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Q-boson model and relations with integrable hierarchies

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ABSTRACT

This work investigates the intricate relationship between the q-boson model, a quantum integrable system, and classical integrable systems such as the Toda and KP hierarchies. Initially, we analyze scalar products of off-shell Bethe states and explore their connections to tau functions of integrable hierarchies. Furthermore, we discuss correlation functions within this formalism, examining their representations in terms of tau functions, as well as their Schur polynomial expansions.

Contents

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1. Introduction

The study of connections between quantum and classical exactly solvable models is an important research program aimed at elucidating the underlying structure of integrable systems. This research program has yielded fruitful insights, as evidenced by [11,10,1,2], to name a few. The present work is situated within this research field.

Here, we examine the emergence of classical integrable structures within correlation functions of the q-boson system. This quantum integrable system describes q-deformed bosons confined to a one-dimensional chain [5–7]. Interestingly, this model is closely related to the AL (Ablowitz-Ladik) model, which is an integrable discretization of the nonlinear Schrödinger equation.

In [7], the author shows that the $q = 0$ limit of the q-boson, known as the *phase model*, is associated with the enumeration of plane partitions. This connection allows for a Schur polynomials expansion for the Bethe states. Subsequently, this result was extended to $q > 0$ in [28], where the Bethe states are expressed in terms of Hall-Littlewood polynomials.

The authors of [9,29] utilize a rich set of dualities between the ring of symmetric functions and q-deformations of the Heisenberg algebra [12,13] to explore the combinatorial properties of this spin chain. As a consequence, it has been shown that the scalar products of off-shell Bethe states are restricted tau functions of the KP (Kadomtsev-Petviashvili) hierarchy.

This underlying combinatorial structure underscores the significance of this model in various physical phenomena, notably in the context of crystal melting and string theory [20,26]. The application of the q-boson model in these contexts has been demonstrated in [25], with the string data recoverable as the chain length approaches infinity.

This paper builds upon these developments and aims to extend some of these results. Firstly, we investigate the connections between the scalar product and tau functions, exploring them in a wider context, particularly in the case of the Toda hierarchy [27]. It is important to note that both the AL and the KP hierarchies are reductions of the Toda hierarchy. This paper takes steps towards resolving this problem, which is instrumental in understanding the finer details of these relationships.

Additionally, we explore other correlation functions within this model and examine how they fit into the classical integrable context. It is noteworthy that even when the object is not a tau function itself, it may have an interesting expansion in terms of different types of Schur polynomials, highlighting some of its combinatorial properties.[1] We also investigate some of these expansions in this paper.

In Section 2, we provide a review of the phase model and q-bosons. This section serves to establish notation and emphasize the key aspects relevant to our analysis. Section 3 delves into the analysis of the phase model and its connections with the KP and Toda integrable hierarchies. We explore the general expansion of a phase model correlation function that yields a Toda system solution. In Section 4, we shift our focus to the q-bosons. We discuss a determinantal formula for the scalar product of two off-shell Bethe states and its expansion in terms of big, supersymmetric Schur functions, as well as in terms of Kostka-Foulkes polynomials. Finally, in Section 5, we conclude with a discussion of our results and an overview of open problems.

2. Phase model and q-bosons

This section introduces the *phase* and *q-boson* models [5–7,28]. It serves as a comprehensive review of existing literature, offering insights into established concepts. While the content does not introduce novel ideas, its presentation adds value, particularly from a pedagogical standpoint. In structuring this discussion, we adopt certain conventions from [29], while aligning our presentation style more closely with that of [28].

2.1. Phase model

Consider the $(M+1)$ set of operators $\{\phi_i, \phi_i^\dagger, \mathcal{N}_i\}_{i=0}^M$ such that

$$[\mathcal{N}_i, \phi_j] = -\phi_i \delta_{i,j} \quad [\mathcal{N}_i, \phi_j^\dagger] = \phi_i^\dagger \delta_{i,j} \quad [\phi_i, \phi_j^\dagger] = \pi_i \delta_{i,j} \tag{1}$$

where $\pi_i = (|0\rangle\langle 0|)_i$ is the vacuum projection operator. These operators can be written as

$$\phi = \sum_{n\geq 0} |n\rangle\langle n+1| \quad \phi^\dagger = \sum_{n\geq 0} |n+1\rangle\langle n| \quad N = \sum_{n\geq 0} n|n\rangle\langle n| \,, \tag{2}$$

and it is easy to see that $\phi^\dagger\phi = \mathbf{1} - |0\rangle|0\rangle$ and $\phi\phi^\dagger = \mathbf{1}$.

The Hamiltonian is given by

$$H = -\frac{1}{2}\sum_{n=0}^{M}\left(\phi_n^\dagger\phi_{n+1} + \phi_n\phi_{n+1}^\dagger\right) + \bar{\mathcal{N}} \,, \tag{3}$$

where $\bar{\mathcal{N}} = \sum_{i=0}^M \mathcal{N}_i$ is the total number operator, and we also impose periodic boundary conditions $\phi_{M+1} = \phi_0$ and $\phi_{M+1}^\dagger = \phi_0^\dagger$.

These operators appear in the context of quantum optics, and for this reason, this model is referred to as the *phase model*. It corresponds to the strongly correlated limit of the q-bosons model [6], which we will define shortly.

[1] Results involving Schur and Hall-Littlewood polynomials have been confirmed using the software [3].

2.1.1. Representation

The representation of the phase model algebra is constructed using the vacuum state defined by $|0\rangle_i$ by $\phi_i|0\rangle_i = 0$. In this context, the state with n_i bosons (oscillators) is given by $|n_i\rangle_i = (\phi_i^\dagger)^{n_i}|0\rangle_i$.

Given the vacuum $|\mathbf{0}\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$, the Fock space is defined as

$$\mathcal{F} = \bigotimes_{i=0}^{M} \mathcal{F}_i = \operatorname{span}\left\{ |\vec{n}\rangle = |n_0\rangle \otimes |n_1\rangle \otimes \cdots \otimes |n_M\rangle \mid n_i \in \mathbb{N} \right\} , \tag{4}$$

where the states $|\vec{n}\rangle$ are defined as

$$|\vec{n}\rangle = |n_0\rangle \otimes |n_1\rangle \otimes \cdots \otimes |n_M\rangle \equiv (\phi_0^\dagger)^{n_0}(\phi_1^\dagger)^{n_1} \cdots (\phi_M^\dagger)^{n_M}|\mathbf{0}\rangle , \tag{5}$$

with $\phi_i^\dagger \equiv \mathbf{1} \otimes \cdots \otimes \phi_i^\dagger \otimes \cdots \otimes \mathbf{1}$. Moreover, it is easy to see that these states are normalized, that is $\langle \vec{n}|\vec{m}\rangle = \delta_{\vec{n},\vec{m}}$. Finally, the actions of the operators $\mathcal{N}_i$ and π_i are

$$\pi_i|\vec{n}\rangle = \delta_{n_i,0}|\vec{n}\rangle \qquad \mathcal{N}_i|\vec{n}\rangle = n_i|\vec{n}\rangle . \tag{6}$$

Given a state $|\vec{n}\rangle = |n_0, n_1, \ldots, n_M\rangle$, we associate a partition $\lambda = (1^{n_1}2^{n_2} \cdots M^{n_M})$. It's worth noting that this correspondence is not unique, as the partition λ does not account for the number of particles n_0. If the total number of particles N is known, we can determine $n_0 = N - \ell(\lambda)$, where $\ell(\lambda)$ represents the number of rows in the Young diagram defined by partition λ. This aspect is crucial because, in our subsequent considerations, the N particle sector remains fixed owing to the integrability of the model. Consequently, the value of n_0 becomes known once we specify the partition λ.

Finally, based on the correspondence we mentioned in the introduction, Wheeler [29] defines a map $\mathcal{M}_\psi : \mathcal{F} \to \mathcal{F}_\psi^{(0)}$, where $\mathcal{F}_\psi^{(0)}$ denotes the Fock space of charged free fermions constructed from the neutral (fermionic) vacuum.

2.1.2. Bethe ansatz

The L-matrix is given by

$$L_{an} = \begin{pmatrix} x^{-1/2} & \phi_n^\dagger \\ \phi_n & x^{1/2} \end{pmatrix}_a , \tag{7}$$

where $x \in \mathbb{S}^1$. We also have the monodromy matrix

$$T_a(x) = L_{aM}(x) \cdots L_{a0}(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}_a . \tag{8}$$

With these expressions, one can finally build the Bethe states

$$|\Psi(y_1, \ldots, y_N)\rangle = \prod_{j=1}^{N} \mathbb{B}(y_j)|\mathbf{0}\rangle \qquad \langle\Psi(y_1, \ldots, y_N)| = \langle\mathbf{0}| \prod_{j=1}^{N} \mathbb{C}(y_j) \tag{9}$$

where $\mathbb{B}(y) = y^{M/2}B(y)$ and $\mathbb{C}(y) = y^{M/2}C(1/y)$.

When the coordinates $\{y_j \mid j = 1, \ldots, N\}$ satisfy the Bethe equations

$$y_i^{N+M} = (-1)^{N-1} \prod_{\substack{j=1 \\ j\neq i}}^{N} y_j , \qquad i = 1, \ldots, N , \tag{10}$$

we say that the Bethe states are *on-shell*; otherwise, we have *off-shell* states. In what follows, we will only consider *off-shell* states.

2.2. q-bosons

The set of operators $\{b_i, b_i^\dagger, \mathcal{N}_i\}_{i=0}^M$ constitute $M+1$ independent q-boson algebras defined by

$$[\mathcal{N}_i, b_j^\dagger] = \delta_{i,j} b_i^\dagger , \quad [\mathcal{N}_i, b_j] = -\delta_{i,j} b_i , \quad [b_i, b_j^\dagger] = \delta_{i,j} q^{-2\mathcal{N}_i} \equiv \delta_{i,j} Q^{\mathcal{N}_i} , \tag{11}$$

where we denote the deformation parameter as $Q = q^{-2}$.

The q-boson model is characterized by its Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{n=0}^{M} \left(b_n^\dagger b_{n+1} + b_n b_{n+1}^\dagger \right) + \bar{\mathcal{N}} , \tag{12}$$

where $\bar{\mathcal{N}} = \sum_{i=0}^{M} \mathcal{N}_i$, and we also impose periodic boundary conditions $b_{M+1} = b_0$ and $b_{M+1}^\dagger = b_0^\dagger$.

In the limit $Q \to 1$, the q-bosons behave as ordinary bosons, while the limit $Q \to 0$ ($q \to \infty$) corresponds to the phase model discussed earlier.

2.2.1. Representation

Let us define the i-th vacuum $|0\rangle_i$ such that $b_i|0\rangle_i = 0$. The representation of this Hilbert space, denoted by $\mathcal{F}_i^Q$, is defined by the states $|n_i\rangle_i \propto (b_i^\dagger)^{n_i}|0\rangle_i$.

Given the vacuum $|\mathbf{0}\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$, the Fock space is defined as

$$\mathcal{F}_Q = \bigotimes_{i=0}^{M} \mathcal{F}_i^Q = \mathrm{span}\left\{ |\vec{n}\rangle = |n_0\rangle \otimes |n_1\rangle \otimes \cdots \otimes |n_M\rangle \mid n_i \in \mathbb{N} \right\} , \tag{13}$$

where the actions of the operators $\{b_i, b_i^\dagger\}$ are given by the following relations

$$b_0|n_0\rangle = (1-\delta_{0,n_0})\frac{(1-Q^{n_0})}{(1-Q)^{1/2}}|n_0-1\rangle \qquad b_0^\dagger|n_0\rangle = \frac{1}{(1-Q)^{1/2}}|n_0+1\rangle \tag{14a}$$

$$b_i|n_i\rangle = \frac{(1-\delta_{0,n_i})}{(1-Q)^{1/2}}|n_i-1\rangle \qquad b_i^\dagger|n_0\rangle = \frac{(1-Q^{n_i+1})}{(1-Q)^{1/2}}|n_0+1\rangle \quad i \neq 0\,. \tag{14b}$$

Consequently, the states $|\vec{n}\rangle \in \mathcal{F}_Q$ are

$$|\vec{n}\rangle = |n_0\rangle \otimes |n_1\rangle \otimes \cdots \otimes |n_M\rangle := (b_0^\dagger)^{n_0}(b_1^\dagger)^{n_1}\cdots(b_M^\dagger)^{n_M}|\mathbf{0}\rangle\,, \tag{15}$$

where we write $b_i^\dagger \equiv \mathbf{1} \otimes \cdots \otimes b_i^\dagger \otimes \cdots \otimes \mathbf{1}$. Moreover, this space has an inner product that satisfies

$$\langle\vec{n}|\vec{m}\rangle = \frac{[m_0]!}{\prod_{i=1}^{M}[m_i]!}\delta_{\vec{n}\vec{m}} \qquad [n]! = \begin{cases} \prod_{i=1}^{n}(1-Q^i) & \text{if } n\neq 0 \\ 1 & \text{otherwise} \end{cases} . \tag{16}$$

Similar to the phase model, we can associate to a given states $|\vec{n}\rangle = |n_0, n_1, \ldots, n_M\rangle$, a Young diagram $\lambda = (1^{n_1}2^{n_2}\cdots M^{n_M})$. It is useful to define a proportionality factor relating these two objects

$$|\lambda\rangle_Q = b_\lambda(Q)|\vec{n}\rangle \ , \qquad b_\lambda(Q) = \prod_i [p_i(\lambda)]! \tag{17}$$

where $p_i(\lambda)$ denotes the number of parts of size i in the partition. These partition states satisfy $\langle\lambda|\mu\rangle_Q = b_\lambda(Q)\delta_{\lambda,\mu}$. Once again, we see that the correspondence is not unique, as the number of oscillators at site $i=0$ is completely ignored in the partition notation. Finally, if the number of particles is fixed, then $n_0 = N - \ell(\lambda)$.

2.2.2. Bethe ansatz

The L-operator for the q-boson is given by

$$L_{an}(x,Q) = \begin{pmatrix} x^{-1/2} & (1-Q)^{\frac{1}{2}}b_n^\dagger \\ (1-Q)^{\frac{1}{2}}b_n & x^{1/2} \end{pmatrix}_a , \tag{18}$$

and the monodromy matrix is

$$T_a(x,Q) = L_{am}(x,Q)\ldots L_{a0}(x,Q) = \begin{pmatrix} A(x,Q) & B(x,Q) \\ C(x,Q) & D(x,Q) \end{pmatrix}_a . \tag{19}$$

As before, the eigenstates of the Hamiltonian have the form

$$|\Psi(y_1,\ldots,y_N;Q)\rangle = \prod_{j=1}^{N}\mathbb{B}(y_j,Q)|\mathbf{0}\rangle \qquad \langle\Psi(y_1,\ldots,y_N;Q)| = \langle\mathbf{0}|\prod_{j=1}^{N}\mathbb{C}(y_j,Q)\,, \tag{20}$$

where $\mathbb{B}(y,Q) = y^{M/2}B(y,Q)$ and $\mathbb{C}(y,Q) = y^{M/2}C(1/y,Q)$. When the parameters $\{y_j \mid j=1,\ldots,N\}$ satisfy the Bethe equations given by

$$y_i^{N+M} = \prod_{\substack{j=1\\ j\neq i}}^{N}\frac{Qy_i - y_j}{y_i - Qy_j}\,, \qquad i = 1,\ldots,N\,, \tag{21}$$

we have *on-shell* Bethe states; otherwise, they are *off-shell* states.

3. Tau functions in the phase model

This section explores the presence of integrable hierarchies in the phase model. We demonstrate its relations with the Toda hierarchy tau function and discuss some implications, particularly its connection to a matrix model. We also argue that these results imply that the scalar products in the model also serve as KP hierarchy tau functions, consistent with the findings of Wheeler [29]. Additionally, we highlight the existence of correlation functions in this model that satisfy the KP hierarchy equations.

Bogoliubov [7] has demonstrated that the scalar product of two vectors in the N-particle sector of a chain with length $M+1$ is

$$\begin{aligned}\mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{y}) &= \langle\mathbf{0}|\prod_{i=1}^{N}\mathbb{C}(x_i)\prod_{j=1}^{N}\mathbb{B}(y_j)|\mathbf{0}\rangle \\ &= \frac{\det H(\boldsymbol{x},\boldsymbol{y})}{\prod_{i<j}(x_i-x_j)(y_i-y_j)}\,,\end{aligned} \tag{22}$$

where $H\equiv[H_{ij}]_{i,j=1}^{N}$ is an $N\times N$ matrix with components

$$H_{ij}=H(x_i,y_j)=\frac{1-(x_iy_j)^{M+N}}{1-x_iy_j}\,. \tag{23}$$

3.1. Toda tau functions

We now argue that the scalar product defined above is a tau function of the Toda hierarchy. Let us first write the function $H(z,w)$ as the geometric sum

$$H(z,w)=\frac{1-(zw)^{M+N}}{1-zw}=\sum_{k=1}^{M+N}(zw)^{k-1}\,. \tag{24}$$

Hence, we express the determinant of the H matrix as

$$\det_{i,j}\left(H(x_i,y_j)\right)=\det_{i,j}\left(\sum_{k=1}^{M+N}x_i^{k-1}y_j^{k-1}\right) \tag{25}$$

Furthermore, we can interpret this expression as the result of multiplying an $N\times(N+M)$ matrix $\mathcal{X}$ by another $(M+N)\times N$ matrix $\mathcal{Y}$, which are given by

$$\mathcal{X}=\begin{pmatrix} x_1^0 & x_1^1 & \dots & x_1^{M+N-1}\\ x_2^0 & x_2^1 & \dots & x_2^{M+N-1}\\ \vdots & & & \\ x_N^0 & x_N^1 & \dots & x_N^{M+N-1}\end{pmatrix} \quad\text{and}\quad \mathcal{Y}=\begin{pmatrix} y_1^0 & y_2^0 & \dots & y_N^0\\ y_1^1 & y_2^1 & \dots & y_N^1\\ \vdots & \vdots & & \vdots\\ y_1^{N+M-1} & y_2^{M+N-1} & \dots & y_N^{M+N-1}\end{pmatrix}, \tag{26}$$

therefore

$$\det_{i,j}\left(H(x_i,y_j)\right)\equiv\det_{i,j}(\mathcal{X}\mathcal{Y})=\sum_{0\le\ell_N\le\cdots\le\ell_1\le N+M}\det_{ik}(x_i^{\ell_k})\det_{ik}(y_j^{\ell_k})\,. \tag{27}$$

If we now define $\ell_j=\lambda_k-k+N$, and using (22), we have

$$\mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{y})=\sum_{\lambda\subseteq[N,M]}\frac{\det_{ik}(x_i^{\lambda_k-k+N})}{\Delta(\boldsymbol{x})}\frac{\det_{ik}(y_j^{\lambda_k-k+N})}{\Delta(\boldsymbol{y})}=\sum_{\lambda\subseteq[N,M]}s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{y})\,, \tag{28}$$

where $\Delta(\boldsymbol{x})$ and $\Delta(\boldsymbol{x})$ are Vandermonde determinants. This formula agrees with the Schur expansion defined in [7].

Define two sets of Miwa coordinates $\boldsymbol{t}=(t_1,t_2,\dots)$ and $\boldsymbol{t}'=(t'_{-1},t'_{-2},\dots)$ as

$$t_q=\frac{1}{q}\sum_{j=1}^{N}x_j^p \qquad t'_{-q}=\frac{1}{q}\sum_{j=1}^{N}y_j^p\,, \tag{29}$$

where $p_q(\boldsymbol{x})=qt_q$ are power sums. One can write the inner product in terms of these coordinates, that is

$$\mathcal{I}(N,M|\boldsymbol{t},\boldsymbol{t}')=\sum_{\lambda\subseteq[N,M]}s_\lambda(\boldsymbol{t})s_\lambda(\boldsymbol{t}')\,. \tag{30}$$

This expression is known to be a tau function for $M,N\to\infty$. As such, utilizing the free fermions representation [4], it can be written as

$$\lim_{M,N\to\infty}\mathcal{I}(N,M|\boldsymbol{t},\boldsymbol{t}')=\langle\mathbf{0}|e^{\boldsymbol{J}_+(\boldsymbol{t})}e^{-\boldsymbol{J}_-(\boldsymbol{t}')}|\mathbf{0}\rangle\,. \tag{31}$$

It is a tau function of the Toda hierarchy with trivial element $\mathbb{1}\in GL(\infty)$, and it is nothing but the Cauchy's identity

$$\sum_\lambda s_\lambda(\boldsymbol{t})s_\lambda(\boldsymbol{t}')=\exp\left(\sum_{m\ge1}mt_mt'_{-m}\right)\,. \tag{32}$$

Bringing all these facts together, the truncation for finite M and N also yields tau functions of the Toda hierarchy. More specifically, according to [4,14,31], the truncation of the tau function corresponds to the inclusion of a projection operator in the expectation value of the tau function written in the fermionic representation.

As a final remark, we can also write

$$H(z,w) = \sum_{k=0}^{M+N-1} h_k (zw)^k , \tag{33}$$

with $h_k = 1$ if $k \in [0, M+N-1]$ and 0 otherwise. In this case, one can define a diagonal $(N+M)\times(N+M)$ matrix $\mathcal{H} = \text{diag}(h_0, \dots, h_{M+N-1})$. Consequently, if we repeat the arguments above, we find

$$\mathcal{I}(N,M|\boldsymbol{t},\boldsymbol{t}') = \sum_{\lambda\subseteq[N,M]} h_\lambda s_\lambda(\boldsymbol{t}) s_\lambda(\boldsymbol{t}') . \tag{34}$$

Here, h_λ is equal to 1 if $\lambda \in [N,M]$, and it is zero otherwise.

In this case, we find that this tau function is a trivial example of the tau functions considered in [22]. We anticipate that in the analysis of more general correlation functions, the diagonal terms h_λ will be more interesting. We will revisit this discussion soon.

3.1.1. Matrix models

It is also interesting to note the particular case when $M \to \infty$, but with a finite number of particles N. In this case, the partitions $\lambda \in [N,\infty]$ satisfy the condition $\ell(\lambda) \le N$. Therefore, the scalar product (28) becomes

$$\mathcal{I}_N(\boldsymbol{t},\boldsymbol{t}') \equiv \lim_{M\to\infty} \mathcal{I}(N,M|\boldsymbol{t},\boldsymbol{t}') = \sum_{\substack{\lambda \\ \ell(\lambda)\le N}} s_\lambda(\boldsymbol{t}) s_\lambda(\boldsymbol{t}') . \tag{35}$$

From [31], we know that this expression can be written as the following integral

$$\mathcal{I}_N(\boldsymbol{t},\boldsymbol{t}') = \frac{1}{N!}\prod_{\ell=1}^{N} \oint_{\Gamma_\ell} \frac{dz_\ell}{2\pi i z_\ell} e^{\xi(\boldsymbol{t},z_\ell)-\xi(\boldsymbol{t}',z_\ell^{-1})} \Delta(z)\Delta(z^{-1}) , \tag{36}$$

where

$$\xi(\boldsymbol{t},z) = \sum_{k\ge 0} t_k z^k \qquad \xi(\boldsymbol{t}',1/z) = \sum_{k\ge 0} t'_{-k} z^{-k} . \tag{37}$$

See also [14] for a detailed proof of this relation, and [24] for other details.

From the results of [31], see citations therein, we have an interesting consequence of this representation. Impose the Bethe equations (10) to one set of variables, say $x_j = e^{-ip_j} \in \mathbb{S}^1$. Additionally, let us set $\boldsymbol{t}' = -\boldsymbol{t}^\star$. In this particular case, we have

$$\xi(\boldsymbol{t},z) - \xi(\boldsymbol{t}',1/z) = 2\,\text{Re}\left(\sum_k t_k z^k\right) . \tag{38}$$

Then, the phase model is equivalent to an ensemble of N 2D Coulomb particles on a circle. In this case, we find that the quantities z_ℓ are eigenvalues of a matrix U.

Furthermore, according to Zabrodin [31], see citations therein, we also know that under the rescaling $t_k \to T_k/\hbar$, $t'_k \to T_{-k}/\hbar$ and $N = T_0/\hbar$, we obtain the dispersion limit tau function

$$F_0(\boldsymbol{T}) = \log \mathcal{I}_{T_0}(\boldsymbol{T},\boldsymbol{T}') + \mathcal{O}(\hbar) , \tag{39}$$

that is a free energy, from the viewpoint of the matrix integral partition function.

It remains unclear how one can use this fact to determine properties of the integrable model, but it might be possible to study the analytic structure of the free energy F_0 to gain some understanding of the Bethe roots $\boldsymbol{x}$. This problem is currently under further investigation, and we hope to report new results elsewhere.

3.1.2. KP tau function

Lastly, one may also observe that if we fix one set of coordinates, say $\boldsymbol{y}$, then we can write

$$\lim_{M,N\to\infty} \mathcal{I}(N,M|\boldsymbol{t}) = \sum_\lambda s_\lambda(\boldsymbol{y}) s_\lambda(\boldsymbol{t}) \equiv \sum_\lambda c_\lambda(\boldsymbol{y}) s_\lambda(\boldsymbol{t}) , \tag{40}$$

with coefficients $c_\lambda(\boldsymbol{y}) = \det(h_{\lambda_i - i + j}(\boldsymbol{y}))$. But now, it is trivial to notice that these are Plücker coordinates in the Jacobi-Trudi form, as seen in [16,4].

Hence, the following expression

$$\mathcal{I}(N,M|\boldsymbol{t}) = \sum_{\lambda\subseteq[N,M]} s_\lambda(\boldsymbol{y}) s_\lambda(\boldsymbol{t}) , \tag{41}$$

is also a KP tau function, a fact that we already know from [29], where the author proved this statement using the free fermions formalism.

3.2. Correlation functions

Bogoliubov has also shown, in [7], that the correlation functions

$$\begin{aligned} A_m(N,M|\boldsymbol{x},\boldsymbol{y}\setminus\{y_N\}) &= \langle 0|\prod_{j=1}^{N}\mathbb{C}(x_j)\prod_{k=1}^{N-1}\mathbb{B}(y_k)\phi_m^\dagger|0\rangle \\ &= \prod_{j=1}^{N}x_j^{M/2}\prod_{k=1}^{N-1}y_j^{M/2}\langle 0|\prod_{j=1}^{N}C(1/x_j)\prod_{k=1}^{N-1}B(y_k)\phi_m^\dagger|0\rangle \end{aligned} \tag{42a}$$

can be written as

$$A_m(N,M|\boldsymbol{x},\boldsymbol{y}\setminus\{y_N\}) = \frac{(-1)^{N-1}}{y_N^{(N-1)/2}}\prod_{j=1}^{N}x_j^{M/2}\prod_{k=1}^{N-1}y_j^{M/2}\left(\prod_{t<N}\frac{y_N-y_t}{y_t}\right)\frac{\det Q}{\det H}\mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{y}) \tag{42b}$$

where Q is an $N\times N$ matrix with components

$$Q_{jN} = x_j^{(M+N-1-2m)/2} \quad \text{and} \quad Q_{jk} = H_{jk}\,, \tag{43}$$

and $H_{jk} = H(x_j, y_k)$ are the components of the matrix H in (23). The components Q_{jN} are independent of the coordinates y; therefore, we cannot express the above expression as a Toda hierarchy tau function.

We already know from (22) that

$$\frac{\mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{y})}{\det H} = \frac{1}{\Delta(x)\Delta(y)}\,, \tag{44}$$

then

$$A_m(N,M|\boldsymbol{x},\boldsymbol{y}\setminus\{y_N\}) \propto \frac{\det Q}{\Delta(x)} \equiv \mathcal{A}_m(N,M|\boldsymbol{x},\boldsymbol{y})\,, \tag{45}$$

and we treat the coordinates $\{\boldsymbol{y}\}$ as a set of N fixed parameters.

Furthermore, we define vector field $\boldsymbol{F}(z) = (F_1,\ldots,F_N)$, where its components are given by

$$F_j(z) = H(z,y_j) \quad \text{if } j\neq N\,, \qquad \text{and} \qquad F_N(z) = z^{(M+N-1-2m)/2}\,. \tag{46}$$

As before, we expand $F_j(z)$, $j\neq N$, as the geometric sum

$$F_j(z) = \frac{1-(y_jz)^{M+N}}{1-y_jz} = \sum_{n=0}^{N+M-1}(y_jz)^n \equiv \sum_{n=0}^{M+N-1}f_{j,n}z^n\,, \quad f_{j,n} = y_j^n\,. \tag{47}$$

We can also express the function $F_N(z)$ in this form by setting $f_{N,n} = \delta_{n,(M-N-1-2m)/2}$ and requiring $(M-N-1)$ to be an even integer.

With these definitions, we conclude that

$$\mathcal{A}_m(N,M|\boldsymbol{x},\boldsymbol{y}) = \frac{\det_{jk}F_j(x_k)}{\Delta(x)}\,. \tag{48}$$

From the expansion

$$\det_{jk}F_j(x_k) = \det_{jk}\left(\sum_{n=0}^{M+N-1}f_{j,n}x_k^n\right)\,, \tag{49}$$

and utilizing the Cauchy-Binet formula, we get

$$\mathcal{A}_m(N,M|\boldsymbol{x},\boldsymbol{y}) = \sum_{0\le\ell_N\le\cdots\le\ell_1\le N+M}\frac{\det_{jk}(f_{j,\ell_k})\det_{jk}(x_k^{\ell_j})}{\Delta(x)}\,. \tag{50}$$

We now use the definition of the Schur polynomials and the Jacobi-Trudi expression of Plücker coordinates, as detailed in [4], leading to the following expression:

$$\mathcal{A}_m(N,M|\boldsymbol{x},\boldsymbol{y}) = \sum_{\lambda}c_\lambda(\boldsymbol{y})s_\lambda(\boldsymbol{x})\,, \tag{51}$$

where $c_\lambda \equiv \det_{jk}(y_j^{\ell_k})$, $\ell_k = \lambda_k - k + N$. Putting all these facts together, we conclude that this expression is also a KP tau function.

This expression underscores the non-trivial nature of these tau functions within the model. However, it also implies the existence of other intriguing examples awaiting exploration. Let us now briefly investigate other cases.

3.2.1. *Skew Schur polynomials expansion*

It has also been demonstrated in [7,28] that the Bethe states exhibit a coordinate expansion

$$\prod_{j=1}^{N} \mathbb{B}(x_j)|\lambda\rangle = \sum_{\substack{\mu\supset\lambda \\ \mu\subseteq[N,M]}} s_{\mu/\lambda}(\boldsymbol{x})|\mu\rangle \qquad \langle\lambda|\prod_{j=1}^{N} \mathbb{C}(x_j) = \sum_{\substack{\mu\supset\lambda \\ \mu\subseteq[N,M]}} s_{\mu/\lambda}(\boldsymbol{x})\langle\mu| \,, \tag{52}$$

where $s_{\mu/\lambda}$ are skew Schur polynomials [15].

Therefore, we can write the correlation function (42a) as

$$\begin{aligned} A_m(N,M|\boldsymbol{x},\boldsymbol{y}\setminus\{y_N\}) &= \sum_{\lambda}\sum_{\substack{\mu\supset(m) \\ \mu\subseteq[N-1,M]}} s_\lambda(\boldsymbol{x})s_{\mu/(m)}(\boldsymbol{y}\setminus\{y_N\}) \\ &= \sum_{\substack{\mu\supset(m) \\ \mu\subseteq[N-1,M]}} s_{\mu/(m)}(\boldsymbol{y}\setminus\{y_N\})s_\mu(\boldsymbol{x}) \,, \end{aligned} \tag{53}$$

where we have used that $\phi^\dagger|\mathbf{0}\rangle = |(m)\rangle$, and in the second line we sum over all partitions, since $s_{\mu/(m)} = 0$, $\forall\ \mu \not\supset (m)$. This expression also shows that the expression A_m can also be written as a tau function.

More generally, let us consider the correlation functions

$$\begin{aligned} A_{\lambda_1\lambda_2}(N',N,M|\boldsymbol{x},\boldsymbol{y}) &= \langle\lambda_1|\prod_{j=1}^{N}\mathbb{C}(x_j)\prod_{k=1}^{N'}\mathbb{B}(y_k)|\lambda_2\rangle \\ &= \sum_{\mu\subseteq[\min(N,N'),M]} s_{\mu/\lambda_1}(\boldsymbol{x})s_{\mu/\lambda_2}(\boldsymbol{y}) \,, \end{aligned} \tag{54}$$

where $N' + \ell(\lambda_2) + n_0 = N' + \ell(\lambda_2) + n_0'$ and we have also used that the skew Schur polynomial for any Young diagram μ that does not contain λ_1 and/or λ_2 vanishes.

In the limit $M, N, N' \to \infty$, we can use the elementary properties of skew Schur polynomials [15]

$$\sum_\mu s_{\mu/\lambda_1}(\boldsymbol{x})s_{\mu/\lambda_2}(\boldsymbol{y}) = \prod_{i,j}\frac{1}{1-x_iy_j}\sum_\nu s_{\lambda_1/\nu}(\boldsymbol{x})s_{\lambda_2/\nu}(\boldsymbol{y}) \,, \tag{55}$$

and the Cauchy's identity, we have that

$$\begin{aligned} \lim_{M,N,N'\to\infty} A_{\lambda_1\lambda_2}(N',N,M|\boldsymbol{x},\boldsymbol{y}) &= \sum_\mu s_\mu(\boldsymbol{x})s_\mu(\boldsymbol{y})\sum_\nu s_{\lambda_1/\nu}(\boldsymbol{x})s_{\lambda_2/\nu}(\boldsymbol{y}) \\ &= \sum_\mu s_\mu(\boldsymbol{x})s_\mu(\boldsymbol{y})\sum_\nu s_{\lambda_1/\nu}(\boldsymbol{x})s_{\lambda_2/\nu}(\boldsymbol{y}) \,. \end{aligned} \tag{56}$$

Therefore, we write the finite case as

$$A_{\lambda_1\lambda_2}(N',N,M|\boldsymbol{x},\boldsymbol{y}) = \mathcal{I}(\min(N',N),M,\boldsymbol{x},\boldsymbol{y})\sum_\nu s_{\lambda_1/\nu}(\boldsymbol{x})s_{\lambda_2/\nu}(\boldsymbol{y}) \,. \tag{57}$$

We observe that this correlation function can be expressed as the product of the off-shell norm (22) and a finite sum over skew Schur functions.

3.2.2. *Schur polynomials expansion of some correlation functions*

Indeed, it is worth noting that other quantities, which are not tau functions, may have interesting Schur polynomial expansions. Consider the state calculated in [7]

$$|\mathcal{Y}\rangle = \sum_{\vec{n}}\prod_{j=0}^{M}|n_j\rangle = \sum_\mu|\mu\rangle \qquad \sum_j n_j = N \,. \tag{58}$$

Then

$$\langle\mathcal{Y}|\prod_{j=1}^{N}B(\boldsymbol{x})|\nu\rangle = \sum_{\lambda\subseteq[N,M]} s_{\lambda/\nu}(\boldsymbol{x}) \,. \tag{59}$$

While it may not be a tau function, it has an interesting expansion as a sum of skew Schur polynomials.

3.3. General tau functions

Based on the discussion we have had so far, and on the general mapping between the phase model and free fermions, one can grasp the general form of tau functions in the context of this integrable chain.

Let us consider the vertex operator construction [19], as also discussed in [4,29]. We consider the vacuum state $|\mathbf{0}\rangle$, often referred to as a "Fermi sea", defined by the conditions $\psi_m|\mathbf{0}\rangle = \psi_n^\star|\mathbf{0}\rangle = 0$ for $m<0$ and $n\geq 0$, where ψ_n are components of a holomorphic free fermionic field. In this formalism, the partition states are given by

$$|\mu\rangle = \mathrm{sign}(\sigma)\prod_{j=1}^{d}\psi_{a_j}\psi^\star_{-b_j}|\mathbf{0}\rangle\,, \tag{60}$$

where $\mathrm{sign}(\sigma)=\pm 1$ are defined in a such a way that the Shur coefficients of the vertex operators (defined below) have positive coefficients.

The pairs $\{(a_j|b_j)\}_{j=1}^d$ define the Frobenius notation of the partition $\mu=(\mu_1,\mu_2,\ldots,\mu_\ell)$. In this notation, a_j is given by $\mu_j - j$ and b_j is given by $\mu'_j - j$, where d represents the number of boxes in the diagonal of the Young diagram, and μ' is its conjugate, or transpose, diagram. From these definitions, we have the equivalence

$$\prod_{k\geq 1}(\phi_k^\dagger)^{n_k} \mapsto \mathrm{sign}(\sigma)\prod_{j=1}^{d}\psi_{a_j}\psi^\star_{-b_j} \qquad |\mu| = \sum_k k n_k = \sum_j (a_j+b_j)+d\,. \tag{61}$$

Finally, the $\mathfrak{gl}(\infty)$ algebra has generators given by the bilinears $X=\sum_{j,j\in\mathbb{Z}} x_{ij} :\psi_i\psi_j^\star: + c$, where $c\in\mathbb{C}$, $x_{ij}=0$ for large $|j-i|$, say $\geq M$, and the colons denote the normal ordering

$$:\psi_i\psi_j^*: = \psi_i\psi_j^* - \langle 0|\psi_i\psi_j^*|0\rangle\,. \tag{62}$$

The group $GL(\infty)$ is defined through the exponential map $\exp:\mathfrak{gl}(\infty)\to GL(\infty)$ as usual.

Note that these elements have only finitely many non-zero entries: the diagonal terms represent number operators $\mathcal{N}$, the upper triangular terms represent annihilation operators ϕ, and the lower triangular terms represent creation operators $\phi^\dagger$. The central charge corresponds to the vacuum projection $\pi=|\mathbf{0}\rangle\langle\mathbf{0}|$. Therefore,

$$GL(\infty)\ni G\mapsto \mathcal{G}=\exp\left[\sum_{i=1}^{M}\left(\sum_{a=1}^{3}c_{i,a}T_i^a + c\pi_i\right)\right]\in \mathrm{Aut}(\mathcal{F})\,, \tag{63}$$

where $T_i^a\in\{\mathcal{N}_i,\phi_i,\phi_i^\dagger\}_{i=1}^M$.

The operators $\mathbb{B}(x)$ and $\mathbb{C}(x)$, for a large enough chain $M\to\infty$, are related to the vertex operators $\Gamma_-(x)$ and $\Gamma_+(x)$, respectively, as

$$\mathbb{B}(x)\mapsto\Gamma_-(x)=\exp\left(\sum_{n\geq 1}\frac{1}{n}x^n J_{-n}\right) \qquad \mathbb{C}(x)\mapsto\Gamma_+(x)=\exp\left(\sum_{n\geq 1}\frac{1}{n}x^n J_{n}\right)\,, \tag{64}$$

where J_n is written in terms of free fermions as $J_n=\sum_{j\in\mathbb{Z}} :\psi_j\psi^*_{j+n}:$. This set of operators generates a Heisenberg subalgebra $\widehat{\mathfrak{gl}}(1)\subset\mathfrak{gl}(\infty)$

$$[J_m,J_n]=m\delta_{n+m,0}\,. \tag{65}$$

Putting all these facts together, we have that the tau functions of the Toda hierarchies, given by

$$\tau_s(\boldsymbol{x},\boldsymbol{y})=\langle s|\prod_i\Gamma_+(x_i)G\prod_j\Gamma_-(x_j)|s\rangle\,, \tag{66}$$

are mapped into objects of the form

$$\begin{aligned}\tau(\boldsymbol{x},\boldsymbol{y}) &= \langle\Psi(\boldsymbol{x})|\mathcal{G}|\Psi(\boldsymbol{y})\rangle\\ &= \langle\mathbf{0}|\prod_i\mathbb{C}(x_i)\mathcal{G}\prod_j\mathbb{B}(y_j)|\mathbf{0}\rangle\,,\end{aligned} \tag{67}$$

where we necessarily have $s=0$ in the phase model

3.3.1. Hypergeometric tau functions

From these expressions, we can conclude that if we consider a diagonal group element $\mathcal{G}=\exp\left(\sum_{i\geq 0}c_i\mathcal{N}_i\right)$, we have that (67) becomes

$$\begin{aligned}\tau(\boldsymbol{x},\boldsymbol{y}) &= \langle \mathbf{0}|\prod_i \mathbb{C}(x_i) e^{\sum_i c_i \mathcal{N}_i}\prod_j \mathbb{B}(y_j)|\mathbf{0}\rangle \\ &= \sum_{\mu,\nu\subseteq[N,M]} c_{\mu\nu} s_\mu(\boldsymbol{x}) s_\nu(\boldsymbol{y})\end{aligned} \tag{68}$$

where

$$c_{\mu\nu} = \langle \mu | e^{\sum_i c_i \mathcal{N}_i} |\nu\rangle . \tag{69}$$

From the representation of the phase model, we have that for the partition $\mu = (1^{n_1} 2^{n_2} \dots M^{n_M})$, we have

$$\mathcal{N}_i |\mu\rangle = n_i |\mu\rangle \qquad n_0 = N - \ell(\mu) . \tag{70}$$

Consequently, we have

$$c_{\mu\nu} \equiv \delta_{\mu\nu} c_\mu = \delta_{\mu\nu} \prod_i e^{c_i n_i} , \tag{71}$$

and we conclude that

$$\tau(\boldsymbol{x},\boldsymbol{y}) = \sum_{\mu,\nu\subseteq[N,M]} c_\mu s_\mu(\boldsymbol{x}) s_\mu(\boldsymbol{y}) . \tag{72}$$

We observe that this tau function has diagonal coordinates c_λ. These tau functions belong to the hypergeometric type considered in [21–24].

4. Tau functions in the Q-bosons model

We now shift our focus to the case of q-bosons. The analysis parallels what we did before, but the specific details are markedly different. We begin by examining the norm of two off-shell Bethe states.

It has been shown [28] (see also [25,29]) that the Bethe states $|\Psi(\boldsymbol{x})\rangle$ have coordinate expansions

$$|\Psi(\boldsymbol{x})\rangle = \sum_{\mu\subseteq[N,M]} P_\mu(\boldsymbol{x},Q)|\mu\rangle_Q \qquad \langle\Psi(\boldsymbol{x})| = \sum_{\mu\subseteq[N,M]} P_\mu(\boldsymbol{x},Q)\,{}_Q\langle\mu| , \tag{73}$$

where P_λ denote Hall-Littlewood polynomials. In this form, the scalar product of two off-shell Bethe states in the q-boson model can be easily calculated to be

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = \langle\mathbf{0}|\prod_{j=1}^N \mathbb{C}(x_j,Q)\prod_{k=1}^N \mathbb{B}(y_k,Q)|\mathbf{0}\rangle = \sum_{\lambda\subseteq[N,M]} b_\lambda(Q) P_\lambda(\boldsymbol{x},Q) P_\lambda(\boldsymbol{y},Q) \tag{74}$$

where we use that $\langle\lambda|\mu\rangle_Q = b_\lambda(Q)\delta_{\lambda,\mu}$ and the completeness relation

$$\sum_\lambda \frac{1}{b_\lambda(Q)} |\lambda\rangle_Q {}_Q\langle\lambda| = \mathbb{1} . \tag{75}$$

It turns out that this expansion is the Cauchy identity for Hall-Littlewood polynomials [15]

$$\sum_\lambda b_\lambda(Q) P_\lambda(\boldsymbol{x},Q) P_\lambda(\boldsymbol{y},Q) = \prod_{j,k=1}^\infty \frac{1-Qx_j y_k}{1-x_j y_k} . \tag{76}$$

Our goal is to explore some properties of Hall-Littlewood polynomials to gain insight into aspects of this expansion.

4.1. Scalar product: determinant formula

We aim to refine the expressions above. First, we consider a determinant expression for the scalar product (74). We proceed with the scenario where M and N are very large but finite. In this case, we express this scalar product as

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = \prod_{j_1,j_2}(1-Qx_{j_1}y_{j_2})\prod_{k_2,k_2}(1-x_{k_1}y_{k_2})^{-1} . \tag{77}$$

Using the Cauchy's identity for Schur polynomials, that is

$$\sum_\lambda s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \prod_{i,j}\frac{1}{1-x_i y_j} , \tag{78}$$

and from the results derived in the phase model, we find that

$$\prod_{i,j}\frac{1}{1-x_i y_j} \equiv \mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{y}) = \frac{\det_{j,k} H(\boldsymbol{x},\boldsymbol{y})}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})} , \tag{79}$$

where H is the matrix (23). Consequently, the scalar product (77) becomes

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = \frac{\mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{y})}{\mathcal{I}(N,M|\boldsymbol{x},Q\boldsymbol{y})}\,, \tag{80}$$

and we see that it is the quotient of scalar products of the phase model. Hence, it is the quotient of two Toda tau functions. Observe that in the case $Q=0$, we have $\mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{0})=1$, then $\mathcal{I}_0(N,M|\boldsymbol{x},\boldsymbol{y}) = \mathcal{I}(N,M|\boldsymbol{x},\boldsymbol{y})$, as expected.

Additionally, we write

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = \frac{\Delta(Q\boldsymbol{y})}{\Delta(\boldsymbol{y})}\frac{\det H(\boldsymbol{x},\boldsymbol{y})}{\det H(\boldsymbol{x},Q\boldsymbol{y})}\,. \tag{81}$$

Using $\Delta(Q\boldsymbol{y})/\Delta(\boldsymbol{y}) = \prod_{j=1}^{N-1} Q^j$, we have

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = Q^{N(N-1)/2}\frac{\det H(\boldsymbol{x},\boldsymbol{y})}{\det H(\boldsymbol{x},Q\boldsymbol{y})}\,. \tag{82}$$

Let us denote $H(\boldsymbol{x},Q\boldsymbol{y}) = H_Q$, therefore

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = Q^{N(N-1)/2}\det H_Q^{-1}\det H = Q^{N(N-1)/2}\det \mathcal{H}(\boldsymbol{x},\boldsymbol{y})\,, \tag{83}$$

where $\mathcal{H} = H_Q^{-1}H$ is a matrix with components $\mathcal{H}_{i,j} \equiv \mathcal{H}(x_i,y_j)$ where

$$\mathcal{H}(z,w) = \sum_k \frac{(1-zy_k)}{(1-Qzy_k)}\frac{(1-Qx_kw)}{(1-x_kw)}\,. \tag{84}$$

Note that from this expression, we cannot decompose this function as in (24) since the coefficients in this expansion depend on $\boldsymbol{x}$ and $\boldsymbol{y}$.

4.2. Big Schur functions expansion

Now, let's revisit the result originally derived in [9] which demonstrates that the scalar product (74) is a tau function of the KP hierarchy. According to [15, Chapter 3, Section 4, Equation (4.7)], we can expand the Cauchy identity (76) as

$$\prod_{j,k=1}^{\infty}\frac{1-Qx_jy_k}{1-x_jy_k} = \sum_\lambda \mathcal{S}_\lambda(\boldsymbol{x},Q)s_\lambda(\boldsymbol{y}) = \sum_\lambda \mathcal{S}_\lambda(\boldsymbol{y},Q)s_\lambda(\boldsymbol{x})\,. \tag{85}$$

Here, the polynomials $\mathcal{S}_\lambda$, which we will refer to as the big-Schur functions, are defined by a Jacobi-Trudi formula:

$$\mathcal{S}_\lambda(\boldsymbol{y},Q) = \det(q_{\lambda_i-i+j}(\boldsymbol{y},Q))\,, \tag{86}$$

where the coefficients q_m are obtained from the expression:

$$\sum_m q_m(\boldsymbol{y},Q)z^m = \prod_j \frac{1-Qy_jz}{1-y_jz}\,, \tag{87}$$

and z is a formal variable. It has been argued [9] in that if we interpret the big-Schur functions as Plücker coordinates, the expression

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = \sum_{\lambda\subseteq[N,M]} \mathcal{S}_\lambda(\boldsymbol{x},Q)s_\lambda(\boldsymbol{y}) = \sum_{\lambda\subseteq[N,M]} \mathcal{S}_\lambda(\boldsymbol{y},Q)s_\lambda(\boldsymbol{x})\,, \tag{88}$$

is a restricted KP tau function with respect to both set of coordinates, that is $\boldsymbol{x}$ and $\boldsymbol{y}$.

Based on these findings, we conclude that the inner product can be expressed as a quotient of two Toda tau functions or as a KP tau function with coefficients given by the big Schur polynomials. Nevertheless, further investigation of these results is necessary.

4.3. Kostka-Foulkes expansion

In a Schur polynomial basis, we write

$$P_\lambda(\boldsymbol{x},Q) = \sum_\mu K^{-1}_{\lambda\mu}(Q)s_\lambda(\boldsymbol{x})\,, \tag{89}$$

where $K^{-1}_{\mu\nu}(Q) \in \mathbb{Z}[Q]$ are inverse Kostka-Foulkes polynomials [15,30].

If we now insert this expansion in the inner product, we have

$$\sum_\lambda b_\lambda(Q)P_\lambda(\boldsymbol{x},Q)P_\lambda(\boldsymbol{y},Q) = \sum_{\mu,\nu}\left(\sum_\lambda K^{-1\,T}_{\mu\lambda}b_\lambda(Q)K^{-1}_{\lambda\nu}\right)s_\mu(\boldsymbol{x})s_\nu(\boldsymbol{y}) \tag{90}$$

and we have the coefficients

$$\tilde{c}_{\mu\nu}(Q) = \sum_{\lambda} K^{-1\,T}_{\mu\lambda} b_\lambda(Q) K^{-1}_{\lambda\nu} \,, \tag{91}$$

for the double expansion of the inner product in terms of Schur polynomials. Based on our discussion so far, we know that these coefficients are not Plücker coordinates of the Toda hierarchy.

But we can say something interesting about these coefficients. Let us now expand the big-Schur functions as

$$\mathcal{S}_\mu(\boldsymbol{x}, Q) = \sum_{\lambda} C^{\mu}_{\lambda}(Q) s_\lambda(\boldsymbol{x}) \,, \tag{92}$$

we can fix the coefficients $C^{\mu}_{\lambda}(Q)$ in terms of Kostka-Foulkes polynomials using the orthogonality relations of these polynomials [15]. In particular, there is an inner product in the ring of symmetric functions such that

$$b_\mu \langle P_\mu(\boldsymbol{x}, Q), P_\nu(\boldsymbol{x}, Q)\rangle = \langle \mathcal{S}_\mu(\boldsymbol{x}, Q), s_\nu(\boldsymbol{x})\rangle = \delta_{\mu\nu} \,. \tag{93}$$

Therefore,

$$\begin{aligned}\delta_{\mu\nu} &= \sum_{\lambda} C^{\mu}_{\lambda} \langle s_\lambda(\boldsymbol{x}), s_\nu(\boldsymbol{x})\rangle = \sum_{\lambda}\sum_{\pi,\sigma} C^{\mu}_{\lambda} K_{\lambda\pi} K_{\nu\sigma} \langle P_\pi(\boldsymbol{x}, Q), P_\sigma(\boldsymbol{x}, Q)\rangle \\ &= \sum_{\lambda} C^{\mu}_{\lambda} \sum_{\sigma} K_{\lambda\sigma} b_\sigma^{-1} K_{\nu\sigma} \,,\end{aligned} \tag{94}$$

and we conclude that

$$(C^{-1})^{\lambda}_{\nu} = \sum_{\sigma} K_{\lambda\sigma} b_\sigma^{-1} (K_{\sigma\nu})^T \,, \tag{95}$$

and consistency with (91) implies that $C^{\lambda}_{\nu} \equiv \tilde{c}_{\lambda\nu}$.

Hence, we conclude that

$$\mathcal{S}_\mu(\boldsymbol{t}, Q) = \sum_{\nu} \tilde{c}_{\mu\nu}(Q) s_\mu(\boldsymbol{t}) \,, \tag{96}$$

where we have written the big Schur polynomial in terms of the Miwa coordinates $\boldsymbol{t}$.

Remark 1. This expression also reveals something interesting. We can formulate this problem in terms of partition states defined in the phase model. Alternatively, we can utilize the conventional vertex operator construction rather than the q-deformed version proposed by Jing [12,13], which is much more challenging to handle.

4.4. Supersymmetric Schur polynomials expansion

Based on this result, we argue that although Equation (74) is not a Toda tau function with respect to the coordinates $\{\boldsymbol{t}; \boldsymbol{t}'\}$, we can define a new set of coordinates $\{\boldsymbol{t}; \boldsymbol{T}\}$ such that the scalar product becomes a trivial Toda tau function.

Let us first decompose

$$\sum_{m} q_m(\boldsymbol{y}, Q) z^m = \prod_{j} (1 + y_j(-Qz)) \prod_{k} (1 - y_k z)^{-1} = \sum_{j,k} e_j(\boldsymbol{y}) h_k(\boldsymbol{y}) (-Q)^j z^{j+k} \,. \tag{97}$$

Reorganizing this sum, we conclude that

$$q_m(\boldsymbol{y}, Q) = \sum_{j=0}^{m} e_j(\boldsymbol{y}) h_{m-j}(\boldsymbol{y}) (-Q)^j = \sum_{j=0}^{m} e_j(-Q\boldsymbol{y}) h_{m-j}(\boldsymbol{y}) \,, \tag{98}$$

where in the last equality we have used the homogeneity of the elementary symmetric polynomials, and $Q\boldsymbol{y} \equiv (Qy_1, Qy_2, \ldots)$.

It is easy to see that

$$\begin{aligned}\prod_{j=1}^{N} (1 - (Qy_j)z) &= \prod_{j=1}^{N} e^{\ln(1-(Qy_j)z)} = \prod_{j=1}^{N} \exp\left(-\sum_{n>0} \frac{Q^n y_j^n}{n} z^n\right) \\ &= \exp\left(-\sum_{n>0}\sum_{j=1}^{N} \frac{Q^n y_j^n}{n} z^n\right) = \exp\left(-\sum_{n>0} t_n^{(Q)} z^n\right)\end{aligned} \tag{99a}$$

where $t_n^{(Q)} = \frac{Q^n}{n} \sum_j y_j^n = Q^n t'_{-n}$. Additionally

$$\prod_{j=1}^{N} \frac{1}{(1 - y_j z)} = \prod_{j=1}^{N} e^{-\ln(1-y_j z)} = \prod_{j=1}^{N} \exp\left(\sum_{n>0} \frac{y_j}{n} z^n\right) = \exp\left(\sum_{n>0} t'_{-n} z^n\right) \,. \tag{99b}$$

Consequently

$$\sum_m q_m(\boldsymbol{y},Q)z^m = \prod_j \frac{1-Qy_j z}{1-y_k z} = \exp\left(\sum_{n\geq 1}(t'_{-n} - t_n^{(Q)})z^n\right) \equiv \exp\left(\sum_{n\geq 1} T_n z^n\right), \tag{100}$$

with $T_n = (1-Q^n)t'_{-n}$. Then, we conclude that $q_m(\boldsymbol{y},Q)$ are homogeneous polynomials with respect the Miwa coordinates $\boldsymbol{T} = (T_1, T_2, \ldots)$. Then

$$S_\lambda(\boldsymbol{t}',Q) = \det\left(h_{\lambda_i - i + j}(\boldsymbol{T})\right) = s_\lambda(\boldsymbol{T}). \tag{101}$$

All in all, we conclude that the big Schur functions $S_\lambda(\boldsymbol{t}',Q)$ are ordinary Schur functions with respect to the coordinates $\boldsymbol{T}$.

Moreover, a more refined approach is also possible. Supersymmetric (or Hook) Schur functions [8], as discussed in works such as [15,17], denoted by $s_\lambda(\boldsymbol{\alpha}/\boldsymbol{\beta})$, are defined as ordinary Schur functions evaluated at Miwa coordinates of the form

$$T_n = \frac{1}{n}\left(\sum_{i=1}^{\dim(\alpha)} \alpha_i^n - \sum_{i=1}^{\dim(\beta)} (-\beta_i)^n\right). \tag{102}$$

Comparing this expression with the results above, we can see that the big Schur functions $S_\lambda(\boldsymbol{t}',Q)$ correspond to the supersymmetric Schur functions for $\boldsymbol{\alpha} = \boldsymbol{y}$ and $\boldsymbol{\beta} = -Q\boldsymbol{y}$, which is

$$S_\lambda(\boldsymbol{t}',Q) = s_\lambda[\boldsymbol{y}/(-Q\boldsymbol{y})]. \tag{103}$$

Putting these facts together, we immediately conclude that

$$\lim_{M,N\to\infty} \mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) = \sum_\lambda s_\lambda(\boldsymbol{T})s_\lambda(\boldsymbol{t}) \tag{104}$$

is a Toda hierarchy tau function with respect to $\{\boldsymbol{t};\boldsymbol{T}\}$. As a result, we obviously have that

$$\mathcal{I}_Q(N,M|\boldsymbol{x},\boldsymbol{y}) \equiv \mathcal{I}_Q(N,M|\boldsymbol{t},\boldsymbol{T}) = \sum_{\lambda\subseteq[N,M]} s_\lambda(\boldsymbol{T})s_\lambda(\boldsymbol{t}) \tag{105}$$

is also a restricted tau function of the Toda hierarchy with respect to $\boldsymbol{t}$ and $\boldsymbol{T}$.

Let us also express the supersymmetric Schur polynomials in terms of ordinary Schur functions, as shown in [15, Sec. I.5, exerc. 23]:

$$s_\lambda(\boldsymbol{x}/\boldsymbol{y}) = \sum_\mu s_{(\lambda/\mu)'}(\boldsymbol{y})s_\mu(\boldsymbol{x}), \tag{106}$$

where the prime denotes the conjugate diagram. Compare this expansion with (92). Then

$$\sum_\lambda s_\lambda(\boldsymbol{T})s_\lambda(\boldsymbol{t}) = \sum_{\lambda,\mu} s_{(\lambda/\mu)'}(-Q\boldsymbol{y})s_\lambda(\boldsymbol{t})s_\mu(\boldsymbol{t}'). \tag{107}$$

From this expression we conclude (and speculate) the following.

Remark 2. Since the skew Schur polynomials have determinant expressions, we can deduce that the $\boldsymbol{y}$-dependent coefficients $c_{\mu\lambda} = s_{(\lambda/\mu)'}(-Q\boldsymbol{y})$ have Jacobi-Trudi expressions. It is tempting to regard these objects as $\boldsymbol{y}$-dependent Plücker coordinates. In this sense, we would have a curve in the infinite Grassmannian instead of a point. Evidently, it is not a tau functions on any known integrable hierarchy, but it might suggest some new generalizations that are worth investigating.

Remark 3. It is worth noting that one of the simplest nontrivial solutions of the KP hierarchies are the Schur functions themselves [32]. As we have just demonstrated, the big Schur polynomials can be expressed as supersymmetric Schur polynomials, which are essentially ordinary Schur polynomials in a specific choice of Miwa coordinates. Therefore, we conclude that the big Schur polynomials are also KP tau functions. This direct conclusion serves as an alternative proof of this fact, originally derived in [18] using the KP bilinear identity.

Combining these observations, we conclude that the left-hand side of equation (107) is a tau function with respect to the coordinates $\{\boldsymbol{t};\boldsymbol{T}\}$. Furthermore, by employing the vertex operator formalism, we can express the Schur polynomials as

$$s_\lambda(\boldsymbol{t}') = \langle\lambda|e^{J_-(\boldsymbol{t}')}|\boldsymbol{0}\rangle \qquad s_\mu(\boldsymbol{t}) = \langle\boldsymbol{0}|e^{J_+(\boldsymbol{t})}|\mu\rangle. \tag{108}$$

Hence

$$\sum_{\lambda,\mu} c_{\mu\lambda}(Q,\boldsymbol{y})s_\mu(\boldsymbol{t}')s_\lambda(\boldsymbol{t}) = \sum_{\lambda,\mu}\langle\boldsymbol{0}|e^{J_+(\boldsymbol{t})}|\mu\rangle c_{\mu\lambda}(Q,\boldsymbol{y})\langle\lambda|e^{J_-(\boldsymbol{t}')}|\boldsymbol{0}\rangle. \tag{109}$$

It is important to reiterate that this expression does not constitute a tau function with respect to $\{\boldsymbol{t}, \boldsymbol{t}'\}$. Despite this crucial distinction, we proceed under the assumption that the coefficients can be expressed as

$$c_{\mu\lambda}(Q, \boldsymbol{y}) = \langle \mu | G_Q(\boldsymbol{y}) | \lambda \rangle \,, \tag{110}$$

where $G_Q(\boldsymbol{y}) \in GL(\infty)$. Hence

$$\lim_{N,M\to\infty} \mathcal{I}_Q(N, M|\boldsymbol{t}, \boldsymbol{t}'(\boldsymbol{y})) = \langle \boldsymbol{0}| e^{J_+(t)} G_Q(\boldsymbol{y}) e^{J_-(t')} |\boldsymbol{0}\rangle \,, \tag{111}$$

where $t'_{-n} = \frac{1}{n}\sum_i y_i^n$.

To derive the expression for this group element, recall that $T_m = t'_{-m} - Q^m t'_{-m}$. With this in mind, we can express the scalar product as follows:

$$\langle \boldsymbol{0}| e^{J_+(t)} e^{J_-(T)} |\boldsymbol{0}\rangle = \langle \boldsymbol{0}| e^{J_+(t)} e^{J_-(t') - J_-(t^{(Q)})} |\boldsymbol{0}\rangle \,. \tag{112a}$$

From the Heisenberg algebra, we deduce that $[J_-(\boldsymbol{t}), J_-(\boldsymbol{t}')] = 0$, therefore

$$\langle \boldsymbol{0}| e^{J_+(t)} e^{J_-(T)} |\boldsymbol{0}\rangle = \langle \boldsymbol{0}| e^{J_+(t)} e^{-J_-(t^{(Q)})} e^{J_-(t')} |\boldsymbol{0}\rangle \,. \tag{112b}$$

We finally conclude that

$$G_Q(\boldsymbol{y}) = \exp\left(-J_-(\boldsymbol{t}^{(Q)})\right) \qquad t_n^{(Q)} = \frac{Q^n}{n}\sum_j y_j^n \,. \tag{113}$$

Generally, the expectation values involving coordinate-dependent group elements $G_Q(\boldsymbol{y})$ do not form Toda hierarchy tau functions. However, in the example above, the coordinates combine in such a way that they generate a tau function with respect to the coordinates $\boldsymbol{T}$ and $\boldsymbol{t}$.

5. Discussion, conclusions and perspectives

In this work, we have explored the connections between a quantum integrable system, the q-boson model, and solutions of a classical integrable system, the Toda hierarchy. Our investigation has extended some early findings in this field and has also unveiled new research avenues, which we aim to explore in future studies. Let us now discuss some of these promising directions.

In Section 3, we explored various aspects of the phase model, presenting the scalar product of two off-shell Bethe states as an elementary example of a Toda system tau function. This section overlaps with existing literature, particularly a paper by Wheeler and Foda [29,9]. In their work, they demonstrate that the inner products of two Bethe states are KP hierarchy tau functions. Our current work builds on this by arguing that these scalar products are also Toda hierarchy tau functions, as they can be expanded as a Cauchy identity, with the coefficients being Schur functions. When one set of coefficients is fixed in this expression, Wheeler's results are recovered.

We also extend this analysis to other correlation functions, determining when these functions satisfy integrable hierarchy equations. Using expressions derived by Bogoliubov and collaborators [6], we reformulated them to check if the correlation functions are KP/Toda tau functions. Although these objects cannot generally be written as Toda tau functions, with some adjustments, they can be expressed as KP tau functions. We have elucidated how various correlation functions align with this framework, revealing a remarkably rich structure.

The main results are discussed in Section 3.3, where we examine the general form of tau functions associated with this phase model. Notably, we explored a mapping between the phase model data and the vertex operator representation of free bosons. An intriguing avenue for future research is to delve into the properties of the hypergeometric tau functions uncovered in Section 3.3.1. Exploring these functions is expected to provide significant understanding, and comparing them with existing literature will further enrich our comprehension of their characteristics and implications within the context of integrable systems. This problem is currently under investigation.

Additionally, we revealed an intriguing alternative portrayal of these scalar products using a matrix integral framework, which corresponds to an ensemble of Coulomb particles. This discovery opens up promising avenues for further inquiry. Specifically, it would be intriguing to further explore this subject and investigate whether the matrix model description provides valuable insights into the phase model and its Bethe roots. Such an investigation promises to illuminate the underlying dynamics of the phase model and its relationship with classical integrable systems.

In Section 4, we addressed the same problem within the context of the q-boson model, attempting to replicate the analysis done in Section 3. There are few known results in the literature regarding the relationship between these objects and KP/Toda tau functions, so we used established formulas to investigate if the scalar products are also tau functions.

Initially, we derived a determinant formula for the q-boson scalar products and discussed their expression in terms of the phase model data. This result establishes a connection between the q-boson and phase model quantities. Furthermore, we explored different expansions for these scalar products. Notably, we observed that they can be expanded in terms of Big Schur and supersymmetric tau functions. Consequently, we introduced a new set of coordinates, demonstrating that scalar products can be precisely expressed as Toda tau functions within this transformed coordinate system.

We also write these formulas in terms of simpler orthogonal polynomials. Firstly, we show that these scalar products cannot be naively written as Toda tau functions. However, using some known expansions, we expand the inner products in different forms. In one of these expansions, we define a new set of coordinates, making the scalar product of q-bosons Toda tau functions with respect to these new coordinates. This is the main result of this section.

One of our most pressing challenges lies in elucidating the intricate connection between our findings and the Ablowitz-Ladik hierarchy. Since this hierarchy arises as a reduction of the Toda hierarchy, it becomes imperative to understand how we can capture the structure of the AL hierarchy using the results we have derived. Considering that the q-boson model effectively quantizes the Ablowitz-Ladik equation, it follows that we should detect some resemblance of this classical problem within the q-boson system. This understanding holds the promise of shedding significant light on the interplay between classical and quantum integrable systems.

We hope to address some of these challenges in future publications.

CRediT authorship contribution statement

Thiago Araujo: Conceptualization, Methodology, Writing – review & editing.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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