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Review

Interactions between arbitrary electromagnetic shaped beams and circular and elliptical infinite cylinders: A review

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ABSTRACT

The theories describing the interactions between arbitrary electromagnetic shaped beams and infinite cylinders, both in cylindrical and elliptical coordinates, are reviewed. Two main approaches are considered (i) an approach in terms of usual functions named the plane-wave spectrum approach and (ii) an approach in terms of Schwartz distributions, the latter leading to a formalism more general than the former. The relationship between both approaches, in cases when the plane-wave spectrum approach is feasible, is discussed. The attention is strongly focused on the description of the illuminating beams, in particular when using localized approximations in circular and elliptical coordinates, similar to the ones already developed in the case of spherical coordinates.

1. Introduction

The generalized Lorenz-Mie theory (GLMT) describes the interaction between electromagnetic arbitrary shaped beams and homogeneous spheres, relying on two ingredients (i) the decomposition of incident waves, scattered waves and waves internal to the sphere in terms of partial waves and (ii) the fact that spherical coordinates allow one to use a method of separation of variables to deal with Maxwell's equations [1–3]. This GLMT has been called GLMT in the strict sense (*stricto sensu*) and the name GLMT has been used generically to designate other theories of interactions between electromagnetic arbitrary shaped beams and a family of particles when waves are decomposed into partial waves, and when the particles are regular enough to allow one to use a method of separation of variables, e.g. GLMT for multilayered spheres [4], assemblies of spheres and aggregates [5], and spheres with eccentrically located spherical inclusions [6].

Two other GLMTs have been developed, namely for infinite cylinders, either (i) with a circular or (ii) with an elliptical cross-section. The development of these theories ranged from 1994 to 2000, which is a fairly long span, particularly when it is noted that the authors were already trained with the GLMT *stricto sensu*. The explanation lies in the fact that, when trying to build these theories in the case of illuminating beams described by the Maxwellian contributions to Gaussian Davis beam descriptions, it has been found that the use of usual functions was relentlessly leading to a failure. It has then been recognized that the most general compulsory framework to be used was the one of Schwartz distributions. This fact renders the access of the newcomer to GLMTs for infinite cylinders fairly difficult. A current revival of the interest for such theories, for instance in the framework of studies devoted to photophoretic forces, e.g. [7–9], and other issues, e.g. [10–13], then provided a motivation to review the GLMTs for infinite cylinders, expounding the published material under a single roof, with an effort of pedagogic skill allowing the newcomer to use an efficient inroad to explore the issue.

The paper is organized as follows. Section 2 describes the GLMT for circular cylinders in terms of usual functions. It is to be noted that the material presented in this section has never been explicitly published in the literature, although numerical results were published, but these numerical results have been preceded by a discussion in terms of distributions which made the access to the newcomer fairly opaque. Section 2 describes the GLMT for circular cylinders in terms of distributions. It explains why the use of distributions has been necessary, how to use them and how we can pass from a formulation to the other (when this passage is possible). Section 3 deals with the case of elliptical infinite cylinders, both in terms of usual functions and in terms of distributions. Section 4 describes localized approximations which may be used

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to speed up numerical computations associated with the description of the illuminating beams. Section 5 complements the paper by discussing other worldwide contributions devoted to the interactions between arbitrary shaped beams and cylinders. Section 6 is a conclusion.

2. Formalism in terms of usual functions

The configuration to be studied is described from Fig. 1 adapted from [14]. The scatterer is an infinite cylinder with a circular cross-section of radius a. The axis of the cylinder is chosen to coincide with the axis $(O_c z)$ of a Cartesian coordinate system (x, y, z). The material of the cylinder is assumed to be non-magnetic, linear, isotropic and homogeneous with respect to both space and time. A cylindrical coordinate system $(x_1, x_2, x_3) = (z, \rho, \varphi)$ is attached to the Cartesian system (x, y, z).The cylinder is illuminated by an arbitrary incident shaped beam having an $\exp(i\omega t)$-time harmonic dependence which will be omitted in the sequel, according to an usual practice. The incident wave is defined by its electric field components $(E_z^i, E_\rho^i, E_\varphi^i)$ and by its magnetic field components $(H_z^i, H_\rho^i, H_\varphi^i)$. It propagates in the surrounding medium, assumed to be non-absorbing. The complex refractive index M of the cylinder material is taken relatively to the surrounding medium. The problem is to solve Maxwell's equations in order to determine the scattered wave and the cylinder (or internal) wave.

2.1. The Bromwich method and generating functions

In spherical coordinates, the GLMT *stricto sensu* has been originally built using scalar potentials, more particularly Bromwich Scalar Potentials (BSPs). It has later been converted to the use of Vector Spherical Wave Functions (VSWFs), see[15]., useful to use translational theorems in the case of assemblies of spheres and aggregates [5] or in the case of spherical particles with an eccentric spherical inclusion [6]. VSWFs are furthermore useful in EBCM (Extended Boundary Condition Method, e.g. [16–18]) devoted to the study of scattering by irregularly shaped particles, e.g. review in Section 8.1 of [19]. The use of scalar potentials may be preferred because it leads to more explicit and more readable formulae, although, historically, it has been a matter of contingency [20]. Therefore the GLMT for circular infinite cylinders, to which this section is devoted, has been developed as well using BSPs. The use of BSPs is exhaustively reviewed in [3], Sections 2.2 and 2.3, see as well [21,22]. For the sake of completeness, the theory of BSPs in circular cylindrical coordinates is reviewed below, following [14].

At point P of the cylindrical coordinate system $(x_1, x_2, x_3) = (z, \rho, \varphi)$, Pythagora's theorem reads as:

$$ds^2 = (e_1)^2 dz^2 + (e_2)^2 d\rho^2 + (e_3)^2 d\varphi^2 \tag{1}$$

in which ds is the infinitesimal distance between two points P and $P + dP$, with:

$$e_1 = e_2 = 1,\ e_3 = \rho \tag{2}$$

leading to:

$$\left.\begin{array}{c} e_1 = 1 \\ \frac{\partial}{\partial x_1}\left(\frac{e_2}{e_3}\right) = \frac{\partial}{\partial z}\left(\frac{1}{\rho}\right) = 0 \end{array}\right\} \tag{3}$$

For the considered non-magnetic, linear, local, isotropic and homogeneous media, the general Maxwell's equations reduce to a simpler form that we shall call the Special Maxwell's Equations (SMEs). When Eq. (3) is satisfied, SMEs can be solved by using the Bromwich method relying on BSPs. A counter-example is the case of spheroidal coordinates, e.g. [23,24]. When using BSPs, any solution to the SMEs is the summation of two special solutions, the TM wave (Transverse Magnetic Wave) and the TE wave (Transverse Electric Wave). The special solutions may be found by first solving a partial differential equation for BSPs U_{TM} and U_{TE}. In the system (z, ρ, φ), this equation, valid for both U_{TM} and U_{TE}, reads as:

$$\frac{\partial^2 U}{\partial z^2} + k^2 U + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} = 0 \tag{4}$$

in which k is the wavenumber in the considered material (i.e. it must be replaced by k_c in the cylinder) and U stands either for U_{TM} and U_{TE}. Once U_{TM} and U_{TE} are determined, all TM and TE field components may be evaluated by using the following set of equations:

$$E_{z,TM} = \frac{\partial^2 U_{TM}}{(\partial z)^2} + k^2 U_{TM} \tag{5}$$

$$E_{\rho,TM} = \frac{\partial^2 U_{TM}}{\partial z \partial \rho} \tag{6}$$

$$E_{\varphi,TM} = \frac{1}{\rho}\frac{\partial^2 U_{TM}}{\partial z \partial \varphi} \tag{7}$$

$$H_{z,TM} = 0 \tag{8}$$

$$H_{\rho,TM} = \frac{i\omega\varepsilon}{\rho}\frac{\partial U_{TM}}{\partial \varphi} \tag{9}$$

$$H_{\varphi,TM} = -i\omega\varepsilon \frac{\partial U_{TM}}{\partial \rho} \tag{10}$$

$$E_{z,TE} = 0 \tag{11}$$

$$E_{\rho,TE} = -\frac{i\omega\mu}{\rho}\frac{\partial U_{TE}}{\partial \varphi} \tag{12}$$

$$E_{\varphi,TE} = i\omega\mu \frac{\partial U_{TE}}{\partial \rho} \tag{13}$$

$$H_{z,TE} = \frac{\partial^2 U_{TE}}{(\partial z)^2} + k^2 U_{TE} \tag{14}$$

$$H_{\rho,TE} = \frac{\partial^2 U_{TE}}{\partial z \partial \rho} \tag{15}$$

$$H_{\varphi,TE} = \frac{1}{\rho}\frac{\partial^2 U_{TE}}{\partial z \partial \varphi} \tag{16}$$

in which μ and ε denote the permeability and the permittivity of the medium respectively (ε must be replaced by ε_c inside the cylinder).

As usual, solutions of Eq. (4) are searched by using coordinate separability according to:

$$U(z, \rho, \varphi) = Z(z) R(\rho) \phi(\varphi) \tag{17}$$

For use in the sequel, this equation must be commented. A coordinate system in which solutions of the form of Eq. 17 exist is called a separable coordinate system. There exist only eleven separable coordinate systems [25,26], including the spherical coordinate system and the cylindrical coordinate systems, for both circular and elliptical cylinder coordinates. Following [26], all solutions of the partial differential Eq. (17) can be built up from linear combinations of the members of the family of separable solutions. We shall later however find that this is true, in general, only if we extend the formalism from usual functions to Schwartz distributions. For the time being, let us be satisfied when working with usual functions.

Inserting the separability Eq. (17) in the BSP Eq. (4), it is found that the (usual) functions $Z(z)$, $R(\rho)$, $\phi(\varphi)$ satisfy:

$$\frac{d^2\phi}{d\varphi^2} + b\phi = 0 \tag{18}$$

$$\frac{d^2Z}{dz^2} + aZ = 0 \tag{19}$$

$$\rho\frac{d}{d\rho}\rho\frac{dR}{d\rho} + (k^2\rho^2 - a\rho^2 - b)R = 0 \tag{20}$$

The solutions of the harmonic Eq. (18) must satisfy a continuity equation $\phi(0) = \phi(2n\pi)$, n integer, and therefore take the form $\exp(im\varphi)$, $m \in \mathbf{Z}$, in which we have set $b = m^2$. Writing down the general solution of Eq. (19) and requiring that solutions must remain finite when $z \to \pm\infty$, it is found that a must be a real number. Solutions may then be given the form $\exp(ik\gamma z)$, $(k\gamma) \in \mathbf{R}$, in which we have set $a = k^2\gamma^2$. Let us introduce:

$$r = k\rho\sqrt{1-\gamma^2} \tag{21}$$

Then, using $R(\rho) = \mathcal{R}(r)$, it is found that Eq. (20) becomes the Bessel equation:

$$r\frac{d}{dr}r\frac{d\mathcal{R}}{dr} + (r^2 - m^2)\mathcal{R} = 0 \tag{22}$$

Two independent solutions of this equation are a Bessel function of the first kind denoted $J_m(r)$ and a Bessel function of the second kind denoted $Y_m(r)$, also called a Neumann function $N_m(r)$, e.g. [27]. From these functions, a set of two other linearly independent solutions is formed by two Hankel functions:

$$H_m^{(1)}(r) = J_m(r) + iY_m(r) \tag{23}$$

$$H_m^{(2)}(r) = J_m(r) - iY_m(r) \tag{24}$$

According to the separability theorem expressed in terms of usual functions, the general BSPs may then be obtained as a linear combination of generating functions $G(z, \rho, \varphi)$ reading as:

$$G(z, \rho, \varphi) = \begin{pmatrix} J_m(r) \\ Y_m(r) \\ H_m^{(1)}(r) \\ H_m^{(2)}(r) \end{pmatrix} \exp(im\varphi)\exp(ik\gamma z),\ m \in Z \tag{25}$$

2.2. Incident wave expansion and Beam Shape Coefficients

Among the set $\left\{J_m(r), Y_m(r), H_m^{(1)}(r), H_m^{(2)}(r)\right\}$, only the functions $J_m(r)$ do not diverge at $r = 0$. Therefore, these functions must be chosen among the generating functions $G(z, \rho, \varphi)$ to express the incident wave expansion. The BSPs for the incident wave, denoted U_{TM}^i and U_{TE}^i, then read as:

$$U_{TM}^i = \frac{E_0}{k^2}\sum_{m=-\infty}^{+\infty}(-i)^m e^{im\varphi}\int I_{m,TM}(\gamma)J_m(k\rho\sqrt{1-\gamma^2})e^{ik\gamma z}d\gamma \tag{26}$$

$$U_{TE}^i = \frac{H_0}{k^2}\sum_{m=-\infty}^{+\infty}(-i)^m e^{im\varphi}\int I_{m,TE}(\gamma)J_m(k\rho\sqrt{1-\gamma^2})e^{ik\gamma z}d\gamma \tag{27}$$

Eqs. (26) and (27) generalize Eqs. (26) and (27) of [14] by introducing an integral over the continuous separation constant γ which, in a preliminary step, was omitted in [14] (we shall return to this issue later). Furthermore, prefactors in Eqs. (26) and (27) have been introduced for later convenience. Also, using the same terminology than in spherical coordinates, e.g. for the GLMT *stricto sensu*, $I_{m,TM}(\gamma)$ and $I_{m,TM}(\gamma)$ are called Beam Shape Coefficients (BSCs), independently of the fact that, depending on the context, they can be genuine coefficients as in [14], Beam Shape Functions (BSFs) as above, or Beam Shape Distributions (BSDs) as in Section 3.

From Eqs. (5) and (26), we then obtain:

$$E_z^i = E_{z,TM}^i = E_0\sum_{m=-\infty}^{+\infty}(-i)^m e^{im\varphi}\int(1-\gamma^2)I_{m,TM}(\gamma)J_m(k\rho\sqrt{1-\gamma^2})e^{ik\gamma z}d\gamma \tag{28}$$

The limits of the integral are not specified. They depend on whether we intend to preserve or not the evanescent waves. In the absence of evanescent waves, these limits are $(-1, +1)$, e.g. [28], [29] in which γ is to be replaced by $C = \cos\Gamma$ with Γ being a tilt angle ranging from 0 to π. Otherwise, the limits may be extended to $(-\infty, +\infty)$. To isolate the BSF $I_{m,TM}(\gamma)$, we may successively use a representation of the so-called Dirac function (better called a Dirac distribution, see again Section 3), and an orthogonality relation for $\exp(im\varphi)$, according to:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(\gamma-\gamma')Z}dZ = \delta(\gamma - \gamma') \tag{29}$$

$$\int_{-\infty}^{+\infty} e^{i(m-m')\varphi} d\varphi = 2\pi \delta_{mm'} \tag{30}$$

in which $Z = kz$, and $\delta_{mm'}$ is the Kronecker symbol. We then obtain:

$$I_{m,TM}(\gamma) = \frac{i^m}{4\pi^2(1-\gamma^2)J_m(k\rho\sqrt{1-\gamma^2})} \int_0^{2\pi} e^{-im\varphi} \int_{-\infty}^{+\infty} \frac{E_z^i}{E_0} e^{-i\gamma Z} dZ d\varphi \tag{31}$$

Working with H_z^i instead of E_z^i, we similarly establish:

$$I_{m,TE}(\gamma) = \frac{i^m}{4\pi^2(1-\gamma^2)J_m(k\rho\sqrt{1-\gamma^2})} \int_0^{2\pi} e^{-im\varphi} \int_{-\infty}^{+\infty} \frac{H_z^i}{H_0} e^{-i\gamma Z} dZ d\varphi \tag{32}$$

A few comments are now useful as follows:

(i) In spherical coordinates, we have two discrete separation constants so that BSPs and fields are expressed in terms of two discrete summations, a fact which is reflected in the notations $g_{n,TM}^m$ and $g_{n,TE}^m$ (n from 1 to ∞, $-n \le m \le +n$) for the BSCs in the GLMT *stricto sensu*. In the present case, we have one discrete separation constant and one continuous separation constant, so that BSPs and fields are expressed in terms of a discrete summation and of an integral, a fact which is reflected in the notations $I_{m,TM}(\gamma)$ and $I_{m,TE}(\gamma)$ for the BSCs. As a result, the extraction of the expressions of BSFs in terms of incident fields required the use of a distribution in Eq. (29), although that, from a physicist point of view, it can be viewed as a "function" without any damage. This anticipates the fact that a more rigorous (and more general) formulation, in terms of distributions, will have to be developed.

(ii) We see that the BSFs $I_{m,TM}(\gamma)$ and $I_{m,TE}(\gamma)$ are determined only in terms of z-components E_z^i and H_z^i respectively. In spherical coordinates, BSCs are similarly deduced only from radial components E_r^i and H_r^i (in a spherical coordinate system).

(iii) The BSFs are seemingly dependent on the coordinate ρ in contrast with the fact that they should be coordinate-independent complex functions. If the beam perfectly satisfies Maxwell's equations, this dependence over ρ must therefore be exactly cancelled out by the quadrature process. An analogous feature has been observed in spherical coordinates, e.g. [30–32] for plane waves and [33,34] for arbitrary shaped Maxwellian beams.

From Eq. (5), we have derived the expansion of $E_z^i = E_{z,TM}^i$ versus the BSF $I_{m,TM}(\gamma)$. Similarly, the other incident field components may be derived by using Eqs. (6)-(16), leading to:

$$E_{\rho,TM}^i = E_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \gamma\sqrt{1-\gamma^2} I_{m,TM}(\gamma) J_m'(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{33}$$

$$E_{\rho,TE}^i = \frac{E_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int I_{m,TE}(\gamma) J_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{34}$$

$$E_{\varphi,TM}^i = \frac{-E_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int \gamma I_{m,TM}(\gamma) J_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{35}$$

$$E_{\varphi,TE}^i = E_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \sqrt{1-\gamma^2} I_{m,TE}(\gamma) J_m'(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{36}$$

$$H_z^i = H_{z,TE}^i = H_0 \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int (1-\gamma^2) I_{m,TE}(\gamma) J_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{37}$$

$$H_{\rho,TM}^i = \frac{-H_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int I_{m,TM}(\gamma) J_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{38}$$

$$H_{\rho,TE}^i = H_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \gamma\sqrt{1-\gamma^2} I_{m,TE}(\gamma) J_m'(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{39}$$

$$H_{\varphi,TM}^i = -H_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \sqrt{1-\gamma^2} I_{m,TM}(\gamma) J_m'(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{40}$$

$$H_{\varphi,TE}^i = \frac{-H_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int \gamma I_{m,TE}(\gamma) J_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{41}$$

in which we have used:

$$\frac{H_0}{E_0} = \frac{\omega\varepsilon}{k} = \frac{k}{\omega\mu} \tag{42}$$

2.3. Scattered wave expansion, including the far-field case

In the generating functions $G(z, \rho, \varphi)$, we must now choose $H_m^{(2)}(r)$ to represent an outgoing wave (see asymptotic expression later). If the harmonic time dependence were $\exp(-i\omega t)$ instead of $\exp(+i\omega t)$, we would have conversely chosen to use $H_m^{(1)}(r)$. For convenience in the sequel, we shall simply note $H_m^{(2)}(r) = H_m(r)$. Then, we may write the BSPs of the scattered wave as:

$$U_{TM}^s = \frac{-E_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int S_{m,TM}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{43}$$

$$U_{TE}^s = \frac{H_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int S_{m,TE}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{44}$$

in which prefactors are again chosen for later convenience. The scattered field components are then given by:

$$E_z^s = E_{z,TM}^s = -E_0 \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int (1-\gamma^2) S_{m,TM}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{45}$$

$$E_{\rho,TM}^s = -E_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \gamma\sqrt{1-\gamma^2} S_{m,TM}(\gamma) H_m^{'}(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{46}$$

$$E_{\rho,TE}^s = \frac{E_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int S_{m,TE}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{47}$$

$$E_{\varphi,TM}^s = \frac{E_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int \gamma S_{m,TM}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{48}$$

$$E_{\varphi,TE}^s = E_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \sqrt{1-\gamma^2} S_{m,TE}(\gamma) H_m^{'}(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{49}$$

$$H_z^s = H_{z,TE}^s = H_0 \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int (1-\gamma^2) S_{m,TE}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{50}$$

$$H_{\rho,TM}^s = \frac{H_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int S_{m,TM}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{51}$$

$$H_{\rho,TE}^s = H_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \gamma\sqrt{1-\gamma^2} S_{m,TE}(\gamma) H_m^{'}(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{52}$$

$$H_{\varphi,TM}^s = H_0 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \sqrt{1-\gamma^2} S_{m,TM}(\gamma) H_m^{'}(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{53}$$

$$H_{\varphi,TE}^s = \frac{-H_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int \gamma S_{m,TE}(\gamma) H_m(k\rho\sqrt{1-\gamma^2}) e^{ik\gamma z} d\gamma \tag{54}$$

When the far-field condition is satisfied, e.g. Section 8 in [14], we may simplify the expressions of the scattered fields by using the following asymptotic expressions for the Hankel function $H_m(r)$ and its derivative, according to:

$$H_m(r) \to H_m^\infty(r) = \sqrt{\frac{2}{\pi r}} e^{-ir} i^m e^{i\pi/4} \tag{55}$$

$$H_m^{'}(r) \to \frac{-(1+2ir)}{2r} H_m^\infty(r) \tag{56}$$

The resulting simplified expressions for the scattered fields are left to the reader.

2.4. Internal wave expansion

As for the incident wave, the generating function must again use J_m to avoid divergence at $\rho = 0$. The wavenumber k must furthermore be replaced by the wavenumber k_c in the cylinder material, and ε by ε_c. Therefore, instead of depending on r given by Eq. (21), J_m must depend on $r_c = kM\rho\sqrt{1-\gamma^2}$ in which we used $M = k_c/k$.

We then introduce BSPs for the cylinder wave according to:

$$U_{TM}^c = \frac{E_0}{k^2 M} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TM}(\gamma) J_m(k_c\rho\sqrt{1-\gamma^2}) e^{ik_c\gamma z} d\gamma \tag{57}$$

$$U^c_{TE} = \frac{iH_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TE}(\gamma) J_m(k_c \rho \sqrt{1-\gamma^2}) e^{ik_c \gamma z} d\gamma \tag{58}$$

in which prefactors are taken to be the same than in [14]. With $k_c = Mk$, these BSPs may be rewritten as:

$$U^c_{TM} = \frac{E_0}{k^2 M} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TM}(\gamma) J_m(Mk\rho \sqrt{1-\gamma^2}) e^{iMk\gamma z} d\gamma \tag{59}$$

$$U^c_{TE} = \frac{iH_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TE}(\gamma) J_m(Mk\rho \sqrt{1-\gamma^2}) e^{iMk\gamma z} d\gamma \tag{60}$$

However, in [14], $\exp(ik\gamma z)$ instead of $\exp(iMk\gamma z)$, and $J_m(k\rho\sqrt{M^2-\gamma^2})$ instead of $J_m(Mk\rho\sqrt{1-\gamma^2})$, have been used. Indeed, let us set $M\gamma = \gamma'$ in Eqs. (59)–(60), we obtain:

$$U^c_{TM} = \frac{E_0}{k^2 M} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TM}(\gamma'/M) J_m(k\rho \sqrt{M^2-\gamma'^2}) e^{ik\gamma' z} d(\gamma'/M) \tag{61}$$

$$U^c_{TE} = \frac{iH_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TE}(\gamma'/M) J_m(k\rho \sqrt{M^2-\gamma'^2}) e^{ik\gamma' z} d(\gamma'/M) \tag{62}$$

We may then relabel γ' to γ, and afterward relabel the BSFs by making the changes $C_{m,TM}(\gamma/M)/M \to C_{m,TM}(\gamma)$ and $C_{m,TE}(\gamma/M)/M \to C_{m,TE}(\gamma)$, leading to:

$$U^c_{TM} = \frac{E_0}{k^2 M} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TM}(\gamma) J_m(k\rho \sqrt{M^2-\gamma^2}) e^{ik\gamma z} d\gamma \tag{63}$$

$$U^c_{TE} = \frac{iH_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TE}(\gamma) J_m(k\rho \sqrt{M^2-\gamma^2}) e^{ik\gamma z} d\gamma \tag{64}$$

This was the option taken in [14]. To let the two options open (to be discussed later), let us complement Eqs. (63) and (64) with two constants α and β later to be discussed, according to:

$$U^c_{TM} = \frac{E_0}{k^2 M} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TM}(\gamma) J_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{iMk\beta\gamma z} d\gamma \tag{65}$$

$$U^c_{TE} = \frac{iH_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int C_{m,TE}(\gamma) J_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{iMk\beta\gamma z} d\gamma \tag{66}$$

Option 1 of Eqs. (59)–(60) correspond to $\alpha = \beta = 1$ while option 2 of Eqs. (63)–(64), i.e. the option taken in [14], correspond to $\alpha = 1/M^2$ and $\beta = 1/M$. Also, in the set of Eqs. (5)–(16), we must use the material permittivity ε_c instead of ε, according to e.g. Eq.(1.100) in [3]:

$$\varepsilon_c = M^2 \varepsilon \tag{67}$$

We must also take care of changing k to k_c in Eqs. (5) and (14). The internal field components then read as:

$$E^c_z = E^c_{z,TM} = E_0 M \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int (1-\beta^2\gamma^2) C_{m,TM}(\gamma) J_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{68}$$

$$E^c_{\rho,TM} = E_0 M \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \beta\gamma \sqrt{1-\alpha\gamma^2} C_{m,TM}(\gamma) J'_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{69}$$

$$E^c_{\rho,TE} = \frac{iE_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int C_{m,TE}(\gamma) J_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{70}$$

$$E^c_{\varphi,TM} = \frac{-E_0}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int \beta\gamma C_{m,TM}(\gamma) J_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{71}$$

$$E^c_{\varphi,TE} = -E_0 M \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int \sqrt{1-\alpha\gamma^2} C_{m,TE}(\gamma) J'_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{72}$$

$$H^c_z = H^c_{z,TE} = iH_0 M^2 \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int (1-\beta^2\gamma^2) C_{m,TE}(\gamma) J_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{73}$$

$$H^c_{\rho,TM} = \frac{-H_0 M}{k\rho} \sum_{m=-\infty}^{+\infty} (-i)^m m e^{im\varphi} \int C_{m,TM}(\gamma) J_m(Mk\rho \sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{74}$$

$$H^c_{\rho,TE} = -H_0 M^2 \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} \int \beta\gamma \sqrt{1-\alpha\gamma^2} C_{m,TE}(\gamma) J'_m(Mk\rho\sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{75}$$

$$H^c_{\varphi,TM} = -H_0 M^2 \sum_{m=-\infty}^{+\infty} i(-i)^m e^{im\varphi} \int \sqrt{1-\alpha\gamma^2} C_{m,TM}(\gamma) J'_m(Mk\rho\sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{76}$$

$$H^c_{\varphi,TE} = \frac{-H_0 M}{k\rho} \sum_{m=-\infty}^{+\infty} i(-i)^m m e^{im\varphi} \int \beta\gamma C_{m,TE}(\gamma) J_m(Mk\rho\sqrt{1-\alpha\gamma^2}) e^{ikM\beta\gamma z} d\gamma \tag{77}$$

in which we have again used Eq. (42).

2.5. Use of boundary conditions

The boundary conditions for $\rho = a$ read as:

$$E^i_z + E^s_z = E^c_z \tag{78}$$

$$E^i_\varphi + E^s_\varphi = E^c_\varphi \tag{79}$$

$$H^i_z + H^s_z = H^c_z \tag{80}$$

$$H^i_\varphi + H^s_\varphi = H^c_\varphi \tag{81}$$

in which each component is the sum of the corresponding TM- and TE -components. We also introduce the following notations:

$$H_m(ka\sqrt{1-\gamma^2}) = H_m \tag{82}$$

$$[H'_m(k\rho\sqrt{1-\gamma^2})]_{\rho=a} = \widetilde{H_m} \tag{83}$$

$$J_m(ka\sqrt{1-\gamma^2}) = J^i_m \tag{84}$$

$$[J'_m(k\rho\sqrt{1-\gamma^2})]_{\rho=a} = \widetilde{J^i_m} \tag{85}$$

$$J_m(Mka\sqrt{1-\alpha\gamma^2}) = J^c_m \tag{86}$$

$$[J'_m(Mk\rho\sqrt{1-\alpha\gamma^2})]_{\rho=a} = \widetilde{J^c_m} \tag{87}$$

The boundary conditions then read as:

$$(1-\gamma^2)[I_{m,TM}(\gamma)J^i_m - S_{m,TM}(\gamma)H_m]e^{ik\gamma z} = M(1-\beta^2\gamma^2)C_{m,TM}(\gamma)J^c_m e^{ikM\beta\gamma z} \tag{88}$$

$$\left\{\frac{m\gamma}{ka}[S_{m,TM}(\gamma)H_m - I_{m,TM}(\gamma)J^i_m] + i\sqrt{1-\gamma^2}[S_{m,TE}(\gamma)\widetilde{H_m} + I_{m,TE}(\gamma)\widetilde{J^i_m}]\right\}e^{ik\gamma z} = \left[\frac{-m}{ka}\beta\gamma C_{m,TM}(\gamma)J^c_m - M\sqrt{1-\alpha\gamma^2}C_{m,TE}(\gamma)\widetilde{J^c_m}\right]e^{ikM\beta\gamma z} \tag{89}$$

$$(1-\gamma^2)[I_{m,TE}(\gamma)J^i_m + S_{m,TE}(\gamma)H_m]e^{ik\gamma z} = iM^2(1-\beta^2\gamma^2)C_{m,TE}(\gamma)J^c_m e^{ikM\beta\gamma z} \tag{90}$$

$$\left\{\frac{m\gamma}{ka}[S_{m,TE}(\gamma)H_m + I_{m,TE}(\gamma)J^i_m] - i\sqrt{1-\gamma^2}[S_{m,TM}(\gamma)\widetilde{H_m} - I_{m,TM}(\gamma)\widetilde{J^i_m}]\right\}e^{ik\gamma z} = \left[\frac{imM}{ka}\beta\gamma C_{m,TE}(\gamma)J^c_m + iM^2\sqrt{1-\alpha\gamma^2}C_{m,TM}(\gamma)\widetilde{J^c_m}\right]e^{ikM\beta\gamma z} \tag{91}$$

From Eqs. (88) and (90), we respectively obtain:

$$C_{m,TM}(\gamma) = \frac{(1-\gamma^2)[I_{m,TM}(\gamma)J^i_m - S_{m,TM}(\gamma)H_m]e^{ik\gamma z}}{M(1-\beta^2\gamma^2)J^c_m e^{ikM\beta\gamma z}} \tag{92}$$

$$C_{m,TE}(\gamma) = \frac{(1-\gamma^2)[I_{m,TE}(\gamma)J^i_m + S_{m,TE}(\gamma)H_m]e^{ik\gamma z}}{iM^2(1-\beta^2\gamma^2)J^c_m e^{ikM\beta\gamma z}} \tag{93}$$

Inserting Eqs. (92)–(93) into Eq. (89), we obtain:

$$A_{m,TM}S_{m,TM}(\gamma)+A_{m,TE}S_{m,TE}(\gamma)=B_{m,TM}I_{m,TM}(\gamma)+B_{m,TE}I_{m,TE}(\gamma) \tag{94}$$

in which:

$$A_{m,TM}=\frac{mH_m\gamma}{ka}\left[1-\frac{\beta(1-\gamma^2)}{M(1-\beta^2\gamma^2)}\right] \tag{95}$$

$$A_{m,TE}=i\sqrt{1-\gamma^2}\widetilde{H_m}-\frac{i\sqrt{1-\alpha\gamma^2}(1-\gamma^2)}{M(1-\beta^2\gamma^2)}\frac{\widetilde{J_m^c}H_m}{J_m^c} \tag{96}$$

$$B_{m,TM}=\frac{mJ_m^i\gamma}{ka}\left[1-\frac{\beta(1-\gamma^2)}{M(1-\beta^2\gamma^2)}\right] \tag{97}$$

$$B_{m,TE}=\frac{i\sqrt{1-\alpha\gamma^2}(1-\gamma^2)}{M(1-\beta^2\gamma^2)}\frac{\widetilde{J_m^c}J_m^i}{J_m^c}-i\sqrt{1-\gamma^2}\widetilde{J_m^i} \tag{98}$$

Similarly, inserting Eqs. (92)–(93) into Eq. (91), we obtain:

$$E_{m,TM}S_{m,TM}(\gamma)+E_{m,TE}S_{m,TE}(\gamma)=F_{m,TM}I_{m,TM}(\gamma)+F_{m,TE}I_{m,TE}(\gamma) \tag{99}$$

in which:

$$E_{m,TM}=\frac{iM\sqrt{1-\alpha\gamma^2}(1-\gamma^2)}{(1-\beta^2\gamma^2)}\frac{\widetilde{J_m^c}H_m}{J_m^c}-i\sqrt{1-\gamma^2}\widetilde{H_m} \tag{100}$$

$$E_{m,TE}=\frac{mH_m\gamma}{ka}\left[1-\frac{\beta(1-\gamma^2)}{M(1-\beta^2\gamma^2)}\right] \tag{101}$$

$$F_{m,TM}=\frac{iM\sqrt{1-\alpha\gamma^2}(1-\gamma^2)}{(1-\beta^2\gamma^2)}\frac{\widetilde{J_m^c}J_m^i}{J_m^c}-i\sqrt{1-\gamma^2}\widetilde{J_m^i} \tag{102}$$

$$F_{m,TE}=\frac{mJ_m^i\gamma}{ka}\left[\frac{\beta(1-\gamma^2)}{M(1-\beta^2\gamma^2)}-1\right] \tag{103}$$

We know enough to discuss the values of the constants α and β. It is then clear that the constants occurring in Eqs. (95)-(98) and 100–103 simplify if we take the option in which $\alpha=\beta=1$, i.e. option 1. Unfortunately, in this option, the z -dependent term $\exp(iMk\gamma z)$ does not remain finite at $z\to\pm\infty$ if M is complex (justifying that option 2 has been chosen in [14]). As a by-product issue, it would then be interesting to check whether options 1 and 2 are numerically equivalent (as expected) when M is real. In this last case, option 1 is however appealing because Eqs. (92)–(93),(95)–(98) and (100)–(103) simplify to:

$$C_{m,TM}(\gamma)=\frac{[I_{m,TM}(\gamma)J_m^i-S_{m,TM}(\gamma)H_m]e^{ik\gamma z}}{MJ_m^c e^{ikM\beta\gamma z}} \tag{104}$$

$$C_{m,TE}(\gamma)=\frac{[I_{m,TE}(\gamma)J_m^i+S_{m,TE}(\gamma)H_m]e^{ik\gamma z}}{iM^2J_m^c e^{ikM\beta\gamma z}} \tag{105}$$

$$A_{m,TM}=\frac{mH_m\gamma}{ka}\left[1-\frac{1}{M}\right] \tag{106}$$

$$A_{m,TE}=i\sqrt{1-\gamma^2}\left[\widetilde{H_m}-\frac{1}{M}\frac{\widetilde{J_m^c}H_m}{J_m^c}\right] \tag{107}$$

$$B_{m,TM}=\frac{mJ_m^i\gamma}{ka}\left[1-\frac{1}{M}\right] \tag{108}$$

$$B_{m,TE}=i\sqrt{1-\gamma^2}\left[\frac{1}{M}\frac{\widetilde{J_m^c}J_m^i}{J_m^c}-\widetilde{J_m^i}\right] \tag{109}$$

$$E_{m,TM}=i\sqrt{1-\gamma^2}\left[M\frac{\widetilde{J_m^c}H_m}{J_m^c}-\widetilde{H_m}\right] \tag{110}$$

$$E_{m,TE}=\frac{mH_m\gamma}{ka}\left[1-\frac{1}{M}\right] \tag{111}$$

$$F_{m,TM}=i\sqrt{1-\gamma^2}\left[M\frac{\widetilde{J_m^c}J_m^i}{J_m^c}-\widetilde{J_m^i}\right] \tag{112}$$

$$F_{m,TE} = \frac{mJ_m^i \gamma}{ka}\left[\frac{1}{M} - 1\right] \tag{113}$$

We now solve the set of Eqs. (94)–(99), as follows. We express $S_{m,TE}(\gamma)$ from Eq. (94) and insert the result in Eq. (99) to obtain $S_{m,TM}(\gamma)$ reading as:

$$S_{m,TM}(\gamma) = \frac{1}{S^1_{m,TM}}[S^2_{m,TM}I_{m,TM}(\gamma) + S^3_{m,TM}I_{m,TE}(\gamma)] \tag{114}$$

in which:

$$S^1_{m,TM} = 1 - \frac{A_{m,TM}E_{m,TE}}{A_{m,TE}E_{m,TM}} \tag{115}$$

$$S^2_{m,TM} = \frac{F_{m,TM}}{E_{m,TM}} - \frac{B_{m,TM}E_{m,TE}}{A_{m,TE}E_{m,TM}} \tag{116}$$

$$S^3_{m,TM} = \frac{F_{m,TE}}{E_{m,TM}} - \frac{B_{m,TE}E_{m,TE}}{A_{m,TE}E_{m,TM}} \tag{117}$$

Similarly, we express $S_{m,TM}(\gamma)$ from Eq. (99) and insert the result in Eq. (94) to obtain $S_{m,TE}(\gamma)$ reading as:

$$S_{m,TE}(\gamma) = \frac{1}{S^1_{m,TE}}[S^2_{m,TE}I_{m,TM}(\gamma) + S^3_{m,TE}I_{m,TE}(\gamma)] \tag{118}$$

in which:

$$S^1_{m,TE} = 1 - \frac{A_{m,TM}E_{m,TE}}{A_{m,TE}E_{m,TM}} = S^1_{m,TM} \tag{119}$$

$$S^2_{m,TE} = \frac{B_{m,TM}}{A_{m,TE}} - \frac{A_{m,TM}F_{m,TM}}{A_{m,TE}E_{m,TM}} \tag{120}$$

$$S^3_{m,TE} = \frac{B_{m,TE}}{A_{m,TE}} - \frac{A_{m,TM}F_{m,TE}}{A_{m,TE}E_{m,TM}} \tag{121}$$

This completes the formulation. Such a formulation has been used by Ren et al. [35] and by Méès et al. [36] (to which we shall return later) in the case of Gaussian beams described by using a localized approximation (see Section 5). However, the paper by Ren et al. introduced a dictionary to translate a formulation in terms of distributions (see Section 3) to a formulation in terms of usual functions and therefore, to save room, did not explicitly repeat the formulation in terms of usual functions. It then happens that the formulation above is not explicitly available, in this form, in the archival literature, although it could be deduced from the dictionary in Ren et al. [35] and although it is discussed in the thesis dissertation by Méès [37]. Hence, the formulation presented above may be viewed as new. At least it is certainly useful for the reader to have it available under a single roof.

Let us add that the formulation above has been given the name of plane-wave spectrum approach. e.g. [35], a denomination (i) which is convenient to oppose it to the approach in terms of distributions soon discussed and (ii) which is motivated by the plane wave term of the form $\exp(ikz)$ occurring in the formulation. This should not be confused with the Angular Spectrum Representation (ASR) or Decomposition (ASD) used to evaluate BSCs in the framework of the GLMT *stricto sensu* (i.e. when the scatterer is a homogeneous sphere, or more generally in spherical coordinates), e.g. review in Sections 3 and 4 of [38] and a recent paper by Shen et al. [39]. In the GLMT for cylinders, the term $\exp(ikz)$ is essential to the formulation because it is a consequence of the structure of the separability equation in the *spatial domain* while, conversely, the ASD in spherical coordinates is the result of the description of the illuminating beam in terms of a plane-wave spectrum in the *spectral domain*.

2.6. A simplified formulation and a trivial example

The formulation in [14] is a simplified formulation in which the integrals over the separability constant is omitted. Then the BSCs (BSFs) of Eqs. (26)–(27) simplify to:

$$U^i_{TM} = \frac{E_0}{k^2}\sum_{m=-\infty}^{+\infty} I_{m,TM}(-i)^m e^{im\varphi}J_m(k\rho\sqrt{1-\gamma^2})e^{ik\gamma z} \tag{122}$$

$$U^i_{TE} = \frac{H_0}{k^2}\sum_{m=-\infty}^{+\infty} I_{m,TE}(-i)^m e^{im\varphi}J_m(k\rho\sqrt{1-\gamma^2})e^{ik\gamma z} \tag{123}$$

in which the BSFs $I_{m,TM}(\gamma)$ and $I_{m,TE}(\gamma)$ do not depend explicitly on γ any more, in the form of functions, and become genuine BSCs. To extract them, Eq. (29) which involves the Dirac function, more properly said the distribution $\delta(\gamma - \gamma')$, is not required any more and the use of Eq. (30) is sufficient to obtain:

$$I_{m,TM} = \frac{\exp(-ik\gamma z)}{2\pi(1-\gamma^2)(-i)^m J_m(k\rho\sqrt{1-\gamma^2})}\int_0^{2\pi}\frac{E^i_z}{E_0}e^{-im\varphi}d\varphi \tag{124}$$

$$I_{m,TE} = \frac{\exp(-ik\gamma z)}{2\pi(1-\gamma^2)(-i)^m J_m(k\rho\sqrt{1-\gamma^2})}\int_0^{2\pi}\frac{H^i_z}{H_0}e^{-im\varphi}d\varphi \tag{125}$$

in which the γ-spectrum has reduced to a single value to be determined.

As a trivial example, focusing only on the description of the illuminating beam and letting the rest of the formulation in the hands of the reader, let us assume that the cylinder is illuminated by a plane wave propagating perpendicularly to the cylinder axis, from negative to positive x, with the electric field vibrating in parallel to the cylinder axis, a simple case well documented in the literature, e.g. [25,40]. We then have:

$$E_z^i = E_0 \exp(-ikx) = E_0 \exp(-ik\rho\cos\varphi) \tag{126}$$

$$H_z^i = 0 \tag{127}$$

From Eq. (124), the $TM-$BSCs then read as:

$$I_{m,TM} = \frac{\exp(-ik\gamma z)}{2\pi(1-\gamma^2)(-i)^m J_m(k\rho\sqrt{1-\gamma^2})} \int_0^{2\pi} \exp(-ik\rho\cos\varphi) e^{-im\varphi} d\varphi \tag{128}$$

Since the BSCs are constant numbers which should not depend on z, we immediately require $\gamma = 0$, so that Eq. (128) becomes:

$$I_{m,TM} = \frac{1}{2\pi(-i)^m J_m(k\rho)} \int_0^{2\pi} \exp[-i(m\varphi + k\rho\cos\varphi)] d\varphi \tag{129}$$

But the integral in Eq. (129) is a classical integral equal to $2\pi(-i)^m J_m(k\rho)$ so that:

$$I_{m,TM} = 1, \ \forall m \tag{130}$$

We also readily find that $I_{m,TE} = 0, \forall m$ so that the incident wave is a pure TM-wave. It is interesting to note, as previously announced, that the ρ-dependent prefactor was only apparent. Indeed, in this plane case, we can see how it is cancelled by the integral term. This is an example of a completely general result already mentioned, according to which the BSCs (or BSFs) are complex numbers which do not depend on the coordinates.

The reader is referred to [14] for a complete analysis of this case. What is important here is to note that the result $\gamma = 0$ is mathematically completely incompatible with the plane-wave spectrum approach in which we would integrate over γ. This fact, which would hold as well if the illumination was not perpendicular (but for a different value of γ), together with the use of a "Dirac function" in Eq. (29), indicates that something is not completely satisfactory in the plane-wave spectrum approach as well as in its simplified formulation.

3. Formalism in terms of distributions: Why and how

Let us consider a Gaussian beam described by using the Davis scheme of approximation [41–43]. In this framework, a potential vector $\mathbf{A} = (A_x, 0, 0)$ is introduced in which the non-zero component A_x reads as:

$$A_x = \psi(x, y, z) \exp(-ikz) \tag{131}$$

in which ψ is expanded as:

$$\psi = \psi_0 + s^2\psi_2 + s^4\psi_4 + \ldots \tag{132}$$

in which s is the beam confinement parameter equal to $(1/kw_0)$ in which w_0 is the beam waist radius.

The lowest order term ψ_0 represents the fundamental mode of the Gaussian beam. It is called the first-order Davis beam. The second and third terms are called the third-order and the fifth-order modes of the Davis beam [42]. Explicit expressions are known only for these three first modes, with the fifth mode available from [43]. The electromagnetic fields are afterward deduced from the potential vector.

None of these modes exactly satisfy Maxwell's equations but they can be shown to be the summation of a first term perfectly satisfying Maxwell's equations, called the Maxwellian term, complemented by a supplementary non-Maxwellian contribution [44,45].

We now consider a Cartesian coordinate system $O_G uvw$ attached to a Gaussian beam, according to the configuration displayed in Fig. 2 (adapted from [46], see as well [14]). The beam waist center is located at the origin O_G. The beam propagates along the w -axis, from negative to positive w's. In the first-order Davis approximation, the field components read as [1,41]:

$$E_u = E_0 \psi_0 \exp(-ikw) \tag{133}$$

$$E_v = 0 \tag{134}$$

$$E_w = \frac{-2Qu}{kw_0^2} E_u \tag{135}$$

$$H_u = 0 \tag{136}$$

$$H_v = H_0 \psi_0 \exp(-ikw) \tag{137}$$

$$H_w = \frac{-2Qv}{kw_0^2} H_v \tag{138}$$

in which:

$$\psi_0 = iQ \exp\left(-iQ\frac{u^2+v^2}{w_0^2}\right) \tag{139}$$

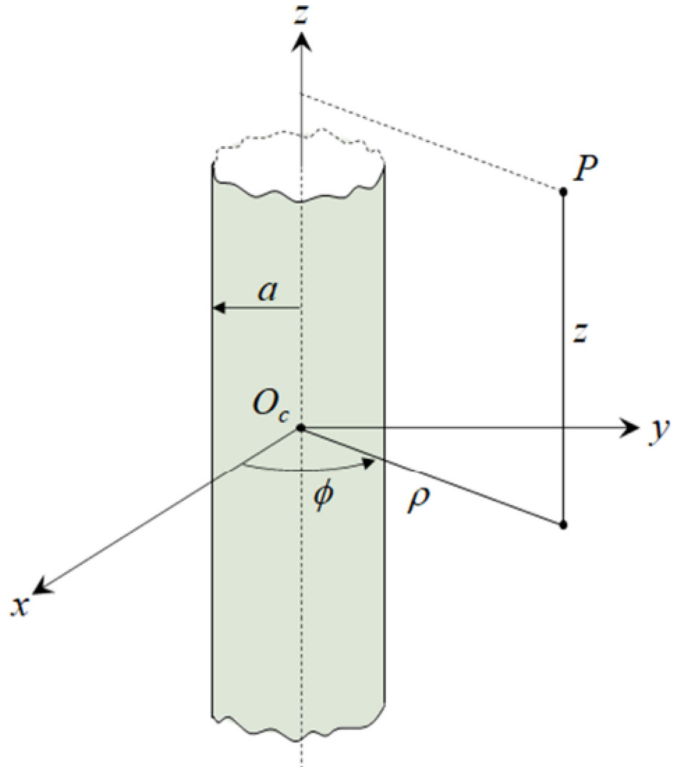

Fig. 1. Geometry and coordinate system of the problem. Attached to an infinite cylinder of radius a there is a cylindrical coordinate sysytem $(x_1, x_2, x_3) = (z, \rho, \phi)$ with origin at O_c. The main axis of the cylinder coincides with the z axis.

$$Q = \frac{1}{i + 2\frac{w}{kw_0^2}} \tag{140}$$

We now assume that the Gaussian beam illuminates the infinite cylinder perpendicularly to its axis with the beam waist center located on the axis. The leading electric field polarization (i.e. electric field polarization at the waist location) is perpendicular to the plane defined by the cylinder axis z and by the incident unit vector e_i, i.e. the cylinder axis z coincides with the $O_G v$ axis. In cylindrical coordinates defined in Fig. 1, limiting ourselves to the z-components of the field required to evaluate the BSCs, we have:

$$E_z^i = 0 \tag{141}$$

$$H_z^i = -H_0 \psi_0 \exp(ik\rho \cos\varphi) \tag{142}$$

with:

$$\psi_0 = iQ \exp\left[-iQ\frac{\rho^2 \sin^2\varphi + z^2}{w_0^2}\right] \tag{143}$$

$$Q = \frac{1}{i - 2\frac{\rho\cos\varphi}{kw_0^2}} \tag{144}$$

Focusing on H_z^i, the Maxwellian contribution to this field reads as [14,44]:

$$H_z^i = H_0[-1 + s^2(R^2 \sin^2\varphi + 2iR\cos\varphi + Z^2)]\exp(iR\cos\varphi) \tag{145}$$

in which we have used $R = k\rho$ and $Z = kz$. This configuration is extensively analyzed in Section 10 of [14]. We here provide a more expedient argument. The $O(s^0)$ term of Eq. (145) is a plane wave component, i.e. it corresponds to the case when $s = 0$, i.e. w_0 is infinite. Indeed, from Eq. (125), we have $\gamma = 0$ and, evaluating trivially the integral, we obtain $I^0_{m,TE} = (-1)^{m+1}$ for the $O(s^0)$-term. Therefore, in utmost rigor, the extended plane-wave spectrum approach is not valid since the separation constant has one precise value, forbidding the use of an integral with a differential element $d\gamma$. For consistency, the $O(s^2)$-term must therefore be studied as well in the restricted plane-wave spectrum approach, with $\gamma = 0$. From Eq. (125), we then obtain that the corresponding BSP $U^2_{m,TE}$ (in which the superscript "2" refers to the fact that we are dealing with an $O(s^2)$-term) simplifies to:

$$U^2_{m,TE} = \frac{H_0}{k^2}\sum_{m=-\infty}^{+\infty}(-i)^m I^2_{m,TM} J_m(R) e^{im\varphi} \tag{146}$$

leading to:

$$I^2_{m,TM} = \frac{1}{2\pi(-i)^m J_m(R)}\int_0^{2\pi}[s^2(R^2\sin^2\varphi + 2iR\cos\varphi + Z^2)]\exp(iR\cos\varphi)e^{-im\varphi}d\varphi \tag{147}$$

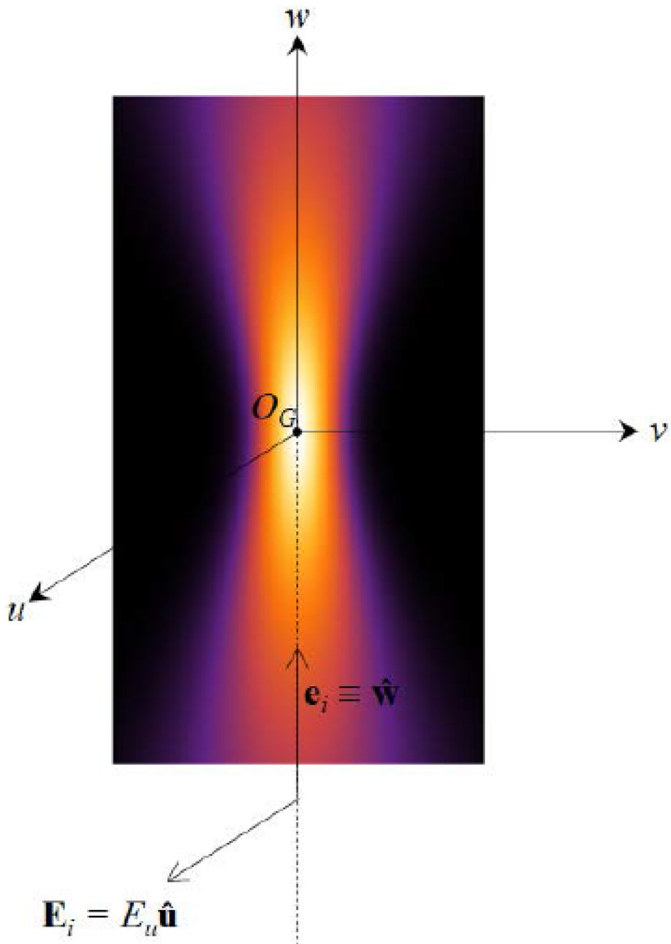

Fig. 2. Geometry and coordinate system for the illuminating beam. The beam is assumed to propagate along the w axis, with the v axis. paralell to the z axis of Fig. 1 The electric field vibrates along the u axis.

A dramatic consequence of Eq. (147) is that $I^2_{m,TM}$ depends on Z and that there is no satisfactory way to get rid of this dependence which is in contrast with the fact that BSCs should not depend on coordinates. However, let us consider the quantity:

$$U^2 = \frac{H_0 s^2}{k^2} \sum_{m=-\infty}^{+\infty} i^m (Z^2 + m^2 - 2 + iR\cos\varphi) J_m(R) e^{im\varphi} \tag{148}$$

which is proposed as an Ansatz from which we shall draw a few conclusions (the reader does not need to know how the expression for U^2 in Eq. (148) has been obtained; it is sufficient to examine thereafter its properties). It may then be shown that applying Eqs. (5)–(16) to U^2, we correctly recover the Maxwellian contributions of the first-order Davis beam in cylindrical coordinates, see [14] for details. Therefore U^2 behaves as a proper BSP. However, it is the summation of U^S and U^{NS} according to:

$$U^S = \frac{H_0 s^2}{k^2} \sum_{m=-\infty}^{+\infty} i^m (m^2 - 2) J_m(R) e^{im\varphi} \tag{149}$$

$$U^{NS} = \frac{H_0 s^2}{k^2} (Z^2 + iR\cos\varphi) \sum_{m=-\infty}^{+\infty} i^m J_m(R) e^{im\varphi} \tag{150}$$

The term U^S is a sum of generating functions as displayed in (25) (with $\gamma = 0$), therefore satisfying the separability theorem expressed by Eq. (17), and is then called a $\sum$-separable potential. Conversely, U^{NS} is not a sum of generating functions as displayed in Eq. (25). Therefore, it does not satisfy the separability theorem expressed by Eq. (17) and is then called a non-$\sum$-separable potential. In order to solve the problems raised by the above discussed difficulties, a first attempt tried, without any success, to build a theory for Gaussian beams (in the case of a first-order Davis approximation) in terms of non-$\sum$-separable potentials [47].

It has soon later been found that the source of all these difficulties lied in the fact that the framework in terms of usual functions was not general enough to correctly handle the Maxwellian contributions of the Davis beam and that a correct framework is the one of distributions, allowing one to deal simultaneously with the plane-wave spectrum approach, its simplified version, and the Maxwellian contributions of Davis beam approximations. In [48], a constructive procedure has been established allowing one to systematically generate non-$\sum$-separable potentials, showing that these potentials provide a specific class of functions which are problematic if we want to use separable functions to solve a problem, and which then would require further investigation if it is wanted to include them, in particular, in light scattering theories. One year later, it was understood that the separability theorem could be established in terms of distributions, allowing one to convert the non-$\sum$-separable potentials in terms of functions to $\sum$-separable potentials in terms of distributions [49]. In other words, the separability approach must be expressed, in a more general framework, in terms of distributions, rather than in terms of functions.

An easy-to-read introduction to the theory of distributions for physicists is available from Roddier [50]. A still more easy-to-read tutorial devoted specifically to the issue of beam parametrization in light scattering is available from [51], in which the reader is kindly requested to correct a missprint, namely changing $\delta(a)$ to $\varphi(a)$ in the r.h.s. of Eq. (73). In the present paper, we shall be content to provide a flavour on the use of the theory of distributions in light scattering, with the attention being paid only to the description of the illuminating beams. Before proceeding, let us state that the distributions form a generalization of functions (therefore they are sometimes viewed as generalized functions), i.e. any usual function is a distribution but some distributions are not usual functions. The essential fact is that distributions solve the problem of differentiation, namely (in contrast with functions) any distribution is infinitely differentiable, and each derivative is a distribution, a property that we shall soon use. Furthermore, when a problem is solved in terms of distributions, it is often interesting, in particular for computer programming and computations, to check whether the solution in terms of distributions can be converted to a solution in terms of usual functions. For an history of the theory of distributions, the reader may refer to [52]. The emblematic example is the "Dirac function" $\delta(x)$ satisfying, in an usual language:

$$\int \delta(x) f(x) dx = f(0) \tag{151}$$

Unfortunately, this equation is a mathematical non-sense since $\delta(x)$ is everywhere equal to 0 but for $x = 0$, so that the differential element dx has no meaning. At best, it is a symbolic notation which should better be written as:

$$< \delta_0, f >= f(0) \tag{152}$$

in which the subscript 0 in δ_0 means that it is 0 everywhere but for 0. From Eq. (152), we see that $\delta(x)$ now receives the status of an operator. More generally, the notation $< T, \varphi >$ is the standard notation to use a distribution T which is defined when we know how it acts on f, this latter function being called a test function. Let us then return to Eq. (150) which, omitting the prefactor $H_0 s^2/k^2$, is from now on simply denoted by U. Writing the cosine function in terms of exponentials, the $\cos\varphi$-term may be rewritten as:

$$\begin{aligned} U_{\cos} &= \frac{R}{2} \sum_{m=-\infty}^{+\infty} i^m e^{im\varphi} [J_{m-1}(R) - J_{m+1}(R)] \\ &= R \sum_{m=-\infty}^{+\infty} i^m e^{im\varphi} J'_m(R) \end{aligned} \tag{153}$$

in which we have used [53]:

$$J'_m(R) = \frac{1}{2}[J_{m-1}(R) - J_{m+1}(R)] \tag{154}$$

leading to:

$$U = \sum_{m=-\infty}^{+\infty} i^m e^{im\varphi} [Z^2 J_m(R) + R J'_m(R)] \tag{155}$$

Returning to Eq. (27) of the plane-wave spectrum approach, we may then search for a beam shape "coefficient" $A_m(\gamma)$ satisfying:

$$U = \sum_{m=-\infty}^{+\infty} e^{im\varphi} \int A_m(\gamma) J_m(R\sqrt{1-\gamma^2}) e^{i\gamma Z} d\gamma \tag{156}$$

which is an equation which actually has no meaning, as Eq. (151), since the separation constant must have the same value $\gamma = 0$ than in Eq.(149). Similarly as for the passage from Eq. (151) to Eq. (152), Eq. (156) must then be rewritten in terms of distributions as:

$$U = \sum_{m=-\infty}^{+\infty} e^{im\varphi} < A_m(\gamma), J_m(R\sqrt{1-\gamma^2}) e^{i\gamma Z} > \tag{157}$$

in which $A_m(\gamma)$ is now a Beam Shape Distribution (BSD), and which must be solved in a rigorous way in the framework of the theory of distributions, using its theorems. Now, the fact that $\gamma = 0$ in terms of functions means that the support of the distribution $A_m(\gamma)$ is zero as well. However, we possess a theorem telling us that a distribution of support {0} is a linear combination of the Dirac distribution and of its derivatives of the form:

$$T = \sum_{k=0}^{\infty} a_k \delta^{(k)} \tag{158}$$

in which $a_k \in \mathbf{C}$ and $\delta^{(k)}$ is the kth derivative of the Dirac distribution. We then rely on the definition of the jth derivative of a distribution according to [50]:

$$< T^{(m)}, \varphi >= (-1)^m < T, \varphi^{(m)} > \tag{159}$$

to establish:

$$A_m(\gamma) = -i^m \delta''(\gamma) \tag{160}$$

Indeed, we then have:

$$< A_m(\gamma), J_m(R\sqrt{1-\gamma^2}) e^{i\gamma Z} >= -i^m < \delta''(\gamma), J_m(R\sqrt{1-\gamma^2}) e^{i\gamma Z} > \tag{161}$$

$$= -i^m (J_m(R\sqrt{1-\gamma^2})e^{i\gamma Z})''_{\gamma=0}$$
$$= i^m [Z^2 J_m(R) + RJ'_m(R)]$$

from which we recover Eq. (155). Hence, the most general framework relies on Beam Shape Distributions which, from Eqs. (26)–(27), intervene in equations which must be rewritten as:

$$U^i_{TM} = \frac{E_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} < I_{m,TM}(\gamma), J_m(k\rho\sqrt{1-\gamma^2})e^{ik\gamma z} > \tag{162}$$

$$U^i_{TE} = \frac{H_0}{k^2} \sum_{m=-\infty}^{+\infty} (-i)^m e^{im\varphi} < I_{m,TE}(\gamma) J_m(k\rho\sqrt{1-\gamma^2})e^{ik\gamma z} > \tag{163}$$

A fortunate fact is that all the knowledge required to deal with distributions in the framework of GLMTs is essentially contained above in this section. It is another fortunate fact that the passage from distributions in which the symbolic notation < . > must be used to usual functions (when this is possible) amounts to the replacement of the symbolic notation by an integral. And it is still another fortunate fact that, “for all practical purposes”, the original plane-wave spectrum approach (or its simplified version) is sufficient to deal with scattering problems in the framework of GLMTs, if we dismiss “exotic” cases such as the Maxwellian contributions to Davis beams.

The GLMT for infinite cylinders in terms of distributions up to $O(s^2)$, i.e. for the Maxwellian contribution to a first-order Davis beam is discussed in [46], for the case when the cylinder is perpendicularly illuminated by the beam. The case of arbitrary location and arbitrary orientation for a first-order Davis beam is discussed in [28]. Such a GLMT for cylinders up to $O(s^2)$ has been used for cylindrical particle characterization by phase-Doppler anemometry [54]. A theory up to $O(s^{10})$ is discussed in [55], still in terms of distributions, but with results eventually expressed in terms of usual functions, allowing classical implementation in computer programs. An approach for the case of arbitrary shaped beams, still in terms of distributions, is provided in [29].

Numerical results for the case of Gaussian beams at normal incidence, with the beam waist center of the beam located upon the cylinder axis, have been displayed in [35]. This paper summarized the passage from a theory in terms of distributions to a theory in terms of usual functions (when it is possible). Three different kinds of beam descriptions were considered (i) Maxwellian beams at limited order relying on the Davis scheme of approximation, (ii) a plane-wave spectrum approach as discussed in Section 1 for quasi-Gaussian beams, and (iii) a plane-wave spectrum approach using a cylindrical localized approximation. Concerning (i), the description used the Maxwellian contributions to the first- and third-Davis beam approximations, and numerical results are accordingly provided up to $O(s^6)$. Concerning (ii), a quasi-Gaussian beam is defined from a complete first order Davis beam which is non-Mawellian whose BSFs then depend on R, and by remodeling them by fixing the variable R to a prescribed value R_p, then producing a remodeled Maxwellian beam which, based on a first order approximation, cannot be defined exactly as being a Gaussian beam, but provides a fairly satisfactory approximation of it. Concerning (iii), the reader should refer to Section 4 below in which localized approximations are discussed. Numerical results for the case of arbitrary location and of arbitrary orientation of a Gaussian beam, still using a localized approximation, is discussed in [36]. This paper also contains some considerations devoted to the use of geometrical optics, comparisons with plane wave scattering, mode separation in the case of tilted Gaussian beams (exhibiting a wave guiding effect in which the incident beam propagates along the cylinder), and properties of the rainbow.

At this point, the reader would like to know what should be the angle of attack he/she should choose when dealing with a problem of scattering of arbitrary shaped beams by infinite cylinders. Our recommendation is then as follows. First, if he/she is not familiar with the theory of distributions, the expedient choice is to use the plane-wave spectrum approach of Section 1, or its simplified version, assuming that one of these frameworks is sufficient for the problem at hand. It must however here be recalled that this plane-wave spectrum approach is not always feasible, and that the most general framework, valid for all cases is the one of distributions, as illustrated and demonstrated above in this section. Therefore the approach in terms of distributions is analytically superior to the plane-wave spectrum approach.

Conversely, if he/she is familiar with the theory of distributions, the approach in terms of distributions might be preferred because (i) it is more general and could deal with any case (such as Maxwellian contributions to non-Maxwellian beams) (ii) the final conversion from distributions to usual functions for implementation in computer programs is very easy, or even trivial (when it is allowed), as extensively discussed in [35] and (iii) although this may be a subjective statement, this more general approach is more aesthetic. Following these recommendations, we are now going to deal with the case of elliptical infinite cylinders using the theory of distributions.

4. Elliptical infinite cylinders

As for the case of circular infinite cylinders discussed in Sections 1 and 2, the GLMT for elliptical infinite cylinders relies on the use of BSPs. The description of the illuminating beam in terms of distributions is extensively discussed in [56]. The BSPs for the incident wave are found to read as:

$$U^i_{TM} = \frac{E_0}{k^2} \sum_{n=0}^{\infty} [< A_{n,TM}(\gamma), ceh_n(\mu, q^2) ce_n(\theta, q^2) e^{ik\gamma z} > + < B_{n,TM}(\gamma), seh_n(\mu, q^2) se_n(\theta, q^2) e^{ik\gamma z} >] \tag{164}$$

$$U^i_{TE} = \frac{H_0}{k^2} \sum_{n=0}^{\infty} [< A_{n,TE}(\gamma), ceh_n(\mu, q^2) ce_n(\theta, q^2) e^{ik\gamma z} > + < B_{n,TE}(\gamma), seh_n(\mu, q^2) se_n(\theta, q^2) e^{ik\gamma z} >] \tag{165}$$

in which the prefactors E_0/k^2 and H_0/k^2 have been chosen for convenience, and $A_{n,TM}(\gamma)$, $B_{n,TM}(\gamma)$, $A_{n,TE}(\gamma)$ and $B_{n,TE}(\gamma)$ are the incident BSDs (let us remark that, in contrast with the case of circular cylinders, the BSPs here depend on two BSDs). The quantities z, μ and θ are elliptical cylinder coordinates, while $ce_n(\theta, q^2)$ and $se_n(\theta, q^2)$ are even and odd Mathieu functions of the first kind respectively, assuming for convenience in the notation that $se_0(\theta, q^2) = 0$ although, in utmost rigor, it is not defined. Similarly, $ceh_n(\theta, q^2)$ and $seh_n(\theta, q^2)$ are modified Mathieu functions of the first kind which are as well even and odd respectively. As for the circular cylinder case, rules of derivation allow one afterward to obtain the various components of the incident wave. The BSDs are found to be obtained from the electric E_z and magnetic H_z components, although using a formulation more complicated than in the circular case, in any case too complicated to be summarized here. Let us only mention that the BSDs are linear combinations of the Dirac distribution and of its derivatives with test functions based on the Mathieu functions. To illustrate the formalism, the case of the Maxwellian contribution to a first-order Gaussian beam is discussed as well in [56].

Maxwellian contributions to higher-order Gaussian beams in the Davis formulation, up to $O(s^6)$, are studied in [57]. After being established, the BSPs are converted from distributions to usual functions, and checked by showing that they allow one to recover the original expressions of the field components. Furthermore, it is shown that the plane wave description may be recovered from the $O(s^0)$ -contribution. Next, in [58], the emphasis relies on the description of arbitrary shaped beams in elliptical cylinder coordinates by using a plane-wave spectrum approach, similar to the one used in Section 1 for circular infinite cylinders. This plane-wave spectrum description is presented as the result of a conversion from the description in terms of distributions. The distributions $A_{n,TM}(\gamma)$, $B_{n,TM}(\gamma)$, $A_{n,TE}(\gamma)$ and $B_{n,TE}(\gamma)$ are obtained in terms of usual functions expressed by double or triple quadratures, a situation completely similar to the one encountered in the GLMT *stricto sensu* (for homogeneous spheres) when the BSCs may be expressed as well by using double or triple quadratures [59]. A summary of the structure of the GLMT for elliptical infinite cylinders is available from [60]. An erratum to correct a few expressions in [56–58,60] is available from [61].

A complete exposition of the GLMT for elliptical infinite cylinders is finally available from [62]. The theory is presented using the plane-wave spectrum approach in which Eqs. (164)–(165) are translated to:

$$U^i_{TM} = \frac{E_0}{k^2}\sum_{n=0}^{\infty}\int [A_{n,TM}(\gamma), ceh_n(\mu, q^2)ce_n(\theta, q^2) + B_{n,TM}(\gamma), seh_n(\mu, q^2)se_n(\theta, q^2)]e^{ik\gamma z}d\gamma \tag{166}$$

$$U^i_{TE} = \frac{H_0}{k^2}\sum_{n=0}^{\infty}\int [A_{n,TE}(\gamma), ceh_n(\mu, q^2)ce_n(\theta, q^2) + B_{n,TE}(\gamma), seh_n(\mu, q^2)se_n(\theta, q^2)]e^{ik\gamma z}d\gamma \tag{167}$$

and partial wave expansions for both the scattered and the internal waves are displayed. Eq. (51) in [62] indicates that an option similar to option 2 in Section 1, e.g. Eqs. (63)–(64) has been chosen, with the question, similar to the one for circular infinite cylinders, to study the newly introduced option 1 when M pertains to **R**. Numerical results have been published in a thesis dissertation [37] but have not been released in the archival literature because, in the case of plane wave incidence, they do not perfectly reproduce the results published by Yeh [63]. Before publishing these numerical results, an independent computer code would be welcome to check the results of [62] and the equivalence between options 1 and 2 when M is real should be examined. Let us furthermore note that the case of circular cylinders may be recovered from the case of elliptical cylinders, e.g. Section 3.6 in [57].

5. Localized approximations

5.1. Reminder in spherical coordinates

In spherical coordinates, in particular to deal with the GLMT *stricto sensu*, several methods have been designed to evaluate BSCs. When applying the GLMT to the case of Gaussian beams, it has originally been found that the numerical evaluation of quadratures was very time-consuming. This difficulty has been solved by introducing localized approximations (with several variants) allowing one to describe the incident wave by using a localized beam model. History and features of localized approximations are reviewed in [64], to be complemented by [65] and [66]. A localized approximation essentially takes the form of a localization operator (or localization procedure). Let us exemplify the localization procedure in the case of TM-BSCs $g^m_{n,TM}$, with an $\exp(i\omega t)$ convention, where it reads as follows, e.g. [67] and references therein.

(i) Expand the radial component of the electric field in terms of m -waves, proportional to $\exp(im\varphi)$, according to:

$$E_r = \sum_{m=-\infty}^{m=+\infty} E^m_r \tag{168}$$

(ii) Extract the non-plane-wave contribution $\mathcal{E}^m_r(R = kr, \theta)$ of E^m_r.

(iii) Then, the localized approximation $\overline{g^m_{n,TM}}$ of the BSC $g^m_{n,TM}$ reads as:

$$\overline{g^m_{n,TM}} = (\frac{-i}{L^{1/2}})^{|m|-1}\mathcal{E}^m_r(L^{1/2}, \pi/2) \tag{169}$$

In the original localized approximation, $R = L^{1/2}$, called the radial evaluation point, was taken to be equal to $R = (n + 1/2)$. In an improved procedure [42], the radial evaluation point was better taken to be $R = [(n-1)(n+2)]^{1/2}$. In a still improved modified localization procedure, we rather have:

$$L = (n - |m|)(n + |m| + 1) = (n + 1/2)^2 - (|m| + 1/2)^2 \tag{170}$$

a variant whose precise implementation requires a specific care when $n = |m|$ to avoid divergence of the term $1/L^{1/2}$ in Eq. (169), see discussion in [66]. There also exist another variant named the integral localized approximation [68] originally relying on $R = (n + 1/2)$ but which could take advantage of the improvements mentioned above, such as exhibited in Eq. (170). We shall below refer to these localized approximations under the name of SLAs (spherical localized approximations).

5.2. Localized approximation in circular cylindrical coordinates

The localized approximation for the GLMT for circular infinite cylinders (CLA: Cylindrical Localized Approximation) was proposed in [35], in analogy with SLAs, but without any firm mathematical basis. A rigorous justification of the CLA, for perpendicular illumination by a Gaussian beam, has been afterward published in [69]. This paper rigorously justifies the CLA in the plane-wave spectrum approach and demonstrates that, although based on a first-order description of the Gaussian beam, the CLA anticipates well the rigorous formulation arising from the use of higher-order beams (a property already shared by SLAs), and introduces as well the CLA in the approach in terms of distributions. The CLA is then justified by many satisfactory numerical checks. The case of arbitrary location and orientation of the scatterer, still for Gaussian beams, has been published in [70]. Again, many satisfactory numerical checks have been displayed. These numerical checks compare original profiles concerning incident fields expressed in terms of coordinates with those obtained by reconstructing the same quantities from the BSPs. Finally, the CLA has been validated, not only for Gaussian beams, but for "arbitrary shaped beams" as well [71], following the same strategy than the one used to validate SLAs in the case of "arbitrary shaped beams" [67].

The CLA procedure is as follows. Let us set:

$$(E_z, H_z) = (E_{z0}, H_{z0}) \exp(iR \sin\Gamma \cos\varphi) \tag{171}$$

in which Γ is a tilt angle specifying the orientation of the cylinder axis with respect to the axis of propagation of the illuminating beam, and $R = k\rho$. We afterward introduce:

$$(\overline{E_{z0}}, \overline{H_{z_0}}) = \widehat{G}(E_{z0}, H_{z0}) \tag{172}$$

in which $\widehat{G}$ is a localization operator which changes R to $(-m)/\sin\Gamma$ and φ to $\pi/2$. Then, the BSCs (or BSFs) in the CLA-framework read as, again with $Z = kz$:

$$\overline{I_{m,TM}(\gamma)} = \frac{(-1)^m}{2\pi(1-\gamma^2)} \int_{-\infty}^{+\infty} \frac{\overline{E_{z0}}}{E_0} \exp(-i\gamma Z) dZ \tag{173}$$

$$\overline{I_{m,TE}(\gamma)} = \frac{(-1)^m}{2\pi(1-\gamma^2)} \int_{-\infty}^{+\infty} \frac{\overline{H_{z0}}}{H_0} \exp(-i\gamma Z) dZ \tag{174}$$

5.3. Localized approximation in elliptical cylindrical coordinates

An Elliptical Cylinder Localized Approximation (ECLA) has afterward been proposed in the case of Gaussian beams in [72] relying on the use of third-order Davis beams, in the case when the cylinder is perpendicularly illuminated by a Gaussian beam, whose beam waist center is identified with a point on the axis of the elliptical infinite cylinder. The case of "arbitrary shaped beams" has been treated in [73]. The procedure to obtain the localized approximation $\overline{A_{n,TM}(\gamma)}$ to $A_{n,TM}(\gamma)$ is then as follows.

(i) Define:

$$E_{z0}(z, i\mu, \theta) = E_z(z, i\mu, \theta)/\mathcal{E}_1 \tag{175}$$

in which:

$$\mathcal{E}_1 = \exp(-iL\eta \sin\Gamma) \tag{176}$$

in which L is a rescaled semifocal length defined for the elliptical coordinates used, Γ is still a tilt angle specifying the orientation of the illuminating beam, and:

$$\eta = \cos\mu \cos\theta \cos\beta + i \sin\mu \sin\theta \sin\beta \tag{177}$$

in which μ, θ are angular elliptical coordinates, and β is an angle defining an arbitrary polarization of the beam.

(ii) Define:

$$\overline{\mathcal{E}_1} = \frac{(-i)^p}{\pi p_n} ce_n(\beta, q^2) \tag{178}$$

in which p is 0 (1) for n even (odd), and p_n designates quantities sometimes called joining factors [63], [74].

(iii) Then:

$$\overline{A_{n,TM}(\gamma)} = \frac{\overline{\mathcal{E}_1}}{1-\gamma^2} \int_{-\infty}^{+\infty} \frac{E_{z0}(z, i\mu_0, \theta_0)}{E_0} \exp(-i\gamma Z) dZ \tag{179}$$

in which $\theta_0 = (\beta - \pi/2)$ and μ_0 is defined by a validity condition discussed in [73], second column of Page 2952.

A similar procedure is valid for $\overline{B_{n,TM}(\gamma)}$ and the TE-BSCs are afterward obtained from the TM-BSCs by changing electric fields to magnetic fields. Numerical validations for Gaussian beams are available from [72] and also from [37]. The reader wanting to deal with the ECLA for "arbitrary shaped beams" is recommended to begin with the case of Gaussian beams as a training.

5.4. Additional remarks

The "arbitrary shaped beams" discussed in [67,71,73 excluded the case of beams whose description involves an axicon angle and/or a topological charge, so that a few warnings are required as listed in this subsection. It has then been shown that SLAs are less accurate in the case of beams exhibiting axicon angles, e.g. [75–79], and/or topological charges [80–82]. Finite series, pertaining to the arsenal of methods to evaluate BSCs in spherical coordinates could then been used to speed up numerical computations [83–86]. Otherwise, we may be content with the use of SLAs leading to localized beam models, even if they depart from the intended beams. It is likely that such limitations occur as well for CLA and ECLA, opening new roads for research beyond the use of numerical quadratures and localized approximations, including the analytical evaluation of quadratures, the design (if possible) of finite series methods, or the use of ASD, all methods successful in spherical coordinates which would be worth to investigate as well in circular and/or elliptical cylindrical coordinates.

6. Worldwide contributions

6.1. The main stream

This section reports on worldwide contributions concerning the interaction between arbitrary electromagnetic shaped beams and infinite cylinders, excluding plane wave illumination, geometrical optics, "semi-analytical" (e.g. EBCM or the use of surface integrals, e.g. the projection method, e.g. [87]) and numerical methods, according essentially to a chronological order (most of the time). It may then certainly be stated that we all have precursors. Two of them, relevant to the framework of the present review paper, are Alexopoulons and Park [88]. In their work, published in 1972, they consider a perfectly conducting and radially inhomogeneous dielectric cylinder. The incident wave exhibits a Gaussian amplitude distribution of the form $\exp(-R^2/w_0^2)$ which results in a non-Maxwellian beam, corresponding to a fairly crude representation of the beam, still simpler than the one of the first-order Davis beam. This beam illuminates the cylinder perpendicularly. The plane wave case is recovered as a special case by making the radius of the scatterer much smaller than the size of the beam. Partial wave expansions similar to the ones used in the simplified plane-wave spectrum of Section 1, e.g. Eqs. (28) and (37) and others, i.e. without the γ-integral, are used. Numerical results are displayed for size parameters ka (with k the wave-number and a the cylinder radius) equal to 1 and 5. It is interesting to compare these values with the ones which could be reached about three decades after, namely more than 500 in [36], and much likely more nowadays. Also, as far as we understand, the results are not considered as accurate when $w_0 \leq a$, i.e. in the case of strongly focused beams. Furthermore, only far fields are considered.

In 1979, Kojima and Yanagiuchi [89] used a more sophisticated description of a two-dimensional Gaussian beam (whose fields depend on the propagation location), with perpendicular incidence, and offset location (off both the beam axis and the beam waist), and partial wave expansions allowed them to describe the various fields relevant to the problem at hand. BSCs are obtained under the form of 1D-definite integrals. Numerical results are displayed and the numerical scheme is validated by reference to the simpler plane wave scattering problem.

In 1980, still relying on the use of partial wave expansions, Iannarella dealt with the case of an inhomogeneous fiber (built from a concentric division into layers, allowing one to approximate the index of refraction dependency to an arbitrary degree of accuracy) perpendicularly illuminated by a "transverse" Gaussian beam in an off-axis configuration. The beam description is still non-Maxwellian but more accurate than in [88], exhibiting a waist radius depending on the propagation coordinate [90].

In 1982, we afterward have a series of three papers by Kozaki. In [91], the scattering of a Gaussian beam by a homogeneous dielectric cylinder (instead of a conducting cylinder) is discussed (the author also quoted papers which, however, are not related to cylinders but to spherical objects). The Gaussian beam still illuminates the cylinder perpendicularly and the beam description is still cruder than the one of the first-order Davis beam. The formalism relies on the use of an ASD, on a far field expression, and on an integral representation of the incident beam which is difficult to evaluate exactly and which, as a consequence, is evaluated approximately. Eventually, the incident, scattered and cylinder fields are expressed as partial wave expansions to deal afterward with the boundary conditions. Results are applicable to the microwave, millimeter range. Numerical results are compared with a laboratory experiment carried out at a frequency equal to 9.6 GHz, leading to a "good" agreement. This study is extended, with a quite similar formulation, to the case of an inhomogeneous dielectric cylinder (i.e. in which the permittivity ε of the cylinder material depends on the coordinate ρ) in [92]. The homogeneous cylinder case is recovered as a special case. Numerical results using the wave theory are compared with results obtained from geometrical optics. Finally, Kozaki returned to the case of a conducting cylinder and, with a formulation similar to the one he previously used, he obtained new simple expressions for the scattering of a Gaussian beam by the cylinder [93]. The expression for the Poynting vector is described as well. Many numerical calculations and experiments in the microwave range have been performed, still at 9.6 Ghz, leading again to a "good" agreement.

In 1989, adapting a simplification proposed by Kozaki and more generally relying on a similar framework [91–93], Rao studied the scattering of a Gaussian beam by a radially inhomogeneous cylinder, more specifically exhibiting a cylindrical dielectric shell between an internal radius a and an external radius b, with the permittivity ε depending on ρ in the region inside the shell [94]. The beam illuminates the cylinder perpendicularly. Numerical results are displayed and discussed.

In 1995, Zimmermann et al. [95] dealt with the scattering of an off-axis Gaussian beam by a dielectric cylinder, still with a perpendicular illumination, and with a partial wave expansion method similar to the one used by previous authors. Comparisons are displayed between geometrical optics and the wave model, and as well between wave model calculations and experiments. Discrepancies between geometrical optics and wave model are exhibited and discussed.

The works quoted above all dealt with Gaussian beams and circular cylinders, and none of them reached the degree of generality of the approaches discussed in Sections 1 and 2 devoted to general approaches usable in the case of arbitrary shaped beams. In parallel with such works of general scope, in the framework of exchanges and efficient collaborations between Normandie University, France, and Cleveland University, USA, a thorough study of the scattering of a diagonally incident focused Gaussian beam by an infinitely long homogeneous circular cylinder has been published by Lock in 1997 [96]. The approach used relies on an ASD, i.e. let us recall, on the modeling of the incident beam by an angular spectrum of plane waves, such as the one already used in spherical coordinates. Beside the methods usable to compute BSCs, e.g. Section 5.4, the ASD (as in spherical coordinates) provides an alternative way of computing the BSCs in order, afterward, to deal with a general GLMT formulation. The formulation presented in Section 2 of [96] is valid for arbitrary shaped beams (in cases when the use of distributions is not compulsory) and is specified in Section 3 for the case of a diagonally incident plane wave, while Section 4 is devoted to the evaluation of BSCs for a diagonally incident Gaussian beam using (i) a Davis model, (ii) an ASD and (iii) a localized approximation. It is recognized that problems associated with the use of a Davis model may be circumvented by using the theory of distributions, as anticipated in Section 3 when dealing with an incident plane wave. Finally, Section 5 discussed far-field scattering properties. As a complementary work, another paper by Lock [97] focused the attention on morphology-dependent resonances (MDRs) for an infinitely long circular cylinder illuminated either by a plane wave or a focused Gaussian beam, using again an ASD approach to evaluate the BSCs, but expressing the scattering processes in terms of a S-matrix analogous to the one used in quantum-mechanical scattering problems. One of the results obtained insists on the difference between MDRs in spheres and MDRs in cylinders. For light scattering by a sphere, the size parameters of the MDRs do not depend on the beam shape profile, in contrast with the case of cylinders in which there is an influence of the shape of the beam on MDRs depending on its angle of incidence. More specifically, the resonant size parameters of the cylinder increase as the angle of incidence of an illuminating plane wave increases, an effect which is conveyed to the case of Gaussian beam insofar as its ASD depends on the angle of incidence. This work is completed by another paper, still in 1997, deriving and discussing the Debye-series expansion of the partial-wave scattering and interior amplitudes for the interaction between a diagonally incident beam of arbitrary profile with an infinitely long homogeneous dielectric circular cylinder, and examining the first-order rainbow extinction transition. Theoretical results are compared with experimental ones [98].

In 2000, plane-wave and Gaussian beam scattering on an infinite non absorbing cylinder are compared by Mroczka and Wysoczanski [99] in the GLMT framework of Section 1, in the case of perpendicular illumination. Numerous numerical results allow the discussion of the effects of wavelength, cylinder diameter, refractive index, polarization, and off-axis location. Next, in 2000 again, Guo and Wu [100] dealt with the problem of an off-axis Gaussian beam perpendicularly incident on an infinitely long multilayered cylinder, using a recursive scheme in terms of three logarithmic derivatives of Bessel functions (fairly similar to a recursive scheme used for multilayered spheres [4,101]), and applied to a discussion of rainbow scattering for homogeneous and inhomogeneous fibers.

In 2006, measurement and analysis of angle-resolved scatter from small particles in a cylindrical microchannel were carried out by Venkatapathi et al. [102]. The scattering theory is used following Lock, i.e. with the BSCs evaluated with an ASD approach. Numerical results for scattered and internal fields are provided in the case of perpendicular incidence, and comparisons with experiments are carried out.

In 2007, a complementary paper, by Venkatapathi and Hirleman, still with an ASD approach to the evaluation of BSCs, in the case of an elliptical Gaussian beam with a perpendicular incidence, has been published [103]. It focused on the examination of the effect of beam size parameters on the internal fields properties, in particular displaying resonance-dominated internal fields when the parameters defining the interaction are well adjusted. Zhang et al. [104] stated that the theory of distributions leads to “excessive difficulties” (it is hoped that the present review paper will demystify the use of the theory of distributions in light scattering). Then, they introduced an original and interesting approach to the problem, by using a bridge between a GLMT in spherical coordinates and a GLMT in cylindrical coordinates. For this, they expressed VSWFs in terms of Vector Cylindrical Wave Functions (VCSWFs) and consequently expressed the BSCs (or BSFs) $\mathrm{I}_{m,TM}$ and $I_{m,TE}$ in cylindrical coordinates in terms of BSCs in spherical coordinates. This is an extrinsic method according to the following definition: “Intrinsic methods evaluate beam shape coefficients of a GLMT posed in a certain coordinate system in terms of quantities pertaining to the same coordinate system. Conversely, extrinsic methods evaluate the beam shape coefficients of a GLMT posed in a certain coordinate system in terms of beam shape coefficients in a different coordinate system” [105]. As examples of the extrinsic method, BSCs in cylindrical coordinates are evaluated in the cases of plane wave and Gaussian beam illuminations. Let us insist of the fact that the authors used VCWFs instead of BSPs in cylindrical coordinates. If necessary, it might be interesting to establish the relationship between these VCWFs and BSPs, as already done and used in spherical coordinates [5,6,15]. Also, still in 2007, Novitsky dealt with the light scattering by multilayer bianisotropic cylindrical particles using a matrix approach in a variant of the plane-wave spectrum approach, and applied it to case of a polarized Gaussian beam interacting with a two-layer bianisotropic core cylinder [106].

In 2008, a complementary paper by Zhang and Han [107], using a similar approach than in [104], with an extrinsic evaluation of BSCs, dealt with the scattering of an arbitrarily oriented shaped beam by an infinite cylinder. In the same year, Wu and Li presented a theory of interaction between an off-axis 2D Gaussian beam in which the electric field is of the form $E_0 \exp(-\rho^2/w_0^2)$ and a multilayered cylinder, by using an ASD to evaluate the BSCs, and performed a Debye series analysis [108].

In 2009, Wang et al. [109] dealt with the scattering of shaped beam by a conducting infinite cylinder with dielectric coating relying on a GLMT framework for cylinders. They however used an extrinsic approach in which cylindrical BSCs are expressed in terms of spherical BSCs, the latter being evaluated by using a SLA (this is an indirect way of implementing localized approximations in a GLMT for cylinders, rather than using a CLA in cylindrical coordinates). Wei et al. [110] investigated the scattering of a shaped beam by an arbitrarily oriented conducting infinite cylinder within a GLMT framework, and the cases of conducting and dielectric cylinders were compared. Li et al[111]. derived Debye Series Expansion (DSE) for infinitely long multilayered cylinders perpendicularly illuminated by an arbitrary shaped beam. The case of Gaussian beam illumination is detailed, with BSCs calculated by using the CLA. Afterward DSE is used to discuss the rainbow scattering by a graded-index polymer optical fiber (GI-POF). DSE approach and GLMT are compared, leading to an excellent agreement. To complete the year 2009, we mention as well a review paper by Han et al. [112] who dealt with a review devoted to the scattering of typical particles obliquely illuminated by arbitrary shaped beams, with BSCs determined by using an extrinsic method for the cases of spheroidal and cylindrical particles.

In 2010, Normatov et al. [113] focused on resonance scattering in the case of cylindrical nanowire illuminated by beams exhibiting a line phase irregularity, i.e. a wavefront dislocation. The theoretical approach is presented as being a “rigorous modeling” which “employs a 2D version of the Richard-Wolf focusing method and the source model technique”. An interesting issue would be to revisit such singular beams in a GLMT framework. Li et al. [114] studied the relation between DSE and GLMT for laser beam scattering by a multilayer cylinder, using recursive relations for Bessel functions already mentioned above. The consistency between DSE and GLMT is proved in detail. As an example, rainbow phenomenon on the scattering of a two-layer cylinder is discussed by using the DSE. We end the year with Sun and Wang [115] who dealt with the scattering of an infinite cylinder illuminated by a couple of Gaussian beams, using an expansion of the beam in terms of VCWFs for arbitrary orientation, in a GLMT framework. Cylindrical BSCs are evaluated using an extrinsic approach in terms of spherical BSCs, the latter being evaluated using a SLA (still an indirect way of dealing with a “CLA”).

In 2011, Pawliuk and Yedlin [116] dealt with the case of scattering between parallel cylinders, accounting for the multiple scattering between the individual scatterers, using a two-dimensional plane wave spectrum to describe the beam.

In 2012, Zhai et al. [117] dealt with the on-axis Gaussian beam scattering by an eccentrically coated conducting cylinder, in a GLMT framework expressed in terms of VCWFs, for oblique incidence. The BSCs are evaluated with an extrinsic approach in which the spherical BSCs are evaluated by using a SLA. An addition theorem for the Bessel functions is invoked to implement the fact that there is an eccentrically located cylinder inside a host cylinder. Numerical results are provided in the case of perpendicular illumination by a tightly focused Gaussian beam, in the far-zone. A similar approach is used in the same year to deal with the scattering by a chiral cylinder [118], still for on-axis oblique incidence and using an extrinsic method with spherical BSCs evaluated by using a SLA. Numerical results are again provided in the far-zone for a tightly focused Gaussian beam under perpendicular illumination.

In 2013, Zhang et al. [119] dealt with the study of internal and near-surface electromagnetic fields for a uniaxial anisotropic cylinder obliquely illuminated by an on-axis Gaussian beam, in a GLMT framework, using an extrinsic approach as previously described in Zhai et al. papers [117,118]. Yang et al. [120] studied the far-field scattering of single walled nanotubes illuminated by a Gaussian beam, in a GLMT framework, in which the nanotubes are assimilated to infinitely long circular cylinders. Hyde IV et al. [121] dealt with the interaction between a circular cylinder and a partially-coherent wave, using a plane-wave spectrum of electromagnetic fields.

In 2014, Chen et al. [122] used a GLMT to study the transmission of a Gaussian beam through a gyrotropic cylinder. The theory is formulated in terms of VCWFs, and relies on an extrinsic method in which the spherical BSCs are evaluated by using a SLA. Gagnon et al[123]. theoretically investigated laser thresholds in a photonic molecule composed of two coupled active cylinders of slightly different radii. The theory relies on two ingredients (i) steady-state *ab initio* laser theory used to study the effect of the underlying gain transition on lasing frequencies and thresholds and (ii) a GLMT approach which may be used for the computation of the scattering by a complex arrangement of dielectric cylinders.

In 2015, Gagnon and Dubé [124] published a tutorial devoted to a GLMT to compute the interaction of light with arrays of cylindrical scatterers, using a GLMT framework together with an addition theorem for cylindrical functions. Emphasis is placed on the derivation of BSCs and on the computation of resonant modes. Let us note that cylinders embedded in other cylinders are also within the reach of the method. Yan et al. [125]. studied the transmission of a Gaussian beam through a coated chiral cylinder in a GLMT framework, using VCWFs and an extrinsic method in which spherical BSCs are evaluated using a SLA.

In 2016, Mitri [126] studied cylindrical particle manipulation and negative spinning under nonparaxial Hermite-Gaussian light-sheet beams, using an ASD. The BSCs are expressed under the form of quadratures evaluated by a standard Simpson's rule for numerical integration. Furthermore, the analysis is extended to the evaluation of longitudinal and transverse radiation forces as well as the spin torque on an absorptive dielectric cylindrical particle, and the dynamics of the particle are examined relying on Newton's second law of motion. Laguerre-Gauss light-sheet beams are considered as well.

In 2017, Mitri [127] presented a generalized nonparaxial analytical solution for a transverse Airy light-sheet, using an ASD, with BSCs expressed by an integral which is evaluated by using a standard trapezoidal rule. The formulation obtained allows applications to the computations of optical scattering, radiation force and torque, in cylindrical coordinates. This is followed by Mitri [128] to be complemented by an addendum [129], dealing with radiation forces and torques of light-sheets, in a GLMT framework, where BSCs are evaluated using an ASD. Numerical examples are displayed concerning dielectric absorptive circular cylinders illuminated by different wavefronts, ranging from plane waves to non-paraxial scalar Airy and Gaussian light-sheet beams. In the same year, Swirniak and Mroczkra [130] proceeded to a theoretical analysis of primary rainbows produced by infinite homogeneous circular cylinders for different types of optical fibers (with step- and graded-index profiles) under low-coherent light illumination. In particular, it is explored how the scattering varies when the incident light is changed from a monochromatic source to a broader spectrum source. A broad source is treated as a superposition of elementary monochromatic waves, each of them being processed in a GLMT framework and using as well DSEs. A complementary study is devoted to $(p = 2)$- scattering (in the DSE-terminology) when the scatterer is illuminated by an ultrashort light pulse (the characteristic time duration of the pulse in free space is taken to be shorter than the propagation time within a fiber, allowing one to isolate various p=2 orders in time when incident on a detector). Next, Han et al. [131] published a review devoted to light wave propagation and scattering through particles. GLMT approaches to deal with the scattering by typical particles with regular shapes (spheres, spheroids, cylinders) are reviewed as well as numerical methods for scattering by complex particles with arbitrary shapes and structures.

In 2018, Shiloah [132] dealt with an algorithmic issue, namely the study of canonical scattering coefficients upward recursion process for long cylinders (and multilayered spheres) with large size parameters.

In 2019, Mitri [133] dealt with a study of optical radiation force exerted on a cylinder material of circular cross section exhibiting circular dichroism (i.e. rotary polarization), illuminated by an electric line source, using a simplified plane-wave spectrum approach.

In 2020, the previous work on dichroism is complemented by studying the radiation force on a cylinder exhibiting dichroism illuminated by a circularly polarized light [134]. Chen et al. [135] studied the Gaussian beam propagation through a biaxial anisotropic circular cylinder, using electromagnetic expansions in terms of appropriate VCWFs, following an extrinsic GLMT scheme where the cylindrical BSCs are evaluated in terms of spherical BSCs, the latter being obtained from a SLA. Zhang et al. [136] dealt with the Gaussian beam scattering by an infinite cylinder with a spherical inclusion. This problem possesses the originality that both VCWFs and VSWFs are used, depending on the region considered in the scatterer. The beam is described by a localized beam model in order to numerically investigate the scattering properties.

In 2021, Mitri [137] used a GLMT-like approach (i.e. using partial wave expansions in cylindrical coordinates, with introduction of BSCs) to discuss longitudinal and transverse optical scattering asymmetry parameters for a dielectric cylinder illuminated by light-sheets of arbitrary wavefronts and polarization.

None of the papers cited in this subsection dealt with the theory of distributions, although this theory provides the most general framework to deal with the interaction between arbitrary shaped beams and cylinders. It must however be recognized or stated that, in most cases (when the theory in terms of distributions may as well be expressed rigorously in terms of usual functions), the choice of a distribution approach or of an usual function approach may be viewed as a "matter of notation". Also, none of these papers dealt with a GLMT for elliptical cylinders whatever its variant, meaning that this issue is under-developed. We may nevertheless quote a paper by Mitri [138] who dealt with optical radiation force exerted on an elliptical conducting cylinder having a smooth or ribbed surface. In deep contrast with the GLMT for elliptical cylinders, which relies on the use of Mathieu functions, this paper relies on expansions in terms of cylindrical Bessel functions which, as stated by the author, are "entirely appropriate to treat elliptical objects. However, the convergence of the solutions requires an extra-check because it is expectedly slower than in the more classical approach of GLMT, in particular due to the fact that the cylindrical wave functions used are not orthogonal to elliptical surfaces, in contrast with elliptical wave fiunctions". Afterward, the same author, using a similar non conventional approach, studied the radiation force and torque exerted on a perfect electrically conducting elliptical cylinder illuminated by a focused Gaussian light sheet with arbitrary incidence [139]. As stated by the author, such an approach however may be viewed as "a semi-analytical" method, which requires a single numerical integration procedure. Complementary studies concern the case of cylinders with arbitrary geometrical cross-section, dealing with dichroism and, in particular, with the evaluation of scattering, extinction and absorption cross sections [140] or of radiation forces and torques [141] in the case of light-sheet illumination. From these examples, we may then conclude that studies devoted to the conventional GLMT for elliptical cylinders are severely under-developed.

6.2. Complementary side-issues

This subsection deals with two side-issues (i) the case of acoustical beams and (ii) the use of VCFWs for particles other than cylindrical particles.

For issue (i), let us first mention that electromagnetic GLMTs deal with vectorial scatterings, i.e. scatterings of electric and magnetic vector fields. GLMT-like approaches may also be developed for scalar fields. An example is quantum mechanics, e.g. [142] and references therein. Another very important scalar case is the one of acoustical waves. Indeed, electromagnetic and acoustical wave scatterings share many common features, in any case a sufficient number of them, such as the use of acoustical BSCs, to allow one to use the denomination of acoustical GLMTs to denote the scattering of acoustical waves by various regular acoustical scatterers possessing enough symmetries to allow one to use the method of separation of variables. These statements are well illustrated by Baresch et al. [143]. who dealt with the modeling of acoustic radiation force exerted on an arbitrarily located elastic sphere placed in an inviscid fluid, by Thomas et al. [144] who reviewed analogies between electromagnetic and scalar scatterings, and comprehensively discussed acoustical and optical radiation force pressures in relation with the development of single beam acoustical tweezers, with a strong emphasis on the mutual enrichment which arose from a long common history, Thomas et al. again [145] who reviewed the similarities and differences between optical and acoustical radiation pressure, with a focus on single-beam acoustical tweezers, and Baudoin and Thomas [146] dealing with an extended review on the issue of acoustic scattering and trapping.

Returning to cylinders, we then have Mitri [147] who studied the interaction between an acoustical beam with an elastic cylinder arbitrarily located in non-viscous fluid, using a partial wave series expansion method in cylindrical coordinates, with a particular attention paid to resonance effects, Zhang et al. [148] who investigated acoustical radiation force exerted on cylindrical particles in water, using a finite series expansion (similar to the ones used in spherical coordinates, e.g. [149,150]) to model an incident acoustical Gaussian beam, and obtaining the BSCs of the beam expanded as VCWFs, and Zhang et al. [151] who dealt with the computation of the acoustic radiation force exerted on a rigid cylinder in an off-axial Gaussian beam, with the expressions for the BSCs given in a closed, analytical form. Wang et al. [152] discussed long-range and robust acoustic pulling which provides a new mechanism for acoustic manipulation apart from levitation, trapping and binding, all phenomena which have their electromagnetic counterpart, a fact which is somehow pointed in an Appendix entitled “Lorenz-Mie for acoustic force”. The pulling effect is achieved by using a pair of one-way chiral surface waves supported on the interface between two phononic crystals composed of spinning cylinders with equal but opposite spinning velocities embedded in water.

For issue (ii), let us note that VCWFs which are natural to the study of cylinders may be used as well for studying the interactions between arbitrary electromagnetic beams and slabs just as exemplified by Wang et al. [153] who expressed the cylindrical BSCs of an incident Gaussian beam by using an extrinsic method with the spherical BSCs evaluated with a SLA, Li et al. [154] who studied the deformations of circularly polarized Bessel vortex beams reflected and transmitted by a uniaxial anisotropic slab, Zhang et al. [155] who dealt with the propagation characteristics of a focused electromagnetic beam in a uniaxial anisotropic slab, Lu et al. [156] who compared differently polarized Bessel vortex beams propagating through a uniaxial anisotropic slab, Liu et al. [157] who investigated the reflection and transmission of a Bessel vortex beam by a stratified uniaxial anisotropic slab, taking a three-layered slab as an example, and Yan et al. [158] dealing with the electromagnetic wave beam propagation through a chiral slab. VCWFs are used as well by Zhang et al. [159] to study the Gaussian beam scattering by a particle above a plane surface, using again an extrinsic method to evaluate the cylindrical BSCs in terms of spherical BSCs evaluated by a SLA, not only for cylinders however, but also for spheres and spheroids. Theoretical results obtained from this formulation in the case of a polystyrene sphere illuminated by a plane wave were compared with EBCM simulations. See as well Yuan et al. [160] for a similar study devoted to the case of Gaussian beam scattering by a particle on or near a plane surface.

7. Conclusion

Since a few decades, there has been a vigorous effort to study the interaction between arbitrary electromagnetic shaped beams and scattering particles, either possessing a sufficient degree of symmetry to allow one to use the method of separation of variables (generically named GLMTs) or irregular particles (e.g. EBCM). The most developed theory of this kind has been the GLMT *stricto sensu* when the scatterer is a homogeneous sphere defined by its diameter and its complex refractive of refraction, which is now about four decades old. Another more recent GLMT has been developed for the case when the scatterer is an infinite cylinder with a cross section which is either circular or elliptical. Original works related to the case of infinite cylinders soon demonstrated that the most general framework to handle these cases requires the use of the theory of distributions which has afterward been found discouraging in the mind of newcomers. Motivated by the revival of applications to the scattering by cylinders (both from electromagnetic or acoustic waves), this paper presents a review, expected to be fairly exhaustive, concerning the theories of interactions between waves and cylinders. More specifically, this review deals with the GLMT for circular and elliptical infinite cylinders, both using usual functions and distributions, with a particular attention paid to the relationship between both approaches. A section is devoted to worldwide contributions to this field of research, including the case of acoustical wave interactions with infinite cylinders. It must furthermore be noted that, although devoted to the case of infinite cylinders, the approaches discussed in the present paper may be used as well to finite cylinders when the transversal size of the illuminating beam is sufficiently small with respect to the length of the cylinder, at least when the angle of incidence is sufficiently close to perpendicular illumination to neglect wave guiding effect inside the cylinder.

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Declaration of Competing Interest

No conflict of interest.

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