UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE FÍSICA DE SÃO CARLOS

Pedro Antônio Soares de Alcântara

Localization of symbol correspondences for spin systems

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Pedro Antônio Soares de Alcântara

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Localization of symbol correspondences for spin systems

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Orientador: Pedro Paulo de Magalhães Rios
ICMC – Instituto de Ciências Matemáticas
e de Computação

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“Tenta uma outra coisa, um curso de informática
E você vai continuar fazendo música?
E você vai continuar fazendo música?

“Velhos e criancinhas, todos te acham maluco
E você vai continuar fazendo música?
E você vai continuar fazendo música?

“A tua vida se afunda, por isso eu te pergunto:
Você vai continuar fazendo música?
E você vai continuar fazendo música?”
(SKYLAB, Rogério. “Você vai continuar fazendo música?”)

Abstract

This work presents some results on the correspondence principle applied for spin systems ($SU(2)$-symmetric mechanical systems). After a short introduction, followed by a concise presentation of the relation between Lie groups $SU(2)$ and $SO(3)$, the quantum spin-j systems and the classical spin system are defined. Then, we present the definition of symbol correspondences for spin systems and list some important facts and properties about them, which are detailed in [8]. For our first original result, we prove an important conjecture found in [8]. Then, as our main original work, we propose a treatment to study asymptotic localization of symbols of pure states, which may provide a more physically intuitive criterion for the asymptotic emergence of classical dynamics from quantum dynamics.

KEYWORDS: Quantum-classical correspondences; Spin systems ($SU(2)$); Poisson dynamics and localization.

1 Introduction

Galileo, Newton and many others laid the foundations for what is now called Classical Mechanics. Centuries of development set the mathematical basis of the theory to its state of art: the Hamiltonian or Poisson formalism. However, in the 1900's, everything we used to know showed to be insufficient faced with the behavior of matter at microscopic level. It was found out that atoms (and its components) were "rebellious against" the laws of nature known at the time. This problem gave rise to Quantum Mechanics, as developed by Planck, Einstein, Bohr, Heisenberg and Schrödinger, among others.

Whereas the Hamiltonian formalism uses Poisson algebras of smooth functions on symplectic manifolds for describing dynamics, the Jordan-Heisenberg-Schrödinger-von Neumann formalism deals with algebras of bounded operators on complex Hilbert spaces. So a natural question came up: how can these two different formalisms be related? More precisely, how to associate quantum measurable quantities and classical measurable quantities in order to consistently relate the dynamics of both theories? The statement that such a relation should exist is known as the correspondence principle (Bohr). Thus, since the mid 20's, mathematical physicists have spent effort to investigate the correspondence principle, but just recently we had a complete study about the case of spin systems [8].

Spin systems, or pure spin systems, are characterized by having 1 degree of freedom and by being symmetric under the action of the Lie group $SU(2)$ (or effectively $SO(3)$, as we shall see). Since $SU(2)$ is compact, its unitary irreducible representations are all finite dimensional, thus the operator spaces of quantum systems are all isomorphic to matrix algebras. For classical system, this symmetry implies that the phase space is the 2-sphere and its Poisson algebra can be decomposed into invariant subspaces spanned by spherical harmonics. Matrix spaces and spherical harmonics are studied even in the first years of undergraduate studies in Physics, so this extremely simplifies our work.

In this work, we will further investigate symbol correspondences for spin systems, as developed in [8], particularly with respect to the relation between quantum and classical dynamics in the asymptotic limit of high spin numbers, when Bohr's correspondence principle would be more relevant. This work is organized as follows. In section 2, we review the groups $SU(2)$ and $SO(3)$. In section 3, we present the quantum spin-j systems and the classical spin system. In section 4, we present the definition of symbol correspondences, and some of their various important properties, emphasizing the description of three special cases. Sections 3 and 4 are fully based on [8], but at the end of section 4 we prove an important conjecture stated in [8]. Section 5 consists entirely of original research work. There, we study the localization of symbols of pure states and show how this leads to a new criterion for the emergence of Poisson dynamics from quantum dynamics in the asymptotic limit of high spin numbers.

2 The Lie Groups $SU(2)$ and $SO(3)$

Let $GL_2(\mathbb{C})$ be the general linear group of degree 2 over $\mathbb{C}$. The special unitary group of degree 2, denoted $SU(2)$, is the group of special unitary transformation on $\mathbb{C}^2$, *ie*, the subgroup of $GL_2(\mathbb{C})$ satisfying the following properties: $\forall g \in SU(2)$, $\det g = 1$ and $gg^* = g^*g = e$, where e is the identity and g^* is the Hermitian conjugate of g. From these properties, we have that

$$g = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \in SU(2) \ , \tag{2.1}$$

for $z_1, z_2 \in \mathbb{C}$ such that $|z_1|^2+|z_2|^2 = 1$. Thus, $SU(2) \simeq S^3$ topologically, so it is compact and simply connected. The Lie algebra of $SU(2)$, denoted $\mathfrak{su}(2)$, is generated by $i\sigma_k$, for $k = 1, 2, 3$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

are the Pauli matrices, satisfying the following commutation rule

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c. \tag{2.3}$$

Now, let $GL_3(\mathbb{R})$ be the general linear group of degree 3 over $\mathbb{R}$. The special orthogonal group of degree 3, denoted $SO(3)$, is the group of special orthogonal transformations, or rotations, on $\mathbb{R}^3$, *ie*, the subgroup of $GL_3(\mathbb{R})$ satisfying the following properties: $\forall g \in SO(3)$, $\det g = 1$ and $gg^T = g^Tg = e$. A simple way of write any element of $SO(3)$ is decomposing it in three rotations around two chosen axes (*cf.* [10]). The Lie algebra of $SO(3)$, denoted $\mathfrak{so}(3)$, is generated by

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

satisfying the commutation rule

$$[L_a, L_b] = \epsilon_{abc} L_c. \tag{2.5}$$

Thus, we have an isomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{so}(2)$ given by $-i\sigma_k \mapsto 2L_k$. Since $SU(2)$ is simply connected, there is a covering homomorphism $\psi : SU(2) \to SO(3)$ such that $\rho = d\psi$ (*cf.* [5]). We refer to [8] for the explicit expression of ψ, whose kernel is $\mathbb{Z}_2$, so that $SO(3) = SU(2)/\mathbb{Z}_2$.

Since $SU(2)$ is compact, its irreducible unitary representations are finite dimensional (*cf.* [7]). A representation of this kind is labeled by j such that $2j+1 \in \mathbb{N}$ is its dimension, so we will denote it by φ_j. A representation φ_j is also a representation of $SO(3)$ if and only if j is an integer.

3 Spin Systems

3.1 Quantum Systems

As exposed in the last section, the irreducible unitary representations of $SU(2)$ are finite dimensional, so are the quantum spin systems. Let $n = 2j$ be a nonnegative integer.

Definition 3.1 *A* quantum spin-j system *is a complex Hilbert space $\mathcal{H}_j \simeq \mathbb{C}^{n+1}$ with an irreducible unitary representation φ_j of $SU(2)$, together with its operator algebra $\mathcal{B}(\mathcal{H}_j)$.*

The spin operators $J_k \in \mathcal{B}(\mathcal{H}_j)$ given by $d\varphi_j(\sigma_k) = 2J_k$ correspond to x, y and z components of the total angular momentum or spin[1]. The usual approach for a spin-j system is to diagonalize the operator J_3, which has eigenvalues $m = -j, -j+1, ..., j-1, j$ (*cf* [1]). Thus, we denote the vectors of an orthonormal basis of $\mathcal{H}_j$ comprised of eigenvectors of J_3 by[2] $\boldsymbol{u}(j,m)$. Then, we define the ladder operators (creation and annihilation, respectively) $J_\pm = J_1 \pm iJ_2$ and the norm of spin operator $J^2 = J_1^2 + J_2^2 + J_3^2 = J_\mp J_\pm + J_3(J_3 \pm I)$, where I is the identity operator. Via straightforward calculations, one can compute the commutation rules

$$[J_a, J_b] = i\epsilon_{abc} J_c, \qquad [J_+, J_-] = 2J_3, \qquad [J_3, J_\pm] = \pm J_\pm \tag{3.1}$$

and, from them, $J^2 = j(j+1)I$, $J_+(\boldsymbol{u}(j,m)) = \alpha_{j,m}\boldsymbol{u}(j,m+1)$ and $J_-(\boldsymbol{u}(j,m)) = \beta_{j,m}\boldsymbol{u}(j,m-1)$, where $\alpha_{j,m}$ and $\beta_{j,m}$ are non zero constants, except for $\alpha_{j,j}$ and $\beta_{j,-j}$. However, this is not enough to determine a unique basis, there still is an individual phase factor for each vector. To (almost) eliminate this irresolution, we choose a highest weight vector $\boldsymbol{u}(j,j)$ and fix all the other phases so that the constants $\beta_{j,m}$ are nonnegative real numbers. Thus, for such a basis, called standard, there is just one free phase on the choice of $\boldsymbol{u}(j,j)$ and the following expressions hold:

$$\alpha_{j,m} = \sqrt{(j-m)(j+m+1)}, \qquad \beta_{j,m} = \sqrt{(j+m)(j-m+1)}. \tag{3.2}$$

[1] We are taking $\hbar = 1$.
[2] In Dirac's notation, $\boldsymbol{u}(j,m) = |j,m\rangle$.

Now, let's take a look at the structure of quantum dynamics.

Definition 3.2 *Chosen a Hermitian operator H, the dynamics of a smoothly time-dependent operator M is defined by Heisenberg's equation*

$$\frac{dM}{dt} = \frac{1}{i}[M, H] + \frac{\partial M}{\partial t}.$$

Such operator H is called a Hamiltonian operator. *If M is Hermitian, it is an* observable.

We can see the operator space $\mathcal{B}(\mathcal{H}_j) = Hom(\mathcal{H}_j, \mathcal{H}_j) \simeq \mathcal{H}_j \otimes \mathcal{H}_j^*$ as the space of square matrices of order $(n+1)$ with complex entries, $M_{\mathbb{C}}(n+1)$. Let $\mathcal{E}_{k,l} \in M_{\mathbb{C}}(n+1)$ such that $(\mathcal{E}_{k,l})_{p,q} = \delta_{k,p}\delta_{l,q}$ and $\boldsymbol{u}^*(j,m)$ be the dualization of $\boldsymbol{u}(j,m)$ by the inner product[3], then $\boldsymbol{u}(j,m_1)\otimes\boldsymbol{u}^*(j,m_2) \leftrightarrow \mathcal{E}_{j-m_1+1,j-m_2+1}$ and the composition of operators become the matrix product. This is an isometry since for the inner product of operators and trace of matrices holds $\langle P, Q\rangle = tr(P^*Q)$. Now, to extend the representation φ_j to $\mathcal{B}(\mathcal{H}_j)$, we will make use of its natural extension on dual space $\check{\varphi}_j$. By means of the identification $\mathcal{H}_j^* \leftrightarrow \mathcal{H}_j$, we have $\check{\varphi}_j \leftrightarrow \varphi_j$ so that $\check{\varphi}_j(g) \leftrightarrow \varphi_j(g)^{-1}$ for all $g \in SU(2)$. Therefore, for the spin operators, $\check{J}_1 \leftrightarrow -J_1$, $\check{J}_2 \leftrightarrow J_2$ and $\check{J}_3 \leftrightarrow -J_3$, so $\check{J}_+ \leftrightarrow -J_-$ and $\check{J}_- \leftrightarrow -J_+$. Thus, a standard basis of the dual space is formed by vectors $\check{\boldsymbol{u}}(j,m) = (-1)^{j+m}\boldsymbol{u}^*(j,-m)$, so that $\boldsymbol{u}(j,m_1) \otimes \check{\boldsymbol{u}}(j,m_2)$ represents the matrix $(-1)^{j+m_2}\mathcal{E}_{j-m_1+1,j+m_2+1}$. Finally, $\varphi_j \otimes \check{\varphi}_j \leftrightarrow \varphi_j \otimes \varphi_j$ so that for $P \in \mathcal{B}(\mathcal{H}_j)$ and $g \in SU(2)$ we have $P^g = \varphi(g)P\varphi_j(g)^{-1}$. As consequence, the spin operators $\boldsymbol{J}_k$ acting on $\mathcal{B}(\mathcal{H}_j)$ can be identified with the spin operators J_k on $\mathcal{H}_j$ so that the action on P is the adjoint action, given by the commutator $[J_k, P]$.

From the Clebsch-Gordan series of $SU(2)$, we state the following result:

Theorem 3.1 (*cf. eg.* [6])

$$\varphi_j \otimes \varphi_j = \bigoplus_{l=0}^{n} \varphi_l$$

So the induced action of $SU(2)$ on $\mathcal{B}(\mathcal{H}_j)$ is effectively an action of $SO(3)$. For each φ_l, we find a standard orthonormal basis of vectors $\boldsymbol{e}^j(l,m)$ as we did for $\mathcal{H}_j$. The basis consisting of $\boldsymbol{u}(j,m_1)\otimes\check{\boldsymbol{u}}(j,m_1)$ is called uncoupled, whereas the basis consisting of $\boldsymbol{e}^j(l,m)$, s.t. $\{\boldsymbol{e}^j(l,m), -l \leq m \leq l\}$ is a basis for each $SO(3)$-invariant φ_l subspace, is called coupled[4]. The coefficients of the change of basis

$$\boldsymbol{u}(j,m_1) \otimes \check{\boldsymbol{u}}(j,m_2) = \sum_{l=0}^{n}\sum_{m=-l}^{l} C_{m_1,m_2,m}^{\,j,\ \ j,\ \ l}\boldsymbol{e}^j(l,m) \tag{3.3}$$

are called Clebsch-Gordan coefficients. They are unique up to a phase that is related to the choice of the phase of the coupled basis. We will adopt a convention in which all the coefficients are real.

[3] In Dirac's notation, $\boldsymbol{u}^*(j,m) = \langle j,m|$.

[4] If we take $\mathcal{H}_j^*$ as an independent spin-j system, this is equivalent to addition (or subtraction) of spin. So, in Dirac's notation, we can write $\boldsymbol{u}(j,m_1) \otimes \check{\boldsymbol{u}}(j,m_2) = |j,m_1,j,m_2\rangle$ and $\boldsymbol{e}^j(l,m) = |(j,j)l,m\rangle$.

3.2 Classical System

Let $\Pi = x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x + z\partial_x \wedge \partial_y$ be the angular *momentum* bi-vector field in $\mathbb{R}^3$. It induces a Poisson algebra such that for two smooth functions $f_1, f_2 : \mathbb{R}^3 \to \mathbb{C}$, we have $\{f_1, f_2\} = \Pi(df_1, df_2)$. One can easily verify that, removing the origin, the remainder of the Poisson manifold can be decomposed into isomorphic symplectic 2-spheres homogeneous under the action of $SU(2)$, which is an effective action of $SO(3)$, *ie*, $\mathbb{R}^3\backslash\{0\} \simeq S^2 \times \mathbb{R}^+$ and its Poisson algebra is a sum of $\{C^\infty_\mathbb{C}(S^2), \omega_r\} \simeq \{C^\infty_\mathbb{C}(S^2), \omega\}$, $\forall r \in \mathbb{R}^+$, where ω is the canonical surface element on S^2 and the action of $g \in SO(3)$ on $f \in C^\infty_\mathbb{C}(S^2)$ is given by $f^g(\mathbf{n}) = f(g^{-1}\mathbf{n})$. This action is unitary with respect the usual inner product of function $\langle f, h\rangle = \frac{1}{4\pi}\int_{S^2} \overline{f} h dS$. In addition to this brief explanation, we recall that a spin system must have 1 degree of freedom, so we already found our phase space.

Definition 3.3 *The* classical spin system *is the* 2*-sphere with its Poisson algebra* $\{C^\infty_\mathbb{C}(S^2), \omega\}$*, where the symplectic form (closed, but not exact) has the local expression* $\omega = \sin\varphi\, d\varphi \wedge d\theta$ *in spherical coordinates with respect to the north pole.*

An interpretation of the classical spin system could be borrowed from an orbital *angular momentum* $\vec{L}$ with constant norm $\|\vec{L}\| = L \neq 0$. The points reached by the vector $\vec{L}$ form a hollow sphere of radius L. By rescaling of L, we can set the radius as unitary, matching the phase space defined above. However, we must point out that, for the classical spin system, the 2-sphere S^2 is itself the phase space, whereas the phase space of an orbital angular momentum of constant norm is T^*S^2.

As in the previous subsection, let's explore the structure of classical dynamics.

Definition 3.4 *Chosen a smooth function* $H : S^2 \to \mathbb{R}$*, the dynamics of a smooth function* $f : S^2 \times \mathbb{R} \to \mathbb{C}$*, where* $\mathbb{R} \ni t$ *is the time parameter, is defined by Hamilton's equation*

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

Such function H *is called a* Hamiltonian function. *If* f *is real valued, it is an* observable.

The representation of the generators of the Lie algebra $\mathfrak{so}(3)$ on $C^\infty_\mathbb{C}(S^2)$ are

$$L_1 = z\partial_y - y\partial_z, \qquad L_2 = x\partial_z - z\partial_x, \qquad L_3 = y\partial_x - x\partial_y.$$

From them, we write spin operators analogous to a spin-j system: $J_k = iL_k$, for $k = 1, 2, 3$. Also, $J_\pm = J_1 \pm iJ_2$ and $J^2 = J_1^2 + J_2^2 + J_3^2 = J_\mp J_\pm + J_3(J_3 \pm I)$. The normalized solutions that diagonalize J_3 and J^2 are orthogonal polynomials known as spherical harmonics and denoted by Y_l^m, where $l \in \mathbb{N}$ and $m \in \mathbb{Z}$ are such that $|m| \leq l$, $J_3(Y_l^m) = mY_l^m$, l is the proper degree of the polynomial[5] and

[5] It is important to remember we are under the relation $x^2 + y^2 + z^2 = 1$.

$J^2(Y_l^m) = l(l+1)Y_l^m$. Futhermore, $J_+(Y_l^m) = \alpha_{l,m}Y_l^{m+1}$ and $J_-(Y_l^m) = \beta_{l,m}Y_l^{m-1}$, for $\alpha_{l,m}$ and $\beta_{l,m}$ given by (5.11). The expressions for spherical harmonics in spherical coordinates are

$$Y_l^m(\boldsymbol{n}) = \sqrt{2l+1}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\varphi)e^{im\theta}, \tag{3.4}$$

where P_l^m are the associated Legendre polynomials. Hence, for a fixed l, the spherical harmonics Y_l^m form an orthonormal basis of the $(2l+1)$-dimensional $SO(3)$-invariant subspace of $C_{\mathbb{C}}^\infty(S^2)$.

4 Symbol Correspondences

In this section, we enunciate some important results from [8] about symbol correspondences for spin-j systems, which are defined below. Moreover, in subsection 4.2, we present a proof of an important conjecture stated in the same reference.

Definition 4.1 *A map $W^j : \mathcal{B}(\mathcal{H}_j) \to C_{\mathbb{C}}^\infty(S^2)$ is a* symbol correspondence for a spin-j system *if, $\forall P, Q \in \mathcal{B}(\mathcal{H}_j)$, $\forall g \in SO(3)$ and $\forall a \in \mathbb{C}$, it satisfies: (i) linearity, $W^j_{aP+Q} = aW^j_P + W^j_Q$, (ii) injectivity, $P \neq Q \Rightarrow W^j_P \neq W^j_Q$, (iii) equivariance, $W^j_{P^g} = (W^j_P)^g$, (iv) reality, $W^j_{P^*} = \overline{W^j_P}$, and (v) normalization, $\frac{1}{4\pi}\int_{S^2} W^j_P dS = \frac{1}{n+1}tr(P)$.*

Schur's lemma (*cf.* [5]) implies that a symbol correspondence is an isomorphism between the subspaces spanned by $\boldsymbol{e}^j(l,m)$ and Y_l^m for fixed l. So we can turn the injectivity requirement *(ii)* into bijectivity by taking $W^j : \mathcal{B}(\mathcal{H}_j) \to Poly_{\mathbb{C}}(S^2)_{\leq n}$, where $Poly_{\mathbb{C}}(S^2)_{\leq n}$ is the space of complex polynomials of proper degree less or equal to n on S^2. A simple way to characterize all symbol correspondences is the following theorem:

Theorem 4.1 (*cf.* [8]) *A map $W^j : \mathcal{B}(\mathcal{H}_j) \to C_{\mathbb{C}}^\infty(S^2)$ is a symbol correspondence iff there is a diagonal matrix $K \in M_{\mathbb{C}}(n+1)$ such that $tr(K) = 1$ and*

$$W^j_P(g\boldsymbol{n}_0) = tr(PK^g),$$

where $g \in SO(3)$ and $\boldsymbol{n}_0$ is the north pole. K is called an operator kernel and has the form

$$K = \frac{1}{n+1}I + \sum_{l=0}^{n} c_l^n \sqrt{\frac{2l+1}{n+1}}\boldsymbol{e}^j(l,0),$$

where $c_l^n \in \mathbb{R}^*$, for $l = 1, ..., n$, are called the characteristic numbers *of the correspondence and of the operator kernel. In particular, W^j provides the mapping $\sqrt{n+1}\boldsymbol{e}^j(l,m) \mapsto c_l^n Y_l^m$.*

So any n-tuple of non-zero real (characteristic) numbers c_l^n uniquely determines a symbol correspondence for a spin-j system. On the other hand, the product of operators in $\mathcal{B}(\mathcal{H}_j)$ induces an algebraic structure on $Poly_{\mathbb{C}}(S^2)_{\leq n}$ through a given symbol correspondence, as follows:

Definition 4.2 *Given a symbol correspondence* W^j, *the* twisted product of symbols *is the operation* $\star^n : Poly_{\mathbb{C}}(S^2)_{\leq n} \times Poly_{\mathbb{C}}(S^2)_{\leq n} \to Poly_{\mathbb{C}}(S^2)_{\leq n}$ *defined by* $W^j_P \star^n W^j_Q = W^j_{PQ}$, $\forall P, Q \in \mathcal{B}(\mathcal{H}_j)$.

Theorem 4.2 (*cf.* [8]) *For any symbol correspondence* W^j*, the twisted product of symbols defines a* $SO(3)$*-invariant associative unital star algebra on* $Poly_{\mathbb{C}}(S^2)_{\leq n}$.

4.1 Some Special Types of Symbol Correspondences

In this section, we will see three special types of symbol correspondences, as examples. To define the first one, we must establish the normalized inner product of operators: for $P, Q \in \mathcal{B}(\mathcal{H}_j)$, it is given by $\langle P, Q\rangle_j = \frac{1}{n+1}\langle P, Q\rangle = \frac{1}{n+1}tr(P^*Q)$. Thus, if $\langle f, q\rangle = \int_{S^2} \bar{f} g dS$ denotes the normalized inner product of functions on S^2, we define the first special type of symbol correspondence:

Definition 4.3 *A* Stratonovich-Weyl correspondence *is a symbol correspondence that is an isometry with respect the normalized inner products, that is,* $\langle P, Q\rangle_j = \langle W_P, W_Q\rangle, \forall P, Q \in \mathcal{B}(\mathcal{H}_j)$.

Theorem 4.3 (*cf.* [8]) *A symbol correspondence is a Stratonovich-Weyl correspondence iff all of its characteristic numbers, henceforth denoted* ε^n_l*, have unitary norm.*

Thus, all $|\varepsilon^n_l| = 1$, but in particular, we have the following more special cases:

Definition 4.4 *The* standard Stratonovich-Weyl correspondence *is the symbol correspondence with all characteristic numbers given by* $\varepsilon^n_l = 1$. *The* alternate Stratonovich-Weyl correspondence *is the symbol correspondence with all characteristic numbers given by* $\varepsilon^n_l = (-1)^l$.

We now define a second special type of symbol correspondence:

Definition 4.5 *A* Berezin correspondence *is a symbol correspondence with a projector* $\Pi_k = \mathcal{E}_{k,k}$ *as operator kernel.*

From (3.3) and general properties of Clebsch-Gordan coefficients (*cf.* [1]), we obtain

$$\Pi_k = \frac{1}{n+1} I + (-1)^{k+1} \sum_{l=0}^{n} C^{j,\ \ j,\ l}_{m,-m,\ 0} \boldsymbol{e}^j(l, 0), \tag{4.1}$$

where $m = j - k + 1$. Π_k does not always provide a symbol correspondence, for every n, because some Clebsch-Gordan coefficient in the decomposition may vanish. However, there are at least two Berezin correspondences that exist for every $n \in \mathbb{N}$. To state this as a theorem we first evoke the maps

$$\begin{gathered} \Phi_j : \mathbb{C}^2 \to \mathbb{C}^{n+1}, (\mathbb{C}^2 \supset S^3 \to S^{2n+1} \subset \mathbb{C}^{n+1}) \\ (z_1, z_2) \mapsto (z_1^n, ..., \sqrt{\binom{n}{k}} z_1^{n-k} z_2^k, ..., z_2^n), \end{gathered} \tag{4.2}$$

$$\sigma : \mathbb{C}^2 \to \mathbb{C}^2 \quad , \quad (z_1, z_2) \mapsto (-\overline{z}_2, \overline{z}_1) \; , \tag{4.3}$$

$$\pi : \mathbb{C}^2 \supset S^3 \to S^2 \subset \mathbb{R}^3 \quad , \quad (z_1, z_2) \mapsto \boldsymbol{n} = (x, y, z)$$
$$x + iy = 2\bar{z}_1 z_2 \; , \; z = |z_1|^2 - |z_2|^2 \; , \tag{4.4}$$

(π is the Hopf map) and the inner product $h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ conjugate linear in the first variable.

Theorem 4.4 (*cf.* [8]) *The Berezin correspondence with Π_1 as operator kernel, denoted B^j, exists for every $n \in \mathbb{N}$ and is given by*

$$B^j : M_{\mathbb{C}}(n{+}1) \to Pol_{\mathbb{C}}(S^2)_{\leq n}$$
$$P \mapsto B^j_P(\boldsymbol{n}) = h(\Phi_j(z_1, z_2), P\Phi_j(z_1, z_2)),$$

where $\boldsymbol{n} = \pi(z_1, z_2)$. Its characteristic numbers, henceforth denoted b^n_l, are given by

$$b^n_l = n! \sqrt{\frac{n+1}{(n+l+1)!(n-l)!}}.$$

Theorem 4.5 (*cf.* [8]) *The Berezin correspondence with Π_{n+1} as operator kernel, denoted B^{j^-}, exists for every $n \in \mathbb{N}$ and is given by*

$$B^{j-} : M_{\mathbb{C}}(n{+}1) \to Pol_{\mathbb{C}}(S^2)_{\leq n}$$
$$P \mapsto B^{j-}_P(\boldsymbol{n}) = h(\Phi^-_j(z_1, z_2), P\Phi^-_j(z_1, z_2)),$$

where $\Phi^-_j = \Phi_j \circ \sigma$. Its characteristic numbers, henceforth denoted b^n_{l-}, are given by $b^n_{l-} = (-1)^l b^n_l$.

Definition 4.6 *The* standard Berezin correspondence *is the Berezin correspondence with Π_1 as operator kernel and the* alternate Berezin correspondence *is the one with Π_{n+1} as operator kernel.*

We also define a third special type of symbol correspondence:

Definition 4.7 *The* upper-middle correspondence *is the symbol correspondence with $S_{1/2} = \frac{1}{2}(\Pi_{\lfloor j+1/2 \rfloor} + \Pi_{\lfloor j+1 \rfloor})$ as operator kernel.*

Theorem 4.6 (*cf.* [8]) *The upper-middle correspondence exists for every $n \in \mathbb{N}$.*

We refer to [8] for explicit formulae for the characteristic numbers p^n_l of the upper-middle-state correspondence, $\forall n \in \mathbb{N}$, $1 \leq l \leq n$. The symbol correspondence with characteristic numbers $p^n_{l-} = (-1)^l p^n_l$ is called the *lower-middle correspondence* and obviously also exists for every $n \in \mathbb{N}$.

4.2 Proof of an important conjecture

The Berezin and the upper-middle (and lower-middle) correspondences are particular cases of coherent-state correspondences, *cf.* [8], and these satisfy a very nice property, as stated bellow.

Definition 4.8 *A symbol correspondence is called* mapping-positive *if it maps positive (resp. positive-definite) operators into positive (resp. strictly-positive) functions.*

Theorem 4.7 **(*cf.* [8])** *A symbol correspondence is mapping-positive iff its operator kernel is a convex combination of projectors* Π_k*, for* $k = 1, ..., n+1$*.*

The mapping-positive property for a correspondence implies in many nice analytical properties for the symbols and, in particular, the corresponding symbols of (pure or mixed) states are non-negative functions which, upon suitable renormalization, can be seen as probability densities on S^2. However, no Stratonovich-Weyl (i.e. isometric) correspondence possesses this nice mapping-positive property. This statement is stated as a conjecture in [8]. Here we give its proof:

Proposition 4.1 *No mapping-positive correspondence is a Stratonovich-Weyl correspondence.*

Proof: Let c_l^n, $l = 1, ..., n$, be the characteristic numbers of a mapping-positive correspondence. From theorem 4.7 and expression (4.1), we have

$$c_l^n = \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} (-1)^{k+1} a_k C_{m,-m,\ 0}^{\,j,\ \ j,\ l} \tag{4.5}$$

where $m = j - k + 1$, $a_k \geq 0$, for $1 \leq k \leq n+1$, and $\sum_{k=1}^{n+1} a_k = 1$. Since the Clebsch-Gordan coefficients are coefficients of a unitary transformation of basis, they satisfy $|C_{m,-m,\ 0}^{\,j,\ \ j,\ l}| \leq 1$, hence

$$\begin{aligned} |c_l^n| &= \left| \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} (-1)^{k+1} a_k C_{m,-m,\ 0}^{\,j,\ \ j,\ l} \right| \\ &\leq \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} \left| (-1)^{k+1} a_k C_{m,-m,\ 0}^{\,j,\ \ j,\ l} \right| = \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} a_k \left| C_{m,-m,\ 0}^{\,j,\ \ j,\ l} \right| \\ &\leq \sqrt{\frac{n+1}{2l+1}} \sum_{k=1}^{n+1} a_k = \sqrt{\frac{n+1}{2l+1}}\ . \end{aligned} \tag{4.6}$$

So, for $1 \leq l \leq 2j$, we have $|c_l^n| < \sqrt{\frac{n+1}{2l+1}}$. In particular, for $l > j$, we have $|c_l^n| < 1$. Thus, such characteristic numbers do not define a Stratonovich-Weyl correspondence. $\square$

4.3 Sequences of Symbol Correspondences

How to study Bohr's correspondence principle via symbol correspondences for spin systems?

Definition 4.9 *Let* $\Delta^+(\mathbb{N}^2) = \{(n, l) \in \mathbb{N}^2 : n \geq l > 0\}$*. A* sequence of symbol correspondences *is a sequence* $\boldsymbol{W}_{\mathcal{C}} = \{W^j\}_{n\in\mathbb{N}}$*, where* $\mathcal{C} : \Delta^+(\mathbb{N}^2) \to \mathbb{R}^*$ *is a function such that each element* W^j *of the sequence has characteristic numbers* $c_l^n = \mathcal{C}(n, l)$*. In addition,* $\boldsymbol{W}_{\mathcal{C}}$ *is of* limiting type *if* $\lim_{n\to\infty} \mathcal{C}(n, l) = c_l^\infty \in \mathbb{R}$*,* $\forall l \in \mathbb{N}$*.*

Recall that the absence of Planck's constant in Classical Mechanics can be seen as a scale situation. Likewise, here we shall take $\hbar = 1$ and look at the asymptotic limit $j \to \infty$, so that the Poisson bracket may emerge from the commutator of twisted product as the term of first order in $1/n$.

Definition 4.10 *A sequence of symbol correspondences is of* Poisson type *if,* $\forall l_1, l_2 \in \mathbb{N}$*, its twisted products satisfy*

i) $\lim_{n\to\infty}(Y_{l_1}^{m_1} \star^n Y_{l_2}^{m_2} - Y_{l_2}^{m_2} \star^n Y_{l_1}^{m_1}) = 0$,

ii) $\lim_{n\to\infty}(Y_{l_1}^{m_1} \star^n Y_{l_2}^{m_2} + Y_{l_2}^{m_2} \star^n Y_{l_1}^{m_1}) = 2Y_{l_1}^{m_1} Y_{l_2}^{m_2}$,

iii) $\lim_{n\to\infty}(n[Y_{l_1}^{m_1} \star^n Y_{l_2}^{m_2} - Y_{l_2}^{m_2} \star^n Y_{l_1}^{m_1}]) = 2i\{Y_{l_1}^{m_1}, Y_{l_2}^{m_2}\}$.

It is of anti-Poisson type *if the third property is replaced by*

iii') $\lim_{n\to\infty}(n[Y_{l_1}^{m_1} \star^n Y_{l_2}^{m_2} - Y_{l_2}^{m_2} \star^n Y_{l_1}^{m_1}]) = -2i\{Y_{l_1}^{m_1}, Y_{l_2}^{m_2}\}$.

The convergences are taken uniformly.

We emphasize that generic sequences of symbol correspondences are not of Poisson or anti-Poisson type. However, there is a simple algebraic criterion to know if the above definition is satisfied by a sequence of symbol correspondences.

Theorem 4.8 (*cf.* [8]) *A sequence of symbol correspondences* $\boldsymbol{W}_{\mathcal{C}}$ *is of Poisson (resp. anti-Poisson) type iff it is of limiting type and its characteristic numbers satisfy* $c_l^\infty = 1$ *(resp.* $c_l^\infty = (-1)^l$*),* $\forall l \in \mathbb{N}$*.*

Corollary 4.8.1 (*cf.* [8]) *The sequences of standard Stratonovich-Weyl and of standard Berezin correspondences are of Poisson type. The sequences of their alternate correspondences are of anti-Poisson type. The sequence of upper-middle correspondences is neither of Poisson nor of anti-Poisson type. Likewise for the sequence of lower-middle correspondences.*

5 Localization of Symbols of Projectors

The projectors Π_k, for $k = 1, ..., n+1$, are pure states of spin which are J_3-invariant. Motivated by a physical view of these objects, some kind of asymptotic localization of their symbols is a behavior we should look for. In this section, we propose a first approach to study the asymptotic localization of such symbols, especially for the case of mapping-positive correspondences. Our goal is to establish a physically intuitive criterion for when a symbol correspondence is of Poisson or anti-Poisson type.

Let W^j be a symbol correspondence with characteristic numbers c_l^n. From Theorem 4.1 and expressions (4.1) and (3.4), we have

$$W^j_{\Pi_k}(\boldsymbol{n}) = \frac{1}{n+1} + \frac{(-1)^{k-1}}{\sqrt{n+1}} \sum_{l=1}^{n} c_l^n C^{j,\ j,\ l}_{m,-m,\ 0} \sqrt{2l+1} P_l(\cos\varphi)\ , \tag{5.1}$$

where $P_l = P_l^0$ are the Legendre polynomials and $m = j - k + 1$. In accordance with our convention of diagonalizing J_3, these symbols are invariant by rotations around the z-axis. Another property to highlight is the parity between projector symbols of alternate correspondences. Let W^{j-} be the alternate correspondence of W^j, that is, it has characteristic numbers $c_{l-}^n = (-1)^l c_l^n$. Then $W^j_{\Pi_k}(z) = W^{j-}_{\Pi_k}(-z)$, so they are the reflection of each other with respect the equator. Because Clebsch-Gordan coefficients satisfy $C^{j,\ j,\ l}_{m_1,m_2,m} = (-1)^{2j-l} C^{j,\ \ j,\ \ l}_{-m_1,-m_2,-m}$ (*cf.* [1]), the same thing occurs for symbols of projectors in symmetric states under the same correspondence. Let k' corresponds to the eigenvalue $-m$ of J_3, *ie*, $-m = j - k' + 1$. Then, $W^j_{\Pi_k}(z) = W^j_{\Pi_{k'}}(-z)$.

A simple expression for the standard Berezin correspondence can be determined:

Proposition 5.1

$$B^j_{\Pi_k}(\boldsymbol{n}) = \binom{n}{k-1} \frac{(1+z)^{n-k+1}(1-z)^{k-1}}{2^n}$$

$$B^{j-}_{\Pi_k}(\boldsymbol{n}) = \binom{n}{k-1} \frac{(1-z)^{n-k+1}(1+z)^{k-1}}{2^n}$$

Proof: Using theorem 4.4 and taking $P = \Pi_k$, we obtain an expression in terms of $|z_1|^2$ and $|z_2|^2$. Using that $|z_1|^2 + |z_2|^2 = 1$ and, by the Hopf map, $z = |z_1|^2 - |z_2|^2$, we get the expression for $B^j_{\Pi_k}$. For the second expression, we use $B^{j-}_{\Pi_k}(z) = B^j_{\Pi_k}(-z)$. □

In view of the discussion in the beginning of this section, we make the following definitions.

Definition 5.1 *Let* $\boldsymbol{W}_{\mathcal{C}} = \{W^j\}_{n\in\mathbb{N}}$ *be a sequence of symbol correspondences (cf. Definition 4.9) and let* $\{k_n\}_{n\in\mathbb{N}}$ *be a sequence of natural numbers such that* $1 \le k_n \le n+1$*, so that* $\Pi_{k_n} \in \mathcal{B}(\mathcal{H}_j)$*. A* Π-symbol sequence *is a sequence of symbols* $\{W^j_{\Pi_{k_n}}\}_{n\in\mathbb{N}}$*. In addition,* $\{W^j_{\Pi_{k_n}}\}_{n\in\mathbb{N}}$ *is said to be* r-convergent *if* $k_n/n \to r \in [0,1]$ *, as* $n \to \infty$*.*

Definition 5.2 *Let* $\{W^j_{\Pi_{k_n}}\}_{n\in\mathbb{N}}$ *be a* Π*-symbol sequence. A* Π-distribution sequence $\{\rho^j_{k_n}\}_{n\in\mathbb{N}}$ *is a sequence of (quasiprobability) distributions on* $[-1,1]$ *given by* $\rho^j_{k_n} = \frac{n+1}{2}W^j_{\Pi_{k_n}}$ *restricted to the* z*-axis (*J_3*-invariance). In addition,* $\{\rho^j_{k_n}\}_{n\in\mathbb{N}}$ *is said to be* r-convergent *if* $\{W^j_{\Pi_{k_n}}\}_{n\in\mathbb{N}}$ *is* r*-convergent.*

The characteristic numbers of the sequence of symbol correspondences will be also referred as the characteristic numbers of the Π-symbol and the Π-distribuition sequences. Moreover, let ρ^j_k be an element of a Π-distribuition sequence with characteristic numbers c^n_l. From (5.1), we get explicity that

$$\rho^j_k(z) = \frac{1}{2} + \frac{(-1)^{k-1}\sqrt{n+1}}{2}\sum_{l=1}^{n} c^n_l C^{j,\ \ j,\ l}_{m,-m,\ 0}\sqrt{2l+1}P_l(z)\ . \tag{5.2}$$

Definition 5.3 *A* Π*-symbol sequence* localizes *(asymptotically) if its* Π*-distribution sequence converges, as distribution, to Dirac's delta* $\delta(z-a)$ *for some* $a\in[-1,1]$.

To bypass the complicated general case, we will start by taking a special look at Π-distribution sequences that are, in fact, probability distribution sequences.

Definition 5.4 *A* Π*-distribution sequence is* positive *if it is constructed from a sequence of mapping-positives correspondences. We say the same for its* Π*-symbol sequence.*

The following lemma on probability distributions can be proved using Chebyshev's inequality (*cf. eg.* [9]).

Lemma 5.1 *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of probability distributions on* $[-1,1]$ *with mean values* $\{\mu_n\}_{n\in\mathbb{N}}$ *and variances* $\{\sigma^2_n\}_{n\in\mathbb{N}}$*. Then* $f_n\to\delta(z-\mu)$*, as distribution, iff* $\mu_n\to\mu$ *and* $\sigma^2_n\to 0$.

Let $\{\rho^j_{k_n}\}_{n\in\mathbb{N}}$ be a positive Π-distribution sequence with characteristic numbers c^n_l. We denote by E_n the expected value operator defined by $\rho^j_{k_n}$, also $\mu_n = E_n(z)$ and $\sigma^2_n = E_n((z-\mu_n)^2) = E_n(z^2)-\mu^2_n$. To compute these quantities, we must integrate $\int_{-1}^{1} z\rho^j_{k_n}dz$ and $\int_{-1}^{1} z^2\rho^j_{k_n}dz$. Therefore, we need expressions for Clebsh-Gordan coefficients of the form $C^{j,\ \ j,\ 1}_{m,-m,\ 0}$ and $C^{j,\ \ j,\ 2}_{m,-m,\ 0}$. From [10],

$$C^{j,\ \ j,\ 1}_{m,-m,\ 0} = (-1)^{k+1}2(j-k+1)\sqrt{\frac{3}{(n+2)(n+1)n}}, \tag{5.3}$$

$$C^{j,\ \ j,\ 2}_{m,-m,\ 0} = (-1)^{k-1}\sqrt{5}\frac{(n-k+1)(n-k)-4(k-1)(n-k+1)+(k-1)(k-2)}{\sqrt{(n-1)n(n+1)(n+2)(n+3)}}, \tag{5.4}$$

where, as usual, $m=j-k+1$. Hence, we have

$$\mu_n = c^n_1\frac{n-2(k_n-1)}{\sqrt{n(n+2)}}, \tag{5.5}$$

$$\sigma_n^2 = \frac{2c_2^n}{3} \cdot \frac{(n-k_n+1)(n-k_n) - 4(k_n-1)(n-k_n+1) + (k_n-1)(k_n-2)}{\sqrt{(n-1)n(n+2)(n+3)}} + \frac{1}{3} - \frac{(c_1^n)^2(n-2(k_n-1))^2}{n(n+2)}. \tag{5.6}$$

From (5.5)-(5.6) and Lemma 5.1 we obtain in a rather straightforward way:

Proposition 5.2 *Let $\{\rho^j_{k_n}\}_{n\in\mathbb{N}}$ be a positive r-convergent Π-distribution sequence. Iff the characteristic numbers of $\{\rho^j_{k_n}\}_{n\in\mathbb{N}}$ are s.t. $c_1^\infty = 1$, then its mean values converge to $1-2r \in [-1,1]$. Iff $c_1^\infty = -1$, the mean values converge to (the reflected value) $2r-1 \in [-1,1]$.*

Theorem 5.2 *Every positive r-convergent Π-distribution sequence, $\forall r \in [0,1]$, constructed from the same sequence of symbol correspondences have variances converging to 0 iff their characteristic numbers are such that $(c_1^\infty)^2 = c_2^\infty = 1$. If $c_1^\infty = 1$, they converge to $\delta(z-1+2r)$ (localize in $1-2r$), if $c_1^\infty = -1$, they converge to $\delta(z-2r+1)$ (localize in $2r-1$).*

Corollary 5.2.1 *Every positive r-convergent Π-symbol sequence, $\forall r \in [0,1]$, constructed from the same sequence of symbol correspondences of Poisson (resp. anti-Poisson) type converges to $\delta(z-1+2r)$ (resp. $\delta(z-2r+1)$), ie, localizes in $1-2r$ (resp. $2r-1$). In particular, every r-convergent Π-symbol sequence constructed from a sequence of standard or alternate Berezin correspondences localizes.*

Now, to expand our study to the general case, we need Edmonds formula.

Lemma 5.3 (Edmonds formula) *Let $\{k_n\}_{n\in\mathbb{N}}$ be as in Definition 5.1 and $m = j-k_n+1$ (depending implicitly on n). If $k_n/n \to r \in [0,1]$, then*

$$\lim_{n\to\infty} (-1)^{k_n-1} C^{j,\ \ j,\ l}_{m,-m,\ 0} \sqrt{\frac{n+1}{2l+1}} = P_l(1-2r), \qquad \forall l \in \mathbb{N}. \tag{5.7}$$

Proof: At first, we follow Brussard and Tolhoek [2] and Flude [4] to prove that

$$\lim_{j\to\infty} C^{l,\ j-\tau,\ j}_{\mu,m-\mu,\ m} = (-1)^{l-\tau} d^l_{\mu,\tau}(\theta), \tag{5.8}$$

where $d^l_{\mu,\tau}$ is the Wigner (small) d-function and $\theta \in [0,\pi]$ is such that $\cos\theta = \lim_{j\to\infty}(m/j)$, holds for fixed l, μ, τ and if either (i) $j-|m| \to \infty$ or (ii) $j-|m| \to 0$. For (i), we start with (*cf.* [1,8,10]):

$$C^{l,\ j-\tau,\ j}_{\mu,m-\mu,\ m} = \sqrt{\frac{2j+1}{2j-\tau+l+1}} \sqrt{(l-\tau)!(l+\tau)!(l+\mu)!(l-\mu)!} \times \sum_z \frac{(-1)^z \sqrt{\Omega_1\Omega_2\Omega_3}}{z!(l-\tau-z)!(l-\mu-z)!(\tau+\mu+z)!} \tag{5.9}$$

where

$$\Omega_1 = \frac{(2j-\tau-l)!}{(2j-\tau+l)!}, \quad \Omega_2 = \frac{(j-\tau-m+\mu)!(j-m)!}{[(j-l-m+\mu+z)]^2}, \quad \Omega_3 = \frac{(j-\tau+m-\mu)!(j+m)!}{[(j-\tau+m-\mu-z)!]^2} .$$

It is easy to see that the summation over z is finite, so we can take the limit of each term. Thus,

$$\Omega_1\Omega_2\Omega_3 \sim (2j)^{-2l}(j-m)^{2l-\tau-\mu-2z}(j+m)^{\tau+\mu+2z} = \left(\frac{j-m}{2j}\right)^{2l-\tau-\mu-2z} \left(\frac{j+m}{2j}\right)^{\tau+\mu+2z} ,$$

where we have used the Stirling approximation

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n} \tag{5.10}$$

in the expressions for Ω_1, Ω_2, Ω_3. Supposing m/j converges, we can take a $\theta \in [0,\pi]$ such that $\cos\theta = \lim_{j\to\infty}(m/j)$, so that $\sin\frac{\theta}{2} = \sqrt{\frac{j-m}{2j}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{j+m}{2j}}$. Then, we get (*cf.* [10] for the exact expression of the Wigner d-function)

$$\begin{aligned}\lim_{j\to\infty} C^{l,\ j-\tau,\ j}_{\mu,m-\mu,\ m} &= \sqrt{(l-\tau)!(l+\tau)!(l+\mu)!(l-\mu)!} \sum_z \frac{(-1)^z \left(\sin\frac{\theta}{2}\right)^{2l-\tau-\mu-2z} \left(\cos\frac{\theta}{2}\right)^{\tau+\mu+2z}}{z!(l-\tau-z)!(l-\mu-z)!(\tau+\mu+z)!} \\ &= (-1)^{l-\tau} d^l_{\mu,\tau}(\theta) .\end{aligned}$$

Now, we must note that $j-|m|$ is a sequence of integer numbers, so the convergence condition (ii) implies that there is a j_0 such that for all $j > j_0$ we have $j-|m| = 0 \Rightarrow m = \pm j$. For $m = j$, we have $\mu \geq |\tau|$ and $m/j \to 1 \Rightarrow \theta = 0 \Rightarrow d^l_{\mu,\tau} = \delta_{\mu,\tau}$; for $m = -j$, we have $\mu \leq |\tau|$ and $m/j \to -1 \Rightarrow \theta = \pi \Rightarrow d^l_{\mu,\tau} = (-1)^{l-\tau}\delta_{-\mu,\tau}$. Writing, again, the exact formula

$$C^{j,\ j-\tau,\ l}_{j,-j\pm\mu,\pm\mu} = \sqrt{\frac{2l+1}{2j-\tau+l+1}} \sqrt{\frac{(l-\tau)!(l\pm\mu)!}{(l+\tau)!(-\tau\pm\mu)!(l\mp\mu)!}} \sqrt{\frac{(2j)!(2j-\tau\mp\mu)!}{(2j-\tau+l)!(2j-\tau-l)!}}$$

and then using (5.10) in the last factor, plus the symmetry properties of Clebsch-Gordan coefficients (*cf.* [1, 8, 10]), we conclude that

$$\lim_{j\to\infty} C^{l,j-\tau,\ j}_{\mu,j-\mu,\ j} = (-1)^{l-\tau}\delta_{\mu,\tau} \quad , \quad \lim_{j\to\infty} C^{l,\ j-\tau,\ j}_{\mu,-j-\mu,\ -j} = \delta_{-\mu,\tau} .$$

Now, we expand condition (ii) to condition (iii) $j-|m| < a$ for some $a \in \mathbb{N}$ when $\tau = \mu = 0$. We first assume $j-|m| \to a \in \mathbb{N}_0$. From [10], we have the following recursive relation:

$$C^{l,\quad j,\quad j}_{0,\pm(j-a-1),\pm(j-a-1)} = \sqrt{\frac{l(l+1)}{(2j-a)(a+1)}} C^{l,\quad j,\ j}_{\pm1,\pm(j-a-1),\pm(j-a)} + C^{l,\ \ j,\ j}_{0,\pm(j-a),\pm(j-a)} \tag{5.11}$$

Assuming the coefficients in the rhs of (5.11) satisfy the already proved (5.8), we obtain

$$\lim_{j\to\infty} C^{l,\quad j,\quad j}_{0,\pm(j-a-1),\pm(j-a-1)} = d^l_{0,0}(\theta)$$

where $\theta = 0$ for plus sign and $\theta = \pi$ for minus sign. The hypothesis holds for $a = 0$, so, by induction, the formula holds for all $a \in \mathbb{N}_0$. Finally, $j - |m|$ does not need to converge to $a \in \mathbb{N}_0$, it is sufficient for it to remain finite, because all the Clebsch-Gordan coefficients converge to the same value.

Putting together all the results, we have proved that, if $\lim_{j\to\infty}(m/j) = \cos\theta = 1 - 2r$, then

$$\lim_{n\to\infty}(-1)^{k_n-1}C^{j,\ \ j,\ l}_{m,-m,\ 0}\sqrt{\frac{n+1}{2l+1}} = \lim_{n\to\infty}(-1)^l C^{l,\ j,\ j}_{0,m,m} = d^l_{0,0}(\theta) = P_l(1-2r)\ ,$$

which is Edmonds formula (5.7). □

Theorem 5.4 *If every r-convergent Π-symbol sequence, $\forall r \in [0,1]$, constructed from the same sequence of symbol correspondences converges to $\delta(z-1+2r)$ (resp. $\delta(z-2r+1)$), then the sequence of symbol correspondences is of Poisson (rep. anti-Poisson) type.*

Proof: Let $\{\rho^j_{k_n}\}_{n\in\mathbb{N}}$ be a r-convergent Π-distribution sequence such that $\rho^j_{k_n} \to \delta(z-1+2r)$. Then,

$$\lim_{n\to\infty}\int_{-1}^{1}\rho^j_{k_n}(z)P_l(z)dz = P_l(1-2r)\ ,\forall l \in \mathbb{N}\ . \tag{5.12}$$

On the other hand, from (5.2) and Lemma 5.3, we have

$$\lim_{n\to\infty}\int_{-1}^{1}\rho^j_{k_n}(z)P_l(z)dz = \lim_{n\to\infty}(-1)^{k_n-1}c^n_l C^{j,\ \ j,\ l}_{m,-m,\ 0}\sqrt{\frac{n+1}{2l+1}} = c^\infty_l P_l(1-2r). \tag{5.13}$$

So, $\forall l \in \mathbb{N}$ s.t. $P_l(1-2r) \neq 0$, we must have $c^\infty_l = 1$. But, $\forall l \in \mathbb{N}$, there is a $r \in [0,1]$ s.t. $P_l(1-2r) \neq 0$. Similarly, from $P_l(-z) = (-1)^l P_l(z)$, localization in $2r-1$ implies $c^\infty_l = (-1)^l$. □

Corollary 5.4.1 *Every positive r-convergent Π-symbol sequence, $\forall r \in [0,1]$, constructed from the same sequence of symbol correspondences localizes in $1-2r$ (resp. $2r-1$), iff the sequence of symbol correspondences is of Poisson (rep. anti-Poisson) type.*

6 Conclusion

Of all symbol correspondences between classical and quantum spin systems, two kinds are specially relevant: the isometries (Stratonovich-Weyl correspondences) and the mapping-positive ones (coherent-state correspondences). Rios and Straume [8] conjectured that these two sets of spin-j symbol correspondences are disjoint. Here we have proven this statement, *cf.* Proposition 4.1.

In [8] the authors also showed that, given a sequence of symbol correspondences, a necessary and sufficient condition for the emergence of classical Poisson (or anti-Poisson) dynamics for the symbols, from the quantum dynamics of the operators, is the convergence of the symbol correspondence sequence to a special isometry, the standard Stratonovich-Weyl (or its alternate, resp.), *cf.* Theorem

4.8. This is the case for the sequence of standard (resp. alternate) Berezin correspondences, but not for the sequences of upper-middle or lower-middle correspondences (also coherent-state sequences).

Here we have shown that another and more physically intuitive criterion for the asymptotic emergence of Poisson (or anti-Poisson) dynamics for the symbols is as follows: for every r-convergent sequence of J_3-invariant projectors, $r \in [0, 1]$, the corresponding sequence of their symbols localizes at $z = 1 - 2r$ (resp. at $z = 2r - 1$), *cf.* Definitions 5.1-5.2 and Theorem 5.4. This is a necessary and sufficient criterion for sequences of coherent-state correspondences, *cf.* Corollary 5.4.1, but so far we have just shown its sufficiency for general sequences of correspondences.

Numerical analysis lie out of the scope of this work and shall be reported elsewhere, but we point out that Proposition 5.1 provides a simple way to verify the localization of Π-symbol sequences constructed from standard/alternate Berezin correspondences. For standard/alternate Stratonovich-Weyl cases, calculations using expression (5.2) show that the higher j is, the more oscillatory the symbols are, so, if the converse of Theorem 5.4 is to be true, these oscillations have to go to zero in some average, except at the immediate vicinity of the maximum related to the point of localization.

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