescription: we must define an initial clock state as the "beginning", but this state is itself subject to the clock time, here determined by the QIC.

As mentioned, the system state inside this "Universe" is effectively determined by how well we can distinguish between different times. This is reminiscent of the work in [34], which studied the impact of real clocks in quantum mechanics. The description we have obtained already effectively implies that the system state undergoes decoherence. We believe that this effect is captured by the second and third terms of Equation (34), which resemble the Lindblad-type terms also found in [34].

Moreover, we argue that this dependence on the initial conditions is natural: the time is set at the initial moment, but in principle, it could have been defined with respect to another parameter, τ'. Unlike in [34], however, here we encounter compound effects arising from the gravitational interaction between the clock and the system. This interaction produces the time-dilated Hamiltonian H_d, in which the system not only loses coherence but does so while evolving with this time-dilated Hamiltonian.

In addition, the fourth term reflects the impact of gravitational errors, which further implies decoherence of the system state. This form of decoherence, however, depends on the full history of the system, which requires knowledge of its state at all previous times. Finally, we note that, in the limit of an ideal clock, one recovers the usual evolution of the system, although it is still subject to time dilation.

If we expand this potential coefficient in order to show the influence of the error and the gravitational constant, we obtain the following result:

$$\begin{aligned} V(\tau) &\propto \left(1 + \mathbb{G}H_S + \mathbb{G}H_S\frac{d}{T}\right)^2 \\ &= \left[1 + 2\mathbb{G}H_S\left(1+\frac{d}{T}\right) + \left[\mathbb{G}H_S\left(1+\frac{d}{T}\right)\right]^2\right]. \end{aligned} \tag{35}$$

For this potential, the energy given by the Hamiltonian is the energy of the state of the system $|\psi_S(k)\rangle$, related to the time basis. Then, the decoherence term will have contributions related to the order of the energy and the squared energy, in contrast to the equation of motion obtained in Ref. [35], which is only dependent on the square of the energy. We also note that in the case that the energy $|\psi_S(k)\rangle$ is negligible, we still have decoherence due to the non-ideal nature of the clock. Moreover, in Equation (34), there are two new terms, both dependent on the initial state of the system $\rho_S(0)$. Given the dependence on initial conditions, this shows that this equation of motion is non-linear in nature, in contrast to other results, in the context of gravitational interaction [35–37].

As mentioned, the non-unitary and non-linear effects arise due to the non-ideal nature of the chosen clock model, and the "idealness" can be related to its dimension, showing the role that the size of the clock plays. The smaller its size, the more pronounced these new terms become; this case would be associated with an ineffective clock dictating the time inside the universe. Looking at Equation (34), we see that this implies that the system state will decohere very rapidly, and we would not be able to fix an initial system state. Meanwhile, the larger it is, the better the clock is considered to be and the closer to an ideal clock it is; therefore, the non-unitary/non-linear behavior is less noticeable and the loss of coherence is less attenuated.

6. Discussion

Here, we further developed the Page–Wootters formulation by examining the impact of a non-ideal clock as the timekeeper when there is a gravitational interaction between the clock and the system state. To this end, we utilized the quasi-ideal clock, a finite clock that can approximate ideal clock conditions. We then defined an effective state for the system as conditioned by this clock model; the effective state is characterized by how well the QIC basis can be distinguished between different times, which can be improved by time-squeezing the QIC states, or in other words, reducing their uncertainty in time and increasing their uncertainty in energy.

We then considered the interaction mediated by gravity and obtained a Schrödinger-like equation with the error parameters given by the QIC. The error depends on the dimension of the clock and decreases exponentially in relation to it. Then, under very large dimensional limitations, we can neglect the errors of the QIC and show that the equation we obtained leads to a linear Schrödinger-like equation under the influence of a Hamiltonian that seems to be time-dilated.

We then consider a mixed system state and derive an equation of motion accounting for clock errors. This leads to a non-linear equation describing decoherence, which is mainly attributed to the non-ideal nature of the clock. The fact that we defined the system state on an imperfect clock time gives rise to two terms dependent on the initial conditions; hence, depending on the how large this error is, the system state could potentially lose coherence too quickly and the model would be ineffective. Since the error parameter depends on the dimension, we can quantify that the larger the clock and the closer it is to the ideal case, the longer the system state will keep its coherence, and the smaller the clock, the faster this will happen. It is interesting to note that, by using a non-ideal clock and placing it in conditions in which it interacts gravitationally with the system, we have an equation of motion that compounds the effects exhibiting decoherence under the guided evolution using a time-dilated Hamiltonian.

We note here that there are other decoherence models in the literature associated with gravity; a noteworthy example is the one given by Pikoviski et al. [37] In their work, they consider a composite system with a well-defined center of mass and some internal degrees of freedom that may act as clocks for the system. Decoherence is obtained by noting that when the superpositions go through different spacetime trajectories, they will experience different times because of time dilation. This analysis does not consider a timeless condition as is carried out here, but in principle we could define the system state as composed of a center-of-mass state and other internal degrees of freedom and obtain an equation of motion that also takes this effect into account.

Another interesting follow up would be to include thermodynamical considerations. Previous works have established that, for autonomous clocks, i.e., independent systems that evolve according to a time-independent Hamiltonian, there is a certain limit for the entropy generated per tick. Interestingly, entropy was found to be associated with the accuracy of the clock, working as a resource that determines the "quality" of the clock. It would be compelling to understand how this affects the evolution of the system and to see whether the production of entropy impacts the decoherence observed here.

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Appendix A. Derivation of the Global State

We can demonstrate that the natural choice for the global state used in Equation (17) is connected to the global state using the SWP clock when we have the constraint

$$H\,|\Psi\rangle = (H_S + H_C)\,|\Psi\rangle = 0, \tag{A1}$$

which is essentially the same construction as that provided in Ref. [38]. First, we repeat this construction and give it a more appropriate notation.

We start with a generic state for the universe with $|E_m\rangle_S \in \mathcal{H}_S$ and $|E_n\rangle_C \in \mathcal{H}_C$:

$$|\Psi\rangle = \sum_{n=0}^{d-1}\sum_{m=0}^{d_S-1} p_{n,m}\,|E_n\rangle_C\,|E_m\rangle_S\,, \tag{A2}$$

which can be written as

$$
\begin{aligned}
|\Psi\rangle &= \sum_{m=0}^{d_S-1} \tilde{p}_m |E=-E_m\rangle_C |E_m\rangle_S \\
&= \sum_{m=0}^{d_S-1} \sum_{k=0}^{d-1} \tilde{p}_m |\theta_k\rangle \langle\theta_k|E=-E_m\rangle_C |E_m\rangle_S \\
&= \frac{1}{\sqrt{d}} \sum_{m=0}^{d_S-1} \sum_{k=0}^{d-1} \tilde{p}_m e^{-i2\pi mk/d} |\theta_k\rangle |E_m\rangle_S \\
&= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |\theta_k\rangle |\psi_S(k)\rangle \,, \qquad \text{(A3)}
\end{aligned}
$$

where we use the constraint of Equation (A1) in the first line and in the second line we use the resolution of the identity for time states. In order to show how this is linked to our choice, we use the covariant POVM generated by the quasi-ideal states [24], which in the limit of large dimensions is approximately $\{P_{QI}(\tau) := U(\tau)|\psi_{QI}(0)\rangle\langle\psi_{QI}(0)|U^\dagger(\tau)\}_{\tau\in[0,T]}$; hence,

$$
\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \frac{1}{T} \int_0^T d\tau \, |\psi_{QI}(\tau)\rangle \langle\psi_{QI}(\tau)|\theta_k\rangle |\psi_S(k)\rangle \\
&= \frac{1}{\sqrt{d}T} \sum_{k=0}^{d-1} \int_0^T d\tau \, |\psi_{QI}(\tau)\rangle \, \psi^*(\tau;k) |\psi_S(k)\rangle \\
&= \frac{1}{T} \int_0^T d\tau \, |\psi_{QI}(\tau)\rangle \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \psi^*(\tau;k) |\psi_S(k)\rangle \,. \qquad \text{(A4)}
\end{aligned}
$$

Using the definition given in Equation (18), we obtain

$$
|\Psi\rangle = \frac{1}{T} \int_0^T d\tau \, |\psi_{QI}(\tau)\rangle |\psi_S(\tau)\rangle \,. \qquad \text{(A5)}
$$

Appendix B. Time Evolution of the QIC

Here, we provide proof for Equation (19); the idea is to use Lemma 8.0.1 of Ref. [23] to obtain the infinitesimal evolution of the QIC, and then employ the definition of a derivative:

$$
\begin{aligned}
\frac{d}{d\tau} |\psi_{QI}(\tau)\rangle &= \sum_k \frac{d}{d\tau} \psi(\tau;k) |\theta_k\rangle \\
&= \sum_k \left[\lim_{\delta\to 0} \frac{\psi(\tau+\delta;k) - \psi(\tau;k)}{\delta} \right] |\theta_k\rangle \\
&= \lim_{\delta\to 0} \frac{\sum_k \psi(\tau+\delta;k) |\theta_k\rangle - \sum_k \psi(\tau;k) |\theta_k\rangle}{\delta} \\
&= \lim_{\delta\to 0} \frac{|\psi_{QI}(\tau)\rangle - i\delta(T/d) H_C |\psi_{QI}(\tau)\rangle - |\varepsilon\rangle_i - |\psi_{QI}(\tau)\rangle}{\delta} \\
&= -\frac{iTH_C}{d} |\psi_{QI}(\tau)\rangle - \lim_{\delta\to 0} \frac{|\varepsilon\rangle_i}{\delta}. \qquad \text{(A6)}
\end{aligned}
$$

The error will have the form $|\varepsilon\rangle_i = \sum_{k\in S_d(k_0)} \left[\delta(\varepsilon_1+\varepsilon_2+\varepsilon_3)+\delta^2 C\right] |\theta_k\rangle$, where the form of each error can be found in Ref. [23]; therefore, its limit will be

$$
\begin{aligned}
\lim_{\delta\to 0}\frac{|\varepsilon\rangle_i}{\delta} &= \lim_{\delta\to 0}\sum_{k\in S_d(k_0)} \frac{\left[\delta(\varepsilon_1+\varepsilon_2+\varepsilon_3)+\delta^2 C\right]}{\delta}|\theta_k\rangle \\
&= \lim_{\delta\to 0}\sum_{k\in S_d(k_0)} \left[(\varepsilon_1+\varepsilon_2+\varepsilon_3)+\delta C\right]|\theta_k\rangle \\
&= \sum_{k\in S_d(k_0)} (\varepsilon_1+\varepsilon_2+\varepsilon_3)\,|\theta_k\rangle \\
&= \sum_{k\in S_d(k_0)} \varepsilon'\,|\theta_k\rangle = |\varepsilon'\rangle\,.
\end{aligned} \tag{A7}
$$

Hence,

$$
\||\varepsilon'\rangle\| \leq \begin{cases} 2\pi A\left(2\sqrt{d}\left(\frac{1}{2}+\frac{1}{2\pi d}+\frac{1}{1-e^{\pi}}\right)e^{\frac{-\pi d}{4}}+\frac{1}{2}+\frac{1}{2\pi d}+\frac{1}{1-e^{\pi}}\right)e^{\frac{-\pi d}{4}} & \text{if } \sigma=\sqrt{d} \\ 2\pi A\left(2\sigma\left(\frac{\alpha_0}{2}+\frac{1}{2\pi\sigma^2}+\frac{1}{1-e^{\pi\sigma^2\alpha_0}}\right)e^{\frac{-\pi\sigma^2\alpha_0}{4}}+\left(\frac{1}{2\pi d}+\frac{d}{2\sigma^2}+\frac{1}{1-e^{\frac{\pi d}{\sigma^2}}}+\frac{1}{1-e^{\frac{\pi d^2}{\sigma^2}}}\right)e^{\frac{-\pi d^2}{4\sigma^2}}\right) & \text{otherwise} \end{cases}, \tag{A8}
$$

which also decreases exponentially with the dimension, where $\alpha_0 \in (0,1]$ is a parameter used to quantify how close j_0 is from the edge of the energy spectrum [23].

Appendix C. Equation of Motion for the Ideal Clock

Following the usual steps to generalize Equation (6) and taking the state of the system to be $|\psi_S(\tau)\rangle$, we obtain

$$
\rho_S(0) = \sum_{\iota} p_\iota\,|\psi_S^\iota(0)\rangle\,\langle\psi_S^\iota(0)|\,, \tag{A9}
$$

with $\rho_S(\tau) = U(\tau)\rho_S(0)U^\dagger(\tau) = \sum_\iota p_\iota\,|\psi_S^\iota(\tau)\rangle\,\langle\psi_S^\iota(\tau)|$; then

$$
\begin{aligned}
\frac{d\rho_S(\tau)}{d\tau} &= \sum_\iota p_\iota\left(\frac{d}{d\tau}\{|\psi_S^\iota(\tau)\rangle\}\,\langle\psi_S^\iota(\tau)| + |\psi_S^\iota(\tau)\rangle\,\frac{d}{d\tau}\{\langle\psi_S^\iota(\tau)|\}\right) \\
&= -i\sum_\iota p_\iota(\{H_S\,|\psi_S^\iota(\tau)\rangle + H_k\,|\psi_S^\iota(\tau)\rangle\}\,\langle\psi_S^\iota(\tau)| \\
&\quad - |\psi_S^\iota(\tau)\rangle\,\{\langle\psi_S^\iota(\tau)|\,H_S + \langle\psi_S^\iota(\tau)|\,H_k\}).
\end{aligned} \tag{A10}
$$

To proceed, we first need to obtain the action of the integral operator H_k. Here, we consider a gravitational interaction between the clock and the system; therefore, the Hamiltonian of the universe will be

$$
H = H_S + H_C - \frac{G}{c^4 x}H_S\otimes H_C, \tag{A11}
$$

where G is the gravitational constant, x is the distance between the clock and the system, and c is the speed of light. We are using the idealized momentum clock $H_C = P_C$; the kernel $K(\tau,\tau')$ can be computed as follows:

$$
\begin{aligned}
K(\tau,\tau') &= -\frac{G}{c^4 x}H_S\,\langle\tau|\,P_C\,|\tau'\rangle \\
&= -\frac{G}{c^4 x}H_S\left[\int dpp\,\langle\tau|p\rangle\,\langle p|\tau'\rangle\right] \\
&= -\frac{G}{c^4 x}H_S\left[\frac{1}{2\pi}\int dppe^{-ip(\tau'-\tau)}\right] \\
&= -\frac{G}{c^4 x}H_S i\dot{\delta}(\tau'-\tau),
\end{aligned} \tag{A12}
$$

where the dot in the delta was used to indicate a time derivative. The adjoint kernel $K^{\dagger}(\tau, \tau')$ is easily obtained. With this,

$$\begin{aligned} H_k |\psi_S(\tau)\rangle &= -\int \frac{G}{c^4 x} H_S i\dot{\delta}(\tau' - \tau) |\psi_S(\tau')\rangle d\tau' \\ &= i\frac{G}{c^4 x} H_S \int \delta(\tau' - \tau) \frac{d}{d\tau'} |\psi_S(\tau')\rangle d\tau' \\ &= i\frac{G}{c^4 x} H_S \frac{d}{d\tau} |\psi_S(\tau)\rangle \end{aligned} \tag{A13}$$

and its adjoint

$$\begin{aligned} \langle\psi_S(\tau)| H_k &= i\frac{G}{c^4 x} \int \dot{\delta}(\tau' - \tau) \langle\psi_S(\tau')| H_S d\tau' \\ &= -i\frac{G}{c^4 x} \int \delta(\tau' - \tau) \frac{d}{d\tau'} \langle\psi_S(\tau')| H_S d\tau' \\ &= -i\frac{G}{c^4 x} H_S \frac{d}{d\tau} \langle\psi_S(\tau)| H_S. \end{aligned} \tag{A14}$$

Then, defining $\mathbb{G} := \frac{G}{c^4 x}$, we obtain

$$\begin{aligned} \frac{d\rho_S(\tau)}{d\tau} =& -i\sum_\iota p_\iota(\{H_S |\psi_S^\iota(\tau)\rangle + H_k |\psi_S^\iota(\tau)\rangle\} \langle\psi_S^\iota(\tau)| - |\psi_S^\iota(\tau)\rangle \{\langle\psi_S^\iota(\tau)| H_S + \langle\psi_S^\iota(\tau)| H_k\}) \\ =& -i\sum_\iota p_\iota\Big([H_S, |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)|] + \mathbb{G}\Big\{iH_S \frac{d|\psi_S^\iota(\tau)\rangle}{d\tau}\langle\psi_S^\iota(\tau)| + i|\psi_S^\iota(\tau)\rangle \frac{d\langle\psi_S^\iota(\tau)|}{d\tau} H_S\Big\}\Big) \\ =& -i\sum_\iota p_\iota\Big([H_S, |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)|] + \mathbb{G}\Big\{H_S^2 |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)| + H_S H_k |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)| \\ &- |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)| H_S^2 - |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)| H_k H_S\Big\}\Big) \\ =& -i\sum_\iota p_\iota\Big([H_S, |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)|] + \mathbb{G}[H_S^2, |\psi_S^\iota(\tau)\rangle\langle\psi_S^\iota(\tau)|] + \mathcal{O}(\mathbb{G}^2)\Big), \end{aligned} \tag{A15}$$

where in the second line we use Equation (6). As all other terms will be of orders superior to $\mathbb{G}$, which is proportional to the inverse of c^4, we assume these terms to be negligible; therefore, our equation of motion will be approximately

$$\frac{d\rho_S(\tau)}{d\tau} = -i[H_S, \rho_S(\tau)] - i\mathbb{G}[H_S^2, \rho_S(\tau)]. \tag{A16}$$

Appendix D. Time-Dependent Potential

To obtain the potential, we start by defining the normalized version of the error $|\varepsilon_{sg}\rangle = \sqrt{\langle\varepsilon_s|\varepsilon_s\rangle}\, |\varepsilon'_{sg}\rangle$; then, from $|\varepsilon\rangle_s := i\langle\varepsilon'|\Psi\rangle$, we have

$$\begin{aligned} |\varepsilon_{sg}\rangle &:= \Big(1 + \mathbb{G}H_S + \mathbb{G}H_S\frac{d}{T}\Big) i \sum_{k\in S_d(0)} \varepsilon' \langle\theta_k|\Psi\rangle \\ &= \mathbb{G}_H \varepsilon' \sum_{k\in S_d(0)} \langle\theta_k|\Psi\rangle \\ &= \frac{\mathbb{G}_H \varepsilon'}{\sqrt{d}} \sum_{k\in S_d(0)} |\psi_S(k)\rangle\,, \end{aligned} \tag{A17}$$

with $\mathbb{G} := G/c^4 x$, where G is the gravitational constant, c is the speed of light, x is the coordinate distance between the clock and the system, and d is the dimension of the Hilbert space. Then, we can define

$$V := \sqrt{\langle \varepsilon_s | \varepsilon_s \rangle}\, |\varepsilon_{sg}\rangle\, {}_e\langle \psi_S(\tau)| \,, \tag{A18}$$

in such a way that Equation (23), disregarding terms of the second order of $\mathbb{G}$, becomes

$$i\frac{d}{d\tau}\,|\psi_S(\tau)\rangle_e = H_S\frac{T}{d}\,|\psi_S(\tau)\rangle_e + \mathbb{G}H_S^2\frac{T}{d}\,|\psi_S(\tau)\rangle_e + V\,|\psi_S(\tau)\rangle_e \,. \tag{A19}$$

Just as a note, from the equations above we can see that the potential can be written explicitly as

$$\begin{aligned} V &= \frac{\mathbb{G}_H \varepsilon'}{\sqrt{d}} \sum_{k,k'} \psi(\tau;k')\,|\psi_S(k)\rangle\,\langle\psi_S(k')| \\ &= \frac{\mathbb{G}_H \varepsilon'}{\sqrt{d}} V_{k,k'}. \end{aligned} \tag{A20}$$

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