Physica D 271 (2014) 48–63

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

# Metastable periodic patterns in singularly perturbed state-dependent delayed equations

Xavier Pellegrin [a], C. Grotta-Ragazzo [b], C.P. Malta [c], K. Pakdaman [a,*]

[a] Institut Jacques Monod, CNRS, UMR 7592, University of Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France

[b] Instituto de Matemática e Estatística, Universidade de São Paulo, 05508-090, São Paulo, SP, Brazil

[c] Instituto de Física, Universidade de São Paulo, R. do Matão, Travessa R, 187, 05508-090 São Paulo, Brazil

## HIGHLIGHTS

- We show the existence of metastable oscillatory transients in state-dependent DDEs.
- Transition layer equations are introduced for state dependent delay equations.
- Transition layer solutions characterize the occurrences of metastability.
- In the positive feedback case, metastability depends on a symmetry condition.
- State dependent delays can render metastable otherwise short oscillatory transients.

## ARTICLE INFO

*Article history:*
Received 19 March 2012
Received in revised form
13 September 2013
Accepted 16 November 2013
Available online 22 November 2013
Communicated by Y. Nishiura

*Keywords:*
Metastability
State-dependent delayed differential equation
Singular perturbation

## ABSTRACT

We consider the scalar delayed differential equation $\epsilon\dot{x}(t) = -x(t) + f(x(t-r))$, where $\epsilon > 0$, $r = r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$, and $f$ represents either a monotone positive feedback $df/dx > 0$ or a monotone negative feedback $df/dx < 0$, plus some other hypotheses. When the delay is a constant, i.e. $r(x, \epsilon) = 1$, this equation can support metastable rapidly oscillating solutions that are transients whose duration is of order $\exp(c/\epsilon)$, for some $c > 0$. In this paper, combining analytic and numerical techniques, we investigate whether this metastable behavior persists when the delay $r(x, \epsilon)$ depends non trivially on the state variable $x$. More precisely, we proceed in three steps. First we explore theoretically and numerically the existence of branches of rapid periodic solutions for these equations. Then, to predict conditions for metastability, we introduce transition layer equations that depict the asymptotic shape of transient oscillations when $\epsilon$ tends to zero and $\eta(\epsilon) \sim \epsilon$. Finally, these predictions are validated by numerical simulations. Numerical explorations are also performed for other scalings of $\eta(\epsilon)$ with $\epsilon$. Our conclusions are: (i) when $\eta(0) \neq 0$, there are no metastable transients; (ii) when $\eta(0) = 0$ and $\eta'(0)$ is finite, for monotone negative feedback, the metastable oscillations exist irrespective of choices of $f$ and $R$; (iii) for monotone positive feedback, they require that the feedback $f$ and the delay $R$ satisfy extra conditions, such as $f$ being an odd function and the delay $R(x)$ an even function. One novel result is that state-dependent delays may lead to metastable dynamics in equations that cannot support such regimes when the delay is constant.



## 1. Introduction

State-dependent delay differential equations (DDEs) of the form

$$\epsilon\frac{dx}{dt}(t) = \epsilon\dot{x}(t) = -x(t) + f(x(t-r)),$$
$$\text{where } r = r(x, \epsilon) = r_0 + \tilde{r}(x, \epsilon), \quad (1)$$

appear as models in economics, physics and biology [1–7]. Many studies have dealt with the asymptotic dynamics of these equations, establishing existence, stability and profiles of the so-called slowly periodic oscillations when $f$ represents a negative feedback (see, for instance, [8–17]). There are also results on existence and stability of periodic solutions, and on the convergence of most solutions to equilibria in the case of monotone positive feedback [18,19]. For a review on DDEs with state-dependent delays see [20]. In contrast with these previous studies, in addition to the long term behavior of solutions, we are also interested in the transient dynamics of Eq. (1): we explore the occurrence of metastable transient oscillations prior to convergence to an asymptotic regime. Typically, in such a

* Corresponding author. Tel.: +33 0 157278055.
*E-mail addresses:* khashayar.pakdaman@gmail.com, khashayar.pakdaman@ijm.univ-paris-diderot.fr (K. Pakdaman).

0167-2789/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.physd.2013.11.012

long-lasting transient, the system engages in seemingly stable and sustained oscillations for a time of order $\exp(c/\epsilon)$ for some constant $c$, before eventually converging to a stable equilibrium or a stable periodic solution.

It is well known that delayed feedback can lead to oscillations in a system that would otherwise remain at rest. Much focus has been on delay-induced sustained undamped oscillations, such as periodic ones. However, the delay can also produce transient oscillations that end up vanishing with time. While these had been reported in several studies (see for instance [21–24] and references therein), for a long time they had not been considered as a topic of investigation on their own. In [24], the term DITO was coined for this phenomenon both as an acronym for "delay induced transient oscillation" and as a means to emphasize its repetitive nature. In the same paper, it was argued that long-lasting DITOs could strongly alter the dynamics of the system and, for instance, interfere with information retrieval in neural networks. That work together with the ones cited above initiated intrinsic interest in the phenomenon. Since these early studies, DITOs have been reported in other systems, such as for instance [25,26]. In support of claims in [21] and numerical explorations in [24], through analytical treatment of a specific system of DDEs with piecewise constant feedback, it was shown, in [27], that some DITOs could persist for durations of the order $\exp(c/\epsilon)$. Such long transients would outlast any observation window when $\epsilon$ is small enough and would be undistinguishable from (nearly) periodic solutions. In analogy with long-lasting transient oscillations in partial differential equations, DITOs with such exponential lifetimes are referred to as metastable [28].

Metastable oscillations have been thoroughly studied in many systems such as scalar partial differential equations [29,30]. However they have received far less attention in DDEs. Here we examine their occurrence in DDE (1). Our previous works characterized the conditions for metastability in equations with constant delays for both monotone positive and negative feedbacks [28,31]. They further revealed a remarkable difference in metastability between the monotone positive and monotone negative feedback equations, that had escaped earlier reports and explorations. Indeed, a positive feedback DDE presents metastability only when the feedback $f$ satisfies a special symmetry property that holds, for instance, when $f$ is an odd function. This is not a requirement in the negative feedback case (see also [32,33] for related results). One consequence of this result is that not all DITOs are metastable.

All these previous works deal with constant delays. As far as we know, the present paper is the first one to investigate and show the existence of metastability for state-dependent delay equations. Our goal in this paper is to study the transient dynamics of Eq. (1) when the parameter $\epsilon > 0$ is small or, equivalently, $r_0$ is large. The present investigation is based upon available information regarding rapidly oscillating periodic solutions (periodic solutions that have a large number of zeros in a time interval of length $r_0$), and educated guesses supported by systematic numerical investigations.

In previous singular perturbation analyses of DDEs with state-dependent delay [34–36], the small parameter $\epsilon$ appears only on the left hand side of the equation. Here, the novelty resides in the fact that $\epsilon$ appears in Eq. (1) in two different places: multiplying the time derivative of $x$, which characterizes a singular perturbation problem, and as an argument of $r$. One major contribution of this work is to highlight the fact that the dynamics of Eq. (1) strongly depend on the way $r$ depends on $\epsilon$.

This paper presents the first exploratory work in the transient dynamics of state-dependent DDEs. As this topic has not been investigated before, our work unfolds in two major steps. First, inspired by what is known for constant delays, we characterize theoretically, and numerically, conditions for metastability, and then we validate these through numerical simulations. To this end, the paper is organized as described below.

For the sake of self-consistency, in Section 2 we schematically review key results on metastability in DDEs with constant delays. These are used to identify the two essential ingredients of metastability: (i) rapidly oscillating periodic solutions, and (ii) transition layer equations whose heteroclinic orbits capture the shape of transient oscillations as $\epsilon \to 0$. In the following sections, we explore the same ingredients in state-dependent DDEs and their consequences on the transient dynamics. In Section 3, using a Hopf bifurcation theorem, we examine the existence of rapidly oscillating periodic solutions for Eq. (1), regardless of the choice of $r$ and its scaling with the small parameter $\epsilon$. In Section 4 we introduce transition layer equations and compute numerically their heteroclinic orbits to describe metastability. These equations generalize those of DDEs with constant delay [37,38,31]. From our investigation, we postulate that symmetry conditions on heteroclinic solutions of these equations can be used to characterize the conditions required for metastability. In Section 5, we present systematic numerical explorations of the transient dynamics of Eq. (1) that validate the predictions of the previous section. Section 6 contains a discussion and a summary of the results. Notably, one novel observation, specific to equations with state-dependent delay, is that the existence of metastable solutions may depend on the behavior of the function $r$ when $\epsilon \to 0$. So we have considered several possible scalings of $\eta$ with $\epsilon$ for $\epsilon \to 0$.

Prior to Section 2, we go over some assumptions and notations. In most applications the functions $f$ and $r$ are differentiable in $x$; here this is always assumed. Furthermore, in accordance with our previous results [28,31], we assume that $f$ is either a monotonic increasing (positive feedback) or a monotonic decreasing (negative feedback) function satisfying the following hypotheses.

*MPFH*
$f'(x) \geq 0, f(0) = 0, f'(0) > 1$, and there exist $a > 0, b > 0$, such that $f(-a) = -a, f(b) = b, 0 < f'(-a) < 1, 0 < f'(b) < 1$, and $f(x) \neq x$ for $x \in (-a, 0) \cup (0, b)$.

*MNFH*
$f'(x) \leq 0, f(0) = 0, f'(0) < -1$, and there exist $a > 0, b > 0$, such that $f(-a) = b, f(b) = -a, 0 < f'(-a)f'(b) < 1$; $|f(f(x))| > x$ for $x \in (-a, 0) \cup (0, b)$, and $|f(f(x))| < x$ for $x \in (-\infty, -a) \cup (b, \infty)$.

The names MPFH and MNFH stand for "monotone positive feedback hypothesis" and "monotone negative feedback hypothesis", respectively.

In the following, a function $f$ satisfying MPFH or MNFH will be called symmetric if it is an odd function. Under these hypotheses, both the dynamics of the map $f : \mathbb{R} \to \mathbb{R}$, and the DDE Eq. (1), with constant delay are well understood. In particular, the map $f$ satisfying MPFH has only three fixed points: $x = 0$, unstable, and $x = -a, x = b$, stable on the interval $[-a, b]$. For $f$ satisfying MNFH, the map has a single unstable fixed point $x = 0$, and a single attracting 2-periodic orbit $\{-a, b\}$ on the interval $[-a, b]$. Without loss of generality, one can use the time scaling $s = \frac{t}{r_0}$ and consider Eq. (1) with $r_0 = 1$. In the following we shall assume that the delay function is

$$r(x, \epsilon) = 1 + \eta(\epsilon)R(x), \tag{2}$$

where $\eta$ is a non negative and smooth function on $\epsilon \geq 0$, and $R : \mathbb{R} \to \mathbb{R}$. In this way, different scalings between the delay $r$ and the parameter $\epsilon$ can be analyzed by changing the function $\eta$. Typically $R$ will be chosen as a smooth nonnegative function.

## 2. Metastability in DDEs with constant delay

Metastability is a concept that appears in several branches of physics and mathematics. Schematically, a state is called metastable if it is transient, that is, it is eventually transformed into another one, but this transformation is on such a slow time scale that it is not perceived in normal observation windows. One of the first systems of relevance to the present work, in which metastability had a full mathematical treatment, is that of scalar parabolic equations [29,30]. Metastable solutions of DDEs (1) with constant delay $r(x, \epsilon) = 1$ share a number of features with those of the partial differential equations [28,31]. In this section, we provide an overview of metastable solutions in scalar DDEs with constant delay, that will serve as a basis for the comparison and analysis of the case of state-dependent delay.

Throughout the remainder of this section, we refer only to DDEs (1) with constant delay set to one. Roughly speaking, metastable solutions of these equations are trajectories that evolve close to unstable manifolds of unstable periodic orbits, and they seem to be periodic, while in fact they are endowed with a drift of order $\exp(c/\epsilon)$. In Section 2.1 we describe the relationship between rapidly oscillating periodic orbits and metastable oscillations, and in Section 2.2 we show how transition layer equations allow us to quantify the slow drift of such oscillations. For details see [28,31].

### *2.1. Metastability and rapidly oscillating periodic orbits*

To make the description more concrete, consider, for instance, the function $f(x) = \frac{1}{2}\arctan(5x)$ satisfying MPFH. For any value of $\epsilon > 0$, Eq. (1) has exactly three equilibria, $x = -a$, $x = 0$, and $x = a$, where $a$ is the positive solution of $x = \frac{1}{2}\arctan(5x)$. For all values of $\epsilon > 0$, the equilibria $x = -a$ and $x = +a$ are locally asymptotically stable and $x = 0$ is unstable. For all $\epsilon$ sufficiently large the unstable manifold of $x = 0$ is one-dimensional and as $\epsilon$ decreases, $x = 0$ undergoes an infinite number of Hopf bifurcations at $\epsilon_1 > \epsilon_2 > \cdots > 0$ such that the dimension of the unstable manifold of $x = 0$ at $\epsilon \in (\epsilon_n, \epsilon_{n+1})$ becomes $2n + 1$. Let $x_n$ denote the periodic solution that bifurcates from $x = 0$ at $\epsilon = \epsilon_n$. If $x_n(t) = 0$ then $x_n(t + s)$ has exactly $2n$ zeros for $s \in [-1, 0]$. The branches of periodic orbits that appear at these successive Hopf bifurcations can be extended up to $\epsilon = 0$. On a given branch the minimal period converges to a constant, the amplitude converges to $a > 0$ [39], and, for $f$ odd, the oscillations tend to a square-wave-like shape when $\epsilon \to 0$. So, provided that $\epsilon > 0$ is sufficiently small, Eq. (1) admits rapidly oscillating periodic solutions: periodic solutions that have a large number of zeros in a time interval of length one. Given $\epsilon > 0$ the global attractor of Eq. (1) consists of its set of equilibria and periodic orbits, and their finite-dimensional unstable manifolds. These solutions are organized in a peculiar way: the global attractor admits a Morse decomposition, with each Morse set containing a periodic orbit or an equilibrium. The direction of the flow on this decomposition is such that the number of zeros of solutions is a non-increasing function of time [40,19]. A similar description of the periodic orbits, their branches, their shape, and the organization of the trajectories on the global attractor holds for Eq. (1) with negative feedback [14,38].

Now that we have depicted the long term dynamics of DDEs with $f$ either satisfying MNFH or being odd and satisfying MPFH, we can describe the way metastable solutions appear in Eq. (1). For $\epsilon > 0$ small, an initial condition $\varphi : [-1, 0] \to \mathbb{R}$ to Eq. (1) with $2n$ zeros gives rise to a solution $x_t(\cdot, \varphi, \epsilon) : [-1, 0] \to \mathbb{R}$ that after a time of order one is pointwise close to a function in the unstable manifold of $x_n$, that has a square-wave-like shape close to that of $x_n$. The dynamics of $x_n$ is approximately periodically oscillatory, but the slow motion of the solution along the unstable manifold of $x_n$ eventually annihilates a pair of zeros of the solution, which takes a time of order $\exp(c/\epsilon)$. After the annihilation of the two zeros the solution drifts along the unstable manifold of another periodic solution $x_{n-1}$, and this process repeats itself until the solution eventually approaches one of the two stable equilibria $x = -a$ or $x = a$ for an odd function $f$ satisfying MPFH, or a slowly oscillating periodic orbit for $f$ satisfying MNFH. Metastability means here that the solutions have a (fast) oscillatory transient time that grows as $\exp(c/\epsilon)$ when $\epsilon \to 0$.

### *2.2. Metastability and transition layers*

Metastability not only depends on the existence of periodic solutions (i.e. the qualitative geometry of the phase portrait), but also on the quantitative dynamics along their unstable manifolds. Metastability happens only when the rapidly oscillating solutions are square-wave-like, in which case their jumps have been understood as transition layer phenomena, and described and analyzed using transition layer equations.

Suppose that $f$ satisfies MPFH and that $x(t)$ is an oscillatory metastable solution of (1), with $r = 1$, that jumps from $x(t) \approx b > 0$, for $t < 0$, to $x(t) \approx -a < 0$, for $t > 0$, with $x(0) = 0$. The approximately periodic behavior of the metastable solution implies that $x(t) \approx x(t + 1 + \rho\epsilon)$. When $\epsilon$ is small, this and Eq. (1) imply

$$\epsilon \frac{dx}{dt}(t) = -x(t) + f\{x(t-1)\} \approx -x(t) + f\{x(t + \rho\epsilon)\}.$$

Rescaling $x(t)$ as $\phi^-(t) = x(\epsilon t)$, the above equation for $x(t)$ implies that in the limit $\epsilon \to 0$, $\phi^-$ must satisfy the transition layer equation (see [31]):

$$\dot{\phi}^-(t) = -\phi^-(t) + f(\phi^-(t + \rho^-)), \quad (3)$$

where $\rho^- > 0$ is an unknown constant and the function $\phi^-$ must satisfy the boundary conditions $\lim_{t\to-\infty} \phi^-(t) = b$ and $\lim_{t\to\infty} \phi^-(t) = -a$. The same ideas apply to a jump from $x(t) \approx -a < 0$ for $t < 0$, to $x(t) \approx b > 0$ for $t > 0$, and leads to the existence of an increasing transition layer solution, i.e. a solution to

$$\dot{\phi}^+(t) = -\phi^+(t) + f(\phi^+(t + \rho^+)) \quad (4)$$

where $\rho^+ > 0$ is an unknown constant and the function $\phi^+$ must satisfy the boundary conditions $\lim_{t\to-\infty} \phi^+(t) = -a$ and $\lim_{t\to\infty} \phi^+(t) = b$.

The constants $\rho^+$ and $\rho^-$ are drift velocities of ascending and descending sign-changes (zeros) of an oscillatory solution. If an oscillatory solution $x(t)$ of Eq. (1) satisfies $x(t_0^+) = 0$ and $x'(t_0^+) > 0$, then one expects to find a time $t_1^+ \approx t_0^+ + 1 + \epsilon\rho^+$ such that $x(t_1^+) = 0$ and $x'(t_1^+) > 0$; if $x(t_0^-) = 0$ and $x'(t_0^-) < 0$, then one expects to find a time $t_1^- \approx t_0^- + 1 + \epsilon\rho^-$ such that $x(t_1^-) = 0$ and $x'(t_1^-) < 0$. This suggests that the symmetry condition $\rho^+ = \rho^-$ should be associated with metastability of oscillating solutions. Indeed, exponential duration of oscillatory transients is proven in [31] for scalar DDEs, with monotone positive feedback, by an estimate of the type $|t_1^\pm - (t_0^\pm + 1 + \epsilon\rho^\pm)| \le e^{c/\epsilon}$, so that when $\rho^+ = \rho^-$ oscillatory solutions are close to a $1 + \epsilon\rho$ periodic solution up to an exponential order.

Finally, we discuss the transition layer for $f$ satisfying MNFH. Transient oscillations are square-wave-like and they have an approximate period $T \approx 2 + \epsilon C$. If an oscillatory solution $x(t)$ of Eq. (1) satisfies $x(t_0) = 0$ and $x'(t_0) > 0$, then one expects to find a time $t_1 \approx t_0 + 1 + \epsilon\rho^+$ such that $x(t_1) = 0$ and $x'(t_1) < 0$, and then a time $t_2 \approx t_1 + 1 + \epsilon\rho^-$ such that $x(t_2) = 0$ and $x'(t_2) > 0$.

So the transition layer equations for the increasing and decreasing transition layer solutions are coupled:

$$\begin{cases}\dot{\phi}^+(t) = -\phi^+(t) + f(\phi^-[t+\rho^-]),\\ \dot{\phi}^-(t) = -\phi^-(t) + f(\phi^+[t+\rho^+]),\end{cases} \tag{5}$$

where $\phi^+$ is increasing and $\phi^-$ is decreasing on $\mathbb{R}$, with $\lim_{t\to-\infty}\phi^+(t) = -a$, $\lim_{t\to\infty}\phi^+(t) = b$, $\lim_{t\to-\infty}\phi^-(t) = b$ and $\lim_{t\to\infty}\phi^-(t) = -a$, with $\phi^+(0) = \phi^-(0) = 0$ and $\rho^\pm$ are unknown real constants. For $f$ satisfying MNFH, given an oscillatory solution $x(t)$, an (ascending) zero $x(t_0) = 0$ with $x'(t_0) > 0$ gives rise to a (descending) zero $x(t_1) = 0$ with $x'(t_1) > 0$, and then to another ascending zero $x(t_2) = 0$ with $x'(t_2) > 0$, with $t_1 \approx t_0 + 1 + \epsilon\rho^+$ and $t_2 \approx t_1 + 1 + \epsilon\rho^-$. So, in the first order, the drift speeds of ascending and descending zeros are identical and equal to $2 + \epsilon(\rho^- + \rho^+)$. Hence the symmetry condition, equivalent to the condition $\rho^+ = \rho^-$ in the case $f$ satisfies MPFH, is here $\rho^+ + \rho^- = \rho^- + \rho^+$, and it is obviously always true, irrespective of $\rho^+$ and $\rho^-$ [31].

In summary, the analysis using transition layer equations reveals and explains that metastability manifests itself in different ways in positive and negative feedback systems. Indeed, monotone positive feedback DDEs present metastability only when a special symmetry property of the transition layer problems (3) and (4) is satisfied, which holds for instance when $f$ is an odd function, while there is no such restriction in the case of monotone negative feedback equations (5) [28,31].

## 3. Existence and amplitude of periodic solutions

Based on literature results and novel results that follow in this section, we conjecture that, provided the delay satisfies a number of classical technical assumptions [34–36,10,20,41–43], the geometric organization of the phase portrait of DDEs with state dependent delays and monotone feedback is similar to that of equations with constant delays. Namely our three conjectures are: (i) Eq. (1), with $\tilde{r}(x,0) = 0$, sustains branches of periodic solutions, that appear at the 0 equilibrium by successive Hopf bifurcations and exist until $\epsilon \to 0$, with amplitudes and periods that converge to some non zero limits. (ii) A Poincaré–Bendixson like theorem holds, and as a consequence the global attractor of (1) is composed of equilibria, periodic solutions, and their unstable manifolds. (iii) The global attractor of (1) has a Morse decomposition, it is ordered by a discrete Lyapunov functional, and it is composed only of equilibria of (1), the periodic solutions described in the first conjecture, and connections from more rapidly oscillating periodic solutions to less rapidly oscillating periodic solutions.

With few hypotheses on the feedback $f$, we have shown that a local Hopf-bifurcation theorem of [41] applies to (1). This gives an essential element in the proof of conjecture (i): for some decreasing sequence $\epsilon_k \to 0$, a Hopf bifurcation occurs at the zero equilibrium of (1) each time that $\epsilon$ crosses one $\epsilon_k$, which gives rise to an oscillating periodic solution with $2k$ zeros per unit time. The corresponding theorem is rigorously stated in Section 3.1 and proved in Appendix A. Assuming the existence of branches corresponding to these periodic solutions, in Section 3.2, we show that if we have $\eta(\epsilon) = c\epsilon + o(\epsilon)$ in (2) for some $0 < c < +\infty$, then the amplitudes of all periodic solutions converge to the same non zero limit when $\epsilon \to 0$. On the contrary, following a proposition of [35], we show in Section 3.3 that when $\eta(0) \neq 0$, the amplitude of periodic solutions with $2k$ zeros per period (along the branch that appears at $\epsilon_k$), is bounded from above by some constant $C_k$ (independent of $\epsilon$) such that $C_k \underset{k\to+\infty}{\longrightarrow} 0$.

### 3.1. Sequence of Hopf-bifurcations for Eq. (1)

The Ph.D. Thesis of M. Eichmann [41] contains a local Hopf-bifurcation theorem for state-dependent DDEs which implies the following.

**Theorem 1.** *Suppose that $f$ is $C^2(\mathbb{R},\mathbb{R})$, $r$ :]0, 1[$\times C^0([-M,0],\mathbb{R}) \to \mathbb{R}$ is $C^1$, and $r$ :]0, 1[$\times C^1([-M,0],\mathbb{R}) \to \mathbb{R}$ is $C^2$, for $M > 0$. Suppose that $f(0) = 0$ and $|f'(0)| > 1$. Then there is a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging to zero, such that for any $k \geq 0$ there is an open interval $]-\eta_k,\eta_k[$ and $C^1$ mappings $y^*$ :$]-\eta_k,\eta_k[\to C^1([-M,0],\mathbb{R})$, $\epsilon^*$ :$]-\eta_k,\eta_k[\to ]0,1[$ and $w^*$ :$]-\eta_k,\eta_k[\to \mathbb{R}$, with $y^*(0) = x^* = 0$, $\epsilon^*(0) = \epsilon_k$ and $w^*(0) = \beta_k = \mathrm{Im}(\lambda_k)$ such that for any $u \in ]-\eta_k,\eta_k[$ there is a periodic solution to*

$$\epsilon^*(u)x'(t) = -x(t) + f(x(t - r(\epsilon^*(u), x_t)))$$

*with initial condition $x_0 = y^*(u)$ and with frequency $\frac{w^*(u)}{2\pi}$.*

We remark that the delay function $r(\epsilon, x_t)$ in the theorem above is more general than that in Eq. (2), and that the function $f$ does not have to be positive or negative feedback.

Theorem 1 is an immediate consequence of the theorem by Eichmann [41]. In Appendix A we verify that the hypotheses of Eichmann's theorem are satisfied. Here we first justify the existence of the critical values $\epsilon_k$, and show that the associated frequencies $\frac{w_k}{2\pi} = \frac{\beta_k}{2\pi} \underset{k\to\infty}{\longrightarrow} +\infty$. Suppose that $f(0) = 0$ and that $|f'(0)| > 1$, and let $x^* = 0$ be the unstable steady solution of Eq. (1). The linearization of Eq. (1) at $x^* = 0$ is

$$\epsilon y'(t) = -y(t) + f'(0)y(t - r_0)$$

where $r_0 = r(0) = r(x^*) = 1$. The characteristic equation associated with this linearized equation is $1 + \epsilon\lambda = f'(0)e^{-\lambda}$, or, equivalently, with $\lambda = \alpha + i\beta \in \mathbb{C}$

$$\begin{cases}1 + \epsilon\alpha = f'(0)e^{-\alpha}\cos(\beta)\\ \epsilon\beta = -f'(0)e^{-\alpha}\sin\beta.\end{cases}$$

This is the same characteristic equation as for the constant-delay equation, and there exists a sequence $\epsilon_k \underset{k\to+\infty}{\longrightarrow} 0$ such that for each $\epsilon = \epsilon_k$ the characteristic equation has a single pair of solutions on the imaginary axis $\lambda = \pm i\beta_k$, $\beta_k > 0$. Moreover, $\beta_k \to \infty$ as $k \to \infty$. Since $\beta_k$ is the angular frequency of the periodic orbit unfolded at $\epsilon_k$, Theorem 1 implies the existence of rapidly oscillating periodic solutions of Eq. (1) as $\epsilon$ tends to zero.

### 3.2. Case $\eta(0) = 0$ with $0 < \eta'(0) < +\infty$

With the previous assumptions on $\eta$, one has $r(x,\epsilon) = 1 + \eta(\epsilon)R(x) \sim 1 + \epsilon\eta'(0)R(x)$ when $\epsilon \to 0$. Without loss of generality, we assume $\eta'(0) = 1$, so that the delay is $r(x,\epsilon) = 1 + \epsilon R(x)$ in Eq. (1).

Suppose that there is an $\epsilon_0$ for which Eq. (1) has a periodic solution $x_0(t)$ with period $T_0$. Given an integer $n > 0$, Eq. (1) and the periodicity of $x_0(t)$ imply that

$$\epsilon_0\dot{x}_0(t) = -x_0(t) + f(x_0\{t - 1 - nT_0 - \epsilon_0 R[x_0(t)]\}).$$

Then, rescaling time as $\hat{t} = t/(1 + nT_0)$, we obtain that $x_n(t) = x_0[(1 + nT_0)\hat{t}]$ satisfies the equation

$$\frac{\epsilon_0}{1 + nT_0}\frac{dx_n}{d\hat{t}}(\hat{t}) = -x_n(\hat{t}) + f\left(x_n\left\{\hat{t} - 1 - \frac{\epsilon_0}{1 + nT_0}R[x_n(\hat{t})]\right\}\right).$$

Therefore, for $\epsilon = \epsilon_n = \epsilon_0/(1 + nT_0)$ Eq. (1) admits the periodic solution $x_n(t) = x_0[t(1 + nT_0)]$ with period $T_n = T_0/(1 + nT_0)$. It shows that the branches of rapidly oscillating periodic solutions can be obtained from the first branch of periodic solutions. Hence, assuming that the first branch exists up to $\epsilon \to 0$, it follows that all the other branches exist, and the amplitude of periodic solutions along all these branches converges to the same positive limit when $\epsilon \to 0$.

In this sense, Eq. (1) sustains "large amplitude" rapidly oscillating periodic solutions as $\epsilon$ tends to zero when the delay function is of the form $r(\epsilon, x) = 1 + \epsilon R(x)$ or $r(\epsilon, x) = 1 + \epsilon\eta'(0)R(x)$. As mentioned in Section 2, this is one of the signatures of the existence of metastable solutions in the case of DDEs with constant delay. So this strengthens the similarity of DDEs with state dependent delay and DDEs with constant delay, thus giving support to the possibility of metastability in the case $\eta(0) = 0$ with $0 < \eta'(0) < +\infty$.

### 3.3. Case $\eta(0) \neq 0$

The situation for $\eta(0) \neq 0$ is different from the one depicted above, and this can be understood thanks to Proposition 3.4 in [35], which precludes the existence of large amplitude rapidly oscillating periodic solutions as $\epsilon \to 0$. First, we recall that proposition in [35], and then we discuss its consequences in terms of the amplitude of the periodic solutions.

**Proposition 1** (*Mallet-Paret, Nussbaum*). *Suppose that the feedback $f$ is $C^0(\mathbb{R}, \mathbb{R})$ and that the delay function $r(\epsilon, x)$ is Lipschitz regular in $x$. Let $x(t)$ satisfy Eq. (1) for $t \in \mathbb{R}$ for some value of $\epsilon$, and suppose there exist $a_0 < a_1$ and $t_0 < t_1 < t_2 < t_3$ such that $x(t_i) \leq a_0$ for even $i$, and $x(t_i) \geq a_1$ for odd $i$, for $0 \leq i \leq 3$. Then*

$$\max_{[a_0,a_1]} r(\cdot, \epsilon) - \min_{[a_0,a_1]} r(\cdot, \epsilon) \leq 3(t_3 - t_0).$$

**Proposition 2.** *Suppose that the feedback $f$ is $C^0(\mathbb{R}, \mathbb{R})$ and that the delay function $r(\epsilon, x)$ is Lipschitz regular in $x$. Let $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$, and suppose that $R(x)$ is not constant on any interval (e.g. $R(x) = r_k x^k$, $k \geq 1$ and $r_k \neq 0$) and $\eta(\epsilon) \sim \eta(0) \neq 0$.*

*Then, there is a function $\varphi$, depending only on $\eta(0)$ and $R$, with $\varphi(T) \underset{T\to 0}{\longrightarrow} 0$, such that for any periodic solution $x(t)$ of Eq. (1) with period $T$, and $a_0 = \min x(t)$ and $a_1 = \max x(t)$, we have*

$$|a_1 - a_0| \leq \varphi(T).$$

*In particular, if $f$ and $r$ satisfy additionally the hypotheses of Theorem 1, the periodic solutions $x_k$ that appear when $\epsilon = \epsilon_k$ (see Theorem 1) have periods $T_k \underset{k\to\infty}{\longrightarrow} 0$ and amplitudes $\max\{x_k(t)\} - \min\{x_k(t)\} \underset{k\to\infty}{\longrightarrow} 0$.*

**Proof.** To see why Proposition 1 precludes the existence of large-amplitude rapidly oscillating periodic solutions, let $x(t)$ be a periodic solution of Eq. (1) with period $T$, and let $a_0 = \min x(t)$ and $a_1 = \max x(t)$. Choose $t_0$ such that $x(t_0) = a_0$, choose $t_1$ such that $t_0 < t_1 < t_0+T$ and $x(t_1) = a_1$, and $t_2 = t_0+T$ and $t_3 = t_1+T$. As in Section 1, let $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$, and suppose that $R(x)$ is not constant on any interval, e.g. $R(x) = r_k x^k$, $k \geq 1$, $r_k \neq 0$, and $\eta(\epsilon) \sim \eta(0) \neq 0$. Then $6T \geq 3(t_3 - t_0) \geq \eta(0)|r_k|\Delta_R/3$, where $\Delta_R = \max_{x\in[a_0,a_1]} R(x) - \min_{x\in[a_0,a_1]} R(x) > 0$. Given that $\eta(0) \neq 0$, we have $|r_k|\Delta_R/3 \leq \frac{6T}{\eta(0)}$.

This estimate relates the period of oscillations indirectly to their amplitude (through $\Delta_R$): the faster the periodic oscillations are, i.e. the smaller the $T$ is, the smaller their amplitude is. The Hopf bifurcation at $\epsilon = \epsilon_k$ gives rise to a periodic solution with period $T_k = \frac{1}{\beta_k} \underset{k\to\infty}{\longrightarrow} 0$ (see Theorem 1 and Appendix A), and the estimate above implies that so does their amplitudes $a_1 - a_0 \underset{k\to\infty}{\longrightarrow} 0$. □

This result indicates that DDE (1) with state-dependent delay, and $\eta(0) \neq 0$, cannot support fast periodic solutions that resemble those of DDEs with constant delays. This led us to conjecture that these equations cannot support large-amplitude square-wave-like metastable solutions similar to those of DDEs with constant delays. This conjecture is supported by extensive numerical investigations (Section 5).

## 4. Transition layer and metastability

In this section, we refine our previous analysis of the conditions under which DDE (1), with state-dependent delay, can support metastable oscillations through the introduction of transition layer equations. Such equations have been used previously to determine the shape of slowly oscillating periodic solutions for scalar DDEs with constant delay and negative feedback in the singular limit $\epsilon \to 0$ [38]. They have also been instrumental for the analysis of metastable solutions in scalar DDEs with constant delays and monotone feedback in the same singular limit [31].

For $f$ satisfying MNFH, the singular limit as $\epsilon \to 0$ of Eq. (1) in the case $\eta(0) \neq 0$ can be analyzed through the theory developed in [34–36] that replaces transition layer equations with the so-called "Max-Plus" equations. As numerical explorations tend to show that such systems do not support metastability (Section 5), we will not dwell any further in this case. Throughout the remainder of this section, our focus is on the case $\eta(0) = 0$, which we henceforth assume to hold.

For some feedback functions $f$ and state-dependent delay function $r$, solutions of Eq. (1) have an approximately periodic square-wave shape when $\epsilon \to 0$, as in the constant delay case $r(x, \epsilon) = 1$, and the corresponding "jumps" can be analyzed with the help of transition layer equations. The approximate period of metastable oscillations (depending on $\epsilon$) is an essential point in finding transition layer equations. For constant delay, at the first order, this period is $2 + \rho\epsilon$ under MNFH ([38] Theorem 3.2), and $1 + \epsilon\rho$ under MPFH [31]. In Section 5 similar asymptotics are shown to hold in the state-dependent delay case as well when $\eta(0) = 0$ and $0 \leq \eta'(0) < +\infty$. In this section we write appropriate transition layer equations under various hypotheses on $\eta$, show that their solutions exist, and that these equations can be used to characterize metastability, as confirmed by the numerical investigation presented in Section 5.

The case $\eta(0) = 0$ and $\eta'(0) = +\infty$ has also been investigated, using $r(x, \epsilon) = 1 + \epsilon^\alpha R(x)$ with $\alpha = \frac{1}{2}$. We found numerically that oscillations are square-wave-like when $\epsilon$ is small, their period is either $2 + \epsilon^\alpha\rho$ ($f$ satisfies MNFH) or $1 + \epsilon^\alpha\rho$ ($f$ satisfies MPFH), and rescaling time as $\phi(t) = x(\frac{t}{\epsilon^\alpha})$, one observes convergence to a transition layer profile (see Fig. 7 in Section 5). However the scaling argument used in the case $\eta'(0) < +\infty$ does not apply here, and these transition layer profiles are not analyzed in this section.

The remainder of this section is organized as follows. First, in Section 4.1, we show that when $0 < \eta'(0) < +\infty$, appropriately defined transition layer equations can be used to find a symmetry condition that characterizes precisely the cases of metastable oscillatory transients. Details about the existence of transition layer solutions, their numerical construction, and illustrating figures can be found in Appendix B. The case of $\eta'(0) = 0$ is examined in Section 4.2. One can still write a transition layer problem, but it does not depend on the delay function $R$ anymore, leading to incorrect results. To overcome this, we have introduced a one-parameter family of auxiliary transition layer problems, and thanks to the analysis of the corresponding one-parameter family of transition layer solutions we are able to characterize the cases where metastability can occur. Finally, in Section 4.3, we discuss a new phenomenon: the possibility of a state-dependent delay giving rise to metastability in equations that do not exhibit such transients when the delay is constant.

### 4.1. Case $\eta(0) = 0$ and $0 < \eta'(0) < +\infty$

#### 4.1.1. Monotone positive feedback

Let $f$ satisfy MPFH, and consider Eq. (1) where $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$ with $\eta(0) = 0$ and $0 < \eta'(0) < +\infty$ (without loss of generality we assume that $\eta'(0) = 1$ in the remainder of this

**Table 1**
Drift speeds $\rho^+$, $\rho^-$, solutions of Eq. (6) for $f$ satisfying MPFH, $\eta(\epsilon) = \epsilon$ and various choices of delay $R(x)$. Metastability occurs only when $\rho^+ = \rho^-$. Table (a) symmetric positive feedback $f(x) = \frac{1}{2}\arctan(5x)$. Table (b) non-symmetric positive feedback $f(x) = \frac{1}{2}\arctan(5(x - 0.05)) + \frac{1}{2}\arctan(0.25)$ (see Appendix B.1 for details on the numerical method and parameters value used).

| (a) | $R(x) = 0$ | $R(x) = x$ | $R(x) = \cos(x)$ | $R(x) = \frac{1}{2}x(1+x)$ |
|---|---|---|---|---|
| $\rho^+$ | 0.824 | 0.554 | 1.752 | 0.732 |
| $\rho^-$ | 0.824 | 1.158 | 1.752 | 1.158 |
| (b) | $R(x) = 0$ | $R(x) = x$ | $R(x) = \cos(x)$ | $R(x) = \frac{1}{2}x(1+x)$ |
| $\rho^+$ | 1.024 | 0.690 | 1.916 | 0.932 |
| $\rho^-$ | 0.664 | 0.994 | 1.612 | 0.884 |

section). For such delays, our numerical investigations show that metastable oscillations are approximately $1 + \epsilon\rho$ periodic. When $\epsilon$ is small, this and Eq. (1) imply that an oscillatory metastable solution $x(t)$ that jumps from $x(t) \approx b > 0$ for $t < 0$, to $x(t) \approx -a < 0$ for $t > 0$, with $x(0) = 0$, satisfies

$$\epsilon\frac{dx}{dt}(t) = -x(t) + f\{x(t - 1 - \eta(\epsilon)R(x(t)))\}$$
$$\approx -x(t) + f\{x(t + \rho^-\epsilon - \eta(\epsilon)R(x(t)))\}.$$

The same ideas apply to a jump from $x(t) \approx b > 0$ for $t > 0$, to $x(t) \approx -a < 0$ for $t < 0$. Likewise in the constant delay case, rescaling $x(t)$ as $\phi^\pm(t) = x(\epsilon t)$, the above equation for $x(t)$ implies that in the limit $\epsilon \to 0$, $\phi^\pm$ must satisfy the transition layer equations:

$$\dot\phi^\pm(t) = -\phi^\pm(t) + f(\phi^\pm(t - R(\phi^\pm(t)) + \rho^\pm)) \tag{6}$$

where $\rho^- > 0$ and $\rho^+ > 0$ are unknown constants (drift speeds) and the functions $\phi^\pm$ satisfy the boundary conditions $\lim_{t\to-\infty}\phi^-(t) = b$, $\lim_{t\to\infty}\phi^-(t) = -a$, $\lim_{t\to-\infty}\phi^+(t) = -a$, and $\lim_{t\to\infty}\phi^+(t) = b$.

If a solution to Eq. (6) exists, it is called a transition layer solution. In contrast to the constant delay-case, this transition layer equation (6) is a state-dependent equation, and from a theoretical point of view, depending on the values of $\rho$ and $R(x)$, it may be both advanced and delayed. The numerical method used for solving equation (6) is presented in Appendix B.1. We have defined an operator $\mathcal{T}$ whose (stable locally attractive) fixed points are solutions of (6).

In general, the constants $\rho^-$ and $\rho^+$ associated with the decreasing and increasing transition layer solutions are different. In Table 1 we display $\rho^-$ and $\rho^+$ solutions of Eq. (6) for $\eta(\epsilon) = \epsilon$ and various choices of $R(x)$, for both symmetric and non-symmetric positive feedback $f$ (numerical method details given in Appendix B.1). We found that in the state-dependent case of Eq. (6), as in the constant delay case, oscillatory transients are metastable only when $\rho^+ = \rho^-$. Table 1 also shows that $\rho^+ = \rho^-$ is obtained only when the positive feedback function is symmetric and the delay $R(x)$ is even.

#### 4.1.2. Monotone negative feedback

We now discuss the transition layer equation for state-dependent delay equation with $f$ satisfying MNFH. Numerically, transient oscillations are square-wave-like and they have an approximate period $T \approx 2 + \epsilon C$ (see Section 5). Likewise the case of DDEs with constant delay, the transition layer equations for the increasing and decreasing transition layer solutions of DDEs with state-dependent delays are coupled:

$$\begin{cases}\dot\phi^+(t) = -\phi^+(t) + f(\phi^-[t - R(\phi^-(t)) + \rho^-]),\\ \dot\phi^-(t) = -\phi^-(t) + f(\phi^+[t - R(\phi^+(t)) + \rho^+]),\end{cases} \tag{7}$$

where $\phi^+$ is increasing and $\phi^-$ is decreasing on $\mathbb{R}$, with $\lim_{t\to-\infty}\phi^+(t) = -a$, $\lim_{t\to\infty}\phi^+(t) = b$, $\lim_{t\to-\infty}\phi^-(t) = b$ and $\lim_{t\to\infty}\phi^-(t) = -a$, with $\phi^+(0) = \phi^-(0) = 0$ and $\rho^\pm$ are unknown real constants (drift speeds). See Appendix B for numerical solutions of Eq. (7).

**Table 2**
Drift speeds $\rho^+$, $\rho^-$, solutions of Eq. (7) for $f$ satisfying MNFH, $\eta(\epsilon) = \epsilon$ and various choices of delay $R(x)$. Metastability occurs regardless of the equality $\rho^+ = \rho^-$. (a) Symmetric negative feedback $f(x) = -\frac{1}{2}\arctan(5x)$, (b) non-symmetric negative feedback $f(x) = -\frac{1}{2}\arctan(5(x + 0.05)) + \frac{1}{2}\arctan(0.25)$ (see Appendix B.2 for details on the numerical method and parameters value used).

| (a) | $R(x) = 0$ | $R(x) = x$ | $R(x) = \cos(x)$ | $R(x) = \frac{1}{2}x(1+x)$ |
|---|---|---|---|---|
| $\rho^+$ | 0.824 | 1.172 | 1.744 | 1.122 |
| $\rho^-$ | 0.824 | 0.434 | 1.744 | 0.714 |
| (b) | $R(x) = 0$ | $R(x) = x$ | $R(x) = \cos(x)$ | $R(x) = \frac{1}{2}x(1+x)$ |
| $\rho^+$ | −0.702 | −0.616 | 0.280 | −0.646 |
| $\rho^-$ | 2.574 | 2.362 | 3.570 | 2.476 |

For negative feedbacks, the symmetry condition supporting metastability is always satisfied when the delay is constant [31], and we show that the same holds when the delay is state-dependent. In Table 2 we display the drift speeds $\rho^+$, $\rho^-$ that are solutions of Eq. (7) for $\eta(\epsilon) = \epsilon$ and various choices of $R(x)$. Comparison of Tables 1 and 2 shows that when the feedback function $f$ is symmetric, and $R(x)$ is even, $\rho^+ = \rho^-$ for both positive and negative feedbacks. This happens because when $f$ is symmetric, and $R(x)$ is even, the increasing solutions of the transition layer equations for both positive and negative feedback coincide (the same happens for the decreasing solutions).

### 4.2. Case $\eta(0) = 0$ and $\eta'(0) = 0$

#### 4.2.1. Monotone positive feedback

If $\eta(0) = 0$ and $\eta'(0) = 0$, solutions are approximately $1 + \epsilon\rho$ periodic, and the same time rescaling $\phi(t) = x(\epsilon t)$ implies $\dot\phi(t) = -\phi(t) + f(\phi(t - \frac{\eta(\epsilon)}{\epsilon}R(\phi(t)) + \rho))$, and as $\epsilon \to 0$

$$\dot\phi(t) = -\phi(t) + f(\phi(t + \rho)), \tag{8}$$

where the decreasing and increasing solutions $\phi^\mp$ must satisfy the boundary conditions $\lim_{t\to-\infty}\phi^+(t) = -a$, $\lim_{t\to+\infty}\phi^+(t) = b$, $\lim_{t\to-\infty}\phi^-(t) = b$ and $\lim_{t\to+\infty}\phi^-(t) = -a$, with $\phi^\pm(0) = 0$ and $\rho = \rho^\mp > 0$ is an unknown constant. This Eq. (8) is the same one found in the constant delay case, for which the existence of decreasing and increasing transition layer solutions has been proven in [31]. In particular, when $\eta(0) = 0$ and $\eta'(0) = 0$, the drift speeds $\rho^\mp$ are equal to those of the corresponding constant delay case ($R(x) = 0$). As a consequence, we obtain that for $f$ satisfying MPFH, if $\eta(0) = 0$ and $\eta'(0) = 0$, metastability cannot occur if $\rho^+ \neq \rho^-$. However, when $\rho^+ = \rho^-$, metastability may or may not occur.

To obtain the symmetry requirement for metastability in this case $\eta(0) = 0$ and $\eta'(0) = 0$, we introduce the following 1-parameter family of transition layer equations

$$\dot\phi^\pm(t) = -\phi^\pm(t) + f(\phi^\pm[t + \rho_\lambda^\pm - \lambda R(\phi^\pm(t))]), \tag{9}$$

where $\lambda \in \mathbb{R}$ is a real parameter. As previously, $\rho_\lambda^\pm$ are some unknown real constants, and $\phi^\pm$ are the transition layer solutions, that are expected to depend on $\lambda$. The case $\lambda = 0$ reproduces the drift speeds $\rho^\pm$ of the transition layer equation (8). In Fig. 1 we display the constants $\rho_\lambda^\pm$ as a function of $\lambda$ for the symmetric positive feedback function $f(x) = \frac{1}{2}\arctan(5x)$ and various choices of $R(x)$. We found that if $\rho_\lambda^+ = \rho_\lambda^-$ holds only for $\lambda = 0$ (panels (b) and (d) in Fig. 1), metastable DITOs are not observed for positive values of $\epsilon$. Oscillatory transients are metastable only when $\rho_\lambda^+ = \rho_\lambda^-$ on some non-trivial interval $\lambda \in [0, \delta]$ with $\delta > 0$ (panels (a) and (c) in Fig. 1). So this is the new sufficient condition for the existence of metastable oscillatory transients when the positive feedback $f$ is symmetric and $\eta'(0) = \eta(0) = 0$.

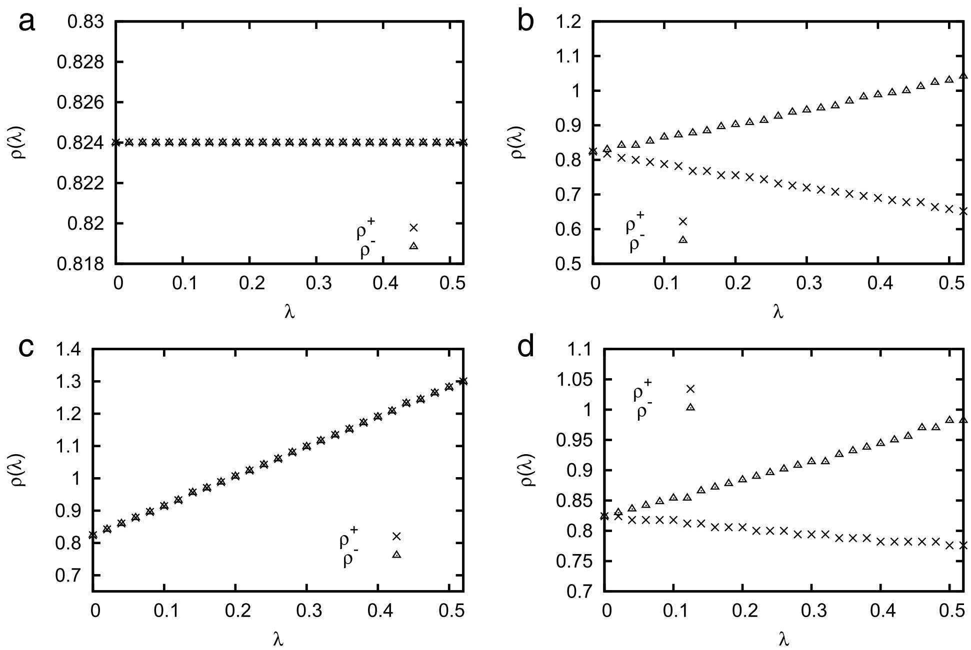



**Fig. 1.** Drift speeds $\rho_\lambda^\pm$, solutions of Eq. (9), for $\lambda \in [0, 0.5]$, using the symmetric positive feedback function $f(x) = \frac{1}{2}\arctan(5x)$ and various choices of $R(x)$: (a) $R(x) = 0$ (constant delay); (b) $R(x) = x$; (c) $R(x) = \cos(x)$; (d) $R(x) = \frac{1}{2}x(1 + x)$. See Appendix B.1 for details on numerical methods and parameters used.

#### 4.2.2. Monotone negative feedback

For $\eta(0) = 0$ and $\eta'(0) \in ]0; +\infty[$, we have seen that the symmetry condition supporting metastability always holds when the feedback is negative. The same is true if $\eta'(0) = \eta(0) = 0$: regardless of the symmetry of the feedback $f$ and of the function $R(x)$ in $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$, rapidly oscillating transients are metastable (see Section 5).

### 4.3. Metastability induced by state-dependent delay

In this section we present a new phenomenon: given a DDE with constant delay that does not display metastability, it is possible to add a state dependence to the delay so that the resulting state-dependent DDE will exhibit metastability. To this end, we consider Eq. (1) with a non-symmetric positive feedback function $f$, and delay function $r_\lambda(x, \epsilon) = 1 + \epsilon\lambda R(x)$, so that $\lambda = 0$ corresponds to constant delay DDE which does not exhibit metastability.

We did a numerical investigation using the non-symmetric positive feedback function $f(x) = \frac{1}{2}\arctan(5(x + 0.05)) - \frac{1}{2}\arctan(0.25)$. We have used the following functions $R(x)$ : $R(x) = x$, $R(x) = \cos(x)$, $R(x) = \frac{1}{2}x(1 + x)$. Solving the transition layer equation (9) for each function $R$ we have numerically computed the $\lambda$ -families of constants $\rho_\lambda^+$ and $\rho_\lambda^-$, the parameter $\lambda$ being varied within the interval $[-1.0, 1.0]$ (it should be remarked that for large $\lambda$ values the numerical solution of the transition layer equation (9) is problematic). Results are displayed in Fig. 2. Metastability will occur for those values of $\lambda$ such that $\rho_\lambda^+ = \rho_\lambda^-$. Fig. 2(c) ($R(x) = \cos(x)$) shows that no solution was found such that $\rho_\lambda^+ = \rho_\lambda^-$, likewise the constant delay case (Fig. 2(a)), indicating that introducing a state dependent delay may not make up for the lack of symmetry of the feedback function $f$. Nevertheless, Fig. 2(b) ($R(x) = x$) and Fig. 2(d) ($R(x) = \frac{1}{2}x(1+x)$) show that, in these cases, adding state dependence to the delay has resulted in metastability. The solution such that $\rho_{\lambda_c}^+ = \rho_{\lambda_c}^-$ is $\lambda_c \approx 0.5$ in the case $R(x) = x$ (see Fig. 2(b)), and $\lambda_c \approx 1.1$ in the case $R(x) = \frac{1}{2}x(1 + x)$ (see Fig. 2(d)).

## 5. Numerical simulations of Eq. (1)

To corroborate the characterization of metastable state-dependent DITOs obtained in Sections 3 and 4, we have carried a numerical investigation of Eq. (1), thus completing the analysis of the transient dynamics.

We shall present the results of numerical solutions of Eq. (1), for functions $f$ satisfying MNFH and MPFH, and delay functions of the form $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$. We have used $\eta(\epsilon) = \epsilon^\alpha$ with $\alpha = 0$ ($\eta(0) \neq 0$), $\alpha = 1$ ($\eta(0) = 0, \eta'(0) > 0$), $\alpha = \frac{1}{2}$ ($\eta(0) = 0, \eta'(0) = \infty$) and $\alpha > 1$ ($\eta(0) = 0, \eta'(0) = 0$). As for $R(x)$, we have considered the following cases: $R(x) = x$, $R(x) = x^2$, $R(x) = \frac{1}{2}x(1 + x)$, $R(x) = \sin(x)$, and $R(x) = \cos(x)$.

For monotone positive feedback, with constant delay, metastability requires an extra symmetry condition such as the feedback function being odd, so in this case we have used $f(x) = \frac{1}{2}\arctan(5x)$. For negative feedback case we have used both symmetric $f(x) = -\frac{1}{2}\arctan(5x)$ and non-symmetric $f(x) = -\frac{1}{2}\arctan(5(x + 0.05)) + \frac{1}{2}\arctan(0.25)$. The results are qualitatively the same for these two functions, so we shall only show the results for the symmetric negative feedback function.

In the case $\eta(0) = 0$ and $\eta'(0) = 0$ ($\alpha > 1$), the observations are qualitatively the same. As $\alpha$ increases, the results are closer and closer to those observed in the constant delay case.

The numerical results were checked using first and second order numerical schemes, using time steps $dt = 5.10^{-5}$ and $dt = 2.10^{-6}$, and with linear interpolation for the state-dependent delay function. Simulations of solutions of Eq. (1) have also been checked using the RADAR-V package in Fortran. The range of $\epsilon$ values we investigated is $\epsilon \in [0.01, 0.1]$. We have used the same initial condition on $t \in [-2, 0]$ for all simulations. Metastability is checked by tracking the zeros of the solutions. In the positive feedback case, we say that transient oscillations end when the last pair of zeros of the solution disappears. In the negative feedback case, we say that transient oscillations end when the solution has at most one pair of zeros in any interval of length two (called "slow oscillations").

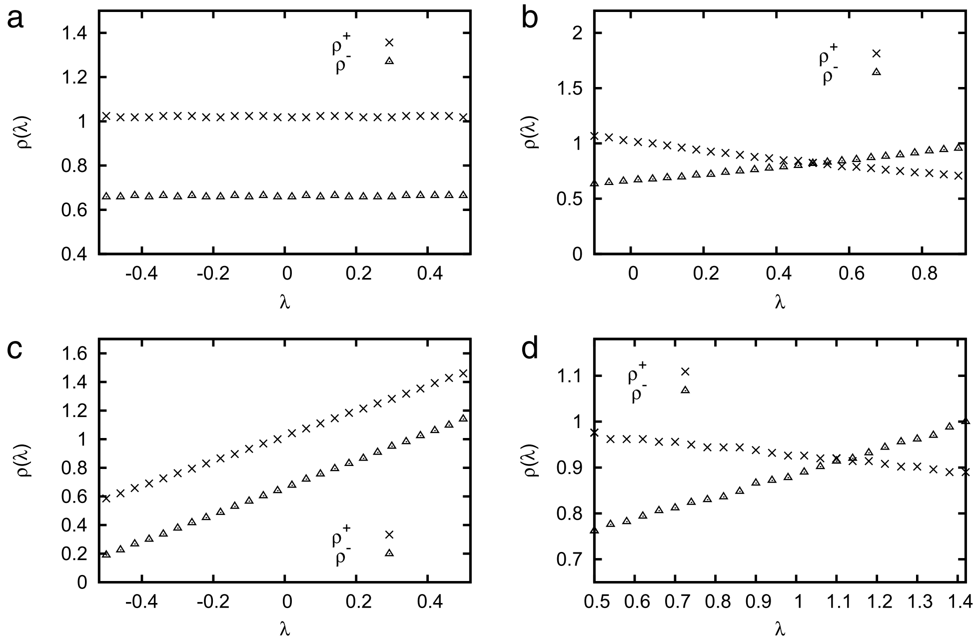


**Fig. 2.** Drift speeds $\rho^{\pm}$ as a function of $\lambda$ for non-symmetric positive feedback function $f(x) = \frac{1}{2}\arctan(5(x+0.05)) - \frac{1}{2}\arctan(0.25)$, and delay function $R(x)$: (a) constant delay case $R(x) = 0$; (b) $R(x) = x$; (c) $R(x) = \cos(x)$; and (d) $R(x) = \frac{1}{2}x(1+x)$. (See Appendix B.1 for details on numerical methods and parameters value used.)

In the following, in order to show whether the transient oscillations time $T_\epsilon$ is of order $\exp(\frac{c}{\epsilon})$, we plot $\epsilon$ Vs $\epsilon\log(T_\epsilon)$. If $y(\epsilon) = \epsilon\log(T_\epsilon)$ satisfies $y(0) > 0$, then $T_\epsilon = e^{\frac{c}{\epsilon}(1+o(1))}$, meaning that oscillatory transients are metastable. On the other hand, if $y(\epsilon) = \epsilon\log(T_\epsilon)$ satisfies $y(0) = 0$, then $T_\epsilon = e^{o\left(\frac{1}{\epsilon}\right)}$, meaning that oscillatory transients are not metastable.

### 5.1. Case $\eta(0) \neq 0$

The key numerical observation in this section is that when $\eta(0) \neq 0$, even when $\epsilon$ is very small, transient oscillations do not last for an exponentially long time. In other words, metastability is precluded.

The results are displayed in Fig. 3, monotone positive feedback on the left panels, and monotone negative feedback on the right panels. The top panels in Fig. 3 exhibit the solution profile for $\epsilon = 0.1, 0.01, 0.001$, and the bottom panels display $\epsilon$ Vs $\epsilon\log(T_\epsilon)$.

The solution profile displayed in Fig. 3(top panels) show that when $\eta(0) \neq 0$, for both positive (top-left panel) and negative (top-right panel) feedbacks, the oscillations are not square-wave-like even for very small $\epsilon$. This contrasts with what happens for constant delay. However, it is consistent with what is known for negative feedback state-dependent DDEs. In fact, as $\epsilon$ tends to zero, profiles take on to the limit profile shapes described by Mallet-Paret and Nussbaum for singularly perturbed state-dependent delays with negative feedback [34–36].

The bottom panels of Fig. 3 show that when $\eta(0) \neq 0$, oscillatory transient duration grows slowly when $\epsilon$ converges to zero. For positive feedback (Fig. 3(c)), when $R(x) = \frac{1}{2}x(1+x)$ (curve $\times$), DITOs' duration never exceeds a few units of time for $\epsilon > 0.001$; for $R(x) = \cos(x)$ (curve $+$) DITOs' duration tends to $+\infty$ but it does not grow as $e^{\frac{c}{\epsilon}}$ when $\epsilon \to 0$ (the function $y(\epsilon) = \epsilon\log(T)$ satisfies $y(\epsilon) \to 0$ when $\epsilon \to 0$), and they are not metastable in this sense. For negative feedback (Fig. 3(d)) the DITOs' duration does not grow as $e^{\frac{c}{\epsilon}}$ when $\epsilon \to 0$, meaning that DITOs are not metastable. Moreover, we can see that in the negative feedback case the DITOs' duration depends very little on $R(x)$, the curves $\epsilon$ Vs $\epsilon\log(T_\epsilon)$ being almost identical for $R(x) = x$ and $R(x) = \frac{1}{2}x(1+x)$ (see Fig. 3(d)).

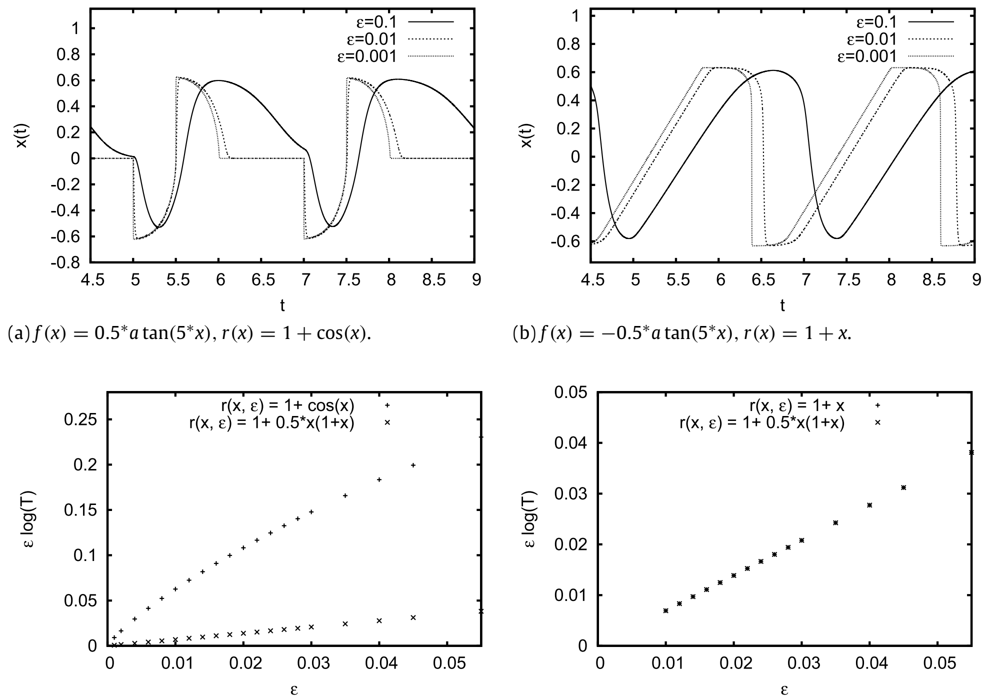
### 5.2. Case $\eta(0) = 0$ with $0 < \eta'(0) < +\infty$

In this case, in addition to the Hopf bifurcation theorem, Cooke's rescaling argument [44] applies to Eq. (1) and one expects that the rapidly oscillating solutions have large amplitude when $\epsilon$ tends to zero. We numerically observed that metastable oscillations are almost $1 + \epsilon\rho$ periodic (the zeros drift-speed $\rho$ depends on the feedback function $f$ and the delay function $r$), and that the constants $\rho$ are coherent with the corresponding constants in Section 4.1. This period estimate is crucial for obtaining the transition layer equation for the state-dependent DDE (see Section 4, Eq. (6)). Here we have taken $\eta(\epsilon) = \epsilon$, so that $r(x, \epsilon) = 1 + \epsilon R(x)$. Metastability is expected for any delay function $R$ and $f$ satisfying MNFH (symmetric and non-symmetric), while for $f$ symmetric satisfying MPFH, the delay function $R$ cannot be arbitrary and must satisfy an extra symmetry assumption, such as being even.

Fig. 4 displays the numerical results for Eq. (1) when $\eta(\epsilon) = \epsilon$, under MPFH (panels on the left) and under MNFH (panels on the right). The top and middle panels of Fig. 4 display the DITOs' profiles for $R(x) = \cos(x)$ and a few $\epsilon$ values. The bottom panels of Fig. 4 display the transient duration ($\epsilon$ Vs $\epsilon\log(T_\epsilon)$) for $R(x) = 0$ (constant delay), $R(x) = \cos(x)$, $R(x) = x$, $R(x) =$, $R(x) = \frac{1}{2}x(1+x)$, $R(x) = x$.

The top panels of Fig. 4 show that for both, positive and negative feedback cases, the oscillatory solutions have a square-wave-like shape when $\epsilon$ goes to zero, likewise the constant delay case. Fig. 4(a) for positive feedback shows that, as $\epsilon \to 0$, the square-wave-like solution has period $1 + c\epsilon$ (at the first order). Fig. 4(b) for negative feedback shows that, as $\epsilon \to 0$, the square-wave-like solution has period $2 + c'\epsilon$ (at first order).

The middle panels in Fig. 4 display the square-wave-like oscillation after rescaling the time, $s = \frac{t}{\epsilon}$. The time-rescaled profiles displayed in the middle panels of Fig. 4 show convergence to a limit profile when $\epsilon$ converges to zero, for both positive and negative feedbacks, in agreement with the results in Section 3. As explained in Section 3, after the appropriate time rescaling $s = \frac{t}{\epsilon}$, the square wave shape converges to some limit profile, which is the solution of the transition layer problem (6). Convergence to




(a) $f(x) = 0.5{*}a\tan(5{*}x)$, $r(x) = 1 + \cos(x)$.

(b) $f(x) = -0.5{*}a\tan(5{*}x)$, $r(x) = 1 + x$.

(c) $f(x) = 0.5{*}a\tan(5{*}x)$.

(d) $f(x) = -0.5{*}a\tan(5{*}x)$.

**Fig. 3.** Profile of solutions of Eq. (1) (top panels), and oscillatory transient duration (bottom panels), when $\eta(0) \neq 0$, with delay $r(x, \epsilon) = 1 + R(x)$, for positive feedback $f(x) = \frac{1}{2}\arctan(5x)$ (panels on the left) and negative feedback $f(x) = -\frac{1}{2}\arctan(5x)$ (panels on the right). Top panels: solution profile for $\epsilon = 0.1, 0.01, 0.001$, (a) positive feedback with $R(x) = \cos(x)$, (b) negative feedback with $R(x) = x$. (c) $\epsilon$ Vs $\epsilon \log(T_\epsilon)$ for positive feedback, and delays $R(x) = \cos(x)$ (curve +), $R(x) = \frac{1}{2}x(1 + x)$ (curve ×); (d) $\epsilon$ Vs $\epsilon \log(T_\epsilon)$ for negative feedback, and delays $R(x) = x$ (curve +), $R(x) = \frac{1}{2}x(1 + x)$ (curve ×).

transition layer profiles occurs for both increasing and decreasing rescaled jumps, for both positive and negative feedbacks.

An approximation of the unknown constant $\rho = \rho^{\pm}$ in problem (6) can be obtained from the numerical approximate period of metastable oscillations: $T \approx 1+\epsilon\rho$. We have checked that the limit profiles agree with the solutions of the transition layer equations obtained as attractive fixed point of an appropriate operator $\mathcal{T}$, up to numerical error of order $O(dt/\epsilon)$, where $dt$ is the discretization parameter (see Appendix B). Due to regularity properties of the operator $\mathcal{T}$, this implies that the corresponding constants $\rho$ also agree at the same order.

Fig. 4(f) shows that in the negative feedback case, metastable oscillatory patterns are observed regardless of the delay function $R$. For negative feedback with constant delay ($R(x) = 0$), the curve + in Fig. 4(f) shows that $T(\epsilon) \sim \exp(\frac{c}{\epsilon})$, for some non-zero constant $c$, as expected. This same panel (f) shows that, for state-dependent delay, $\epsilon \log(T(\epsilon)) \approx c_1 + c_2\epsilon + o(\epsilon)$, implying that $T(\epsilon) = \exp(\frac{c_1}{\epsilon})\exp(c_2)(1 + o(1))$. The value of constants $c_1$ and $c_2$ depends on the delay function $R$, but in all cases $c_1 > 0$ implying that the state-dependent DITOs are metastable.

State-dependent DITOs are metastable if and only if the function $y(\epsilon) = \epsilon \log(T)$ has a non-zero limit when $\epsilon \to 0$. Fig. 4(e) shows that in the positive feedback case, the DITOs are metastable only for odd functions $f$ and even delay functions $R(x)$. The curve + in Fig. 4(e) shows that, for constant delay, $T(\epsilon) \sim \exp(\frac{c}{\epsilon})$, for some non-zero constant $c$, as expected. In the case of state-dependent delay, when the delay function is even, the curves for $R(x) = \cos(x)$, $R(x) = x^2$ in Fig. 4(e) show that $\epsilon \log(T(\epsilon)) \approx c_1 + c_2\epsilon + o(\epsilon)$, meaning that $T = \exp(\frac{c_1}{\epsilon})\exp(c_2)(1 + o(1))$ so that the state-dependent DITOs are metastable. In contrast, when $R(x)$ is not even (cases $R(x) = \sin(x)$, $R(x) = x + \frac{1}{2}x^2$), Fig. 4(e) shows that $\epsilon \log(T(\epsilon)) \underset{\epsilon \to 0}{\longrightarrow} 0$ so that $T(\epsilon)$ is not of order $\exp(\frac{c}{\epsilon})$, implying that the state-dependent DITOs are not metastable.

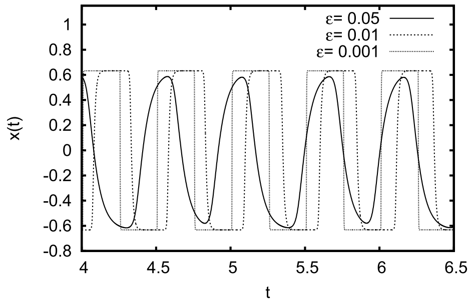
### 5.3. Metastability induced by state-dependent delay

As already emphasized, for $f$ satisfying MPFH, DDE (1) with constant delay exhibits metastability only if $f$ is odd. The constants $\rho^+$ and $\rho^-$ that solve the transition layer problem are equal in the case of odd monotone positive feedback $f$ with constant delay, implying metastability. When $f$ satisfying MPFH is not symmetric, the condition $\rho^+ = \rho^-$ does not hold for constant delays, and DITOs are not metastable. Nevertheless, in Section 4.3 we have shown that for non-symmetric $f$ satisfying MPFH, and state-dependent delay $r(x, \epsilon) = 1 + \lambda\epsilon R(x)$, for a critical value $\lambda = \lambda_c$, the transition layer equations (9) do have solutions such as $\rho^+ = \rho$, provided that $R(x)$ is not even (see Fig. 2). Therefore, metastability has been induced by introducing the appropriate state-dependence to DDE (1) with non-symmetric monotone positive feedback.

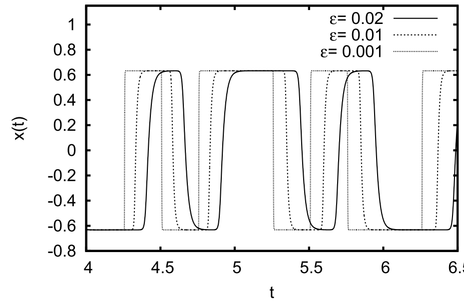
In this section, we present the numerical solutions of Eq. (1) for the same case analyzed in Section 4.3: $f(x) = \frac{1}{2}\arctan(5(x + 0.05)) - \frac{1}{2}\arctan(0.25)$, with state-dependent delay $r(x, \epsilon) = 1 + \lambda\epsilon R(x)$, $R(x)$ not even. Panels (a) and (c) of Fig. 5 display the results for the case $R(x) = \frac{1}{2}\epsilon x(1 + x)$ (corresponding to Fig. 2(d)); panels (b) and (d) display the results for $R(x) = x$ (corresponding to Fig. 2(b)).

The fronts and the values of $\rho^+$ and $\rho^-$ change continuously with $\lambda$. In other words, simulations over fixed durations of DDE solutions for $\lambda$ in the vicinity of $\lambda_c$ are similar because for $\lambda$ close to $\lambda_c$, the difference $\rho^+ - \rho^-$ is small and the DITOs are long lasting. This similarity not withstanding, the transient regime durations scale differently with $\epsilon$ at $\lambda_c$ and nearby values $\lambda$. Only at $\lambda_c$ the system becomes metastable in the sense that the DITOs last for exponentially long times. This difference is illustrated in the figures that show transient regime duration at fixed $\lambda$ for various $\epsilon$ and the reverse, i.e. at fixed $\epsilon$ for various values of $\lambda$.

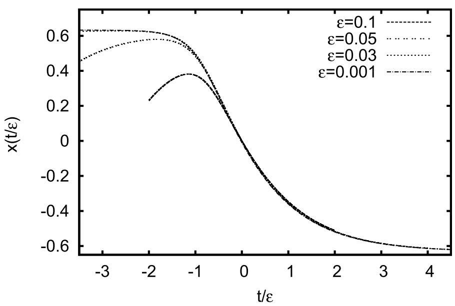
The panels (a) and (b) of Fig. 5 display $\epsilon$ Vs $1/\log(T(\epsilon))$, for three values of $\lambda$, and we can see that when the parameter $\lambda$ is larger or



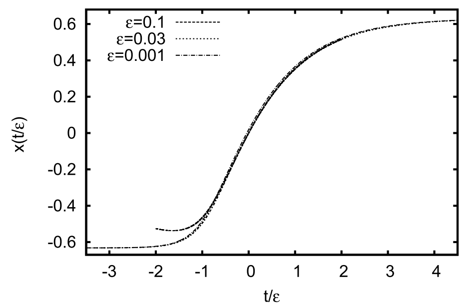

(a) $f(x) = 0.5^* a\tan(5^*x)$, $r(x, \varepsilon) = 1 + \varepsilon\cos(x)$.



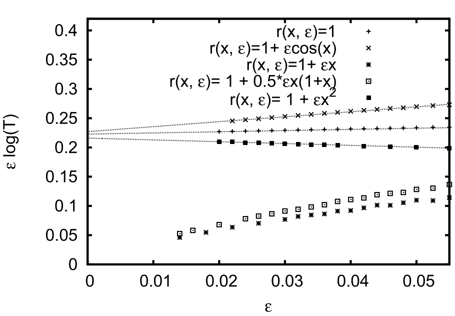

(b) $f(x) = -0.5^* a\tan(5^*x)$, $r(x, \varepsilon) = 1 + \varepsilon\cos(x)$.



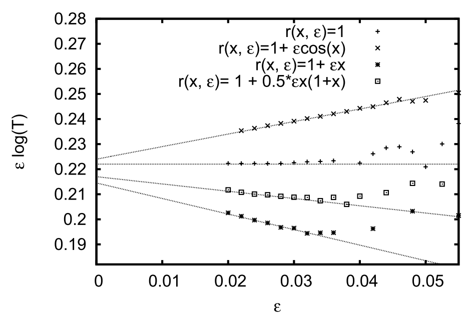

(c) $f(x) = 0.5^* a\tan(5^*x)$, $r(x, \varepsilon) = 1 + \varepsilon\cos(x)$.




(d) $f(x) = -0.5^* a\tan(5^*x)$, $r(x, \varepsilon) = 1 + \varepsilon\cos(x)$.




(e) $f(x) = 0.5^* a\tan(5^*x)$.




(f) $f(x) = -0.5^* a\tan(5^*x)$.

**Fig. 4.** Profile of solutions of (1) (top and middle panels), and oscillatory transient duration (bottom panels), when $\eta(\epsilon) = \epsilon$, for positive feedback $f(x) = \frac{1}{2}\arctan(5x)$ (panels on the left), and negative feedback $f(x) = -\frac{1}{2}\arctan(5x)$ (panels on the right). *Top panels:* solution profile for $R(x) = \cos(x)$ and decreasing $\epsilon$. *Middle panels:* descending and ascending jumps obtained by the time scale $s = \frac{t}{\epsilon}$ of the solutions profile for $R(x) = \cos(x)$, varying $\epsilon$: (c) positive feedback; (d) negative feedback. The curves for $\epsilon = 0.03$ and $\epsilon = 0.001$ are too close to be distinguished. *Bottom panels:* (e) transient duration for positive feedback with delays $R(x) = 0$, $R(x) = \cos(x)$, $R(x) = x$, $R(x) = \frac{1}{2}x(1 + x)$, $R(x) = x$; (f) transient duration for negative feedback with delays $R(x) = 0$, $R(x) = \cos(x)$, $R(x) = x$, $R(x) = \frac{1}{2}x(1 + x)$.

smaller than the critical value $\lambda_c$, the curves $y(\epsilon) = 1/\log(T_\epsilon)$ are very steep when $\epsilon \to 0$, indicating that DITOs are not metastable in those cases. In contrast, when $\lambda = \lambda_c$, the curves $y(\epsilon)$ have a bounded slope as $\epsilon \to 0$, indicating that DITOs are metastable. We have obtained $\lambda_c \approx 1.05$ for the case $R(x) = \frac{1}{2}\epsilon x(1 + x)$ (Fig. 5(a)), and $\lambda_c \approx 0.55$ for the case $R(x) = x$ (Fig. 5(b)). The difference between these values of $\lambda_c$, and those values of $\lambda_c$ found in Section 4.3, is smaller than $10^{-1}$, which is of the order of numerical precision for these parameters.

The panels (c) and (d) of Fig. 5 display $\lambda$ Vs $\epsilon\log(T_\epsilon)$, for three values of $\epsilon$, and we can see that when $\epsilon$ is fixed, there is a unique value $\lambda_\epsilon$ such that DITOs' duration is maximal, and it converges to the critical value $\lambda_c$ as $\epsilon$ decreases.

The results in this subsection agree with the results in Section 4.3, confirming the possibility of state-dependence of the delay inducing metastability when the constant delay case does not exhibit metastability.

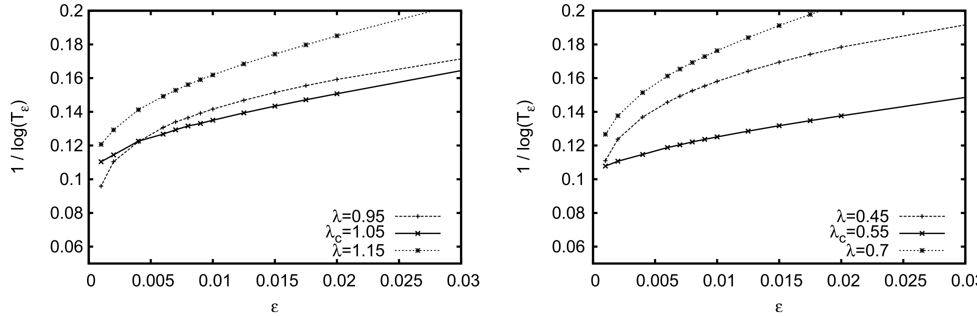
### 5.4. *Case $\eta(0) = 0$ with $\eta'(0) = 0$*

When $\eta'(0) = 0$, likewise the case $0 < \eta'(0) < \infty$, oscillations have a square wave shape, a "period" $T = 1 + \epsilon\rho$, and a rescaled limit transition-layer profile (figures not shown). The corresponding transition layer equation (8) is independent of the function $\eta$, and it is the same transition layer equation as for the constant-delay case $r(x, \epsilon) = 1$.

For given feedback $f$ and delay $R(x)$, if metastability occurs when $0 < \eta'(0) < +\infty$, our numerical investigation indicates that it will also occur for $\eta'(0) = 0$. We have computed the transient duration for the symmetric positive feedback $f(x) = \frac{1}{2}\arctan(5x)$, with delay $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$, $\eta(\epsilon) = \epsilon^\alpha$, $\alpha > 1$.

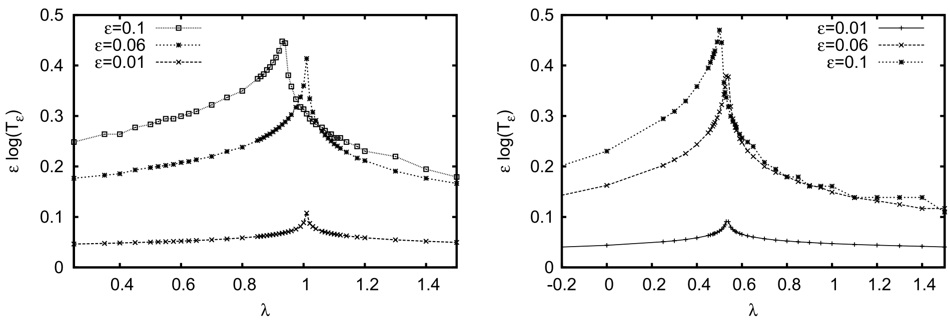
Fig. 6(a) displays $\epsilon$ Vs $\epsilon\log(T_\epsilon)$ for the state-dependent delay $R(x) = \frac{1}{2}x(1 + x)$, and $\alpha = 1.5, 2, 3$. It shows that $\epsilon\log(T)$ converges to zero when $\epsilon \to 0$, regardless of $\alpha$, implying that for symmetric positive feedback the DITOs are not metastable if $R(x)$ is not even.

Fig. 6(b) displays $\epsilon$ Vs $\epsilon\log(T_\epsilon)$ for the state-dependent delay $R(x) = x^2$ (even function), and $\alpha = 2, 3$. It shows that $\epsilon\log(T)$ does not converge to zero when $\epsilon \to 0$, implying that for symmetric positive feedback the DITOs are metastable if $R(x)$ is even. From Fig. 6(b) we can also see that as $\alpha$ increases, the duration $T$ of transient oscillations is closer and closer to the duration of the transient for the constant delay case.




(a) $r(x, \varepsilon) = 1 + 0.5\lambda\varepsilon x(1 + x)$.

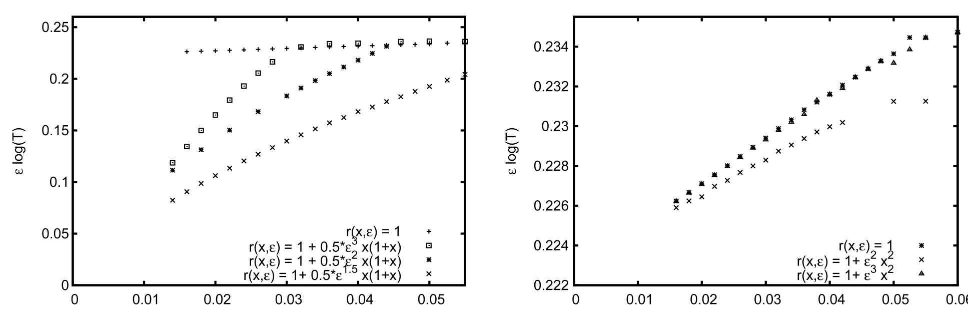
(b) $r(x, \varepsilon) = 1 + \lambda\varepsilon x$.




(c) $r(x, \varepsilon) = 1 + 0.5\varepsilon\lambda x(1 + x)$.

(d) $r(x, \varepsilon) = 1 + \varepsilon\lambda x$.

**Fig. 5.** DITO duration for positive non-symmetric feedback $f(x) = \frac{1}{2}\arctan(5(x + 0.05)) - \frac{1}{2}\arctan(0.25)$, with state-dependent delay $r(x, \epsilon) = 1 + \lambda\epsilon R(x)$. Panels (a) and (c): $R(x) = \frac{1}{2}x(1 + x)$; panels (b) and (d): $R(x) = x$.




(a) $f(x) = 0.5^* a\tan(5^* x)$, Non-even delays.

(b) $f(x) = 0.5^* a\tan(5^* x)$, Even delays.

**Fig. 6.** Oscillatory transient duration for symmetric positive feedback $f(x) = \frac{1}{2}\arctan(5x)$, with constant delay (curve +), and state-dependent delay $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$, $\eta(\epsilon) = \epsilon^\alpha$, $\alpha > 1$: (a) $R(x) = \frac{1}{2}x(1 + x)$, $\alpha = 1.5, 2, 3$; (b) $R(x) = x^2$, $\alpha = 2, 3$.

For $f$ satisfying MNFH, the same result holds: metastable DITOs for the case $0 < \eta'(0) < +\infty$ implies metastable DITOs for the case $\eta'(0) = 0$.

### 5.5. Case $\eta(0) = 0$ with $\eta'(0) = +\infty$

When $\eta(0) = 0$, with $\eta'(0) = +\infty$ oscillations have a square-wave-like shape and an approximate period $T = 1 + \epsilon^\alpha \rho$ when $\epsilon \to 0$ (figures not shown). The usual time rescaling $s = \frac{t}{\epsilon}$ does not give converging profiles, but the scaling $s = \frac{t}{\epsilon^\alpha}$ does. As illustration, we display in Fig. 7 the results for the negative feedback $f(x) = -\frac{1}{2}\arctan(5x)$, with $r = 1 + \epsilon^{1/2}\cos(x)$. At the time scale $t\epsilon^{-1/2}$, we observe the convergence of oscillating solutions jumps to a limit transition layer profile when $\epsilon \to 0$.

Here neither Mallet-Paret and Nussbaum's nor Cooke's argument [44] apply, so that we have no indication on whether rapidly oscillating periodic solutions (which are expected because of Eichmann's Hopf bifurcation theorem) have large or small amplitudes. Numerical simulations have shown that oscillatory transients can

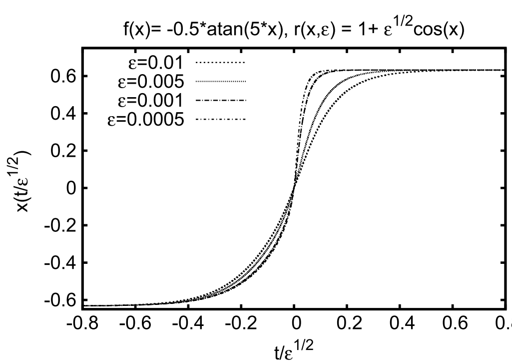



**Fig. 7.** Convergence of rescaled jumps of oscillating solutions in the limit $\epsilon \to 0$, when $\eta(0) = 0$ with $\eta'(0) = +\infty$. We used negative feedback $f(x) = -\frac{1}{2}\arctan(5x)$, with $r = 1 + \epsilon^{1/2}\cos(x)$.

last for a very long time, but due to numerical difficulties it is not clear whether this transient duration is exponential; hence we cannot say whether these long-lasting oscillatory transients are indeed metastable or not.

## 6. Discussion and conclusion

At present time, no global geometric characterization of the organization of the phase portrait of scalar state-dependent DDEs is available. Even basic results, such as the Hopf bifurcation theorem, have only recently been established. However, based on current knowledge and systematic numerical explorations, it is possible as done in Section 3 to conjecture that the phase portraits of DDEs with state-dependent delays have the same geometrical organization as those of constant delays. Furthermore, it is possible to discuss the putative occurrence of metastability based upon the available information regarding rapidly oscillating periodic solutions, and educated guesses supported by numerical investigations.

In this paper we have shown that metastable oscillating solutions can exist in singularly perturbed DDEs with state-dependent delays of type (1). Based both on mathematical and numerical results, we have been able to link the properties of Eq. (1), and its solutions, to the occurrence of oscillatory metastable transients.

Such metastable transients were never observed in numerical explorations when the state-dependent delay $r = r(x)$ does not depend on the singular parameter $\epsilon$. Considering delays of the form $r(x, \epsilon) = 1 + \eta(\epsilon)R(x)$, we found that the scaling parameter value $\eta(\epsilon)$ is crucial for the existence of metastable transients. When $\eta(0) \neq 0$ exponentially long-lasting oscillatory transients were never observed, even for very small $\epsilon$. When $\eta(0) = 0$ and $\eta'(0) = \infty$ long-lasting oscillatory transients can be observed, but due to numerical difficulties one cannot conclude whether these transients are indeed metastable or not. For $\eta(\epsilon) = \epsilon^{\alpha}$, the larger the $\alpha$ is, the longer transient oscillations will last. When $\eta(0) = 0$ and $0 \leq \eta'(0) < \infty$ metastable oscillatory transients can always be observed for monotone negative feedback, while for monotone positive feedback some symmetry condition must be satisfied by the feedback function $f$ and the delay function $R$.

There are two main tools to analyze metastability phenomena. With a geometric approach, one can look for the existence of a global attractor containing a "cascade" of unstable periodic orbits and heteroclinic connections between them, and with a dynamical approach one can investigate transition layer equations that describe the asymptotic shape of the oscillations when $\epsilon$ converges to zero.

We have shown that, with few hypotheses on the functions $f$ and $r$, a Hopf bifurcation theorem of Eichmann applies to Eq. (1). This implies the existence of a sequence of Hopf bifurcations as $\epsilon$ converges to zero, meaning that a global attractor with the previously described structure might exist in many cases. Nevertheless, metastable oscillations cannot be observed in general. Our numerical investigation has shown that when the delay function $r$ does not converge to a constant as $\epsilon$ converges to zero (see Fig. 3 in Section 5.1), the amplitude of periodic solutions has to converge to zero as the period converges to zero, and metastable oscillations were not observed. This suggests that not only rapidly oscillating but also large amplitude periodic solutions are needed to support metastability.

Furthermore, even when the delay function $r$ converges to a constant as $\epsilon$ tends to zero (Section 5, cases $\eta(0) = 0$), the state-dependent DITOs are metastable only in those cases where there exist transition layer equations similar to those found for DDEs of type (1) with constant delay ($\alpha \geq 1$, Sections 5.2 and 5.4). This suggests that metastable state-dependent DITOs cannot exist unless the oscillations have a limiting shape determined by heteroclinic solutions to a transition layer problem of the form (6), as $\epsilon$ converges to zero.

Our numerical analysis of the transient oscillations in DDE (1) induced by state-dependent delay of the form $r(x) = 1 + \eta(\epsilon)R(x)$ has revealed that for $f$ satisfying MNFH, metastable state-dependent DITOs exist in the same way as for the constant delay case, while for $f$ satisfying MPFH, metastable state dependent DITOs exist if and only if $f$ and $R$ satisfy some symmetry conditions. If $f$ is an odd function satisfying MPFH (sufficient condition for metastability in the case of constant delay), then metastable state-dependent DITOs exist when the delay $R(x)$ is an even function. We also have shown that by adding state dependence to the delay, it is possible to obtain metastability for $f$ satisfying MPFH for which the constant delay transient oscillations are not metastable (see Sections 4.3 and 5.3).

An important contribution of this work has been the introduction of a novel class of transition layer equations associated with state dependent delays. From our numerical investigations, we claim that these equations capture two essential aspects of the dynamics of DDEs with state-dependent delays. The first is the shape of the transient oscillations as the parameter $\epsilon$ becomes small. The second is the drift of the oscillations. Our focus has been on monotone feedbacks; nevertheless the transition layer equations we have introduced remain valid for non monotone feedbacks as well. That square-wave like solutions have been reported in other classes of singularly perturbed DDEs (albeit with constant delay) [7] further suggests that similar analyses should hold for broader equations than Eq. (1). Our paper paves the way to investigate the dynamics of DDEs with such feedbacks and state-dependent delays, through the novel transition layer equations.

### Acknowledgments

The authors thank Denis Mestivier for his help in handling RADAR-V codes.

CGR is partially supported by CNPq (Brazil) grant 305089/2009-9, CPM is partially supported by CNPq (Brazil) grant 311022/2009-0.

## Appendix A. Proof of Theorem 1

Theorem 1 is an immediate consequence of a "Hopf bifurcation theorem" proven by Eichmann (Eichmann's Ph.D thesis [41] p 81). For $\epsilon \in ]0; 1[$ and $y \in C^1([-M, 0], \mathbb{R})$, let $g$ be the function $g(\epsilon, y) = \frac{1}{\epsilon}(-y(0) + f(y(-r(\epsilon, y))))$, such that Eq. (1) is equivalent to

$$x'(t) = g(\epsilon, x_t), \tag{10}$$

with the notation $x_t(s) = x(t + s)$ for all $s \in [-M, 0]$.

The Hopf bifurcation theorem of Eichmann has three first order derivatives hypotheses $(H_1)$ $(H_2)$ and $(H_3)$, three second order derivatives hypotheses $(H_4)$ $(H_5)$ $(H_6)$ and three spectral hypothesis $(L_1)$, $(L_2)$ and $(L_3)$. For convenience of the reader we state these hypotheses here before proving that they are satisfied by Eqs. (1) and (2).

Suppose that there is an open subset $U$ of $C^1([-M, 0], \mathbb{R})$ such that

- $(H_1)$ the mapping $g : ]0, 1[\times U \to \mathbb{R}$ is continuously differentiable,
- $(H_2)$ for any $(\epsilon, x) \in ]0, 1[\times U$ the second partial derivative $D_x g(\epsilon, x)$ can be extended to a linear continuous map

  $$D_x g(\epsilon, x) : C^0([-M, 0], \mathbb{R}) \to \mathbb{R};$$
- $(H_3)$ the (extended) mapping

  $$]0, 1[\times U \times C^0([-M, 0], \mathbb{R}) \to \mathbb{R}$$
  $$(\epsilon, x, y) \mapsto D_x g(\epsilon, x)y,$$

  is continuous;

- $(H_4)$ the mapping $g\ :]0,1[\times\left(U\cap C^2\right)\to\mathbb{R}$ is twice continuously differentiable;
- $(H_5)$ for any $(\epsilon,x)\ \in]0,1[\times U\cap C^2$ the second order partial derivative $D^2_x g(\epsilon,x)$ can be extended to a bilinear continuous map

  $D^2_x g(\epsilon,x):\ C^1([-M,0],\mathbb{R})\times C^1([-M,0],\mathbb{R})\to\mathbb{R};$
- $(H_6)$ the (extended) mapping

  $]0,1[\times U\times C^1([-M,0],\mathbb{R})\times C^1([-M,0],\mathbb{R})\to\mathbb{R}$
  $(\epsilon,x,y,z)\mapsto D^2_x g(\epsilon,x)(y,z)$

  is continuous, and

  $]0,1[\times U\times C^1([-M,0],\mathbb{R})\to\mathcal{L}(C^2,\mathbb{R})$
  $(\epsilon,x,y)\mapsto D^2_x g(\epsilon,x)(y,\cdot),$

  is continuous (where $\mathcal{L}(C^2,\mathbb{R})$ is the space of linear functionals from $C^2([-M,0],\mathbb{R})$ to $\mathbb{R}$).

Suppose that $g(\epsilon,0)=0$, for any $\epsilon\in]0,1[$, and let $A(\epsilon)$ be the generator of the strongly continuous semigroup on $C^0([-M,0],\mathbb{R})$ generated by the linearized equation

$$y'_t=D_x g(\epsilon,0)y_t.$$

Suppose that there is a $\epsilon^*\ \in]0,1[$ and some open interval $]\epsilon^*-\eta,\epsilon^*+\eta[\subset]0,1[$ such that

- $(L_1)$ for any $\epsilon\in]\epsilon^*-\eta,\epsilon^*+\eta[$, there is a simple eigenvalue $\lambda(\epsilon)$ of $A(\epsilon)$, such that the mapping $\epsilon\mapsto\lambda(\epsilon)$ is $C^1(]\epsilon^*-\eta,\epsilon^*+\eta[,\mathbb{C})$,
- $(L_2)$ the eigenvalue $\lambda(\epsilon)$ crosses the imaginary axis at $\epsilon^*$ : $\Re(\lambda(\epsilon^*))=0$ and $\Im(\lambda(\epsilon^*))=\omega_0>0$, and $\frac{d\lambda}{d\epsilon}(\epsilon^*)\neq0$,
- $(L_3)$ and for any $k\in\mathbb{Z}-\{-1,1\}$, $\nu=ik\omega_0$ is not an eigenvalue of $A(\epsilon^*)$.

Now we show that the hypotheses of Eichmann's theorem are satisfied so that it immediately implies Theorem 1. We start by proving the following Propositions 3 and 4.

**Proposition 3.** *Let $U$ be an open subset of $C^1([-M,0],\mathbb{R})$. Suppose $f\ :\mathbb{R}\to\mathbb{R}$ is $C^1$ and $r\ :]0;1[\times U\to[0,M]$ is $C^1$. Suppose that for any $(\epsilon,x_t)\in]0;1[\times U$, $\frac{\partial r}{\partial y}:]0;1[\times C^1([-M,0],\mathbb{R})\to\mathbb{R}$ can be extended to a continuous linear map $]0;1[\times C^0([-M,0],\mathbb{R})\to\mathbb{R}$, and that $(\epsilon,x_t,h_t)\to\frac{\partial r}{\partial y}(\epsilon,x_t)\cdot h_t$ is continuous $]0;1[\times U\times C^0([-M,0],\mathbb{R})\to\mathbb{R}$.*

*Then the regularity hypotheses $(H_1)$, $(H_2)$ and $(H_3)$ of Eichmann's Hopf Bifurcation Theorem hold.*

Here, $\frac{\partial}{\partial\epsilon}$ and $\frac{\partial}{\partial y}$ denote Frechet derivatives.

**Proof.** One finds

$$\frac{\partial g}{\partial\epsilon}(\epsilon,y)=\frac{-1}{\epsilon}g(\epsilon,y)-\frac{1}{\epsilon}\times\left(\frac{\partial r}{\partial\epsilon}(\epsilon,y)y'(-r(\epsilon,y))f'(y(-r(\epsilon,y)))\right)$$

$$\frac{\partial g}{\partial y}(\epsilon,x)\cdot h=\frac{1}{\epsilon}\left[-h(0)+\left(-x'(-r(\epsilon,x))\frac{\partial r}{\partial y}\times(\epsilon,x)\cdot h+h(-r(\epsilon,x))\right)f'(x(-r(\epsilon,x)))\right].$$

This shows that $\frac{\partial g}{\partial\epsilon}:]0,1[\times U\to\mathbb{R}$ and $\frac{\partial g}{\partial y}:]0,1[\times U\to\mathbb{R}$ are $C^0$, so that $g\in C^1$, and $(H_1)$ holds. Using the hypotheses made on $r$ in the Theorem 1, the formula shows that $(H_2)$ and $(H_3)$ also hold. □

**Proposition 4.** *Suppose that the hypotheses of Proposition 3 holds. Suppose additionally that $f\ :\mathbb{R}\to\mathbb{R}$ is $C^2$ and that $r\ :]0,1[\times\left(U\cap C^2([-M,0],\mathbb{R})\right)\to[0,M]\subset\mathbb{R}$ is $C^2$. Suppose that for any $(\epsilon,x)\ \in]0,1[\times U$, the function $\frac{\partial^2 r}{\partial y^2}(\epsilon,x)$ has a bilinear continuous extension to $C^1([-M,0],\mathbb{R})\times C^1([-M,0],\mathbb{R})\to\mathbb{R}$, which depends continuously on $(\epsilon,x)\in]0,1[\times(U\cap C^2)$. And suppose that $(\epsilon,x,h)\mapsto\frac{\partial^2 r}{\partial y^2}(\epsilon,x)(h,\cdot)$ is continuous $]0,1[\times(U\cap C^2)\times C^1\to\mathcal{L}(C^2,\mathbb{R})$.*

*Then regularity hypotheses $(H_4)$, $(H_5)$ and $(H_6)$ of Eichmann's Hopf Bifurcation Theorem are true.*

**Proof.** This proposition is a consequence of the following formulas for second order derivatives of function $g$.

$$-\frac{\partial}{\partial\epsilon}\left(\epsilon\frac{\partial g}{\partial\epsilon}+g\right)(\epsilon,x)=\frac{\partial^2 r}{\partial\epsilon^2}(\epsilon,x)x'(-r(\epsilon,x))\times f'(x(-r(\epsilon,x)))-x''(-r(\epsilon,x))\left(\frac{\partial r}{\partial\epsilon}(\epsilon,x)\right)^2f'(x(-r(\epsilon,x)))-\left(x'(-r(\epsilon,x))\frac{\partial r}{\partial\epsilon}(\epsilon,x)\right)^2f''(x(-r(\epsilon,x)))$$

and

$$-\frac{\partial}{\partial y}\left(\epsilon\frac{\partial g}{\partial\epsilon}+g\right)(\epsilon,x)\cdot h=\left(\frac{\partial^2 r}{\partial y\partial\epsilon}(\epsilon,x)\cdot h\right)x'(-r(\epsilon,x))\times f'(x(-r(\epsilon,x)))-\frac{\partial r}{\partial\epsilon}(\epsilon,x)x''(-r(\epsilon,x))\left(\frac{\partial r}{\partial y}(\epsilon,x)\cdot h\right)\times f'(x(-r(\epsilon,x)))+\frac{\partial r}{\partial\epsilon}(\epsilon,x)x'(-r(\epsilon,x))f''(x(-r(\epsilon,x)))\times\left[-x'(-r(\epsilon,x))\left(\frac{\partial r}{\partial y}(\epsilon,x)\cdot h\right)+h(-r(\epsilon,x))\right]$$

and

$$\frac{\partial}{\partial\epsilon}\left(\epsilon\frac{\partial g}{\partial y}(\epsilon,x)\cdot h\right)=f'(x(-r(\epsilon,x)))\left[x''(-r(\epsilon,x))\frac{\partial r}{\partial\epsilon}(\epsilon,x)\left(\frac{\partial r}{\partial y}(\epsilon,x)\cdot h\right)\right]+f'(x(-r(\epsilon,x)))\left[-x'(-r(\epsilon,x))\times\frac{\partial^2 r}{\partial\epsilon\partial y}(\epsilon,x)\cdot h-h'(-r(\epsilon,x))\frac{\partial r}{\partial\epsilon}(\epsilon,x)\right]-\left[-x'(-r(\epsilon,x))\frac{\partial r}{\partial y}(\epsilon,x)\cdot h+h(-r(\epsilon,x))\right]\times f''(x(-r(\epsilon,x)))x'(-r(\epsilon,x))\frac{\partial r}{\partial\epsilon}(\epsilon,x)$$

and

$$\frac{\partial}{\partial y}\left(\epsilon\frac{\partial g}{\partial y}\right)(\epsilon,x)\cdot(h,k)=\left(-\frac{\partial r}{\partial y}(\epsilon,x)\cdot h+h(-r(\epsilon,x))\right)\times\left(-\frac{\partial r}{\partial y}(\epsilon,x)\cdot k+k(-r(\epsilon,x))\right)\times x'(-r(\epsilon,x))f''(x(-r(\epsilon,x)))+f'(x(-r(\epsilon,x)))\left[x''(-r(\epsilon,x))\left(\frac{\partial r}{\partial y}(\epsilon,x)\cdot k\right)\times\left(\frac{\partial r}{\partial y}(\epsilon,x)\cdot h\right)\right]$$

$$+f'(x(-r(\epsilon,x)))\left[-x'(-r(\epsilon,x))\frac{\partial^2 r}{\partial y^2}(\epsilon,x)\cdot(h,k)\right.$$
$$\left.\times\ -h'(-r(\epsilon,x))\frac{\partial r}{\partial y}(\epsilon,x)\cdot k\right].$$

If $r$ and $f$ are $C^2$, then one can check that $g\ :]0,1[\times(U\cap C^2([-M,0],\mathbb{R}))\to\mathbb{R}$ is $C^2$ and this is $(H_4)$. If $\frac{\partial^2 r}{\partial y^2}(\epsilon,x)$ has a continuous extension to $C^1([-M,0],\mathbb{R})\times C^1([-M,0],\mathbb{R})\to\mathbb{R}$, then so does $\frac{\partial^2 g}{\partial y^2}(\epsilon,x)$ and this is $(H_5)$. Since the extension of $\frac{\partial r}{\partial y}(\epsilon,x)$ and $\frac{\partial^2 r}{\partial y^2}(\epsilon,x)$ are continuous in $(\epsilon,x)$ the first part of $(H_6)$ is satisfied. Since $r\ :]0,1[\times U\to\mathbb{R}$ is $C^1$, and due to the last hypothesis on $\frac{\partial^2 r}{\partial y^2}$ in the proposition, the last requirement of $(H_6)$ holds too. □

The hypotheses of Theorem 1: $f$ is $C^2(\mathbb{R},\mathbb{R})$, $r\ \ :]0,1[\times C^0([-M,0],\mathbb{R})\to\mathbb{R}$ is $C^1$, and $r\ :]0,1[\times C^1([-M,0],\mathbb{R})\to\mathbb{R}$ is $C^2$, imply that the regularity hypotheses of Propositions 3 and 4 are verified.

We now turn to the spectral hypotheses of Eichmann's Hopf-bifurcation theorem. We consider the equilibrium $x^*=0$, which satisfies $g(\epsilon,0)=0$ for all $\epsilon\in]0,1[$, and the linearized equation at $x^*: y'(t)=\frac{\partial g}{\partial y}(\epsilon,0)y_t$ with $y_0\in C^1([-M,0],\mathbb{R})$, i.e.

$$\epsilon y'(t)=-y(t)+f'(0)y(t-r_0)$$

where $r_0=r(0)=r(x^*)=1$. The characteristic equation is

$$\begin{cases}1+\epsilon\alpha=f'(0)e^{-\alpha}\cos(\beta)\\ \epsilon\beta=-f'(0)e^{-\alpha}\sin\beta.\end{cases}$$

This is the same characteristic equation for the constant-delay equation, and a standard argument shows that there exists a sequence $\epsilon_k\underset{k\to+\infty}{\longrightarrow}0$ such that for each $\epsilon=\epsilon_k$ the characteristic equation has a single pair of solutions on the imaginary axis $\lambda=\pm i\beta_k$, $\beta_k>0$. This implies in particular that $(L_3)$ of [41] is satisfied.

To check that for each $k$ the eigenvalue $\lambda_k$ can be tracked in a neighborhood of $\epsilon=\epsilon_k$, we use an implicit function theorem. Considering $G\ \ :]0,1[\times\mathbb{R}\times\mathbb{R}^2$ defined by $G(\epsilon,\alpha,\beta)=(1+\epsilon\alpha-f'(0)e^{-\alpha}\cos(\beta);\epsilon\beta+f'(0)e^{-\alpha}\sin(\beta))$, we have $G(\epsilon,\alpha,\beta)=0$ if and only if $\lambda=\alpha+i\beta$ is a characteristic root. We have $G(\epsilon_k,0,\beta_k)=0$ (for any $k$), and the derivative of $G$ with respect to $\alpha$ and $\beta$ at $(\epsilon_k,0,\beta_k)$ is

$$\begin{pmatrix}1+\epsilon_k+f'(0)\cos(\beta_k) & f'(0)\sin(\beta_k)\\ -f'(0)\sin(\beta_k) & \epsilon_k+f'(0)\cos(\beta_k)\end{pmatrix}$$
$$=\begin{pmatrix}2+\epsilon_k & f'(0)\sin(\beta_k)\\ -f'(0)\sin(\beta_k) & 1+\epsilon_k\end{pmatrix}$$

which has a positive determinant and is invertible. Thus (for any $k$) there is an open interval $I_k$ containing $\epsilon_k$ and a $C^1$ function $\nu_k: I_k\to\mathbb{C}$ such that for all $\epsilon\in I_k$, $\nu_k(\epsilon)$ is a characteristic root and $\nu_k(\epsilon_k)=\lambda_k$.

Furthermore, computing the $\epsilon$ derivative of $\nu_k$ at $\epsilon=\epsilon_k$ and using the relations for $\lambda_k$, one finds

$$\begin{cases}(1+\epsilon_k)\alpha'(\epsilon_k)=\epsilon_k\beta_k\beta_k'(\epsilon_k)\\ (1+\epsilon_k)\beta_k'=-(1+\epsilon_k\alpha_k'(\epsilon_k))\beta_k,\end{cases}$$

which gives either $\beta_k'(\epsilon_k)=0$ and $\alpha_k'(\epsilon_k)=\frac{-1}{\epsilon_k}$, or $\beta_k'(\epsilon_k)\neq 0$ and $\alpha_k'(\epsilon_k)=\frac{\epsilon_k\beta_k\beta_k'(\epsilon_k)}{1+\epsilon_k}$, so that one always has $\alpha_k'(\epsilon_k)\neq 0$. Consequently both $(L_1)$ and $(L_2)$ of Eichmann's spectral hypotheses are satisfied, so that Theorem 1 follows.

## Appendix B. A numerical method for solving transition layer equations

In this section we present a method for solving numerically the transition layer equations (6)–(9). This section is divided into two parts: negative and positive feedbacks. Some details are only provided for the positive feedback case, since for the negative feedback case they are similar.

### *B.1. Monotone positive feedback*

The existence of transition layer solutions is usually proven with the help of a fixed point theorem. Suppose that $f$ is smooth, has three fixed points $x=-a$, $x=0$ and $x=b$, and that $f$ is increasing on the interval $[-a,b]$ (positive feedback, with no symmetry hypothesis). Consider the set $\mathcal{C}^+$ of functions $\phi\in C^1(\mathbb{R},\mathbb{R})$ such that $\phi(0)=0$, $\lim_{x\to-\infty}\phi(x)=-a$, $\lim_{x\to+\infty}\phi(x)=b$, $\|\phi'\|_{L^\infty}\le\sup_{x\in[-a;b]}|f(x)|=\max\{a,b\}$, and $\phi$ is strictly increasing on $\mathbb{R}$. On that space we define the operator $\mathcal{T}$ that to $\phi\in\mathcal{C}^+$ associates the function $\mathcal{T}\phi=\psi$ given by

$$\psi(t)=e^{-t}\int_{-\infty}^{t}e^sf(\phi[s+\rho-R(\phi(s))])ds,$$

where $\rho\in[0;+\infty[$ is the only constant such that $\psi(0)=\int_{-\infty}^{0}e^sf(\phi[s+\rho-R(\phi(s))])ds=0$. It can be shown that $\psi\in\mathcal{C}^+$. Then $\psi=\mathcal{T}\phi$ is the unique solution of equation

$$\dot\psi(t)=-\psi(t)+f(\phi[t-R(\phi(t))+\rho]) \tag{11}$$

such that $\lim_{x\to-\infty}\psi(x)=-a$, $\lim_{x\to+\infty}\psi(x)=b$ and $\psi(0)=0$. So an increasing solution of the transition layer equation (6) (case $\eta(0)=0$ and $\eta'(0)=1$) is a fixed point of the operator $\mathcal{T}$ in the set $\mathcal{C}^+$. In the following we show how to numerically solve the fixed point problem $\mathcal{T}\phi^+=\phi^+$. A solution to the problem $\mathcal{T}\phi^-=\phi^-$ is obtained in the same way.

Let $\phi$ be a smooth increasing function such that $\lim_{t\to-\infty}\phi(t)=-a$ and $\lim_{t\to+\infty}\phi(t)=b$. Then, for any $\gamma\in\mathbb{R}$, the function

$$\psi_\gamma(t)=e^{-t}\int_{-\infty}^{t}e^sf(\phi[s+\gamma-R(\phi(s))])ds$$

is also smooth, increasing, and satisfies $\lim_{t\to-\infty}\psi(t)=-a$ and $\lim_{t\to+\infty}\psi(t)=b$. Since, for any $t\in\mathbb{R}$, the map $\gamma\mapsto\psi_\gamma(t)$ is increasing, there is a unique $\gamma=\rho$ such that $\psi_\rho(0)=0$ and so $\psi_\rho=\mathcal{T}\phi$. A numerical approximation to the map $\phi\to\psi_\rho$ is the following. Let $L>0$ and $\phi^L$ be the function that coincides with $\phi$ on the interval $[-L,L]$ and such that $\phi^L(t)=-a$ for $t<-L$ and $\phi^L(t)=b$ for $t>L$. Clearly $\sup_{t\in\mathbb{R}}|\phi^L(t)-\phi(t)|$ can be made arbitrarily small if $L$ is chosen sufficiently large. So, we fix $L>0$ and choose an initial $\phi_0$ that satisfies $\phi_0(t)=-a$ for $t<-L$ and $\phi_0(t)=b$ for $t>L$. Then for a given $\gamma$ we use a first order Euler method to solve the equation

$$\psi_\gamma'(t)=-\psi_\gamma(t)+f\left(\phi\left[t+\gamma-R(\phi_0(t))\right]\right),$$

with the initial condition $\psi_\gamma(-L)=-a$ and time step $dt$. Knowing that $\gamma>\rho$ is equivalent to $\psi_\gamma(0)>0$, we can use a shooting method to find $\gamma_1$ and $\gamma_2$ such that $\rho-dt\le\gamma_1\le\rho\le\gamma_2\le\rho+dt$. Defining $\phi_1(t)=\psi_{\gamma_1}(t)$ for $t\le L$ and $\phi_1(t)=b$ for $x>L$ we obtain the approximation $\mathcal{T}\phi_0\approx\phi_1$. This procedure can be iterated $\phi_{n+1}=\mathcal{T}\phi_n$, $n=1,2,\ldots$, hoping that it converges to a fixed point $\phi^+$. The convergence of this sequence was numerically verified, typically iterating $\phi_{n+1}=\mathcal{T}\phi_n$ up to $n=20$ and checking that

$$|\phi_n(t)-\phi_{n-1}(t)|\le dt.$$

A sample of our results are shown in Table 1(a) (Section 4.1.1) and in Fig. 8.

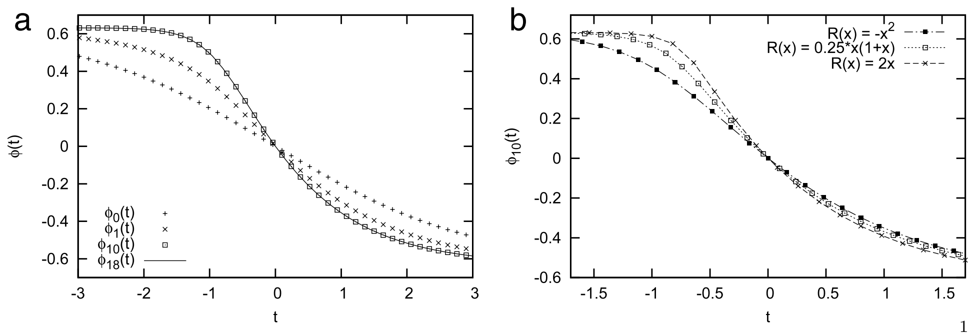



**Fig. 8.** Decreasing transition layer solutions of Eq. (6) ($\eta(0) = 0$ with $\eta'(0) = 1$) with $f(x) = \frac{1}{2}\arctan(5x)$ (Positive feedback case). In (a): $R(x) = \cos(x)$ and $\phi_0(t) = ab\frac{1-e^t}{a+be^t}$. In (b): $\phi_{10}(t) = \mathcal{T}^{10}\phi_0(t)$ is displayed (with $\phi_0(t) = ab\frac{1-e^t}{a+be^t}$), for functions $R(x) = 2x$, $R(x) = -x^2$, and $R(x) = \frac{1}{4}x(1+x)$.

In Table 1(a) the constants $\rho^{\pm}$ were computed for $f(x) = \frac{1}{2}\arctan(5x)$, $L = 100.0$, and $dt = 0.001$. The initial condition used to obtain Table 1(a) was $\phi_0^+(t) = \frac{ab(1-e^{-\frac{2}{3}t})}{a+be^{-\frac{2}{3}t}}$ for the increasing transition layer solution, and $\phi_0^-(t) = \frac{ab(1-e^{\frac{2}{3}t})}{a+be^{\frac{2}{3}t}}$ for the decreasing transition layer solutions.

In Fig. 8(a) several iterates $\phi_n$ are shown to converge to a limit profile $\phi^-$ for $f(x) = \frac{1}{2}\arctan(5x)$, with $R(x) = cos(x)$, and initial condition $\phi_0(t) = ab\frac{1-e^t}{a+be^t}$. In that case the feedback $f$ is symmetric, so that the operator $\mathcal{T}$ is also symmetric, and if we consider an increasing initial profile $\tilde{\phi}_0 = -\phi_0$, the corresponding sequence is $\tilde{\phi}_n = \mathcal{T}^n(-\phi_0) = -\mathcal{T}\phi_0$. In particular, Fig. 8 illustrates the convergence and shapes of iterates for both increasing and decreasing profiles. In Fig. 8(b) the limit profile $\phi_{10}(t) = \mathcal{T}^{10}\phi_0(t) \approx \phi^-(t)$ is shown for $R(x) = 2x$, $R(x) = -x^2$, and $R(x) = \frac{1}{4}x(1+x)$, using the initial condition $\phi_0(t) = ab\frac{1-e^t}{a+be^t}$. For $R(x) = x$, the numerical convergence of the sequence $\phi_{n+1} = \mathcal{T}\phi_n$ was successfully tested for the following initial functions: $\phi_0(t) = -\frac{t}{|t|}$, and $\phi_0(t) = ab\frac{1-e^t}{a+be^t}(1+0.4\cos(t))$. All these results also hold for a non-symmetric positive feedback as well.

For DDEs with state-dependent delays, such as transition layer equations (6), the existence of solutions $\phi$ to the Cauchy problem is known under classical hypotheses, in particular on the delay function $R$, that ensures that $t \mapsto t - R(\phi(t))$ is not decreasing (see [20] for example). We mention that in several numerical examples, when $R$ is too large, the maps $t \mapsto t - R(\phi_n(t))$ are not monotone, the iterative sequences $\phi_n = \mathcal{T}^n\phi_0$ does not converge, and transition layer solutions seem not to exist (see Section 4.2)

In the case $\eta(0) = \eta'(0) = 0$, the transition layer equation (9) is associated with the operator

$$\mathcal{T}_\lambda\phi(t) = e^{-t}\int_{-\infty}^{t} e^s f(\phi[s + \rho^+ - \lambda R(\phi(s))])ds,$$

and the same numerical method presented above can be used to compute its fixed point. For instance, Fig. 1 (Section 4.2.1) shows the constants $\rho^{\pm}$ that were computed using various choices of $R$ and: $f(x) = \frac{1}{2}\arctan(5x)$, $L = 100.0$, $dt = 0.001$, $n = 20$, and the initial condition $\phi_0^+(t) = \frac{ab(1-e^{-\frac{2}{3}t})}{a+be^{-\frac{2}{3}t}}$ for the increasing transition layer solution and $\phi_0^-(t) = \frac{ab(1-e^{\frac{2}{3}t})}{a+be^{\frac{2}{3}t}}$ for the decreasing transition layer.

Finally, in Fig. 2 (in Section 4.3), illustrating that metastability may be induced by state-dependent delay in the case of non-symmetric positive feedback (that does not exhibit metastability for constant delay), the values of $\rho^{\pm}$ were computed using $L = 100.0$, $dt = 0.001$, $n = 20$, and the initial conditions $\phi_0^+(t) = \frac{ab(1-e^{-\frac{2}{3}t})}{a+be^{-\frac{2}{3}t}}$, $\phi_0^-(t) = \frac{ab(1-e^{\frac{2}{3}t})}{a+be^{\frac{2}{3}t}}$.

### *B.2. Monotone negative feedback*

For $f$ satisfying MNFH, one can repeat the procedure above and define the operator $\mathcal{T}$ that has as fixed points the solutions to the transition layer equation (7), namely

$$\phi^+(t) = \mathcal{T}\phi^-(t) = e^{-t}\int_{-\infty}^{t} e^s f(\phi^-[s + \rho^+ - R(\phi^-(s))])ds$$

$$\phi^-(t) = \mathcal{T}\phi^+(t) = e^{-t}\int_{-\infty}^{t} e^s f(\phi^+[s + \rho^- - R(\phi^+(s))])ds$$

where $\phi^+$ (resp. $\phi^-$) are increasing (resp. decreasing) smooth functions with $\lim_{t\to-\infty}\phi^+ = -a$, $\lim_{t\to+\infty}\phi^+ = b$ and $\lim_{t\to-\infty}\phi^- = b$, $\lim_{t\to+\infty}\phi^- = -a$, and $\rho^{\pm}$ are the only constants such that $\phi^+(0) = \phi^-(0) = 0$. Since the function $f$ is decreasing, the operator $\mathcal{T}$ now maps an increasing function to a decreasing one and vice versa. As in the case of positive feedback, if the function $\phi$ is strictly monotone the function $\gamma \mapsto \Psi_\gamma(t)$ is strictly monotone too, so that the constants $\rho^{\pm}$ above are well defined and can be computed by a shooting method. We choose a smooth increasing initial function $\phi_0$ with $\lim_{t\to-\infty}\phi^+ = -a$, $\lim_{t\to+\infty}\phi^+ = b$ and the subsequences $\phi_{2n}$ and $\rho_{2n}$ (resp. $\phi_{2n+1}$ and $\rho_{2n+1}$) converge to the transition layer solution $\phi^+$ and the constant $\rho^+$ (resp. $\phi^-$ and $\rho^-$). Numerically, as in the positive feedback case, we use discretization of step $dt$ on an interval $[-L, L]$, the functions $\Psi_\gamma$ are computed using a first order Euler scheme of step $dt$, the constant $\rho$ are approximated with a precision $dt$, and the convergence after $n$ iterations of $\mathcal{T}$ is checked similarly:

$$|\phi_n(t) - \phi_{n-2}(t)| \le dt,$$
$$|\phi_{I-1}(t) - \phi_{I-3}(t)| \le dt.$$

In Table 2 (Section 4.1.2) the constants $\rho^{\pm}$ were computed using $L = 100.0$, $dt = 0.001$, $n = 40$, and $\phi_0^+(t) = \frac{ab(1-e^{-\frac{2}{3}t})}{a+be^{-\frac{2}{3}t}}$.

## References

[1] J. Bélair, M. Mackey, Consumer memory and price fluctuations in commodity markets: an integrodifferential model, J. Dynam. Differential Equations 1 (1989) 299–325.
[2] M. Mackey, Commodity price fluctuations: price dependent delays and nonlinearities as explanatory factors, J. Econom. Theory 48 (1989) 497–509.
[3] A. Longtin, J. Milton, Modelling autonomous oscillations in the human pupil light reflex using non-linear delay-differential equations, Bull. Math. Biol. 51 (1989) 605–624.
[4] A. Longtin, J. Milton, Insight into the transfer function, gain, and oscillation onset for the pupil light reflex using nonlinear delay-differential equations, Biol. Cybernet. 61 (1989) 51–58.

[5] O. Arino, E. Sànchez, A. Fathallah, State-dependent delay differential equations in population dynamics: modeling and analysis, in: T. Faria, P. Freitas (Eds.), Topics in Functional Differential and Difference Equations, in: Fields Inst. Commun., vol. 29, AMS, Providence, RI, 2001, pp. 19–36.
[6] C. Foley, M. Mackey, Dynamic hematological disease: a review, J. Math. Biol. 58 (2009) 285–322.
[7] E. Liz, G. Röst, Global dynamics in a commodity market model, J. Math. Anal. Appl. 398 (2013) 707–714.
[8] W. Alt, Some periodicity criteria for functional differential equations, Manuscripta Math. 23 (1978) 295–318.
[9] O. Arino, K.P. Hadeler, M.L. Hbid, Existence of periodic solutions for delay differential equations with state-dependent delay, J. Differential Equations 144 (1998) 263–301.
[10] T. Krisztin, O. Arino, The 2-dimensional attractor of a differential equation with state-dependent delay, J. Dynam. Differential Equations 13 (2001) 453–522.
[11] Y. Kuang, H.L. Smith, Slowly oscillating periodic solutions of autonomous state-dependent delay differential equations, Nonlinear Anal. 19 (1992) 855–872.
[12] J. Mallet-Paret, R.D. Nussbaum, P. Paraskevopoulos, Periodic solutions for functional differential equations with multiple state-dependent time lags, Topol. Methods Nonlinear Anal. 3 (1994) 101–162.
[13] H.-O. Walther, Stable periodic motion of a system with state-dependent delay, Differential Integral Equations 15 (2002) 923–944.
[14] J. Mallet-Paret, Morse decompositions for differential delay equations, J. Differential Equations 72 (1988) 270–315.
[15] J. Mallet-Paret, R. Nussbaum, Superstability and rigorous asymptotics in singularly perturbed state-dependent delay-differential equations, J. Differential Equations 250 (2011) 4037–4084.
[16] J. Mallet-Paret, R. Nussbaum, Stability of periodic solutions of state-dependent delay-differential equations, J. Differential Equations 250 (2011) 4085–4103.
[17] R.D. Nussbaum, Limiting profiles for solutions of differential-delay equations, in: Dynamical Systems, in: Lecture Notes in Mathematics, vol. 1822, Springer, Berlin, Heidelberg, 2003, pp. 299–342.
[18] T. Krisztin, H.-O. Walther, J. Wu, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback, in: Fields Institute Monograph Series, AMS, Providence, RI, 1999.
[19] M. Polner, Morse decomposition for delay-differential equations with positive feedback, Nonlinear Anal. 48 (2002) 377–397.
[20] F. Hartung, T. Krisztin, W. Hans-Otto, J. Wu, Functional differential equations with state-dependent delays: theory and applications, in: Handbook of Differential Equations: Ordinary Differential Equations, in: Handb. Differ. Equ., vol. III, Elsevier/North-Holland, Amsterdam, 2006, pp. 435–545.
[21] A.N. Sharkovsky, Yu.L. Maistrenko, E.Yu. Romanenko, Difference Equations and their Applications, Kluwer, Dordrecht, 1993.
[22] G. Giacomelli, R. Meucci, A. Politi, F.T. Arecchi, Defects and spacelike properties of delayed dynamical systems, Phys. Rev. Lett. 73 (1994) 1099–1102.
[23] G. Giacomelli, A. Politi, Multiple scale analysis of delayed dynamical systems, Physica D 117 (1998) 26–42.
[24] K. Pakdaman, C.P. Malta, C. Grotta-Ragazzo, O. Arino, J.-F. Vibert, Transient oscillations in continuous-time excitatory ring neural networks with delay, Phys. Rev. E 55 (1997) 3234–3248.
[25] J. Milton, P. Naik, C. Chan, S.A. Campbell, Indecision in neural decision making models, Math. Model. Nat. Phenom. 5 (2010) 125–145.
[26] J. Milton, A. Quan, I. Osorio, Nocturnal frontal lobe epilepsy: metastability in a dynamic disease?, in: I. Osorio, H.P. Zavari, M.G. Frei, S. Arthurs (Eds.), The Intersection of Neurosciences, Biology, Mathematics, Engineering and Physics, CRC Press, Boca Raton, 2011, pp. 501–510.
[27] K. Pakdaman, C. Grotta-Ragazzo, C.P. Malta, Transient regime duration in continuous-time neural networks with delay, Phys. Rev. E 58 (1998) 3623–3627.
[28] C. Grotta-Ragazzo, K. Pakdaman, C.P. Malta, Metastability for delayed differential equations, Phys. Rev. E 60 (1999) 6230–6233.
[29] J. Carr, R.L. Pego, Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math. XLII (1989) 523–576.
[30] G. Fusco, J.K. Hale, Slow-motion manifolds, dormant instability, and singular perturbations, J. Dynam. Differential Equations 1 (1989) 75–94.
[31] C. Grotta-Ragazzo, C.P. Malta, K. Pakdaman, Metastable periodic patterns in singularly perturbed delayed equations, J. Dynam. Differential Equations 22 (2010) 203–252.
[32] M. Nizette, Stability of square oscillations in a delayed-feedback system, Phys. Rev. E 70 (2004) 056204-1–056204-6.
[33] M. Nizette, Front dynamics in a delayed-feedback system with external forcing, Physica D 183 (2003) 220–244.
[34] J. Mallet-Paret, R. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags, I, Arch. Ration. Mech. Anal. 120 (1992) 99–146.
[35] J. Mallet-Paret, R. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags: II, J. Reine Angew. Math. 477 (1996) 129–198.
[36] J. Mallet-Paret, R. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags: III, J. Differential Equations 189 (2003) 640–692.
[37] S.-N. Chow, J. Mallet-Paret, Singularly perturbed delay-differential equations, in: Math. Stud., vol. 80, North-Holland, 1983, pp. 7–12.
[38] J. Mallet-Paret, R. Nussbaum, Global continuation and asymptotic behavior for periodic solutions of a diff-delay equation, Ann. Mat. Pura Appl. (4) CXLV (1986) 33–128.
[39] O. Arino, R. Benkhalti, Periodic solutions for: $x(t) = \lambda f(x(t), x(t-1))$, Proc. Roy. Soc. Edinburgh Sect. A 109 (1988) 245–260.
[40] O. Arino, P. Séguier, Existence of oscillating solutions for certain differential equations with delay, in: H.-O. Peitgen, H.-O. Walthers (Eds.), Functional Differential Equations and Approximation of Fixed Point, in: Lecture Notes in Mathematics, vol. 430, Springer Verlag, New York, 1979, pp. 46–64.
[41] M. Eichmann, A local Hopf bifurcation theorem for differential equations with state—dependent delays, Ph.D. Thesis, Department of Mathematics, Justus—Liebig—University Giessen, Giessen-Germany, 2006.
[42] M. Bartha, Convergence of solutions for an equation with state-dependent delay, J. Math. Anal. Appl. 254 (2001) 410–432.
[43] M. Bartha, Periodic solutions for differential equations with state-dependent delay and positive feedback, Nonlinear Anal. 53 (2003) 839–857.
[44] K.L. Cooke, W. Huang, On the problem of linearization for state-dependent delay differential equations, Proc. Amer. Math. Soc. 124 (1996) 1417–1426.