Studies in Applied Mathematics

WILEY

STUDIES IN APPLIED MATHEMATICS

ORIGINAL ARTICLE OPEN ACCESS

Existence and Orbital Stability of Standing-Wave Solutions of the Nonlinear Logarithmic Schrödinger Equation On a Tadpole Graph

Jaime Angulo Pava | Andrés Gerardo Pérez Yépez

Department of Mathematics, Cidade Universitária, São Paulo, São Paulo, Brazil

Correspondence: Jaime Angulo Pava (angulo@ime.usp.br)

Received: 18 February 2025 | **Revised:** 18 June 2025 | **Accepted:** 14 July 2025

Funding: J. Angulo was partially funded by CNPq/Brazil Grant and Universal/CNPq project, he would like to thank to Mathematics Department of the Federal University of Pernambuco (UFPE) by the support and the warm stay during the Summer-program/2025, where part of the present project was developed. A. Pérez was partially funded by CAPES/Brazil and UNIVESP/São Paulo/Brazil.

Keywords: extension theory of symmetric operators | nonlinear Schrödinger model | quantum graphs | stability | standing wave solutions | Sturm comparison theorem

ABSTRACT

This work aims to study some dynamical aspects of the nonlinear logarithmic Schrödinger equation (NLS-log) on a tadpole graph, namely, a graph consisting of a circle with a half-line attached at a single vertex. By considering Neumann–Kirchhoff boundary conditions at the junction, we show the existence and the orbital stability of standing wave solutions with a profile determined by a positive single-lobe state. Via a splitting-eigenvalue method, we identify the Morse index and the nullity index of a specific linearized operator around a positive single-lobe state. To our knowledge, the results contained in this paper are the first to study the (NLS-log) on tadpole graphs. In particular, our approach has the prospect of being extended to study stability properties of other bound states for the (NLS-log) on a tadpole graph or other non-compact metric graph such as a looping-edge graphs.

MATHEMATICS SUBJECT CLASSIFICATION (2020) 35Q51, 35Q55, 81Q35, 35R02 (Primary), 47E05 (Secondary)

1 | Introduction

The following Schrödinger model with a logarithmic nonlinearity (NLS-log)

$$i\partial_t u + \Delta u + uLog|u|^2 = 0, \tag{1.1}$$

where $u = u(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $N \geq 1$, was introduced in 1976 by Bialynicki-Birula and Mycielski [20] who proposed a model of nonlinear wave mechanics to obtain a nonlinear equation which helped to quantify departures from the strictly linear regime, preserving in any number of dimensions some fundamental aspects of quantum mechanics, such as separability and additivity of total energy of noninteracting subsystems. The NLS model in (1.1) equation admits

https://doi.org/10.1111/sapm.70085

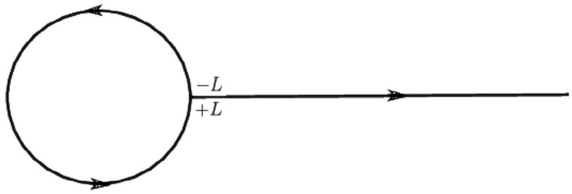

FIGURE 1 | Tadpole graph.

applications to dissipative systems [32], quantum mechanics, quantum optics [21], nuclear physics [31], transport and diffusion phenomena (e.g., magma transport) [27], open quantum systems, effective quantum gravity, theory of superfluidity, and Bose–Einstein condensation (see [31, 45], and references therein).

The analysis of nonlinear evolution PDEs models on metric graphs has potential applicability in the analysis of physical models for modeling particle and wave dynamics in branched structures and networks. Since branched structures and networks appear in different areas of contemporary physics with many applications in electronics, biology, material science, and nanotechnology, the development of effective modeling tools is important for the many practical problems arising in these areas (see [18], and references therein). Nevertheless, real systems can exhibit strong inhomogeneities due to different nonlinear coefficients in different regions of the spatial domain or to a specific geometry of the spatial domain. Thus, we will chose a "simple" metric graph-tool such as the tadpole to discover several characteristics of the NLS-log model.

The NLS-log ($N = 1$) on metric graphs has been studied by several authors in the recent years (see [10, 12, 13, 16, 17], and reference therein). Two basic metric graphs Γ_i, $i = 0, 1$, were examined. For $\Gamma_0 = (-\infty, 0) \cup (0, +\infty)$ with boundary δ-or δ'-interactions at the vertex $\nu = 0$, the existence and orbital (in)stability of standing wave solutions with a Gausson profile were established. A similar study was also conducted for $\Gamma_1 = \bigcup_{j=1}^{N}(0, +\infty)$, known as a star metric graph with the common vertex $\nu = 0$.

In this work, we study issues related to the existence and orbital stability of standing wave solutions of the NLS-log ($N = 1$) on a tadpole graph, specifically the vectorial model

$$i\partial_t U + \Delta U + U \mathrm{Log}|U|^2 = 0. \tag{1.2}$$

defined on a graph comprising a ring with one half-line attached at one vertex point (see Figure 1).

Thus, if in the tadpole graph, the ring is identified by the interval $[-L, L]$ and the semi-infinite line with $[L, +\infty)$, we obtain a metric graph $\mathcal{G}$ with a structure represented by the set $\mathbb{E} = \{e_0, e_1\}$ where $e_0 = [-L, L]$ and $e_1 = [L, +\infty)$, which are the edges of $\mathcal{G}$ and they are connected at the unique vertex $v = L$. $\mathcal{G}$ is also called a lasso graph (see [28], and references therein). In this form, we identify any function U on $\mathcal{G}$ (the wavefunctions) with a collection $U = (u_e)_{e\in E}$ of functions u_e defined on the edge e of $\mathcal{G}$. In the case of the NLS-log in (1.2), we have $U(x_e, t) = (u_e(x_e, t))_{e\in\mathbb{E}}$ and the nonlinearity $U\mathrm{Log}|U|^2$, acting componentwise, i.e., for instance $(U\mathrm{Log}|U|^2)_e = u_e \mathrm{Log}|u_e|^2$. The action of the Laplacian operator Δ on the tadpole $\mathcal{G}$ is given by

$$-\Delta : (u_e)_{e\in E} \to (-u_e'')_{e\in\mathbb{E}}. \tag{1.3}$$

Several domains make the Laplacian operator self-adjoint on a tadpole graph (see [14, 18, 28, 29],). Here, we will consider a domain of general interest in physical applications. In fact, if we denote a wavefunction U on the tadpole graph $\mathcal{G}$ as $U = (\Phi, \Psi)$, with $\Phi : [-L, L] \to \mathbb{C}$ and $\Psi : [L, +\infty) \to \mathbb{C}$, we define the following domains for $-\Delta$:

$$D_Z = \{U \in H^2(\mathcal{G}) : \Phi(L) = \Phi(-L) = \Psi(L), \text{ and, } \Phi'(L) - \Phi'(-L) = \Psi'(L+) + Z\Psi(L)\}, \tag{1.4}$$

with $Z \in \mathbb{R}$ and for any $n \geq 0$, $n \in \mathbb{N}$,

$$H^n(\mathcal{G}) = H^n(-L, L) \oplus H^n(L, +\infty).$$

The boundary conditions in (1.4) are called of δ-interaction type if $Z \neq 0$, and of flux-balanced or Neumann–Kirchhoff condition if $Z = 0$ (with always continuity at the vertex). It is not difficult to see that $(-\Delta, D_Z)_{Z\in\mathbb{R}}$ represents a one-parameter family of self-adjoint operators on the tadpole graph $\mathcal{G}$. We note that it is possible determine other domains where the Laplacian is self-adjoint on a tadpole (see [18]). In particular, using the approach in Angulo&Munōz [14], we

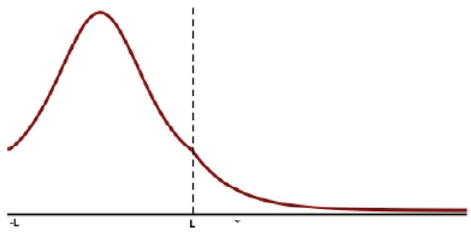

FIGURE 2 | A positive single-lobe state profile for the NLS-log model on $\mathcal{G}$.

can obtain domains that can be characterized by the following family of 6-parameters of boundary conditions

$$\begin{aligned} &\Phi(-L) = \Phi(L), \quad (1 - m_3)\Phi(L) = m_4\Psi(L) + m_5\Psi'(L),\\ &\Phi'(L) - 2\Phi'(-L) = m_1\Phi(-L) + A_1\Psi(L) + A_2\Psi'(L) \text{ and}\\ &\Phi'(L) - A_3\Psi'(L) = m_6\Phi(-L) + m_7\Psi(L), \end{aligned} \tag{1.5}$$

$A_1 = m_4m_6 - m_3m_7$, $A_2 = m_5m_6 - \frac{2m_3+m_7m_5m_3}{m_4}$ and $A_3 = \frac{2+m_7m_5}{m_4}$, $m_1, m_3, m_4, m_5, m_6, m_7 \in \mathbb{R}$ and $m_4 \neq 0$. Note that for $m_3 = m_5 = 0$ and $m_4 = 1$, we get the so-called δ-interaction conditions in (1.4). Moreover, for $m_1 = m_6 = m_7 = 0$, $m_3 = 1$, $m_4 = 2$, and m_5 arbitrary, we get the following δ'-interaction type condition on a tadpole graph

$$\begin{aligned} &\Phi'(-L) = \Phi'(L) = \Psi'(L), \quad \Phi(L) = \Phi(-L) \quad \text{and}\\ &\Psi(L) = -\frac{m_5}{2}\Psi'(L). \end{aligned} \tag{1.6}$$

Now, a problem of general interest is the interaction between standing waves in spatially confined systems and those in large or unbounded reservoirs (so we can say that a tadpole is a configuration that fulfills these characteristics). Our main interest here will be to study some dynamics aspects of (1.2) such as the existence and orbital stability of standing wave solutions given by the profiles $U(x,t) = e^{ict}\Theta(x)$, with $c \in \mathbb{R}$, $\Theta = (\phi, \psi) \in D_Z$, $Z = 0$, and satisfying the NLS-log vectorial equation

$$-\Delta\Theta + c\Theta - \Theta \mathrm{Log}(|\Theta|^2) = 0. \tag{1.7}$$

More explicitly, for ϕ and ψ real-valued we obtain the following system, one on the ring and the other one on the half-line, respectively,

$$\begin{cases} -\phi''(x) + c\phi(x) - \mathrm{Log}(\phi^2(x))\phi(x) = 0, \quad & x \in (-L, L),\\ -\psi''(x) + c\psi(x) - \mathrm{Log}(\psi^2(x))\psi(x) = 0, \quad & x \in (L, +\infty),\\ \phi(L) = \phi(-L) = \psi(L), & \\ \phi'(L) - \phi'(-L) = \psi'(L+). & \end{cases} \tag{1.8}$$

The critical challenge in solving (1.8) lies in the component ϕ on $[-L, L]$ (we note that explicit profiles for ϕ are not known). The component of ψ is given by a translated Gausson-profile (see [11, 23]) of the form

$$\psi_c(x) = e^{\frac{c+1}{2}} e^{\frac{-(x-L+a)^2}{2}}, \quad a \neq 0, \quad c \in \mathbb{R}, \quad x \geq L. \tag{1.9}$$

Among all profiles for (1.8) (see, for instance, Figures 2 and 3 for the case of ϕ- profiles), we focus on *positive single-lobe states*. More precisely, we define (see Figure 2).

Definition 1.1. The standing wave profile $\Theta = (\phi, \psi) \in D_0$ is said to be a positive single-lobe state for (1.8) if each component is positive on every edge of $\mathcal{G}$, the maximum of Θ is achieved at a single internal point symmetrically located on $[-L, L]$, and ϕ is monotonically decreasing on $[0, L]$. Moreover, ψ is strictly decreasing on $[L, +\infty)$.

The study of ground or bound states on general metric graphs for the NLS model with a polynomial nonlinearity

$$i\partial_t U + \Delta U + |U|^{2p}U = 0, \quad p > 0 \tag{1.10}$$

has been investigated in [1–4, 8, 9, 22, 33–35, 40, 41, 43]. To our knowledge, the results contained in this paper are the first in studying the NLS-log on tadpole graphs.

Our focus in this work is to study the existence and the orbital stability for the NLS-log model of positive single-lobe states in the case $Z = 0$ (for the case $Z \neq 0$, we refer the reader to Section 7). For the existence, we use tools from dynamical systems theory for orbits on the plane, based on the period function introduced in [34, 35]. To our knowledge, this approach has not been applied in the literature to logarithmic nonlinearities, and a non-trivial analysis will be required. For the stability, we follow the abstract stability framework by Grillakis et al. [30]. For clarity, we outline the main steps of this framework for standing wave solutions for NLS-log models on a tadpole graph (see Theorem A.7 in the Appendix). Subsequently, we will present our main results.

To begin, we note that the basic symmetry associated with the NLS-log model (1.2) on tadpole graphs is the phase invariance: if U is a solution of (1.2), then $e^{i\theta}U$ is also a solution for any $\theta \in [0, 2\pi)$. Thus, we define orbital stability for (1.2) as follows (see [30]).

Definition 1.2. The standing wave $U(x,t) = e^{ict}(\phi(x), \psi(x))$ is said to be *orbitally stable* in a Banach space X if for any $\varepsilon > 0$ there exists $\eta > 0$ with the following property: if $U_0 \in X$ satisfies $||U_0 - (\Phi, \Psi)||_X < \eta$, then the solution $U(t)$ of (1.2) with $U(0) = U_0$ exists for any $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in[0,2\pi)} ||U(t) - e^{i\theta}(\phi, \psi)||_X < \varepsilon.$$

Otherwise, the standing wave $U(x,t) = e^{ict}(\phi(x), \psi(x))$ is said to be *orbitally unstable* in X.

The space X in Definition 1.2 for the model (1.2) will depend on the domain of the action of $-\Delta$, namely, D_0, and a specific weighted space. Indeed, we will consider the following spaces:

$$\mathcal{E}(\mathcal{G}) = \left\{(f, g) \in H^1(\mathcal{G}) : f(-L) = f(L) = g(L)\right\} \quad \text{"continuous energy-space"}, \tag{1.11}$$

and the Banach spaces $W(\mathcal{G})$ and $\widetilde{W}$ defined by

$$\begin{aligned} W(\mathcal{G}) &= \{(f, g) \in \mathcal{E}(\mathcal{G}) : |g|^2 \mathrm{Log}|g|^2 \in L^1(L, +\infty)\} \\ \widetilde{W} &= \{(f, g) \in \mathcal{E}(\mathcal{G}) : xg \in L^2(L, +\infty)\}. \end{aligned} \tag{1.12}$$

We note that $\widetilde{W} \subset W(\mathcal{G})$ (see Lemma 2.1). Due to our local stability analysis (not variational type, see Section 7 for discussion on this approach), we will consider $X = \widetilde{W}$ in Definition 1.2.

Next, we consider the following two functionals associated with (1.2):

$$E(U) = \|\nabla U\|^2_{L^2(\mathcal{G})} - \int_{-L}^{L} |f_1|^2 \mathrm{Log}(|f_1|^2)dx - \int_{L}^{+\infty} |f_2|^2 \mathrm{Log}(|f_2|^2)dx, \qquad \text{(energy)} \tag{1.13}$$

and

$$Q(U) = \|U\|^2_{L^2(\mathcal{G})}, \qquad \text{(mass)} \tag{1.14}$$

where $U = (f_1, f_2)$. These functionals satisfy $E, Q \in C^1(W(\mathcal{G}) : \mathbb{R})$ (see [23] or Proposition 6.3 in [10]) and, at least formally, E is conserved by the flow of (1.2). The use of the space $W(\mathcal{G})$ is because E fails to be continuously differentiable on $\mathcal{E}(\mathcal{G})$ (a proof of this can be based on the ideas in [23]). Now, as our stability theory is based on the framework of Grillakis et al. [30], E needs to be twice continuously differentiable at the profile $\Theta \in D_0$. To satisfy this condition, we introduced the space $\widetilde{W}$. Moreover, this space naturally appears in the following study of the linearization of the action functional around Θ. We note that $E, Q \in C^1(\widetilde{W})$.

Now, for a fixed $c \in \mathbb{R}$, let $U_c(x,t) = e^{ict}(\boldsymbol{\phi}_c(x), \boldsymbol{\psi}_c(x))$ be a standing wave solution for (1.2) with $(\phi_c, \psi_c) \in D_0$ being a positive single-lobe state. Then, for the action functional

$$\mathbf{S}(U) = E(U) - (c+1)Q(U), \qquad U \in \widetilde{W}, \tag{1.15}$$

we have $\mathbf{S}'(\phi_c, \psi_c) = \mathbf{0}$. Next, for $U = U_1 + iU_2$ and $W = W_1 + iW_2$, where the functions U_j, W_j, $j = 1, 2$, are real. The second variation of $\mathbf{S}$ in (ϕ_c, ψ_c) is

$$\mathbf{S}''(\phi_c, \psi_c)(U, W) = \langle \mathcal{L}_1 U_1, W_1 \rangle + \langle \mathcal{L}_2 U_2, W_2 \rangle, \tag{1.16}$$

where the two 2×2-diagonal operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are given for

$$\begin{aligned} \mathcal{L}_1 &= \operatorname{diag}\big(-\partial_x^2 + (c-2) - \operatorname{Log}|\phi_c|^2, -\partial_x^2 + (c-2) - \operatorname{Log}|\psi_c|^2\big) \\ \mathcal{L}_2 &= \operatorname{diag}\big(-\partial_x^2 + c - \operatorname{Log}|\phi_c|^2, -\partial_x^2 + c - \operatorname{Log}|\psi_c|^2\big) \end{aligned} \tag{1.17}$$

These operators are self-adjoint with domain (see Theorem 3.1)

$$\mathcal{D} := \{(f, g) \in D_0 : x^2 g \in L^2([L, +\infty))\}.$$

Since $(\phi_c, \psi_c) \in \mathcal{D}$ and satisfies system (1.8), $\mathcal{L}_2(\phi_c, \psi_c)^t = \mathbf{0}$, so the kernel of $\mathcal{L}_2$ is non-trivial. Moreover, $\langle \mathcal{L}_1(\phi_c, \psi_c)^t, (\phi_c, \psi_c)^t \rangle < 0$ implies that the Morse index of $\mathcal{L}_1$, $n(\mathcal{L}_1)$, satisfies $n(\mathcal{L}_1) \geqq 1$. Next, from [30] we know that the Morse index and the nullity index of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are a fundamental step in deciding about the orbital stability of standing wave solutions. For the case of the profile $(\boldsymbol{\phi}_c, \boldsymbol{\psi}_c)$ being a positive single-lobe state, our main results are the following:

Theorem 1.3. *Consider the self-adjoint operator $(\mathcal{L}_1, \mathcal{D})$ in (1.17) determined by the positive single-lobe state $(\boldsymbol{\phi}_c, \boldsymbol{\psi}_c)$. Then,*

1. *Perron–Frobenius property: let $\beta_0 < 0$ be the smallest eigenvalue of $\mathcal{L}_1$ with associated eigenfunction $(f_{\beta_0}, g_{\beta_0})$. Then, f_{β_0} is positive and even on $[-L, L]$, and $g_{\beta_0}(x) > 0$ with $x \in [L, +\infty)$,*
2. *β_0 is simple,*
3. *the Morse index of $\mathcal{L}_1$ is one,*
4. *The kernel of $\mathcal{L}_1$ on $\mathcal{D}$ is trivial.*

Theorem 1.4. *Consider the self-adjoint operator $(\mathcal{L}_2, \mathcal{D})$ in (1.17) determined by the positive single-lobe state $(\boldsymbol{\phi}_c, \boldsymbol{\psi}_c)$. Then,*

1. *the kernel of $\mathcal{L}_2$, $ker(\mathcal{L}_2)$, satisfies $ker(\mathcal{L}_2) = span\{(\boldsymbol{\phi}_c, \boldsymbol{\psi}_c)\}$.*
2. *$\mathcal{L}_2$ is a non-negative operator, $\mathcal{L}_2 \geqq 0$.*

The proof of Theorem 1.3 will be based on a *splitting eigenvalue method* applied to $\mathcal{L}_1 := \operatorname{diag}(\mathcal{L}_1^0, \mathcal{L}_1^1)$ in (1.17) on a tadpole graph (see Lemma 4.4). More precisely, we reduce the eigenvalue problem associated with $\mathcal{L}_1$ with domain $\mathcal{D}$ to two classes of eigenvalue problems, one with periodic boundary conditions on $[-L, L]$ for $\mathcal{L}_1^0$ and the other one with Neumann boundary conditions on $= [L, +\infty)$ for $\mathcal{L}_1^1$. Thus, by using tools of the extension theory of Krein–von Neumann for symmetric operators, the theory of real coupled self-adjoint boundary conditions on $[-L, L]$ and the Sturm comparison theorem, will lead to our results.

Our orbital stability result is the following:

Theorem 1.5. *Consider $Z = 0$ in (1.8). Then, there exists a C^1-mapping $c \in \mathbb{R} \to \Theta_c = (\phi_c, \psi_c)$ of positive single-lobe states on $\mathcal{G}$. Moreover, for $c \in \mathbb{R}$, the orbit*

$$\{e^{i\theta} \Theta_c : \theta \in [0, 2\pi)\}$$

is stable.

The proof of Theorems 1.3 and 1.4 are given in Section 4. The orbital stability statement follows from Theorems 1.3 and 1.4 and the abstract stability framework by Grillakis et al. in [30]. For the convenience of the reader, we provide an adaptation

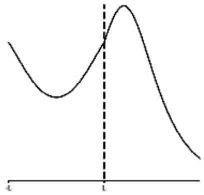

FIGURE 3 | A positive two-lobe state profile on a tadpole graph.

of the abstract results in [30] to the case of tadpole graphs in Theorem A.7 (the Appendix). We note that the main challenge in applying Theorem A.7 lies in spectral analysis, as the Vakhito–Kolokolov condition $\frac{d}{dc}\|(\phi_c, \psi_c)\|^2 > 0$ is essentially trivial for the NLS-log model. Definition 1.2 requires a priori information about the local/global well-posedness of the Cauchy problem for (1.2), which is established in Section 2 for the space $\widetilde{W}$ in (2.1).

The existence of a C^1-mapping for positive single-lobe states in Theorem 1.5 relies on the dynamical systems theory for planar orbits via the period function for second-order differential equations (see [8, 9, 33–35, 40], and reference therein)

We would like to point out that our approach for studying positive single-lobe states for the NLS-log on a tadpole has prospects of being used to study other standing wave profiles, such as positive two-lobe states (see [9] and Figure 3) or the NLS-log on others metric graphs such as looping edge graphs, a graph consisting of a circle with several half-lines attached at a single vertex (see Figure 7 in Section 7).

The paper is organized as follows. In Section 2, we establish local and global well-posedness results for the NLS-log model on a tadpole graph. In Section 3, we show the linearization of the NLS-log around a positive single-lobe state and its relation to the self-adjoint operators $\mathcal{L}_1, \mathcal{L}_2$. In Section 4, we show Theorems 1.3 and 1.4 via our splitting eigenvalue lemma (Lemma 4.4). In Section 5, we provide the proof of the existence of positive single-lobe states and their stability under the NLS-log flow. In the Appendix, we briefly outline tools from the Krein–von Neumann extension theory, a Perron–Frobenius property for δ-interaction Schrödinger operators on the line, and the orbital stability criterion from Grillakis et al. [30] adapted to our framework.

Notation. Let $-\infty \le a < b \le +\infty$. We denote by $L^2(a,b)$ the Hilbert space equipped with the inner product $(u,v) = \int_a^b u(x)\overline{v(x)}dx$. By $H^n(\Omega)$ we denote the classical Sobolev spaces on $\Omega \subset \mathbb{R}$ with the usual norm. We denote by $\mathcal{G}$ the tadpole graph parameterized by the set of edges $\mathbb{E} = \{e_0, e_1\}$, where $e_0 = [-L, L]$ and $e_1 = [L, +\infty)$, and attached to the common vertex $\nu = L$. On the graph $\mathcal{G}$, we define the spaces

$$L^p(\mathcal{G}) = L^p(-L, L) \oplus L^p(L, +\infty), \quad p > 1,$$

with the natural norms. Also, for $U = (u_1, g_1), V = (v_1, h_1) \in L^2(\mathcal{G})$, the inner product on $L^2(\mathcal{G})$ is defined by

$$\langle U, V\rangle = \int_{-L}^{L} u_1(x)\overline{v_1(x)}dx + \int_{L}^{+\infty} g_1(x)\overline{h_1(x)}dx.$$

Let A be a closed densely defined symmetric operator in the Hilbert space H. The domain of A is denoted by $D(A)$. The deficiency indices of A are denoted by $n_\pm(A) := \dim ker(A^* \mp iI)$, with A^* denoting the adjoint operator of A. The Morse index of A, denoted by $n(A)$, is the number of negative eigenvalues counting multiplicities.

2 | Global Well-Posedness in $\widetilde{W}$

In this section, we show that the Cauchy problem associated with the NLS-log model on a tadpole graph is globally well-posed in the space $\widetilde{W}$. From Definition (1.2), this information is crucial for the stability theory. We start with the following technical result.

Lemma 2.1. *Let $W(\mathcal{G})$ and $\widetilde{W}$ be the Banach spaces defined by*

$$\begin{aligned} W(\mathcal{G}) &= \{(f,g) \in \mathcal{E}(\mathcal{G}) : |g|^2 Log|g|^2 \in L^1(L, +\infty)\}, \\ \widetilde{W} &= \{(f,g) \in \mathcal{E}(\mathcal{G}) : xg \in L^2(L, +\infty)\}. \end{aligned} \tag{2.1}$$

Then, $\widetilde{W} \subset W(\mathcal{G})$.

Proof.

1. Let us start by proving that $\widetilde{W} \subset L^1(\mathcal{G})$. For $(\varphi, \xi) \in \widetilde{W}$ and the Cauchy–Schwarz inequality we have

$$\int_{-L}^{L} |\varphi| dx + \int_{L}^{+\infty} |\xi| dx = \int_{-L}^{L} |\varphi| dx + \int_{L}^{+\infty} x|\xi| \frac{1}{x} dx$$
$$\leq 2L \sup_{[-L,L]} |\varphi| + \left(\int_{L}^{+\infty} x^2 |\xi|^2 dx \right)^{\frac{1}{2}} \left(\int_{L}^{+\infty} \frac{1}{x^2} dx \right)^{\frac{1}{2}} < \infty.$$

2. Let again $(\varphi, \xi) \in \widetilde{W}$, then

$$\int_{L}^{+\infty} |\xi|^2 |\mathrm{Log}|\xi|| dx = \int_{\{x \in [L,+\infty) : |\xi(x)| < 1\}} |\xi|^2 |\mathrm{Log}|\xi|| dx + \int_{\{x \in [L,+\infty) : |\xi(x)| \geq 1\}} |\xi|^2 |\mathrm{Log}|\xi|| dx. \tag{2.2}$$

   Note also that

$$|\mathrm{Log}|\xi(x)|| < \frac{1}{|\xi(x)|} \ \text{ for } \ |\xi(x)| < 1, \ \text{ and } \ |\mathrm{Log}|\xi(x)|| < |\xi(x)| \ \text{ for } \ |\xi(x)| \geq 1. \tag{2.3}$$

   Since $\xi \in H^1([L, +\infty))$, there exists $k > L$ such that $|\xi| < 1$ for $[L, +\infty) \setminus [L, k]$. Thus, from (2.2), (2.3), and the inclusion $\widetilde{W} \subset L^1(\mathcal{G})$ we get

$$\int_{L}^{+\infty} |\xi|^2 |Log|\xi|| dx \leq \int_{\{x \in (k,+\infty) : |\xi| < 1\}} |\xi| dx + \int_{\{x \in [L,k] : |\xi| < 1\}} |\xi| dx + \int_{\{x \in [L,k] : |\xi| \geq 1\}} |\xi|^3 dx$$
$$\leq \int_{\{x \in (k,+\infty) : |\xi| < 1\}} |\xi| dx + (k - L) + (k - L) \sup_{[L,k]} |\xi|^3 < \infty.$$

The assertion is proved. □

Theorem 2.2. *For any $U_0 \in \widetilde{W}$, there is a unique solution $U \in C(\mathbb{R}, \widetilde{W})$ of (1.2) such that $U(0) = U_0$. Furthermore, the conservation of energy and mass hold, i.e., for any $t \in \mathbb{R}$, we have $E(U(t)) = E(U_0)$ and $Q(U(t)) = Q(U_0)$.*

Proof. The idea of the proof is to use the strategy proposed in [25]-Section 2 (see also [24, 26]) and adapted to the NLS-log model on the tadpole graph. As the analysis follows the Cazenave's framework which involves multiples steps, we highlight the key modifications for our case. In this way, we introduce the "reduced" Cauchy problem

$$\begin{cases} i\partial_t U_n + \Delta U_n + U_n F_n(|U_n|^2) = 0, \\ U_n(0) = U_0 \in \widetilde{W}, \end{cases} \tag{2.4}$$

where for $U_n = (u_n, v_n)$, $U_n F_n(|U_n|^2) = (u_n f_n(|u_n|^2), v_n f_n(|v_n|^2))$, $f_n(s) = \inf\{n, \sup\{-n, f(s)\}\}$, and $f(s) = \mathrm{Log}(s)$, $s > 0$. Initially, we show that the problem in (2.4) has a unique global solution for any $U_0 \in \mathcal{E}(\mathcal{G})$ and fixed n. To use Theorem 3.3.1 in [24], define the mapping $g_n : L^2(\mathcal{G}) \to L^2(\mathcal{G})$, $g_n(U) = UF_n(|U|^2)$, which is Lipschitz continuous on bounded sets of $L^2(\mathcal{G})$ (since each f_n is Lipschitz on $\mathbb{R}^+$). For $p_n(s) = \int_0^s f_n(s) ds$, define $P_n(w) = p_n(|w|^2)$, yielding $P_n'(w) = |w| f_n(|w|^2)$ formally. For $G_n(U) = (P_n(u), P_n(v))$ with $U = (u, v) \in \mathcal{E}(\mathcal{G})$, we get $G_n'(U) = g_n(U)$ (where the derivative of G_n is computed component-wise). Since Δ with domain D_0 is self-adjoint and non-positive operator in $L^2(\mathcal{G})$ (see Remark 2.3), Theorem 3.3.1 implies that for any $U_0 \in \mathcal{E}(\mathcal{G})$ there exists a unique global solution U_n of (2.4) such that $U_n \in C(\mathbb{R}, \mathcal{E}(\mathcal{G})) \cap C^1(\mathbb{R}, \mathcal{E}(\mathcal{G})')$. Furthermore, for (2.4), the conservation of energy and charge hold, i.e., for all $t \in [-T, T]$

$$\begin{aligned} &Q(U_n(t)) = \|U_n(t)\|^2_{L^2(\mathcal{G})} = \|U_0\|^2_{L^2(\mathcal{G})}, \quad E_n(U_n(t)) = E_n(U_0), \\ &E_n(U) = \|\nabla U\|^2_{L^2(\mathcal{G})} - \textstyle\int_{-L}^{L} P_n(|u|^2) dx - \int_{L}^{+\infty} P_n(|v|^2) dx, \qquad U = (u, v), \end{aligned} \tag{2.5}$$

and so we also obtain that

$$\|U_n\|_{L^\infty([-T,T],\mathcal{E}(\mathcal{G}))} \leqq C, \quad \text{for all} \quad n. \tag{2.6}$$

Note that the last statement is a consequence of $U_0 \in \widetilde{W} \subset W(\mathcal{G})$ by Lemma 2.1 and from Lemmas 2.3.2, 2.3.3, and 2.3.4 in [25]. Now, for $\mathcal{C}_R = \{(f,g) \in L^2((-L,L)\times(L,R)) : f(-L) = f(L) = g(L)\}$ with $R > L$, we consider the Hilbert spaces $\mathcal{E}_R = \{(p,q) : (p,q) \in (H^1(-L,L)\times H^1(L,R)) \cap \mathcal{C}_R\}$ and the dual space of $\mathcal{E}_R$, $\mathcal{E}'_R$. We note that as for any $a, b \in \mathbb{R}$, the embedding $H^1(a,b) \hookrightarrow C([a,b])$ is compact, we get initially that $\mathcal{C}_R$ is closed in $H^1(-L,L)\times H^1(L,R)$ and so $\mathcal{E}_R$ is a Hilbert space. Further, from the embedding relations $\mathcal{E}_R \underset{\text{compact}}{\hookrightarrow} \mathcal{C}_R \hookrightarrow \mathcal{E}'_R$ follows by the Aubin–Lions theorem ([38]) implies that

$$\mathcal{J} = \{F : F \in L^2([-T,T],\mathcal{E}_R), F' = \frac{d}{dt}F \in L^2([-T,T],\mathcal{E}'_R)\},$$

is a Banach space compactly embedded in $L^2([-T,T],\mathcal{C}_R)$. By following a similar analysis to Lemma 2.3.5 in [25] (see also Lemma 9.3.6 in [24]), we get that $U_{n_k} \to U$ in $L^\infty([0,T],\mathcal{E}(\mathcal{G}))$ weak-$\star$. Therefore, we get that U is a solution of (1.2) in the sense of distributions. Moreover, the conservation of charge Q in (1.14) holds and so $U_{n_k}(t) \to U(t)$ strong in $\mathcal{C} = \{(f,g) \in L^2((-L,L)\times(L,+\infty)) : f(-L) = f(L) = g(L)\}$. We have also that energy E in (1.13) is conserved via a standard monotonicity argument (see section 2.4 in [25]). Thus, the inclusion $U \in C(\mathbb{R};\mathcal{E}((\mathcal{G}))$ follows from conservation laws. Lastly, for $U_0 = (g,h)$, the condition $xh \in L^2(L,+\infty)$ implies $xv \in L^2(L,+\infty)$ (for the solution $U = (u,v)$), repeating the argument of Lemma 7.6.2 in [26]. This finishes the proof. □

Remark 2.3. By considering $(-\Delta_Z, D_Z)$, with $-\Delta_Z = -\Delta$ and D_Z in (1.4), it is possible to see the following (see [15, 40]): for every $Z \in \mathbb{R}$, the essential spectrum of the self-adjoint operator $-\Delta_Z$ is purely absolutely continuous and $\sigma_{\text{ess}}(-\Delta_Z) = \sigma_{\text{ac}}(-\Delta_Z) = [0,+\infty)$. If $Z > 0$, then $-\Delta_Z$ has exactly one negative eigenvalue, i.e., its point spectrum is $\sigma_{\text{pt}}(-\Delta_Z) = \{-\varrho_Z^2\}$, where $\varrho_Z > 0$ is the only positive root of the transcendental equation

$$\varrho(2\tanh(\varrho L) + 1) - Z = 0,$$

and with the eigenfunction

$$\mathbf{\Phi}_Z = \begin{pmatrix} \cosh(x\varrho_Z), & x \in [-L,L] \\ \mathrm{e}^{-x\varrho_Z}, & x \in [L,+\infty) \end{pmatrix}.$$

If $Z \leqq 0$, then $-\Delta_Z$ has no point spectrum, $\sigma_{pt}(-\Delta_Z) = \emptyset$. Therefore, for $Z = 0$ we have that $-\Delta_Z$ is a non-negative operator.

3 | Linearization of NLS-Log Equation

For fixed $c \in \mathbb{R}$, let $U(x,t) = \mathrm{e}^{\mathrm{i}ct}\Theta_c(x)$ be a standing wave solution for (1.2) with $\Theta_c(x) = (\phi_c(x), \psi_c(x)) \in D_0$ being a positive single-lobe state. We consider the action functional $S_c = E + (c+1)Q$, thus Θ_c is a critical point of S_c. Next, we determine the linear operator associated with the second variation of S_c calculated at Θ_c, which is crucial for applying the approach in [30]. To express $S''_c(\Theta_c)$, it is convenient to split $u, v \in \widetilde{W}$ into real and imaginary parts: $u = u_1 + \mathrm{i}u_2$, $v = v_1 + \mathrm{i}v_2$. Then, we get $S''_c(\Theta_c)(u,v)$ can be formally rewritten as

$$S''_c(\Theta_c)(u,v) = B_1(u_1,v_1) + B_2(u_2,v_2), \tag{3.1}$$

where

$$B_1((f,g),(h,q)) = \int_{-L}^{L} f'h'dx + \int_{L}^{+\infty} g'q'dx + \int_{-L}^{L} fh[c-2-\mathrm{Log}|\phi_c|^2]dx + \int_{L}^{+\infty} gq[(x-L+a)^2-3]dx, \tag{3.2}$$

$$B_2((f,g),(h,q)) = \int_{-L}^{L} f'h'dx + \int_{L}^{+\infty} g'q'dx + \int_{-L}^{L} fh[c-\mathrm{Log}|\psi_c|^2]dx + \int_{L}^{+\infty} gq[(x-L+a)^2-1]dx,$$

and dom $(B_i) = \widetilde{W} \times \widetilde{W}, i \in \{1,2\}$. Note that the forms $B_i, i \in \{1,2\}$, are bilinear bounded from below and closed. Therefore, by the first representation theorem (see [36], Chapter VI, Section 2.1), they define operators $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ such that for $i \in \{1,2\}$

$$\begin{aligned} &\mathrm{dom}(\widetilde{\mathcal{L}}_i) = \{v \in \widetilde{W} : \exists w \in L^2(\mathcal{G}) \text{ s.t. } \forall z \in \widetilde{W}, B_i(v,z) = \langle w, z\rangle\} \\ &\widetilde{\mathcal{L}}_i v = w. \end{aligned} \tag{3.3}$$

Theorem 3.1. *The operators $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ determined in (3.3) are given by*

$$\widetilde{\mathcal{L}}_1 = diag\left(-\partial_x^2 + (c-2) - Log|\phi_c|^2, -\partial_x^2 + (x-L+a)^2 - 3\right)$$

$$\widetilde{\mathcal{L}}_2 = diag\left(-\partial_x^2 + c - Log|\phi_c|^2, -\partial_x^2 + (x-L+a)^2 - 1\right)$$

on the domain $\mathcal{D} = \{(f,g) \in D_0 : x^2 g \in L^2(L,+\infty)\}$. Thus, $\widetilde{\mathcal{L}}_i = \mathcal{L}_i$ defined in (1.17).

Proof. Since the proof for $\widetilde{\mathcal{L}}_2$ is similar to the one for $\widetilde{\mathcal{L}}_1$, we deal with $\widetilde{\mathcal{L}}_1$. Let $B_1 = B^0 + B^1$, where $B^0 : \mathcal{E}(\mathcal{G}) \times \mathcal{E}(\mathcal{G}) \to \mathbb{R}$ and $B^1 : \widetilde{W} \times \widetilde{W} \to \mathbb{R}$ are defined by

$$B^0((f,g),(h,q)) = \langle (f',g'),(h',q')\rangle, \;\; B^1((f,g),(h,q)) = \langle V_1(f,g),(h,q)\rangle, \tag{3.4}$$

and $V_1(f,g) = ([c-2-\mathrm{Log}|\phi_c(x)|^2]f, [(x-L+a)^2-3]g)$. We denote by $\mathcal{L}^0$ (resp. $\mathcal{L}^1$) the self-adjoint operator on $L^2(\mathcal{G})$ associated (by the first representation theorem) with B^0 (resp. B^1). Thus,

$$\begin{aligned} &\mathrm{dom}(\mathcal{L}^0) = \{v \in \mathcal{E}(\mathcal{G}) : \exists w \in L^2(\mathcal{G}) \text{ s.t. } \forall z \in \mathcal{E}(\mathcal{G}), B^0(v,z) = \langle w, z\rangle\} \\ &\mathcal{L}^0 v = w. \end{aligned}$$

We claim that $\mathcal{L}^0$ is the self-adjoint operator

$$\mathcal{L}^0 = -\Delta = -\frac{d^2}{dx^2}, \;\; \mathrm{dom}(-\Delta) = D_0.$$

Indeed, let $v = (v_1, v_2) \in D_0$ and $w = -v'' \in L^2(\mathcal{G})$. Then for every $z = (z_1, z_2) \in \mathcal{E}(\mathcal{G})$ and using integration by parts, we have

$$\begin{aligned} B^0(v,z) = (v',z') &= \int_{-L}^{L} v_1' z_1' dx + \int_{L}^{+\infty} v_2' z_2' dx \\ &= v_1'(L)z_1(L) - v_1'(-L)z_1(-L) - v_2'(L)z_2(L) - \int_{-L}^{L} v_1'' z_1' dx - \int_{L}^{+\infty} v_2'' z_2' dx \\ &= (-v'', z) = (w, z). \end{aligned}$$

Thus, $v \in$ dom $(\mathcal{L}^0)$ and $\mathcal{L}^0 v = w = -v'' = -\Delta v$. Hence, $-\Delta \subset \mathcal{L}^0$. Since $-\Delta$ is self-adjoint on D_0, $\mathcal{L}^0 = -\Delta$.

Again, by the first representation theorem,

$$\mathrm{dom}(\mathcal{L}^1) = \{v \in \widetilde{W} : \exists w \in L^2(\mathcal{G}) \text{ s.t. } \forall z \in \widetilde{W}, B^1(v,z) = \langle w, z\rangle\}$$
$$\mathcal{L}^1 v = w.$$

Note that $\mathcal{L}^1$ is the self-adjoint extension of the following multiplication operator for $u = (f, g)$

$$\mathcal{M}u = V_1(f,g), \ \mathrm{dom}(\mathcal{M}) = \{u \in \mathcal{E}(\mathcal{G}) : V_1 u \in L^2(\mathcal{G})\}.$$

Indeed, for $v \in \mathrm{dom}\,(\mathcal{M})$ we have $v \in \widetilde{W}$, and we define $w = V_1 v \in L^2(\mathcal{G})$. Then for every $z \in \widetilde{W}$ we get $B^1(v,z) = \langle w, z\rangle$. Thus, $v \in \mathrm{dom}\,(\mathcal{L}^1)$ and $\mathcal{L}^1 v = w = V_1 v$. Hence, $\mathcal{M} \subset \mathcal{L}^1$. Since $\mathcal{M}$ is self-adjoint, $\mathcal{L}^1 = \mathcal{M}$. The proof of the theorem is complete. □

4 | Proof of Theorems 1.3 and 1.4

Let us consider one a priori positive single-lobe state (ϕ_c, ψ_c) solution for (1.8) with $c \in \mathbb{R}$. For convenience, we denote $\phi = \phi_c$ and $\psi = \psi_c$. Thus, the linearized operator $\mathcal{L}_1$ in Theorem 3.1 becomes as

$$\mathcal{L}_1 = \mathrm{diag}(-\partial_x^2 + (c-2) - \mathrm{Log}|\phi|^2, -\partial_x^2 + (c-2) - \mathrm{Log}|\psi|^2) \tag{4.1}$$

with domain $\mathcal{D} = \{(f,g) \in D_0 : x^2 g \in L^2([L, +\infty))\}$.

Next, for $(f,g) \in \mathcal{D}$ define $h(x) = g(x+L)$ for $x > 0$. Then $h(0) = g(L)$ and $h'(0) = g'(L)$. Therefore, the eigenvalue problem $\mathcal{L}_1(f,g)^t = \lambda(f,g)^t$ is equivalent to

$$\begin{cases} \mathcal{L}_{0,1} f(x) = \lambda f(x), & x \in (-L, L), \\ \mathcal{L}_{1,1} h(x) = \lambda h(x), & x \in (0, +\infty), \\ (f,h) \in D_+, \end{cases} \tag{4.2}$$

where

$$\mathcal{L}_{0,1} = -\partial_x^2 + (c-2) - \mathrm{Log}|\phi|^2, \quad \mathcal{L}_{1,1} \equiv -\partial_x^2 + (c-2) - \mathrm{Log}|\psi_{0,a}|^2, \tag{4.3}$$

and $\psi_{0,a}(x) = \mathrm{e}^{\frac{c+1}{2}} \mathrm{e}^{-\frac{(x+a)^2}{2}}$, with $x > 0$, $a > 0$. In this form,

$$\mathcal{L}_{1,1} = -\partial_x^2 + (x+a)^2 - 3. \tag{4.4}$$

The domain D_+ is given by

$$D_+ = \{(f,h) \in X^2(-L,L) : f(L) = f(-L) = h(0), f'(L) - f'(-L) = h'(0), x^2 h \in L^2(0, +\infty)\}, \tag{4.5}$$

with $X^n(-L,L) \equiv H^n(-L,L) \oplus H^n(0,+\infty)$, $n \in \mathbb{N}$. We note that $(\phi, \psi_{0,a}) \in D_+$.

For notational convenience, let $\psi_a = \psi_{0,a}$. Define $\mathcal{L}_+ = \mathrm{diag}(\mathcal{L}_{0,1}, \mathcal{L}_{1,1})$ with domain D_+. The proof of Theorem 1.3 will follow from Sections 4.1 and 4.2.

4.1 | Perron–Frobenius Property and Morse Index for $(\mathcal{L}_+, D_+)$

Initially, we show that $n(\mathcal{L}_+) \geqq 1$. Since $(\phi, \psi_a) \in D_+$ and

$$\langle \mathcal{L}_+(\phi, \psi_a)^t, (\phi, \psi_a)^t\rangle \quad = -2\left[\int_{-L}^{L} \phi^2(x)dx + \int_0^{+\infty} \psi_a^2(x)dx\right] < 0, \tag{4.6}$$

the mini-max principle implies $n(\mathcal{L}_+) \geqq 1$. Now, we note that is possible to show, via the extension theory for symmetric operators of Krein–von Neumann, that $n(\mathcal{L}_+) \leqq 2$.

Theorem 4.1. *The Morse index associated with $(\mathcal{L}_+, D_+)$ is one. Consequently, the Morse index for $(\mathcal{L}_1, \mathcal{D})$ is also one.*

The proof of Theorem 4.1 is based on the following Perron–Frobenius property (PF property) for $(\mathcal{L}_+, D_+)$. Our approach is grounded in ordinary differential equation (ODE) techniques (oscillation theorems) and the extension theory for symmetric operators proposed in [8, 9] for the NLS model (1.10). We note that significant adjustments to this approach are necessary for the NLS-log model. The proof of Theorem 4.1 is given at the end of this section.

4.1.1 | Perron–Frobenius Property for $(\mathcal{L}_+, D_+)$

We start our analysis by defining the quadratic form $\mathcal{Q}$ associated with operator $\mathcal{L}_+$ on D_+, namely, $\mathcal{Q} : D(\mathcal{Q}) \to \mathbb{R}$, with

$$\mathcal{Q}(\eta, \zeta) = \int_{-L}^{L} (\eta')^2 + V_\phi \eta^2 dx + \int_0^{+\infty} (\zeta')^2 + W_\psi \zeta^2 dx, \tag{4.7}$$

$V_\phi = (c-2) - \mathrm{Log}|\phi|^2$, $W_\psi = (c-2) - \mathrm{Log}|\psi_a|^2 = (x+a)^2 - 3$, and $D(\mathcal{Q})$ defined by

$$D(\mathcal{Q}) = \{(\eta, \zeta) \in X^1(-L, L) : \eta(L) = \eta(-L) = \zeta(0), x\zeta \in L^2(0, +\infty)\}. \tag{4.8}$$

Theorem 4.2. *Let $\lambda_0 < 0$ be the smallest eigenvalue for $\mathcal{L}_+$ on D_+ with associated eigenfunction $(\eta_{\lambda_0}, \zeta_{\lambda_0})$. Then, η_{λ_0} and ζ_{λ_0} are positive functions. Moreover, η_{λ_0} is even on $[-L, L]$.*

Proof. The strategy of the proof is based on tools used in [8, 9] for the case of the NLS model in (1.10). For the NLS-log model, significant changes are required, and for the reader's convenience, we highlight in the differences in the approach. We split the proof into several steps.

1. The profile ζ_{λ_0} is not identically zero: indeed, suppose $\zeta_{\lambda_0} \equiv 0$, then η_{λ_0} satisfies

$$\begin{cases} \mathcal{L}_{0,1}\eta_{\lambda_0}(x) = \lambda_0 \eta_{\lambda_0}(x), & x \in (-L, L), \\ \eta_{\lambda_0}(L) = \eta_{\lambda_0}(-L) = 0 \\ \eta'_{\lambda_0}(L) = \eta'_{\lambda_0}(-L). \end{cases} \tag{4.9}$$

   From the Dirichlet condition and oscillation theorems of the Floquet theory, η_{λ_0} must be odd. By the Sturm–Liouville theory, there is an eigenvalue θ for $\mathcal{L}_{0,1}$ such that $\theta < \lambda_0$, with an associated eigenfunction $\xi > 0$ on $(-L, L)$, and $\xi(-L) = \xi(L) = 0$.
   Let $\mathcal{Q}_{\mathrm{Dir}}$ be the quadratic form associated with $\mathcal{L}_{0,1}$ with Dirichlet domain, i.e., $\mathcal{Q}_{\mathrm{Dir}} : H_0^1(-L, L) \to \mathbb{R}$ defined by

$$\mathcal{Q}_{\mathrm{Dir}}(f) = \int_{-L}^{L} (f')^2 + V_\phi f^2 dx. \tag{4.10}$$

   Then, $\mathcal{Q}_{\mathrm{Dir}}(\xi) = \mathcal{Q}(\xi, 0) \geqq \lambda_0 \|\xi\|^2$ and so, $\theta \geqq \lambda_0$. This is a contradiction.
2. $\zeta_{\lambda_0}(0) \neq 0$: suppose $\zeta_{\lambda_0}(0) = 0$ and we consider the odd-extension ζ_{odd} for ζ_{λ_0}, and the even-extension ψ_{even} of the tail-profile ψ_a on all the line. Then, $\zeta_{\mathrm{odd}} \in H^2(\mathbb{R})$ and $\psi_{even} \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R})$. Next, we consider the following unfold operator $\widetilde{\mathcal{L}}$ associated with $\mathcal{L}_{1,1}$,

$$\widetilde{\mathcal{L}} = -\partial_x^2 + (c-2) - \mathrm{Log}|\psi_{\mathrm{even}}|^2 = -\partial_x^2 + (|x| + a)^2 - 3, \tag{4.11}$$

   on the δ-type interaction domain

$$D_{\delta,\gamma} = \{f \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R}) : x^2 f \in L^2(\mathbb{R}), f'(0+) - f'(0-) = \gamma f(0)\} \tag{4.12}$$

for any $\gamma \in \mathbb{R}$. By the extension theory for symmetric operators, the family $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$ represents all the self-adjoint extensions of the following symmetric operator $(\mathcal{M}_0, D(\mathcal{M}_0))$ defined by

$$\mathcal{M}_0 = \widetilde{\mathcal{L}}, \quad D(\mathcal{M}_0) = \{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R}), f(0) = 0\},$$

with deficiency indices given by $n_\pm(\mathcal{M}_0) = 1$. For clarity, we summarize these results (see [12]). Consider the Hilbert spaces scale associated with the self-adjoint operator

$$\mathcal{L} = -\partial_x^2 + (|x| + a)^2, \quad D(\mathcal{L}) = \{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R})\},$$

i.e., $\mathcal{H}_s(\mathcal{L}) = \{f \in L^2(\mathbb{R}) : \|f\|_{s,2} = \|(\mathcal{L} + I)^{s/2} f\| < \infty\}$, $s \geqq 0$, with

$$\cdots \subset \mathcal{H}_2(\mathcal{L}) \subset \mathcal{H}_1(\mathcal{L}) \subset L^2(\mathbb{R}) \subset \mathcal{H}_1(\mathcal{L}) \subset \mathcal{H}_{-2}(\mathcal{L}) \subset \cdots .$$

The dual space of $\mathcal{H}_s(\mathcal{L})$ will be denoted by $\mathcal{H}_{-s}(\mathcal{L}) = \mathcal{H}_s(\mathcal{L})'$. In this form, we get that the δ-functional $\delta : \mathcal{H}_1(\mathcal{L}) \to \mathbb{C}$ defined by $\delta(\psi) = \psi(0)$ belongs to $\mathcal{H}_1(\mathcal{L})' = \mathcal{H}_{-1}(\mathcal{L})$ (by Sobolev embedding) and so $\delta \in \mathcal{H}_{-2}(\mathcal{L})$. By Lemma 1.2.3 in [7], the restriction $\mathcal{M}_0 = \mathcal{L} - 3$ to $D(\mathcal{M}_0)$ is a densely defined symmetric operator with deficiency indices $n_\pm(\mathcal{M}_0) = 1$. Then, by Proposition A.3 (the Appendix) and Remark 4.2—item (iii) in [12], we can characterized all the self-adjoint extensions of $(\mathcal{M}_0, D(\mathcal{M}_0))$ as $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$.

Now, the even tail-profile ψ_{even} satisfies $\psi'_{\text{even}}(x) \neq 0$ for all $x \neq 0$, and so from the well-defined relation

$$\mathcal{M}_0 f = -\frac{1}{\psi'_{\text{even}}} \frac{d}{dx}\left[(\psi'_{\text{even}})^2 \frac{d}{dx}\left(\frac{f}{\psi'_{\text{even}}}\right)\right], \quad x \neq 0 \tag{4.13}$$

we can see easily that $\langle \mathcal{M}_0 f, f\rangle \geqq 0$ for all $f \in D(\mathcal{M}_0)$. By extension theory (Proposition A.4 in the Appendix) we obtain that the Morse index for the family $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ satisfies $n(\widetilde{\mathcal{L}}) \leqq 1$, for all $\gamma \in \mathbb{R}$. Since $\zeta_{\text{odd}} \in D_{\delta,\gamma}$ (for any γ) and $\widetilde{\mathcal{L}}\zeta_{\text{odd}} = \lambda_0 \zeta_{\text{odd}}$ on $\mathbb{R}$, we have $n(\widetilde{\mathcal{L}}) = 1$. Then, λ_0 is the smallest negative eigenvalue for $\widetilde{\mathcal{L}}$ on δ-interactions domains in (4.12), and by Theorem A.6 in the Appendix (the Perron Frobenius property for $\widetilde{\mathcal{L}}$ with δ-interactions domains on the line), ζ_{odd} must be positive, which is a contradiction. Therefore, $\zeta_{\lambda_0}(0) \neq 0$.

3. $\zeta_{\lambda_0} : [0, +\infty) \to \mathbb{R}$ can be chosen strictly positive: Without loss of generality, assume $\zeta_{\lambda_0}(0) > 0$. The condition $\eta'_{\lambda_0}(L) - \eta'_{\lambda_0}(-L) = \zeta'_{\lambda_0}(0)$ implies

$$\zeta'_{\lambda_0}(0) = \left[\frac{\eta'_{\lambda_0}(L) - \eta'_{\lambda_0}(-L)}{\zeta_{\lambda_0}(0)}\right]\zeta_{\lambda_0}(0) \equiv \gamma_0 \zeta_{\lambda_0}(0).$$

Let ζ_{even} denote even extension of ζ_{λ_0} to the entire line. Then, $\zeta_{\text{even}} \in D_{\delta,2\gamma_0}$ and $\widetilde{\mathcal{L}}\zeta_{\text{even}} = \lambda_0 \zeta_{\text{even}}$. A similar analysis as in item 2) above suggests that λ_0 is the smallest eigenvalue for $(\widetilde{\mathcal{L}}, D_{\delta,2\gamma_0})$. Hence, ζ_{even} is strictly positive on $\mathbb{R}$. Therefore, $\zeta_{\lambda_0}(x) > 0$ for all $x \geqq 0$.

4. $\eta_{\lambda_0} : [-L, L] \to \mathbb{R}$ can be chosen strictly positive: initially, we have that η_{λ_0} satisfies the following boundary condition,

$$\eta'_{\lambda_0}(L) - \eta'_{\lambda_0}(-L) = \left[\frac{\zeta'_{\lambda_0}(0)}{\zeta_{\lambda_0}(0)}\right]\zeta_{\lambda_0}(0) \equiv \alpha_0 \zeta_{\lambda_0}(0) = \alpha_0 \eta_{\lambda_0}(L).$$

Consider the eigenvalue problem for $\mathcal{L}_{0,1}$ in (4.3) with real coupled self-adjoint boundary condition determined by $\alpha \in \mathbb{R}$:

$$(RC_\alpha) : \begin{cases} \mathcal{L}_{0,1} y(x) = \beta y(x), & x \in (-L, L), \\ y(L) = y(-L), \\ y'(L) - y'(-L) = \alpha y(L). \end{cases} \tag{4.14}$$

By Theorem 1.35 in Kong et al. [37] or Theorem 4.8.1 in Zettl [44], the first eigenvalue β_0 for (4.14) with $\alpha = \alpha_0$ is simple. Since $(\eta_{\lambda_0}, \lambda_0)$ solves (4.14), $\lambda_0 \geqq \beta_0$. *In the following, we show* $\lambda_0 = \beta_0$. Indeed, we consider the quadratic form associated with the (RC_{α_0})-problem in (4.14), Q_{RC}, where for $h \in H^1(-L, L)$ with $h(L) = h(-L)$,

$$Q_{RC}(h) = \int_{-L}^{L} (h')^2 + V_\phi h^2 dx - \alpha_0 |h(L)|^2. \tag{4.15}$$

Let $\xi = \nu\zeta_{\lambda_0}$ with $\nu \in \mathbb{R}$ chosen such that $\xi(0) = \nu\zeta_{\lambda_0}(0) = h(L)$. Then, $(h, \xi) \in D(Q)$ in (4.8). Using $\mathcal{L}_{1,1}\zeta_{\lambda_0} = \lambda_0\zeta_{\lambda_0}$, we obtain

$$\begin{aligned} Q_{RC}(h) &= Q(h,\xi) - \alpha_0 h^2(L) - \int_0^{+\infty} (\xi')^2 + W_\psi \xi^2 dx \\ &= Q(h,\xi) - \alpha_0 h^2(L) + \nu^2 \zeta'_{\lambda_0}(0)\zeta_{\lambda_0}(0) - \lambda_0 \nu^2 \|\zeta_{\lambda_0}\|^2 \\ &= Q(h,\xi) - \xi'_{\lambda_0}(0)h(L) + h(L)\xi'_{\lambda_0}(0) - \lambda_0\|\xi\|^2 \\ &= Q(h,\xi) - \lambda_0\|\xi\|^2 \geqq \lambda_0[\|h\|^2 + \|\xi\|^2] - \lambda_0\|\xi\|^2 = \lambda_0\|h\|^2. \end{aligned} \tag{4.16}$$

Thus, $\beta_0 \geqq \lambda_0$ and so $\beta_0 = \lambda_0$.

By the analysis above, we get that λ_0 *is the first eigenvalue for the problem* (RC_{α_0}) in (4.14) and thus it is simple. Then, η_{λ_0} is either odd or even. If η_{λ_0} is odd, the condition $\eta_{\lambda_0}(L) = \eta_{\lambda_0}(-L)$ implies $\eta_{\lambda_0}(L) = 0$. However, $\eta_{\lambda_0}(L) = \zeta_{\lambda_0}(0) > 0$. So, we must have that η_{λ_0} is even. Now, from the oscillation theorem for the (RC_{α_0})-problem, the number of zeros of η_{λ_0} on $[-L, L]$ is 0 or 1 (see Theorem 4.8.5 in [44]). Since $\eta_{\lambda_0}(-L) > 0$ and η_{λ_0} is even, we necessarily conclude that $\eta_{\lambda_0} > 0$ on $[-L, L]$. This completes the proof.

□

Corollary 4.3. *Let $\lambda_0 < 0$ be the smallest eigenvalue for $(\mathcal{L}_+, D_+)$. Then, λ_0 is simple.*

Proof. The proof is immediate. Suppose λ_0 is a double eigenvalue. Then, there exists an eigenfunction (f_0, g_0) associated with λ_0 orthogonal to $(\eta_{\lambda_0}, \zeta_{\lambda_0})$. By Theorem 4.2, $f_0, g_0 > 0$. This contradicts the orthogonality of eigenfunctions. □

4.1.2 | Splitting Eigenvalue Method on Tadpole Graphs

In the following, we establish our main strategy for studying eigenvalue problems on a tadpole graph $\mathcal{G}$ as deduced in Angulo [9]. More exactly, we reduce the eigenvalue problem for $\mathcal{L}_1 \equiv \mathrm{diag}(\mathcal{L}_1^0, \mathcal{L}_1^1)$ in (1.17) to two classes of eigenvalue problems: one for $\mathcal{L}_1^0 = -\partial_x^2 + (c-2) - \mathrm{Log}|\phi|^2$ with periodic boundary conditions on $[-L, L]$ and the other for the operator $\mathcal{L}_1^1 = -\partial_x^2 + (c-2) - \mathrm{Log}|\psi|^2$ with Neumann-type boundary conditions on $[L, +\infty)$.

Lemma 4.4. *Let us consider $(\mathcal{L}_1, \mathcal{D})$ in Theorem 1.3. Suppose $(f, g) \in \mathcal{D}$ with $g(L) \neq 0$ and $\mathcal{L}_1(f, g)^t = \gamma(f, g)^t$, for $\gamma \leq 0$. Then, we obtain the following two eigenvalue problems:*

$$\begin{cases} \mathcal{L}_1^0 f(x) = \gamma f(x), \ x \in (-L, L), \\ f(L) = (-L), \quad f'(L) = f'(-L), \end{cases} \qquad \begin{cases} \mathcal{L}_1^1 g(x) = \gamma g(x), \ x > L, \\ g'(L+) = 0. \end{cases}$$

Proof. For $(f, g) \in \mathcal{D}$ and $g(L) \neq 0$, we have

$$f(-L) = f(L), \quad f'(L) - f'(-L) = \left[\frac{g'(L+)}{g(L)}\right] f(L) \equiv \theta f(L),$$

and so f satisfies the real-coupled problem (RC_α) in (4.14) with $\alpha = \theta$ and $\beta = \gamma$. Then, from the proof of Lemma 3.4 in [9] we obtain that $\theta = 0$, and thus we prove the lemma. □

Remark 4.5. From Lemma 4.4, it follows that η_{λ_0} in Theorem 4.2 remains an even-periodic function on $[-L, L]$ and ζ_{λ_0} satisfies a Neumann boundary condition on $[L, +\infty)$.

4.1.3 | Morse Index for $(\mathcal{L}_+, D_+)$

In the following, we provide the proof of Theorem 4.1.

Proof. We consider $\mathcal{L}_+$ on D_+ and suppose $n(\mathcal{L}_+) = 2$ without loss of generality (note that via extension theory we can show $n(\mathcal{L}_+) \leqq 2$). From Theorem 4.2 and Corollary 4.3, the first negative eigenvalue λ_0 for $\mathcal{L}_+$ is simple with an associated eigenfunction $(\eta_{\lambda_0}, \zeta_{\lambda_0})$ having positive components and η_{λ_0} being even on $[-L, L]$. Therefore, for λ_1 being the second negative eigenvalue for $\mathcal{L}_+$, we need to have $\lambda_1 > \lambda_0$.

Let $(f_1, g_1) \in D_+$ be an associated eigenfunction to λ_1. In the following, we divide our analysis into several steps.

1. Suppose $g_1 \equiv 0$: then $f_1(-L) = f_1(L) = 0$ and f_1 is odd (see step 1) in the proof of Theorem 4.2). Now, our profile-solution ϕ satisfies

$$\mathcal{L}_{0,1}\phi' = 0,\ \phi' \text{ is odd},\ \phi'(x) > 0,\ \text{for } x \in [-L, 0),$$

   thus, since $\lambda_1 < 0$ we obtain from the Sturm comparison theorem that there is $r \in (-L, 0)$ such that $\phi'(r) = 0$, which is a contradiction. Then, g_1 is non-trivial.
2. Suppose $g_1(0+) = 0$: we consider the odd-extension $g_{1,\text{odd}} \in H^2(\mathbb{R})$ of g_1 and the unfold operator $\widetilde{\mathcal{L}}$ in (4.11) on the δ-interaction domains $D_{\delta,\gamma}$ in (4.12). Then, $g_{1,\text{odd}} \in D_{\delta,\gamma}$ for any γ and so by the Perron–Frobenius property for $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ (see the Appendix) we must have that $n(\widetilde{\mathcal{L}}) \geqq 2$. But, by step 2) in the proof of Theorem 4.2 we obtain $n(\widetilde{\mathcal{L}}) \leqq 1$ for all γ, and so we get a contradiction.
3. Suppose $g_1(0+) > 0$ (without loss of generality): we will see that $g_1(x) > 0$ for all $x > 0$. Indeed, by Lemma 4.4 we get $g_1'(0+) = 0$. Thus, by considering the even-extension $g_{1,\text{even}}$ of g_1 on all the line and the unfold operator $\widetilde{\mathcal{L}}$ in (4.11) on $D_{\delta,0}$, we have that $g_{1,\text{even}} \in D_{\delta,0}$ and so $n(\widetilde{\mathcal{L}}) \geqq 1$. But, we know that $n(\widetilde{\mathcal{L}}) \leqq 1$ and so λ_1 is the smallest eigenvalue for $(\widetilde{\mathcal{L}}, D_{\delta,0})$. Therefore, by the Perron–Frobenius property for $(\widetilde{\mathcal{L}}, D_{\delta,0})$ (see the Appendix) g_1 is strictly positive. We note that as $D_{\delta,0} = \{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R})\}$, it follows from the classical oscillation theory (see Theorem 3.5 in [19]) that $g_{1,\text{even}} > 0$.
4. Lastly, since the pairs $(\zeta_{\lambda_0}, \lambda_0)$ and (g_1, λ_1) satisfy the eigenvalue problem

$$\begin{cases} \mathcal{L}_{1,1} g(x) = \gamma g(x),\ x > 0, \\ g'(0+) = 0, \end{cases} \tag{4.17}$$

   it follows the property that ζ_{λ_0} and g_1 need to be orthogonal (which is a contradiction). This finishes the proof.

□

4.2 | Kernel for $(\mathcal{L}_1, \mathcal{D})$

In the following, we study the nullity index for $\mathcal{L}_1$ on $\mathcal{D}$ (see item 4) in Theorem 1.3. Using the notation at the beginning of this section, it is sufficient to show the following.

Theorem 4.6. *Let us consider $\mathcal{L}_+ = diag(\mathcal{L}_{0,1}, \mathcal{L}_{1,1})$ on D_+. Then, the kernel associated with $\mathcal{L}_+$ on D_+ is trivial.*

Proof. Let $(f, h) \in D_+$ such that $\mathcal{L}_+(f, h)^t = \mathbf{0}$. Thus, since $\mathcal{L}_{1,1}h = 0$ and $\mathcal{L}_{1,1}\psi_a' = 0$, we obtain from the classical Sturm–Liouville theory on half-lines ([19]) that there is $b \in \mathbb{R}$ with $h = b\psi_a'$ on $(0, +\infty)$. Next, we have the following cases:

1. Suppose $b = 0$: then $h \equiv 0$ and f satisfies $\mathcal{L}_{0,1} f = 0$ with Dirichlet-periodic conditions

$$f(L) = f(-L) = 0, \quad \text{and} \quad f'(L) = f'(-L).$$

   Suppose $f \neq 0$. From the oscillation theory of the Floquet theory and the Sturm–Liouville theory for Dirichlet conditions, $\mathcal{L}_{0,1}\phi' = 0$ on $[-L, L]$, ϕ' is odd and $\phi'(L) \neq 0$, we get $f \equiv 0$ and so $\ker(\mathcal{L}_+)$ is trivial (see proof of Theorem 1.4 in [9]).

2. Suppose $b \neq 0$: then $h(0) \neq 0$ ($h(x) > 0$ without loss of generality with $b < 0$). Hence, from the splitting eigenvalue result in Lemma 4.4, we obtain that f satisfies

$$\begin{cases} \mathcal{L}_{0,1} f(x) = 0, \;\; x \in [-L, L], \\ f(L) = f(-L) = h(0) > 0, \quad \text{and} \quad f'(L) = f'(-L), \end{cases} \tag{4.18}$$

and $h'(0) = 0$. The last equality implies immediately $\psi_a''(0) = 0$, therefore if we have for $\alpha = \psi_a''(0)$ that $\alpha \neq 0$, we obtain a contradiction. Then, $b = 0$ and by item 1) above $\ker(\mathcal{L}_+) = \{\mathbf{0}\}$.
Next, we consider the case $\alpha = 0$. Then, initially, by the Floquet theory and the oscillation theory, we have the following partial distribution of eigenvalues, β_n and μ_n, associated with $\mathcal{L}_{0,1}$ with periodic and Dirichlet conditions, respectively,

$$\beta_0 < \mu_0 < \beta_1 \leqq \mu_1 \leqq \beta_2 < \mu_2 < \beta_3. \tag{4.19}$$

In the following, we will see $\beta_1 = 0$ in (4.19) and it is simple. Indeed, by (4.19) suppose that $0 > \mu_1$. Now, we know that $\mathcal{L}_{0,1}\phi' = 0$ on $[-L, L]$, ϕ' is odd and $\phi'(x) > 0$ for $[-L, 0)$, and the eigenfunction associated with μ_1 is odd, therefore from the Sturm comparison theorem we get that ϕ' needs to have one zero on $(-L, 0)$, which is impossible. Hence, $0 \leqq \mu_1$.
Next, suppose that $\mu_1 = 0$ and let χ_1 be an odd eigenfunction for μ_1. Let $\{\phi', P\}$ be a basis of solutions for the problem $\mathcal{L}_{0,1} g = 0$ (we recall that P can be chosen to be even and satisfying $P(0) = 1$ and $P'(0) = 0$). Then, $\chi_1 = a\phi'$ with $0 = \chi_1(L) = a\phi'(L)$. Hence, $\chi_1 \equiv 0$ which is not possible. Therefore, $0 < \mu_1$ and so $\beta_1 = 0$ is simple with eigenfunction f (being even or odd). We note that β_2 is also simple.
Lastly, since $f(-L) = f(L) > 0$, it follows that f is even and by the Floquet theory f has exactly two different zeros $-a, a$ ($a > 0$) on $(-L, L)$. Hence, $f(0) < 0$. Next, we consider the Wronskian function (constant) of f and ϕ',

$$W(x) = f(x)\phi''(x) - f'(x)\phi'(x) \equiv C, \quad \text{for all} \;\; x \in [-L, L].$$

Then, $C = f(0)\phi''(0) > 0$. Therefore, by the hypotheses ($\alpha = 0$) we obtain

$$C = f(L)\phi''(L) = h(0)\psi_a''(0) = 0, \tag{4.20}$$

which is a contradiction. Then, $b = 0$ and by item 1) above we get again $ker(\mathcal{L}_+) = \{\mathbf{0}\}$. This finishes the proof. □

4.3 | Morse and Nullity Indices for Operator $\mathcal{L}_2$

Theorem 1.4. *We consider a positive single-lobe state (ϕ_c, ψ_c). Then, from (1.8) we get $(\phi_c, \psi_c) \in ker(\mathcal{L}_2)$. Next, we consider*

$$\mathcal{M} = -\partial_x^2 + c - Log(\phi_c^2), \quad \mathcal{N} = -\partial_x^2 + c - Log(\psi_c^2)$$

then for any $V = (f, g) \in D_0$, we obtain

$$\begin{aligned} \mathcal{M} f &= -\frac{1}{\phi_c}\frac{d}{dx}\left[\phi_c^2 \frac{d}{dx}\left(\frac{f}{\phi_c}\right)\right], \qquad x \in (-L, L) \\ \mathcal{N} g &= -\frac{1}{\psi_c}\frac{d}{dx}\left[\psi_c^2 \frac{d}{dx}\left(\frac{g}{\psi_c}\right)\right], \qquad x > L. \end{aligned} \tag{4.21}$$

Thus, we obtain immediately

$$\langle \mathcal{L}_2 V, V\rangle = \int_{-L}^{L} \phi_c^2 \left(\frac{d}{dx}\left(\frac{f}{\phi_c}\right)\right)^2 dx + \int_{L}^{+\infty} \psi_c^2 \left(\frac{d}{dx}\left(\frac{g}{\psi_c}\right)\right)^2 dx \geqq 0.$$

Moreover, since $\langle \mathcal{L}_2 V, V\rangle = 0$ if and only if $f = d_1\phi_c$ and $g = d_2\psi_c$, we obtain from the continuity property at $x = L$ that $d_1 = d_2$. Then, $ker(\mathcal{L}_2) = span\{(\phi_c, \psi_c)\}$. This finishes the proof. □

5 | Existence of the Positive Single-Lobe State

In this section, our focus is on proving the following existence theorem:

Theorem 5.1. *For any $c \in \mathbb{R}$, there is only one single-lobe positive state $\Theta_c = (\phi_c, \psi_c) \in D_0$ that satisfies the NLS-log equation (1.7), is monotonically decreasing on $[0, L]$ and $[L, +\infty)$, and the map $c \in \mathbb{R} \mapsto \Theta_c \in D_0$ is C^1. Moreover, the mass $\mu(c) = Q(\Theta_c)$ satisfies $\frac{d}{dc}\mu(c) > 0$.*

For the proof of Theorem 5.1, we need some tools from dynamical systems theory for orbits on the plane and so for the convenience of the reader, we will divide our analysis into several steps (sub-sections) in the following.

5.1 | The Period Function

We consider $L = \pi$ (without loss of generality) and $\Psi(x) = \psi(x + \pi)$, $x > 0$. Then, the transformation

$$\phi_0(x) = \mathrm{e}^{\frac{1-c}{2}}\Psi(x), \quad \phi_1(x) = \mathrm{e}^{\frac{1-c}{2}}\phi(x) \tag{5.1}$$

implies that system (1.8)with $Z = 0$ is transformed into the following system of differential equations independent of the velocity c:

$$\begin{cases} -\phi_1''(x) + \phi_1(x) - \mathrm{Log}(\phi_1^2(x))\phi_1(x) = 0, & x \in (-\pi, \pi), \\ -\phi_0''(x) + \phi_0(x) - \mathrm{Log}(\phi_0^2(x))\phi_0(x) = 0, & x > 0, \\ \phi_1(\pi) = \phi_1(-\pi) = \phi_0(0), \\ \phi_1'(\pi) - \phi_1'(-\pi) = \phi_0'(0). \end{cases} \tag{5.2}$$

We know that a positive decaying solution to the equation $-\phi_0'' + \phi_0 - \mathrm{Log}(\phi_0^2)\phi_0 = 0$ on the positive half-line is expressed by

$$\phi_0(x) = e\mathrm{e}^{\frac{-(x+a)^2}{2}}, \quad x > 0, \;\; a \in \mathbb{R} \tag{5.3}$$

with $\phi_0(0) = e\mathrm{e}^{\frac{-a^2}{2}}$. If $a > 0$, ϕ_0 is monotonically decreasing on $[0, +\infty)$ (a Gausson tail-profile); and if $a < 0$, ϕ_0 is non-monotone on $[0, +\infty)$ (a Gausson bump-profile). To prove Theorem 5.1, we will only consider Gausson-tail profiles for ϕ_0, therefore, we need to choose $a > 0$.

The second-order differential equations in system (5.2) are integrable with a first-order invariant given by

$$E(\phi, \xi) = \xi^2 - A(\phi), \quad \xi := \frac{d\phi}{dx}, \quad A(\phi) = 2\phi^2 - log(\phi^2)\phi^2. \tag{5.4}$$

Note that the value of $E(\phi, \xi) = E$ is independent of x. Next, for $A(u) = 2u^2 - \mathrm{Log}(u^2)u^2$ we have that there is only one positive root of $A'(u) = 2u(1 - \mathrm{Log}(u^2))$ denoted by r_* such that $A'(r_*) = 0$. In fact, $r_* = \mathrm{e}^{\frac{1}{2}}$. As shown in Figure 4, for $E = 0$ there are two homoclinic orbits: one corresponds to positive $u = \phi$ and the other to negative $u = -\phi$. Periodic orbits exist inside each of the two homoclinic loops and correspond to $E \in (E_*, 0)$, where $E_* = -A(r_*) = -e$, and they correspond to either strictly positive u or strictly negative u. Periodic orbits outside the two homoclinic loops exist for $E \in (0, +\infty)$ and they correspond to sign-indefinite u. Note that

$$E + A(r_*) > 0, \quad E \in (E_*, +\infty). \tag{5.5}$$

The homoclinic orbit with the profile solution (5.3) and $a > 0$ corresponds to $\xi = -\sqrt{A(\phi)}$ for all $x > 0$. Let us define $r_0 := e\mathrm{e}^{\frac{-a^2}{2}}$, that is, the value of $\phi_0(x)$ at $x = 0$. Then, $-\sqrt{A(r_0)}$ is the value of ϕ_0' at $x = 0$. Note that:

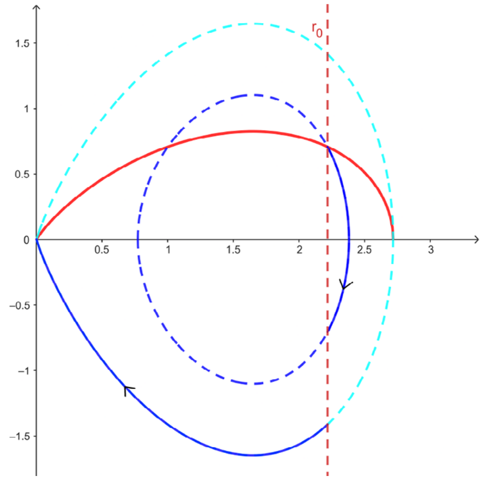

FIGURE 4 | Red line: $s_0(r_0) = \frac{1}{2}\sqrt{A(r_0)} = \frac{1}{2}\phi_0'(0)$. Dashed–dotted vertical line depicts the value of $r_0 = \phi_0(0) = ee^{-a^2/2}$. The blue dashed curve represents the homoclinic orbit at $E = 0$, with the solid part depicting the shifted-tail NLS-log soliton. The level curve $E(\phi, \zeta) = E(r_0, s_0)$, at $r_0 = ee^{-a^2/2}$ and $s_0 = \frac{1}{2}\sqrt{A(p_0)}$, is shown by the blue dashed line. The blue solid parts depict a suitable positive single-lobe state profile solution for the NLS-log.

- $r_0 \in (0, e)$ is a free parameter obtained from $a \in (0, +\infty)$.
- $r_0(a) \longrightarrow e$ when $a \longrightarrow 0$.
- $r_0(a) \longrightarrow 0$ when $a \longrightarrow +\infty$.

Hence, the profile ϕ_1 will be found from the following boundary-value problem:

$$\begin{cases} -\phi_1''(x) + \phi_1(x) - \mathrm{Log}(\phi_1^2(x))\phi_1(x) = 0, \quad x \in (-\pi, \pi), \\ \phi_1(\pi) = \phi_1(-\pi) = r_0, \\ \phi_1'(-\pi) = -\phi_1'(\pi) = \frac{\sqrt{A(r_0)}}{2} \end{cases} \tag{5.6}$$

where $r_0 \in (0, e)$ will be considered as a free parameter of the problem. The positive single-lobe state ϕ_1 will correspond to a part of the level curve $E(\phi, \xi) = E$ which intersects r_0 only twice at the ends of the interval $[-\pi, \pi]$ (see Figure 4 for a geometric construction of a positive single-lobe state for (5.6)).

Let us denote by $s_0 = \frac{\sqrt{A(r_0)}}{2}$ and we define the period function for a given (r_0, s_0):

$$T_+(r_0, s_0) := \int_{r_0}^{r_+} \frac{d\phi}{\sqrt{E + A(\phi)}}, \tag{5.7}$$

where the value E and the turning point r_+ are defined from (r_0, s_0) by

$$E = s_0^2 - A(r_0) = -A(r_+). \tag{5.8}$$

For each level curve of $E(\phi, \xi) = E$ inside the homoclinic loop we have $E \in (E_*, 0)$, and the turning point satisfies:

$$0 < r_* < r_+ < e. \tag{5.9}$$

We recall that ϕ_1 is a positive single-lobe solution of the boundary-value problem (5.6) if and only if $r_0 \in (0, e)$ is a root of the nonlinear equation

$$T(r_0) = \pi, \quad \text{where } T(r_0) = T_+\left(r_0, \frac{\sqrt{A(r_0)}}{2}\right). \tag{5.10}$$

We note that, as $T(r_0)$ is uniquely defined by $r_0 \in (0, e)$, the nonlinear equation (5.10) defines a unique mapping $(0, e) \ni r_0 \mapsto T(r_0) \in (0, +\infty)$. In the following subsection, we show the monotonicity of this function.

5.2 | Monotonicity of the Period Function

In this subsection, we will use the theory of dynamical systems for orbits on the plane and the period function in (5.7) to prove that the mapping $(0, e) \ni r_0 \mapsto T(r_0) \in (0, +\infty)$ is C^1 and monotonically decreasing.
Recall that if $W(\phi, \xi)$ is a C^1 function in an open region of $\mathbb{R}^2$, then the differential of W is defined by

$$dW(\phi, \xi) = \frac{\partial W}{\partial \phi} d\phi + \frac{\partial W}{\partial \xi} d\xi$$

and the line integral of $dW(\phi, \xi)$ along any C^1 contour γ connecting (ϕ_0, ξ_0) and (ϕ_1, ξ_1) does not depend on γ and is evaluated as

$$\int_\gamma dW(\phi, \xi) = W(\phi_1, \xi_1) - W(\phi_0, \xi_0).$$

At the level curve of $E(\phi, \xi) = \xi^2 - A(\phi) = E$, we can write

$$d\left[\frac{2(A(\phi) - A(r_*))\xi}{A'(\phi)}\right] = \left[2 - \frac{2(A(\phi) - A(r_*))A''(\phi)}{[A'(\phi)]^2}\right]\xi d\phi + \frac{2(A(\phi) - A(r_*))}{A'(\phi)} d\xi \tag{5.11}$$

where the quotients are not singular for every $\phi > 0$. In view of the fact that $2\xi d\xi = A'(\phi)d\phi$ on the level curve $E(\phi, \xi) = E$, we can express (5.11) as:

$$\frac{(A(\phi) - A(r_*))}{\xi} d\phi = -\left[2 - \frac{2(A(\phi) - A(r_*))A''(\phi)}{[A'(\phi)]^2}\right]\xi d\phi + d\left[\frac{2(A(\phi) - A(r_*))\xi}{A'(\phi)}\right]. \tag{5.12}$$

The following lemma justifies the monotonicity of the mapping $(0, e) \ni r_0 \mapsto T(r_0) \in (0, +\infty)$.

Lemma 5.2. *The function $r_0 \mapsto T(r_0)$ is C^1 and monotonically decreasing for every $r_0 \in (0, e)$.*

Proof. Since $s_0 = \frac{\sqrt{A(r_0)}}{2}$ in (5.10), for a given $r_0 \in (0, e)$, we have that the value of $T(r_0)$ is obtained from the level curve $E(\phi, \xi) = E_0(r_0)$, where

$$E_0(r_0) \equiv s_0^2 - A(r_0) = -\left(1 - \frac{1}{4}\right)A(r_0) = -\frac{3}{4}A(r_0). \tag{5.13}$$

For every $r_0 \in (0, e)$, we use the formula (5.12) to get

$$\begin{aligned} [E_0(r_0) + A(r_*)]T(r_0) &= \int_{r_0}^{r_+} \left[\xi - \frac{A(\phi) - A(r_*)}{\xi}\right] d\phi \\ &= \int_{r_0}^{r_+} \left[3 - \frac{2(A(\phi) - A(r_*))A''(\phi)}{[A'(\phi)]^2}\right]\xi d\phi + \frac{2(A(r_0) - A(r_*))s_0}{A'(r_0)}, \end{aligned} \tag{5.14}$$

where we use (5.8) to obtain that $\xi = 0$ at $\phi = r_+$ and $\xi = s_0$ at $\phi = r_0$. Because the integrands are free of singularities and $E_0(r_0) + A(r_*) > 0$ due to (5.5), the mapping $(0, e) \ni r_0 \mapsto T(r_0) \in (0, +\infty)$ is C^1. We only need to prove that $T'(r_0) < 0$ for every $r_0 \in (0, e)$.

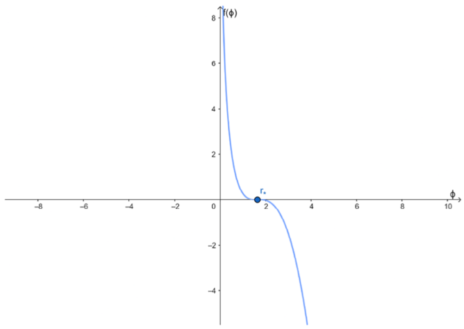

FIGURE 5 | Graph of f in (5.17).

Differentiating (5.14) with respect to r_0 yields

$$
\begin{aligned}
[E_0(r_0)+A(r_*)]T'(r_0) =&([E_0(r_0)+A(r_*)]T(r_0))' - E_0'(r_0)T(r_0)\\
=&\int_{r_0}^{r+}\left[3-\frac{2(A(\phi)-A(r_*))A''(\phi)}{[A'(\phi)]^2}\right]\frac{\partial\xi}{\partial r_0}d\phi-\left[3-\frac{2(A(r_0)-A(r_*))A''(r_0)}{[A'(r_0)]^2}\right]s_0\\
&+\left[2-\frac{2(A(r_0)-A(r_*))A''(r_0)}{[A'(r_0)]^2}\right]s_0+\frac{2(A(r_0)-A(r_*))}{A'(r_0)}s_0'-E_0'(r_0)\int_{r_0}^{r+}\frac{d\phi}{\xi}\\
=&-\frac{A(r_*)}{4s_0}-\frac{3A'(r_0)}{8}\int_{r_0}^{r+}\left[1-\frac{2(A(\phi)-A(r_*))A''(\phi)}{[A'(\phi)]^2}\right]\frac{d\phi}{\xi},
\end{aligned}
\tag{5.15}
$$

where we have used

$$
s_0=\frac{2A(r_0)s_0'}{A'(r_0)},\quad E_0'(r_0)=-\frac{3A'(r_0)}{4},\quad s_0'(r_0)=\frac{A'(r_0)}{8s_0},\quad \frac{\partial\xi}{\partial r_0}=\frac{E_0'(r_0)}{2\xi}.
$$

As $A(\phi)=2\phi^2-\mathrm{Log}(\phi^2)\phi^2$, we obtain that (5.15) is as follows:

$$
[E_0(r_0)+A(r_*)]T'(r_0)=-\frac{A(r_*)}{4s_0}-\frac{3A'(r_0)}{8}\int_{r_0}^{r+}\frac{\phi^2(3-\mathrm{Log}(\phi^2))-e(1+\mathrm{Log}(\phi^2))}{\phi^2(1-\mathrm{Log}(\phi^2))^2\xi}d\phi. \tag{5.16}
$$

Let us define the following function:

$$
f(\phi)=\phi^2(3-\mathrm{Log}(\phi^2))-e(1+\mathrm{Log}(\phi^2)). \tag{5.17}
$$

Differentiating (5.17) with respect to ϕ yields:

$$
\begin{aligned}
&f'(\phi)=2\phi(2-\mathrm{Log}(\phi^2))-\frac{2e}{\phi}\\
\Leftrightarrow\ &\phi f'(\phi)=2\phi^2(2-\mathrm{Log}(\phi^2))-2e \qquad (\phi>0).
\end{aligned}
\tag{5.18}
$$

Note that $f'(\phi)=0 \Leftrightarrow \phi=r_*$ and $f'(\phi)<0$ for any $\phi\in(0,r_*)\cup(r_*,+\infty)$. Since $f(\phi)=0\Leftrightarrow\phi=r_*$ we have that $f(\phi)>0$ for any $\phi\in(0,r_*)$ and $f(\phi)<0$ for any $\phi\in(r_*,+\infty)$.

Next, we know that $A'(r_0)<0$ for any $r_0\in(r_*,e)$ and $f(\phi)<0$ for $\phi\in[r_0,r_+]\subset(r_*,e)$ then follows that $T'(r_0)<0$. Similarly, since $A'(r_*)=0$, we have $T'(r_*)<0$.

Now, we consider the case $r_0\in(0,r_*)$. Then,

$$
\int_{r_0}^{r+}\frac{\phi^2(3-\mathrm{Log}(\phi^2))-e(1+\mathrm{Log}(\phi^2))}{\phi^2(1-\mathrm{Log}(\phi^2))^2\xi}d\phi=I_1+I_2 \tag{5.19}
$$

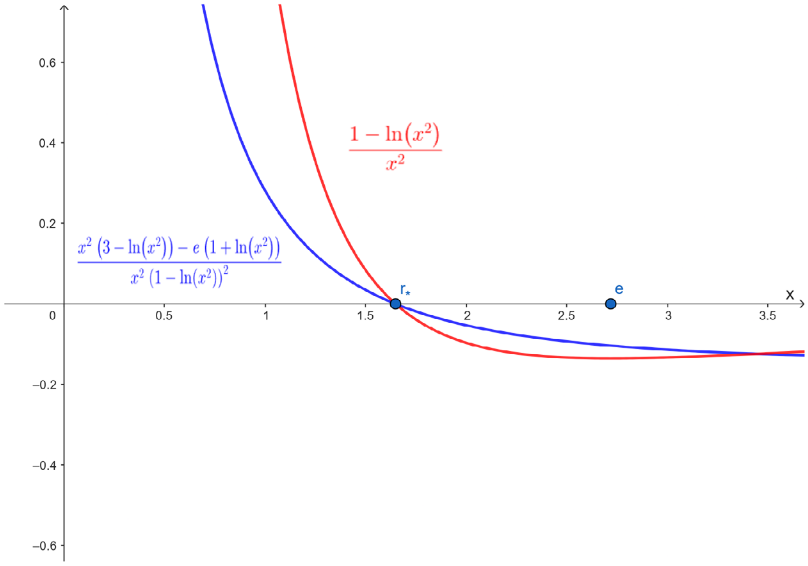

FIGURE 6 | Inequality in (5.21).

where

$$I_1 = \int_{r_0}^{r_*} \frac{\phi^2(3-\mathrm{Log}(\phi^2)) - e(1+\mathrm{Log}(\phi^2))}{\phi^2(1-\mathrm{Log}(\phi^2))^2\xi} d\phi,$$
$$I_2 = \int_{r_*}^{r_+} \frac{\phi^2(3-\mathrm{Log}(\phi^2)) - e(1+\mathrm{Log}(\phi^2))}{\phi^2(1-\mathrm{Log}(\phi^2))^2\xi} d\phi \tag{5.20}$$

Because $f(\phi) > 0$ for any $\phi \in (0, r_*)$, $I_1 > 0$. By using $A'(\phi) = 2\phi(1-\mathrm{Log}(\phi^2))$, we get that (see Figure 6)

$$\frac{\phi^2(3-\mathrm{Log}(\phi^2)) - e(1+\mathrm{Log}(\phi^2))}{\phi^2(1-\mathrm{Log}(\phi^2))^2} > \frac{A'(\phi)}{2\phi^3} \quad \text{for any } \phi \in (r_*, e) \supset (r_*, r_+). \tag{5.21}$$

By (5.21) and proceeding by integration by parts, we have

$$\begin{aligned} I_2 &> \int_{r_*}^{r_+} \frac{A'(\phi)}{2\phi^3\sqrt{E_0(r_0)+A(\phi)}} d\phi \\ &= -\frac{\sqrt{E_0(r_0)+A(r_*)}}{r_*^3} + 3\int_{r_*}^{r_+} \frac{\sqrt{E_0(r_0)+A(\phi)}}{\phi^4} d\phi \\ &= -\frac{\sqrt{E_0(r_0)+A(r_*)}}{r_*^3} + 3\int_{r_*}^{r_+} \frac{\xi}{\phi^4} d\phi \end{aligned} \tag{5.22}$$

Substituting this into (5.16) yields

$$\begin{aligned} [E_0(r_0)+A(r_*)]T'(r_0) &= -\frac{A(r_*)}{4s_0} - \frac{3A'(r_0)}{8} I_1 - \frac{3A'(r_0)}{8} I_2 \\ &< -\frac{A(r_*)}{4s_0} + \frac{3A'(r_0)}{8}\frac{\sqrt{E_0(r_0)+A(r_*)}}{r_*^3} - \frac{3A'(r_0)}{8} I_1 \\ &\quad - \frac{9A'(r_0)}{8}\int_{r_*}^{r_+} \frac{\xi}{\phi^4} d\phi. \end{aligned} \tag{5.23}$$

To evaluate the first two terms we will use that $A(r_*) = e$, $s_0 = \frac{\sqrt{A(r_0)}}{2}$, and $\max_{r_0 \in (0,r_*)} A'(r_0) = 4e^{-\frac{1}{2}}$, so that we get

$$
\begin{aligned}
-\frac{A(r_*)}{4s_0} + \frac{3A'(r_0)}{8}\frac{\sqrt{E_0(r_0)+A(r_*)}}{r_*^3} &= -\frac{e}{2\sqrt{A(r_0)}} + \frac{3A'(r_0)}{8}\frac{\sqrt{E_0(r_0)+e^c}}{e^{\frac{3}{2}}} \\
&< -\frac{e}{2\sqrt{A(r_0)}} + \frac{3\sqrt{E_0(r_0)+e}}{2e^2}.
\end{aligned}
\tag{5.24}
$$

For (5.13) we get

$$
\begin{aligned}
-\frac{e}{2\sqrt{A(r_0)}} + \frac{3\sqrt{E_0(r_0)+e}}{2e^2} < 0 \Leftrightarrow \frac{3\sqrt{E_0(r_0)+e}}{2e^2} < \frac{e}{2\sqrt{A(r_0)}} \Leftrightarrow \frac{9(E_0(r_0)+e)}{4e^4} < \frac{e^2}{4A(r_0)} \\
\Leftrightarrow -\frac{27A(r_0)}{4} + 9e < \frac{e^6}{A(r_0)} \Leftrightarrow 9e < \frac{e^6}{A(r_0)} + \frac{27A(r_0)}{4}.
\end{aligned}
\tag{5.25}
$$

Since $A'(r_0) > 0$ for any $r_0 \in (0, r_*)$, $A(r_0)$ increases monotonically for any $r_0 \in (0, r_*)$, and $\frac{1}{A(r_0)}$ decreases monotonically for any $r_0 \in (0, r_*)$. Differentiating $\frac{e^6}{A(r_0)} + \frac{27A(r_0)}{4}$ with respect to r_0 yields

$$
\begin{aligned}
\left(\frac{e^6}{A(r_0)} + \frac{27A(r_0)}{4}\right)' &= -\frac{e^6A'(r_0)}{(A(r_0))^2} + \frac{27A'(r_0)}{4} < -\frac{e^6A'(r_0)}{(A(r_*))^2} + \frac{27A'(r_0)}{4} \\
&= -e^4A'(r_0) + \frac{27A'(r_0)}{4} = \left(\frac{27}{4} - e^4\right)A'(r_0) < 0.
\end{aligned}
\tag{5.26}
$$

Therefore, the function $\frac{e^6}{A(r_0)} + \frac{27A(r_0)}{4}$ is monotonically decreasing for any $r_0 \in (0, r_*)$. Using this we get that

$$
\frac{e^6}{A(r_0)} + \frac{27A(r_0)}{4} > \frac{e^6}{A(r_*)} + \frac{27A(r_*)}{4} = \left(e^4 + \frac{27}{4}\right)e > 9e
\tag{5.27}
$$

Thus, for (5.25) we get

$$
-\frac{A(r_*)}{4s_0} + \frac{3A'(r_0)}{8}\frac{\sqrt{E_0(r_0)+A(r_*)}}{r_*^3} < 0 \text{ for any } r_0 \in (0, r_*).
$$

As a result of the above calculations, for every $r_0 \in (0, r_*)$ we have $T'(r_0) < 0$. This completes the proof. □

5.3 | Proof of Theorem 5.1

Proof. Due to the monotonicity of the period function $T(r_0)$ in r_0 given by Lemma 5.2 we have a diffeomorphism $(0, e) \ni r_0 \mapsto T(r_0) \in (0, +\infty)$. In fact, we will show that $T(r_0) \longrightarrow 0$ when $r_0 \longrightarrow e$ and $T(r_0) \longrightarrow +\infty$, when $r_0 \longrightarrow 0$. Indeed, from (5.8), (5.13), and (5.9), we have that for $r_0 \in (0, e)$, the equation $E_0(r_0) = -A(r_+)$ determines $r_+ = r_+(r_0)$ from the nonlinear equation:

$$
2r_+^2 - \mathrm{Log}(r_+^2)r_+^2 = \frac{3}{4}[2r_0^2 - \mathrm{Log}(r_0^2)r_0^2],
\tag{5.28}
$$

and $\lim_{r_0\to e}\frac{3}{4}[2r_0^2 - \mathrm{Log}(r_0^2)r_0^2] = 0$. Now for (5.28) we have

$$\begin{aligned}2r_+^2 - \mathrm{Log}(r_+^2)r_+^2 \to 0,\ \text{when}\ r_0 \to e \Leftrightarrow r_+ \to e,\ \text{when}\ r_0 \to e\\ \Leftrightarrow |r_+ - r_0| \to 0,\ \text{when}\ r_0 \to e.\end{aligned} \tag{5.29}$$

Since the weakly singular integrand below is integrable, we have

$$T(r_0) = \int_{r_0}^{r_+} \frac{d\phi}{\sqrt{E + A(\phi)}} = \int_{r_0}^{r_+} \frac{d\phi}{\sqrt{A(\phi) - A(r_+)}} \to 0,\ \text{when}\ r_0 \to e, \tag{5.30}$$

Next, for every $0 < r_0 < r_+ < e$ we obtain

$$\begin{aligned}T(r_0) &= \int_{r_0}^{r_+} \frac{d\phi}{\sqrt{A(\phi) - A(r_+)}} \geq \int_{r_0}^{r_+} \frac{d\phi}{\sqrt{A(\phi)}}\\ &= \int_{r_0}^{r_+} \frac{d\phi}{\sqrt{2\phi^2 - \mathrm{Log}(\phi^2)\phi^2}} = \int_{r_0}^{r_+} \frac{d\phi}{\phi\sqrt{2 - \mathrm{Log}(\phi^2)}}.\end{aligned} \tag{5.31}$$

Since $r_+ \in (r_*, e)$ and

$$\lim_{r_0\to 0}\frac{3}{4}[2r_0^2 - \mathrm{Log}(r_0^2)r_0^2] = 0,$$

by (5.28) we have $r_+ \to e,$ when $r_0 \to 0$. Therefore, as

$$\int_0^e \frac{d\phi}{\phi\sqrt{(2 - \mathrm{Log}(\phi^2)}} = \lim_{\phi\to 0}\sqrt{2 - \mathrm{Log}(\phi^2)} = +\infty$$

we have $T(r_0) \to +\infty$ as $r_0 \to 0$. Then, as the function $T(r_0)$ is monotone decreasing, the codomain of the function $r_0 \mapsto T(r_0)$ is indeed $(0, +\infty)$.

Thus, there is a unique $r_0 \in (0, e)$ such that $T(r_0) = \pi$. Therefore, the boundary-value problem (5.6) has a solution. Next, define $a_0 > 0$ such that $ee^{\frac{-a_0^2}{2}} = r_0$. Then,

$$\phi_c(x) = \mathrm{e}^{\frac{c-1}{2}}\phi_1(x) \quad \text{and} \quad \psi_c(x) = \mathrm{e}^{\frac{c+1}{2}}\mathrm{e}^{\frac{-(x-\pi+a_0)^2}{2}}$$

satisfies (1.8) with $Z = 0$. Obviously, we have that $\mathbb{R} \ni c \mapsto \Theta(c) = (\phi_c, \psi_c) \in D_0$ is a C^1-mapping of positive single-lobe state for the NLS-log equation on the tadpole graph.

Next, the relation

$$\mu(c) = \|\Theta(c)\|^2 = \mathrm{e}^{c-1}\left[\int_{-\pi}^{\pi}\phi_1^2(x)dx + \int_0^{+\infty}\phi_0^2(x)dx\right]$$

with $\phi_0(x) = ee^{\frac{-(x+a_0)^2}{2}}$ and ϕ_1 independent of c, implies $\mu'(c) > 0$ for all $c \in \mathbb{R}$. The proof of the theorem is completed. □

6 | Proof of the Stability Theorem

In this section, we show Theorem 1.5 based on the stability criterion in Theorem A.7 (the Appendix).

Proof. By Theorem 5.1, we obtain the existence of a C^1-mapping $c \in \mathbb{R} \to \Theta_c = (\phi_c, \psi_c)$ of positive single-lobe states for the NLS-log model on a tadpole graph. Moreover, the mapping $c \to \|\Theta_c\|^2$ is strictly increasing. Next, from Theorem 1.3 we get that the Morse index for $\mathcal{L}_1$ in (1.17), $n(\mathcal{L}_1)$, satisfies $n(\mathcal{L}_1) = 1$ and $\ker(\mathcal{L}_1) = \{\mathbf{0}\}$. Moreover, Theorem 1.4 establishes that $\mathrm{Ker}(\mathcal{L}_2) = \mathrm{span}\{(\phi_c, \psi_c)\}$ and $\mathcal{L}_2 \geqq 0$. Thus, by Theorem 2.2 and stability criterion in Theorem A.7, we obtain that $e^{ict}(\phi_c, \psi_c)$ is orbitally stable in $\widetilde{W}$. This finishes the proof. □

7 | Discussion and Open Problems

In this paper, we have established the existence and orbital stability of standing wave solutions for the NLS-log model on a tadpole graph with a profile being a positive single-lobe state. To that end, we use tools from dynamical systems theory for orbits on the plane and we use the period function for showing the existence of such a state with Neumann–Kirchhoff condition at the vertex $\nu = L$ ($Z = 0$ in (1.8)). We believe that the existence (and simultaneous stability properties) of positive single-lobe solutions on the tadpole graph can be obtained via variational analysis applied to the constrained problem

$$I_\lambda = \inf\{E(U) : U \in W(\mathcal{G}),\ Q(U) = \lambda > 0\},$$

For this analysis, we would to use the approach in Cazenave [23] combined with symmetric rearrangement strategies on metric graphs (see Adami et al. [1] or Ardila [16]). Furthermore, we note that via this approach for the existence, the phase velocity c in the vectorial NLS-log equation (1.7) is determined by the Lagrange multiplier associated with the minimization problem I_λ. Additionally, stability information (after proving the globally well-posed theory in $W(\mathcal{G})$) is only established for the following minimizing set:

$$G_\lambda = \{U \in W(\mathcal{G}) : E(U) = I_\lambda,\ Q(U) = \lambda > 0\}.$$

Thus, the advantage of our approach using the period function is that we can demonstrate the existence of positive single-lobe states for any $c \in \mathbb{R}$.

The orbital stability of the positive single-lobe states established in Theorem 5.1 is based on the framework of Grillakis et al. in [30] adapted to the tadpole graph and so via a splitting eigenvalue method and tools of the extension theory of Krein–von Neumann for symmetric operators and the Sturm comparison theorem we identify the Morse index and the nullity index of a specific linearized operator around a positive single-lobe state which is a fundamental ingredient in this endeavor. For the case $Z \neq 0$ in (1.8) and by supposing the existence of a positive single-lobe state, is possible to obtain similar results for $(\mathcal{L}_2, D_Z)$ as in Theorem 1.4. Statements $(1) - (3)$ in Theorem (1.3) are also true for $(\mathcal{L}_1, D_Z)$ (see Section 3 in [8]). Moreover, if we define the quantity for the shift $a = a(Z)$

$$\alpha(a) = \frac{\psi_c''(L)}{\psi_c'(L)} + Z = \frac{1 - a^2}{a} + Z,$$

then, the kernel associated with $(\mathcal{L}_1, D_Z)$ is trivial in the following cases: for $\alpha \neq 0$ or $\alpha = 0$ in the case of admissible parameters Z satisfying $Z \leqq 0$ (see Theorem 1.3 and Lemma 3.5 in [8]). The existence of these positive single-lobe state profiles with $Z \neq 0$ is more challenging and will be addressed in future work. Our approach has a prospect of being extended to study stability properties of other standing wave states for the NLS-log on a tadpole graph (by instance, to choose boundary conditions in the family of 6-parameters given in (1.5)) or on another non-compact metric graph such as

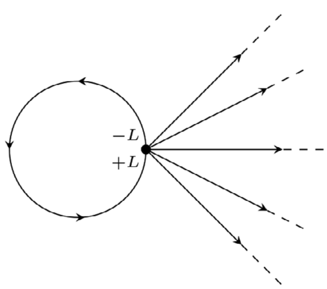

FIGURE 7 | A looping edge graph with $N = 5$ half-lines.

a looping edge graph (see Figure 7), namely, a graph consisting of a circle with several half-lines attached at a single vertex (see [8, 9, 14]).

Acknowledgments

The authors thank the anonymous referees for their insightful comments and suggestions which improved the quality of the paper. J. Angulo was partially funded by CNPq/Brazil Grant and Universal/CNPq project, he would like to thank to Mathematics Department of the Federal University of Pernambuco (UFPE) by the support and the warm stay during the Summer-program/2025, where part of the present project was developed. A. Pérez was partially funded by CAPES/Brazil and UNIVESP/São Paulo/Brazil.

The Article Processing Charge for the publication of this research was funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) (ROR identifier: 00x0ma614).

Data Availability Statement

No data were used for the research described in the paper.

References

1. R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, "Variational Properties and Orbital Stability of Standing Waves for NLS Equation on a Star Graph," *Journal of Differential Equations* 257 (2014): 3738–3777.

2. R. Adami, E. Serra, and P. Tilli, "NLS Ground States on Graphs," *Calculus of Variations* 54, no. 1 (2015): 743–761.

3. R. Adami, E. Serra, and P. Tilli, "Threshold Phenomena and Existence Results for NLS Ground States on Graphs," *Journal of Functional Analysis* 271 (2016): 201–223.

4. R. Adami, E. Serra, and P. Tilli, "Negative Energy Ground States for the L^2-Critical NLSE on Metric Graphs," *Communications in Mathematical Physics* 352 (2017): 387–406.

5. J. P. Albert, J. L. Bona, and D. Henry, "Sufficient Conditions for Stability of Solitary-Wave Solutions of Model Equations for Long Waves," *Physica D: Nonlinear Phenomena* 24 (1987): 343–366.

6. S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed. (AMS Chelsea Publishing, Providence, RI, 2005).

7. S. Albeverio and P. Kurasov, *Singular Perturbations of Differential Operators*, London Mathematical Society Lecture Note Series, vol. 271 (Cambridge University Press, Cambridge, 2000).

8. J. Angulo, "Stability Theory for the NLS Equation on Looping Edge Graphs," *Mathematische Zeitschrift* 308, no. 1 (2024): 19.

9. J. Angulo, "Stability Theory for Two-Lobe States on the Tadpole Graph for the NLS Equation," *Nonlinearity* 37, no. 4 (2024): 045015.

10. J. Angulo and A. H. Ardila, "Stability of Standing Waves for Logarithmic Schrödinger Equation with Attractive Delta Potential," *Illinois Journal of Mathematics* 67 (2018): 471–494.

11. J. Angulo and M. Cavalcante, "Nonlinear Dispersive Equations on Star Graphs," 32° *Colóquio Brasileiro de Matemática* (IMPA, 2019).

12. J. Angulo and N. Goloshchapova, "Stability of Standing Waves for Nonlinear Schrödinger Logarithmic Equation with δ-Interaction," *Nonlinear Differential Equations and Applications* 24 (2017): 27.

13. J. Angulo and N. Goloshchapova, "Extension Theory Approach in the Stability of the Standing Waves for the Nonlinear Schrödinger Equation with Point Interactions on a Star Graph," *Advances in Differential Equations* 23, no. 11–12 (2018): 793–846.

14. J. Angulo and A. Munoz, "Airy and Schrödinger-Type Equations on Looping Edge Graphs," arXiv:2410.11729v2 (2024).

15. J. Angulo and R. Plaza, "Dynamics of the Sine-Gordon on Tadpole Graphs," arXiv:2502.16726v1 (2025).

16. A. H. Ardila, "Logarithmic Nonlinear Schrödinger Equation on Star Graphs: Existence and Stability of Standing Waves," *Differential and Integral Equations* 30 (2017): 735–762.

17. A. H. Ardila, "Stability of Ground States for Logarithmic Schrödinger Equation with a δ'-Interaction," *Evolution Equations and Control Theory* 6 (2017): 155–175.

18. G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, Mathematical Surveys and Monographs, vol. 186 (American Mathematical Society, Providence, RI, 2013).

19. F. A. Berezin and M. A. Shubin, *The Schrödinger Equation*, translated from the 1983 Russian edition by Yu. Rajabov, D. A. Leites, and N. A. Sakharova, revised by Shubin, Mathematics and its Applications (Soviet Series), vol. 66 (Kluwer, Dordrecht, 1991).

20. I. Bialynicki-Birula and J. Mycielski, "Nonlinear Wave Mechanics," *Annals of Physics* 100 (1976): 62–93.

21. H. Buljan, A. Siber, M. Soljacic, T. Schwartz, M. Segev, and D. N. Christodoulides, "Incoherent White Light Solitons in Logarithmically Saturable Noninstantaneous Nonlinear Media," *Physical Review E* 68 (2003): 036607.

22. C. Cacciapuoti, D. Finco, and D. Noja, "Topology Induced Bifurcations for the NLS on the Tadpole Graph," *Physical Review E* 91, no. 1 (2015): 013206.

23. T. Cazenave, "Stable Solutions of the Logarithmic Schrödinger Equation," *Nonlinear Analysis* 7, no. 10 (1983): 1127–1140.

24. T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, vol. 10 (American Mathematical Society, Courant Institute of Mathematical Sciences, 2003).

25. T. Cazenave and A. Haraux, "Équations d'évolution avec non linéarité logarithmique," *Annales de la Faculté des Sciences de Toulouse* 5, no. 2 (1980): 21–51.

26. T. Cazenave, A. Haraux, and Y. Martel, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and its Applications, vol. 13 (Clarendon Press, 1998).

27. S. De Martino, M. Falanga, C. Godano, and G. Lauro, "Logarithmic Schrödinger-like Equation as a Model for Magma Transport," *Europhysics Letters* 63, no. 3 (2003): 472–475.

28. P. Exner, "Magnetoresonance on a Lasso Graph," *Foundations of Physics* 27 (1997): 171.

29. P. Exner and M. Jex, "On the Ground State of Quantum Graphs with Attractive δ-Coupling," *Physics Letters A* 376 (2012): 713–717.

30. M. Grillakis, J. Shatah, and W. Strauss, "Stability Theory of Solitary Waves in the Presence of Symmetry, I," *Journal of Functional Analysis* 74 (1987): 160–197.

31. E. F. Hefter, "Application of the Nonlinear Schrödinger Equation with a Logarithmic Inhomogeneous Term to Nuclear Physics," *Physical Review A* 32 (1985): 1201.

32. E. S. Hernández and B. Remaud, "General Properties of Gausson-Conserving Descriptions of Quantal Damped Motion," *Physica A* 105, no. 1–2 (1981): 130–146.

33. A. Kairzhan, D. Noja, and D. E. Pelinovsky, "Standing Waves on Quantum Graphs," *Journal of Physics A: Mathematical and Theoretical* 55 (2022): 243001.

34. A. Kairzhan, R. Marangell, D. E. Pelinovsky, and K. Xiao, "Standing Waves on a Flower Graph," *Journal of Differential Equations* 271 (2021): 719–763.

35. A. Kairzhan and D. E. Pelinovsky, "Multi-pulse Edge-localized States on Quantum Graphs," *Analysis and Mathematical Physics* 11 (2021): 171.

36. T. Kato, *Perturbation Theory for Linear Operators*, Die Grundlehren der Mathematischen Wissenschaften, vol. 132 (Springer-Verlag, New York, 1966).

37. Q. Kong, H. Wu, and A. Zettl, "Dependence of the nth Sturm–Liouville Eigenvalue Problem," *Journal of Differential Equations* 156 (1999): 328–354.

38. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires* (Dunod Gauthier-Villars, Paris, 1969).

39. M. A. Naimark, *Linear Differential Operators* (Russian), 2nd ed., revised and augmented (Izdat. "Nauka", Moscow, 1969).

40. D. Noja and D. E. Pelinovsky, "Standing Waves of the Quintic NLS Equation on the Tadpole Graph," *Calculus of Variations and Partial Differential Equations* 59 (2020): 173.

41. D. Noja, D. Pelinovsky, and G. Shaikhova, "Bifurcations and Stability of Standing Waves in the Nonlinear Schrödinger Equation on the Tadpole Graph," *Nonlinearity* 28, no. 7 (2015): 2343–2378.

42. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II: Fourier Analysis and Self-Adjointness* (Academic Press, New York, 1978).

43. D. Noja and D. E. Pelinovsky (eds.), *Symmetries of Nonlinear PDEs on Metric Graphs and Branched Networks*, Symmetry (MDPI, Basel, 2019), https://www.mdpi.com/journal/symmetry/special_issues/Symmetries_Nonlinear_PDEs_Metric_Graphs_Branched_Networks.

44. A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, vol. 121 (American Mathematical Society, Providence, RI, 2005).

45. K. G. Zloshchastiev, "Logarithmic Nonlinearity in Theories of Quantum Gravity: Origin of Time and Observational Consequences," *Gravitation and Cosmology* 16 (2010): 288–297.

Appendix A

A.1 | Classical Extension Theory Results

The following results are classical in the extension theory of symmetric operators and can be found in [39, 42]. Let A be a closed densely defined symmetric operator in the Hilbert space H. The domain of A is denoted by $D(A)$. The deficiency indices of A are denoted by $n_\pm(A) := \dim \ker(A^* \mp iI)$, with A^* denoting the adjoint operator of A. The number of negative eigenvalues counting multiplicities (or Morse index) of A is denoted by $n(A)$.

TheoremA.1 (von-Neumann decomposition). *Let A be a closed, symmetric operator, then*

$$D(A^*) = D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}. \tag{A.1}$$

with $\mathcal{N}_{\pm i} = \ker(A^* \mp iI)$. Therefore, for $u \in D(A^*)$ and $u = x + y + z \in D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}$,

$$A^* u = Ax + (-i)y + iz. \tag{A.2}$$

Remark A.2. The direct sum in (A.1) is not necessarily orthogonal.

Proposition A.3. *Let A be a densely defined, closed, symmetric operator in some Hilbert space H with deficiency indices equal to $n_{\pm}(A) = 1$. All self-adjoint extensions A_θ of A may be parametrized by a real parameter $\theta \in [0, 2\pi)$ such that*

$$D(A_\theta) = \{x + c\phi_+ + \zeta e^{i\theta}\phi_- \, : \, x \in D(A), \zeta \in \mathbb{C}\},$$

$$A_\theta(x + \zeta\phi_+ + \zeta e^{i\theta}\phi_-) = Ax + i\zeta\phi_+ - i\zeta e^{i\theta}\phi_-,$$

with $A^*\phi_\pm = \pm i\phi_\pm$, and $\|\phi_+\| = \|\phi_-\|$.

The following proposition provides a strategy for estimating the Morse-index of the self-adjoint extensions (see [39, 42]-Chapter X).

Proposition A.4. *Let A be a densely defined lower semi-bounded symmetric operator (i.e., $A \geq mI$) with finite deficiency indices, $n_{\pm}(A) = k < \infty$, in the Hilbert space H, and let $\widehat{A}$ be a self-adjoint extension of A. Then, the spectrum of $\widehat{A}$ in $(-\infty, m)$ is discrete and consists of, at most, k eigenvalues counting multiplicities.*

The next proposition can be found in Naimark [39] (see Theorem 9).

Proposition A.5. *All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and the same continuous spectrum.*

A.2 | Perron–Frobenius Property for δ-Interaction Schrödinger Operators on the Line

In this section, we establish the Perron–Frobenius property for the unfolded self-adjoint operator $\widetilde{\mathcal{L}}$ in (4.11),

$$\widetilde{\mathcal{L}} = -\partial_x^2 + (c-2) - \mathrm{Log}(\psi_{\mathrm{even}}^2) = -\partial_x^2 + (|x| + a)^2 - 3 \tag{A.3}$$

on δ-interaction domains, namely,

$$D_{\delta,\gamma} = \{f \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R}) \, : \, x^2 f \in L^2(\mathbb{R}), f'(0+) - f'(0-) = \gamma f(0)\} \tag{A.4}$$

for any $\gamma \in \mathbb{R}$. Here, ψ_{even} is the even extension to the whole line of the Gausson tail-soliton profile $\psi_a(x) = \mathrm{e}^{\frac{c+1}{2}}\mathrm{e}^{-\frac{(x+a)^2}{2}}$, with $x > 0$, $a > 0$. Since

$$\lim_{|x|\to+\infty} (|x| + a)^2 = +\infty,$$

operator $\widetilde{\mathcal{L}}$ has a discrete spectrum, $\sigma(\widetilde{\mathcal{L}}) = \sigma_d(\widetilde{\mathcal{L}}) = \{\lambda_k\}_{k\in\mathbb{N}}$ (this statement can be obtained similarly using the strategy in the proof of Theorem 3.1 in [19]). In particular, from Sections 2 and 3 in Chapter 2 in [19] adapted to $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ for γ fixed (see also Lemma 4.8 in [12]) we have the following distribution of the eigenvalues $\lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots$, with $\lambda_k \to +\infty$ as $k \to +\infty$ and from the semi-boundedness of $V_a = (|x| + a)^2 - 3$ we obtain that any solution of the equation $\widetilde{\mathcal{L}}v = \lambda_k v$, $v \in D_{\delta,\gamma}$, is unique up to a constant factor. Therefore, each eigenvalue λ_k is simple.

TheoremA.6 (Perron–Frobenius property). *Consider the family of self-adjoint operators $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$. For γ fixed, let $\lambda_0 = \inf \sigma_p(\widetilde{\mathcal{L}})$ be the smallest eigenvalue. Then, the corresponding eigenfunction ζ_0 of λ_0 is positive (after replacing ζ_0 by $-\zeta_0$ if necessary) and even.*

Proof. This result can be obtained by following the strategy in the proof of Theorem 3.5 in [19]. Here, we give another approach via a slight twist of standard abstract Perron-Frobenius arguments (see Proposition 2 in Albert et al. [5]). The basic point in the analysis is to show that the Laplacian operator $-\Delta_\gamma \equiv -\frac{d^2}{dx^2}$ on the domain $D_{\delta,\gamma}$ has its resolvent $R_\mu = (-\Delta_\gamma + \mu)^{-1}$ represented by a positive kernel for some $\mu > 0$ sufficiently large. Namely, for $f \in L^2(\mathbb{R})$

$$R_\mu f(x) = \int_{-\infty}^{+\infty} K(x, y) f(y) dy$$

with $K(x, y) > 0$ for all $x, y \in \mathbb{R}$. By the convenience of the reader, we show this main point; the remainder of the proof follows the same strategy as in [5]. Thus, for γ fixed, let $\mu > 0$ be sufficiently large (with $-2\sqrt{\mu} < \gamma$ in the case $\gamma < 0$), then from the Krein formula (see Theorem 3.1.2 in [6]) we obtain

$$K(x, y) = \frac{1}{2\sqrt{\mu}}\left[e^{-\sqrt{\mu}|x-y|} - \frac{\gamma}{\gamma + 2\sqrt{\mu}} e^{-\sqrt{\mu}(|x|+|y|)}\right].$$

Moreover, for every x fixed, $K(x, \cdot) \in L^2(\mathbb{R})$. Thus, the existence of the integral above is guaranteed by Holder's inequality. Moreover, $x^2 R_\mu f \in L^2(\mathbb{R})$. Now, since $K(x, y) = K(y, x)$, it is sufficient to show that $K(x, y) > 0$ in the following cases.

1. Let $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$: for $\gamma \geqq 0$, we obtain from $\frac{\gamma}{\gamma+2\sqrt{\mu}} < 1$ and $|x - y| \leqq |x| + |y|$, that $K(x, y) > 0$. For $\gamma < 0$ and $-2\sqrt{\mu} < \gamma$, it follows immediately $K(x, y) > 0$.
2. Let $x > 0$ and $y < 0$: in this case,

$$K(x, y) = \frac{1}{\gamma + 2\sqrt{\mu}} e^{-\sqrt{\mu}(x-y)} > 0$$

   for any value of γ (where again $-2\sqrt{\mu} < \gamma$ in the case $\gamma < 0$).

This finishes the proof. □

A.3 | Orbital Stability Criterion

For the convenience of the reader, in this subsection we adapt the abstract stability results from Grillakis et al. in [30] for the case of the NLS-log on a tadpole graph. This criterion was used in the proof of Theorem 1.5 for the case of standing waves that are positive single-lobe states.

Theorem A.7. *Suppose that there is C^1-mapping $c \to (\phi_c, \psi_c)$ of standing-wave solutions for the NLS-log model (1.2) on a tadpole graph. We consider the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ in (1.17). For $\mathcal{L}_1$ suppose that the Morse index is one and its kernel is trivial. For $\mathcal{L}_2$ suppose that it is a non-negative operator with kernel generated by the profile (ϕ_c, ψ_c). Moreover, suppose that the Cauchy problem associated with the NLS-log model (1.2) is globally well-posed in the space $\widetilde{W}$ in (1.12). Then, $e^{ict}(\phi_c, \psi_c)$ is orbitally stable in $\widetilde{W}$ if $\frac{d}{dc}||(\phi_c, \psi_c)||^2 > 0$.*