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# Some vortex solutions in the extended Skyrme-Faddeev model

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**Abstract.** Analytical and numerical vortex solutions for the extended Skyrme-Faddeev model in a $(3+1)$ dimensional Minkowski space-time are investigated. The extension is obtained by adding to the Lagrangian a quartic term, which is the square of the kinetic term, and a potential which breaks the $SO(3)$ symmetry down to $SO(2)$. The construction of the solutions has been done in twofold: one makes use of an axially symmetric ansatz and solves the resulting ODE by an analytical and a numerical way. The analytical vortices are obtained for special form of the potentials, and the numerical ones are computed using the successive over relaxation method for wider choice of the potentials. Another is based on a simulational technique named the simulated annealing method which is available to treat the non-axisymmetric shape of solutions. The crucial thing for determining the structure of vortices is *the type* of the potential.

## 1. Introduction

The so-called Skyrme-Faddeev model was introduced in the seventies [1] as a generalization to $(3+1)$ dimensions of the $O(3)$ non-linear sigma model in $(2+1)$ dimensions. The Skyrme term, quartic in derivatives of the field, balances the quadratic kinetic term and according to Derrick's theorem, allows the existence of stable solutions with non-trivial Hopf topological charges. Due to the highly non-linear character of the model and the lack of symmetries, the first soliton solutions were only constructed in the late nineties using numerical methods [2, 3, 4, 5]. Since then the interest in the model has increased considerably and it has found applications in many areas of physics due mainly to the knotted character of the solutions [6]. One of the aspects of the model that has attracted considerable attention has been its connection with gauge theories. Faddeev and Niemi have conjectured that it might describe the low energy limit of the pure $SU(2)$ Yang-Mills theory [7]. They based their argument on a decomposition of the physical degrees of freedom of the $SU(2)$ connection, proposed in the eighties by Cho [8], and involving a triplet of scalar fields $\vec{n}$ taking values on the sphere $S^2$ ($\vec{n}^2 = 1$). Gies [9] has calculated the Wilsonian one loop effective action for the pure $SU(2)$ Yang-Mills theory assuming Cho's decomposition, and found that the Skyrme-Faddeev action is indeed part of it, but additional quartic terms in the derivatives of the triplet $\vec{n}$ are unavoidable. In fact, the first numerical Hopf solitons were first constructed for the Skyrme-Faddeev model modified by a quartic term

[2] which is the square of the kinetic term. However, the soliton solutions in [2] were constructed for a sector of the theory where the signs of the coupling constants disagree with those indicated by Gies' calculations. Therefore, it is worth investigating the model with correct sign of the coupling constants.

In this paper we consider an extended Skyrme-Faddeev model (ESF) defined by the Lagrangian

$$\mathcal{L} = M^2\,\partial_\mu\vec{n}\cdot\partial^\mu\vec{n} - \frac{1}{e^2}(\partial_\mu\vec{n}\wedge\partial_\nu\vec{n})^2 + \frac{\beta}{2}\,(\partial_\mu\vec{n}\cdot\partial^\mu\vec{n})^2 - V(n_3) \tag{1}$$

where $\vec{n}$ is a triplet of real scalar fields taking values on the sphere $S^2$, $n_3$ its third component, $M$ is a coupling constant with dimension of $(\text{length})^{-1}$, $e^2$ and $\beta$ are dimensionless coupling constants, and the potential is a functional of the third component $n_3$ of the triplet $\vec{n}$. Note that the potential breaks the $O(3)$ symmetry of the original Skyrme-Faddeev down to $O(2)$, the group of rotations on the plane $n_1\,n_2$, and so eliminating two of the three Goldstone boson degrees of freedom. In this paper the main role of potential is to stabilize the vortex solutions.

The static energy density ($\mathcal{H}_{\text{static}} = -\mathcal{L}$) associated to (1) is positive definite if $V > 0$, $M^2 > 0$, $e^2 > 0$ and $\beta < 0$. That is the sector explored in [2] and where Hopf soliton solutions were first constructed (for $V = 0$). In addition, that is also the sector explored in [10] but with additional terms involving second derivatives of the $\vec{n}$ field, and where Hopf soliton were also constructed. The static energy density of (1) is also positive definite for $V > 0$ if

$$M^2 > 0\,; \qquad e^2 < 0\,; \qquad \beta < 0\,; \qquad \beta\,e^2 \geq 1 \tag{2}$$

That is the sector that agrees with the signature of the terms in the one loop effective action calculated in [9] and it is the sector that we will consider in this paper. Static Hopf solitons were constructed in [11] for the sector (2) (with $V = 0$) and their quantum excitations, including comparison with glueball spectrum, were considered in [12]. An interesting feature of the Hopf solitons constructed in [11] is that they shrink in size and then disappear as $\beta\,e^2 \to 1$, which is exactly the point where the exact vortex solution exist [13].

The aim of the present paper is to investigate if vortex solutions for the model (1) continue to exist when the condition $\beta\,e^2 = 1$ is relaxed, and so if they co-exist with the Hopf solitons of [11]. In order to stabilize the solution, we shall introduce the types of potential

$$V(n_3) = \frac{\mu^2}{2}v_b^a, \qquad v_b^a \equiv (1+n_3)^a(1-n_3)^b \tag{3}$$

where $a + b =$ *non-zero integer*, and $\mu$ is a real coupling constant. A special choice of the parameters $a, b$ we have holomorphic solutions of the model while for the other case we still have numerical solutions.

In this paper, first we discuss the integrable holomorphic solutions of the model and next we shall perform the numerical stuff. The numerical simulations are done by twofold: one is by solving a differential equation which is accomplished in terms of *the standard successive over relaxation*. Another is based on energy minimization scheme called *the simulated annealing*. Especially the latter analysis demonstrates the detailed behavior of the symmetry breaking of the solutions by the change of the structure of the potential.

## 2. The integrable sector of the model

The first exact vortex solutions for the theory (1) were constructed in [13] for the case where the potential vanishes, and by exploring the integrability properties of a submodel of (1). In order to describe those exact vortex solutions it is better to perform the stereographic projection of

the target space $S^2$ onto the plane parameterized by the complex scalar field $u$ and related to $\vec{n}$ by

$$\vec{n} = (u + u^*, -i(u - u^*), u^2 - 1)/(1 + u^2) \tag{4}$$

It was shown in [13] that the field configurations of the form

$$u \equiv u(z, y), \quad u^* \equiv u^*(z^*, y), \quad \text{for} \;\; \beta e^2 = 1, \qquad V = 0 \tag{5}$$

are exact solutions of (1), where $z = x^1 + i\varepsilon_1 x^2$ and $y = x^3 - \varepsilon_2 x^0$, with $\varepsilon_a = \pm 1$, $a = 1, 2$, and $x^\mu$, $\mu = 0, 1, 2, 3$, are the Cartesian coordinates of the Minkowski space-time. The simplest solution is of the form $u = z^n e^{i\,k\,y}$, with $n$ integer, and it corresponds to a vortex parallel to the $x^3$-axis and with waves traveling along it with the speed of light.

In terms of the complex scalar field $u$ introduced in 4 the Lagrangian 1 becomes

$$\mathcal{L} = 4M^2 \frac{\partial_\mu u \partial^\mu u^*}{(1 + u^2)^2} + \frac{8}{e^2}\left[\frac{(\partial_\mu u)^2(\partial_\nu u^*)^2}{(1 + u^2)^4} + (\beta e^2 - 1)\frac{(\partial_\mu u\, \partial^\mu u^*)^2}{(1 + u^2)^4}\right] - V(\mid u \mid^2) \tag{6}$$

where we have used the fact that $n_3$ is a functional of $\mid u \mid^2$ only, and so is the potential. The Euler-Lagrange equations following from (6), or (1), reads

$$(1 + |u|^2)\partial^\mu \mathcal{K}_\mu - 2u^* \mathcal{K}_\mu \partial^\mu u = -\frac{u}{4}(1 + |u|^2)^3\, V' \tag{7}$$

where $V' = \frac{\partial V}{\partial u^2}$, and

$$\mathcal{K}_\mu \equiv M^2 \partial_\mu u + \frac{4}{e^2}\,\frac{\left[(\partial_\nu u \partial^\nu u)\partial_\mu u^* + (\beta\, e^2 - 1)(\partial_\nu u \partial^\nu u^*)\partial_\mu u\right]}{(1 + u^2)^2} \tag{8}$$

We point out that the theory (6) possesses an integrable sector defined by the condition

$$(\partial_\mu u)^2 = 0 \tag{9}$$

Such condition was first discovered in the context of the $CP^1$ model using the generalized zero curvature condition for integrable theories in any dimension [14], and then applied to many models with target space being the sphere $S^2$, or $CP^1$ (see [15] for a review). It leads to an infinite number of local conserved currents. Indeed, (9) together with the equations of motion (7) imply the conservation of the infinity of currents given by

$$J^G_\mu \equiv \mathcal{K}_\mu \frac{\delta G}{\delta u} - \mathcal{K}^*_\mu \frac{\delta G}{\delta u^*} \tag{10}$$

where $G$ is any functional of $\mid u \mid^2$ only. For the case where the potential vanishes, the set of conserved currents is considerably enlarged since $G$ can be an arbitrary functional of $u$ and $u^*$, but not of their derivatives. If in addition to the condition (9) one takes $V = 0$ and $\beta\, e^2 = 1$, then the equations of motion reduce to $\partial^2 u = 0$. It is in that integrable sector that the solutions (5) lie, and were studied in [13].

It is interesting to note that (7) with a special choice of the potential

$$V(n_3) = \frac{\mu^2}{2}(1 + n_3)^{2-2/n}(1 - n_3)^{2+2/n} \tag{11}$$

have an analytical, holomorphic solution for each topological charge as

$$u(\rho, \varphi, z, \tau) = \left(\frac{\rho}{a}\right)^n e^{i[\epsilon n\varphi + k(z+\tau)]} \tag{12}$$

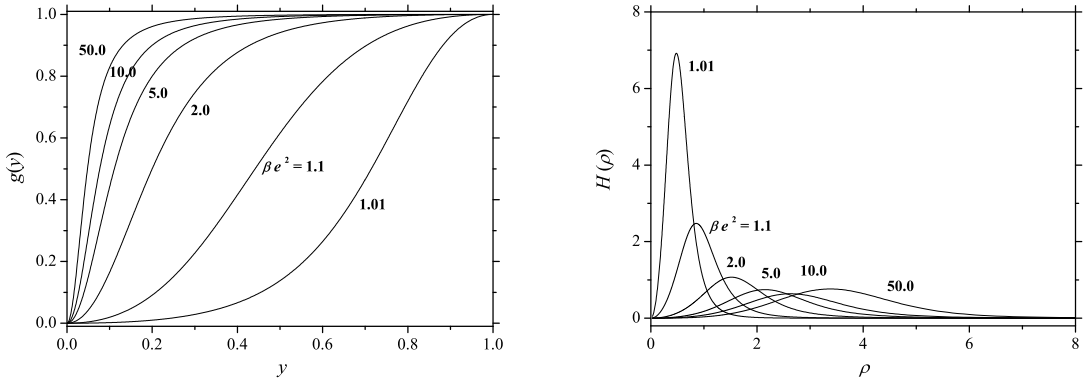



**Figure 1.** The $n = 2$ profile $g(y)$ and the corresponding Hamiltonian density of the real space $H(\rho)$ of $k^2 = 0.0$ for the constant $r_0^2\mu^2/M^2 = 1.0$.

where $\epsilon = \pm 1$ and $a$ describes a scale of the solution. Here we used dimensionless polar coordinates $(\rho, \varphi, z, \tau)$ defined by

$$x^0 = ct = r_0\tau, \quad x^1 = r_0\rho\cos\varphi, \quad x^2 = r_0\rho\sin\varphi, \quad x^3 = r_0 z \tag{13}$$

and where we have introduced a length scale $r_0$ given by

$$r_0^2 = -\frac{4}{M^2\, e^2}. \tag{14}$$

Substituting (12) into (7) the $a$ can be determined such as

$$a = \mid n \mid \left[\frac{M^2(\beta e^2 - 1)}{r_0^2\mu^2}\right]^{1/4} = \mid n \mid \left[\frac{(-e^2)\,(\beta e^2 - 1)M^4}{4\,\mu^2}\right]^{1/4} \tag{15}$$

Clearly, the special solution at $\beta e^2 = 1$ is obtained if we take a proper limit of the vanishing potential, i.e.$\beta e^2 \to 1$ and $\mu^2 \to 0$ with $(\beta e^2 - 1)/\mu^2 =$ constant.

In the case with the potential, the current (10) is still conserved because

$$\begin{aligned}\partial^\mu J_\mu = &\frac{\delta^2 G}{\delta u^2}\partial^\mu u\mathcal{K}_\mu + \frac{\delta G}{\delta u}\partial^\mu\mathcal{K}_\mu + \frac{\delta^2 G}{\delta u^*\delta u}\partial^\mu u^*\mathcal{K}_\mu \\ &-\frac{\delta^2 G}{\delta u^{*2}}\partial^\mu u^*\mathcal{K}^*_\mu - \frac{\delta G}{\delta u^*}\partial^\mu\mathcal{K}^*_\mu - \frac{\delta^2 G}{\delta u\delta u^*}\partial^\mu u^*\mathcal{K}^*_\mu = 0\end{aligned} \tag{16}$$

where we have used the reduced integrable equation

$$\partial^\mu\mathcal{K}_\mu + \frac{\mu^2}{4}(1 + |u|^2)^2\frac{\partial V(|u|^2)}{\partial u^*} = 0. \tag{17}$$

The Hamiltonian density associated to (6) is not positive definite due to the quartic terms in time derivatives. We shall arrange the Legendre transform of each term in (6) to make explicit

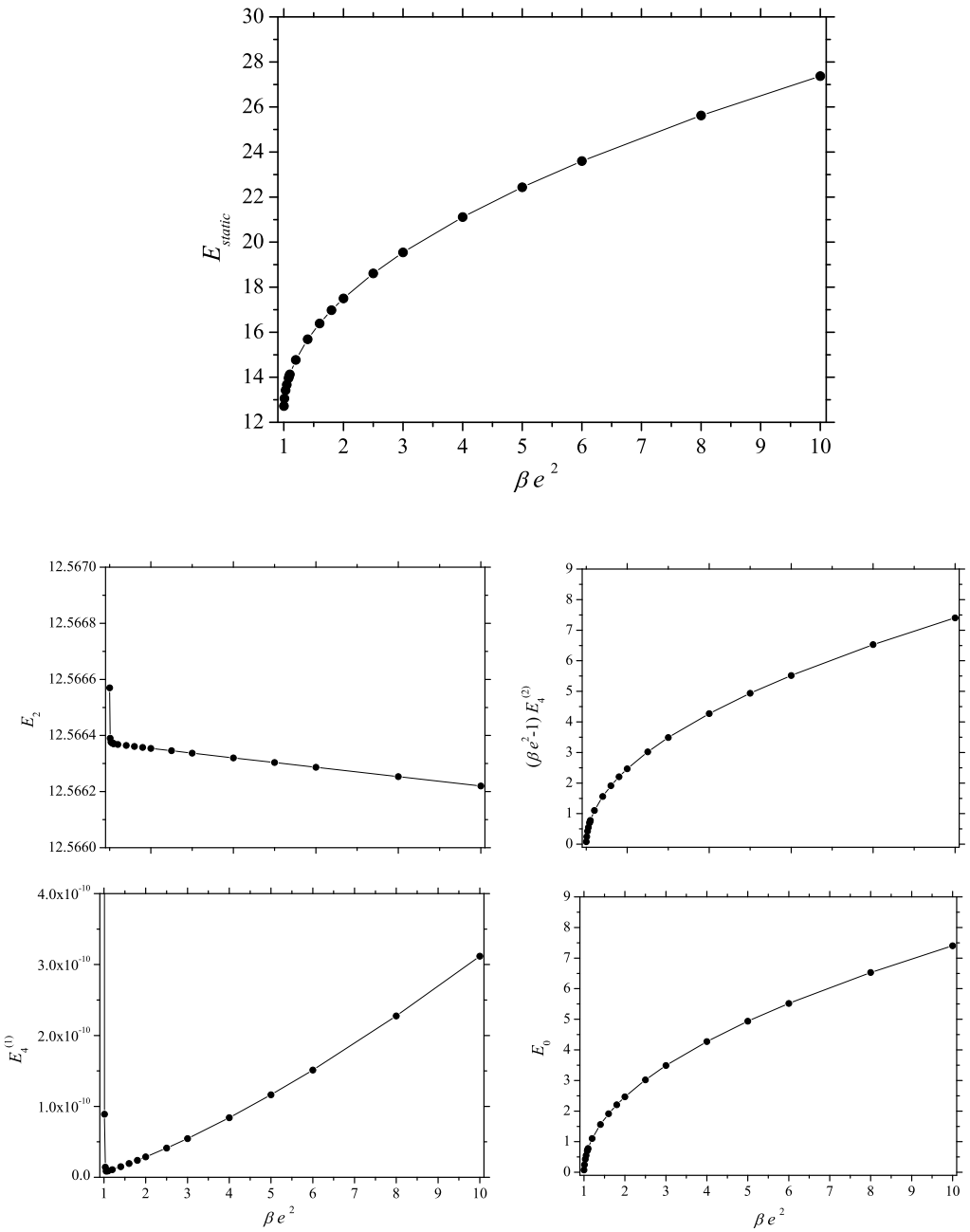



**Figure 2.** The static energy and its components corresponding to the solutions of Fig.1.

such non positive contributions, and write the Hamiltonian density as (see [16] for details)

$$\begin{aligned}\mathcal{H} &= 4M^2\frac{\left[|\,\dot{u}\,|^2+\vec{\nabla}u\cdot\vec{\nabla}u^*\right]}{(1+u^2)^2}-\frac{24}{e^2}\frac{(\vec{\nabla}u)^2(\vec{\nabla}u^*)^2}{(1+u^2 2)^4}\left[\left(\frac{2}{3}\right)^2-F^2\right]\\ &- 24\frac{(\beta e^2-1)}{e^2}\frac{\left[|\,\dot{u}\,|^2+\frac{1}{3}\vec{\nabla}u\cdot\vec{\nabla}u^*\right]\left[\vec{\nabla}u\cdot\vec{\nabla}u^*-|\,\dot{u}\,|^2\right]}{(1+u^2)^4}+V(|\,u\,|^2)\end{aligned}\tag{18}$$

where $\dot{u}$ denotes the $x^0$-derivative of $u$, and $\vec{\nabla}u$ its spatial gradient, and where we have denoted

$$\frac{\dot{u}^2}{(\vec{\nabla}u)^2}\equiv\frac{1}{3}+Fe^{i\,\Phi}\tag{19}$$

with $F>0$ and $0\le\Phi\le 2\pi$, being functions of the space-time coordinates. The most of terms in (18) make positive contribution while the second term has some possibility to be negative. Note also, for static configurations apparently it is positive definite for the range of parameters given in (2).

## 3. The integrable and the non-integable sectors: numerical analysis by the SOR

Although in the previous section we used the polar coordinates, for the numerical study it is more convenient to use a new radial coordinate $y$ ($0\le y\le 1$), defined by $\rho=\sqrt{\frac{1-y}{y}}$. We introduce the solution ansatz of the form

$$u(\rho,\varphi,z,\tau)=\sqrt{\frac{1-g(y)}{g(y)}}e^{i[\epsilon n\varphi+k(z+\tau)]}\tag{20}$$

where the profile function $g(y)$ is defined at the period $0\le g\le 1$. The equation can be written as

$$\begin{aligned}&\frac{d}{dy}\left[\frac{y(1-y)}{g(1-g)}g'R\right]+\left(g-\frac{1}{2}\right)\frac{S}{y(1-y)}\left\{\Omega-\left(\frac{y(1-y)}{g(1-g)}g'\right)^2\right\}\\ &\qquad=-\frac{1}{y^2}\frac{r_0^2\mu^2}{M^2}(1-g)^{1-\frac{2}{n}}g^{1+\frac{2}{n}}\left\{4g-2\left(1+\frac{1}{n}\right)\right\}\end{aligned}\tag{21}$$

where the primes at this time indicate derivatives w.r.t.$y$ and where

$$\begin{aligned}\Omega&=n^2\\ S&=1+\beta e^2 g(1-g)\tfrac{y}{1-y}\left\{\Omega+\left(\tfrac{y(1-y)}{g(1-g)}g'\right)^2\right\}\\ R&=1+g(1-g)\tfrac{y}{1-y}\left\{(\beta e^2-2)\Omega+\beta e^2\left(\tfrac{y(1-y)}{g(1-g)}g'\right)^2\right\}\end{aligned}\tag{22}$$

The energy in the unit of $4M^2$ per unit length for the time-dependent vortex can be estimated in terms of following four parts of integrals of the dimensionless Hamiltonian $H:=\mathcal{H}/4M^2$

$$E = 2\pi\int_0^\infty\rho d\rho H(\rho)=E_2+E_4^{(1)}+(\beta e^2-1)E_4^{(2)}+\frac{r_0^2\mu^2}{M^2}E_0\tag{23}$$

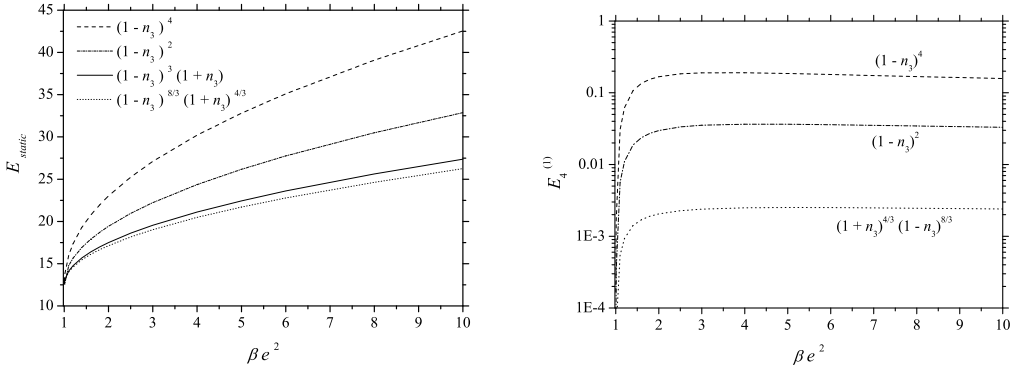



**Figure 3.** The $n = 2$ static energy and the component $E_4^{(1)}$ for several type of potentials $v_j^i$. $k^2 = 0$ and $r_0^2\mu^2/M^2 = 1.0$.

in which the components are defined as

$$E_2 = \pi \int_0^1 \frac{dy}{y(1-y)} \left\{ 2k^2 \frac{1-y}{y} + n^2 + \left( \frac{y(1-y)}{g(1-g)} g' \right)^2 \right\} g(1-g) \tag{24}$$

$$E_4^{(1)} = \pi \int_0^1 \frac{dy}{2(1-y)^2} \left\{ 4k^2 \frac{1-y}{y} + n^2 - \left( \frac{y(1-y)}{g(1-g)} g' \right)^2 \right\} \times \left\{ n^2 - \left( \frac{y(1-y)}{g(1-g)} g' \right)^2 \right\} (g(1-g))^2 \tag{25}$$

$$E_4^{(2)} = \pi \int_0^1 \frac{dy}{2(1-y)^2} \left\{ 4k^2 \frac{1-y}{y} + n^2 + \left( \frac{y(1-y)}{g(1-g)} g' \right)^2 \right\} \times \left\{ n^2 + \left( \frac{y(1-y)}{g(1-g)} g' \right)^2 \right\} (g(1-g))^2 \tag{26}$$

$$E_0 = 2\pi \int_0^1 \frac{dy}{y^2} g^{2+\frac{2}{n}} (1-g)^{2-\frac{2}{n}}. \tag{27}$$

It is easy to see that the $k^2$ term in (26) has positive contributions to the energy while in (25) the sign of the $k^2$ term depends on the spatial structure of the solution $g$. For the holomorphic solution (12), the energy (25) becomes zero and then the energy of the integrable sector keeps positive definite for all values of $k^2$.

For $n = 2$, the explicit form the potential is

$$V_{n=2} = \frac{\mu^2}{2}(1+n_3)(1-n_3)^3. \tag{28}$$

The potential has zero at both the origin and the infinity thus it is so called a *new*-BS (Baby-Skyrmion) type. The equation (21),(22) are solved in terms of the standard successive over relaxation scheme. Fig.1 is the profile function and the Hamiltonian density for $n = 2$. Fig.2 is the energy per unit length and its components for several values of $\beta e^2$ and fixed $\mu^2$. We confirmed that the value of the component $E_1^{(4)}$ is regarded as zero within the numerical uncertainty. This clearly indicates that the solution satisfies the condition (9).

Although we have obtained the analytical solutions for a special form of the potential (3), we have many options for choice of the potential. We can obtain many numerical solutions for

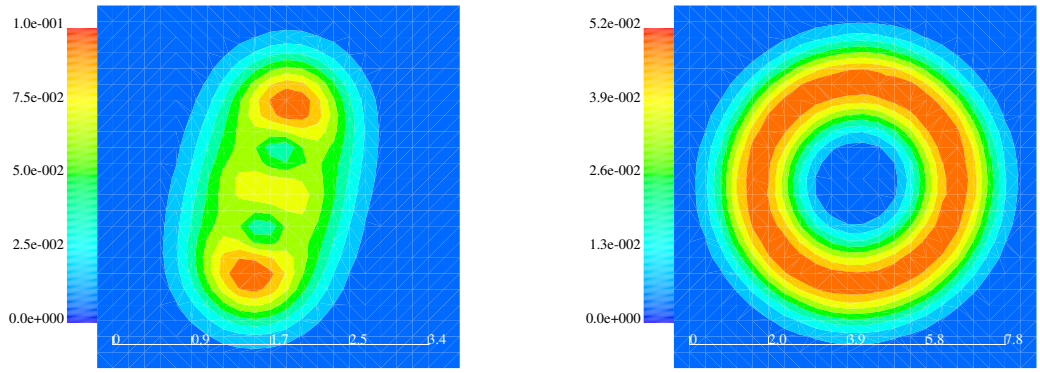
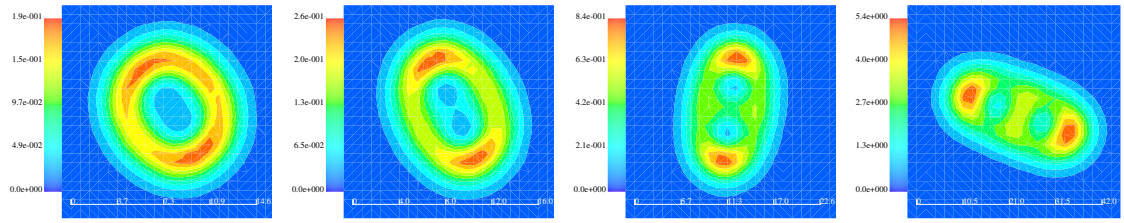



**Figure 4.** The energy density per unit length of the *old*-BS potential $v_1^0$ (*left*) and the *new*-BS potential $v_1^1$ (*right*).



$\beta e^2 = 1.01$ $\beta e^2 = 1.1$ $\beta e^2 = 2.0$ $\beta e^2 = 20.0$

**Figure 5.** The energy density per unit length of the *old*-BS potential $v_1^0$ for several $\beta e^2$.

the several types of the potentials. We show the result of $n = 2$ for the potentials $v_2^0, v_4^0, v_{8/3}^{4/3}$; of course these are not of the form of the analytical solution. Note that for such non-integrable solutions, the second term in (18) survives and has a possibility to be negative. Here we only consider the case for $k^2 = 0$. Fig.3 presents the energies and the component $E_4^{(1)}$ for these potentials. For $n = 2$, the *old*-BS potentials give higher total energy than the *new*-BS. This indicates that the same class of potentials gives the similar energy and then, for $n = 2$ the energy of the *new* type potential $v_{8/3}^{4/3}$ is closest to the integrable sector, which is also plotted in Fig.3 for reference.

## 4. Broken axisymmetric solutions: analysis by the SA

In the previous sections, we have assumed that the solution is invariant under the $O(2)$ internal symmetry $u \to e^{i\phi}u$ as well as the transformations of the Poincaré group given by rotation on the plane $x^1x^2$ and translations in the directions $x^0$ and $x^3$. However, there is a possibility of relaxing some of these symmetries. In [17], the authors generalized the *old*-BS type potential as $V = \frac{\mu^2}{2}(1 - n_3)^s$ $(0 < s \leq 4)$ and saw how the solution deforms from the rotational symmetry on the $x^1x^2$ plane by the change of $\mu$ or $s$.

For the deformed solution, the straightforward generalization of the ansatz (20) is

$$u = f(\rho, \varphi)e^{i\Theta(\rho,\varphi)}e^{ik(z+\tau)} \tag{29}$$

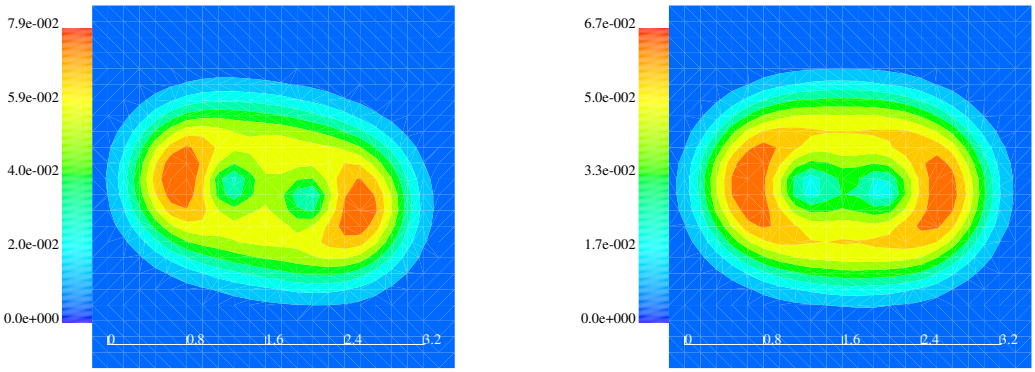



**Figure 6.** The energy density per unit length of $V_\epsilon = \frac{\mu^2}{2}(1-n_3)(1+\epsilon n_3)$ for $\epsilon = 0.25$ (*left*) and $\epsilon = 0.5$ (*right*). The results for $\epsilon = 0, 0, 1.0$ are already presented in Fig.4.

or equivalently,

$$\vec{n} = (\sin F(\rho,\varphi)\cos[\Theta(\rho,\varphi)+k(z+\tau)], \sin F(\rho,\varphi)\sin[\Theta(\rho,\varphi)+k(z+\tau)], \cos F(\rho,\varphi)) \quad (30)$$

where the $\vec{n}$ field in terms of $u$ is given in (4). Here $\Theta(\rho,\varphi) := n\varphi + \Theta_0(\rho,\varphi)$ and $\Theta_0$ is homotopic to the constant map. The method is a kind of the Monte-Carlo method in which one generates random numbers and properly change the value of the fields $\vec{n}$ by the numbers so as to drop the energy. However, a more sophisticated method may be applied to the problem. The simulated annealing method [18] is the application of the Metropolis algorithm which can successfully avoids the unwanted saddle points. The analysis has been done by minimizing the energy per unit length which is obtained by integrating the hamiltonian corresponding to (1) with the ansatz (30) into the $(\rho,\varphi)$ plane. We have confirmed that for the holomorphic solution (12) the hamiltonian is positive definite. For the general solution, however, the energy positivity is supported only for a certain range of $k$. Here we present the results for $k = 0$. Also, we examine the case of $n = 3$ because the solution exhibits obvious symmetry breaking such as from axial to $\mathbb{Z}_2$-symmetry.

We study the following several cases:

(i) We start with two standard cases: the *old*-BS potential $v_1^0$ and the *new*-BS potential $v_1^1$. In Fig.4, we present the energy density per unit length. For the *old*-BS potential, the solution strongly deforms from the axial symmetry while for the *new*-BS the solution keeps the symmetry.

(ii) Next we shall see how the solution behaves for the change of the model parameters. In Fig.5, we plot the energy density per unit length of the *old*-BS potential for several values of the model parameter $\beta e^2$. As is easily observed that for larger value of $\beta e^2$, the deformation is enhanced.

(iii) From the results of (i),(ii), we get a new insight for the symmetry breaking. That is, most crucial thing for the deformation is that the potential is finite or not at the point antipodal to the vacuum. In order to investigate this criterion further, we introduce following two types of one-parameter family of potentials

$$V_\epsilon(n_3) := \frac{\mu^2}{2}(1-n_3)(1+\epsilon n_3), \quad 0 \le \epsilon \le 1 \quad (31)$$

$$V_\delta(n_3) := \frac{\mu^2}{2}(1-n_3)(1+n_3)^\delta, \quad 0 \le \delta \le 1 \quad (32)$$

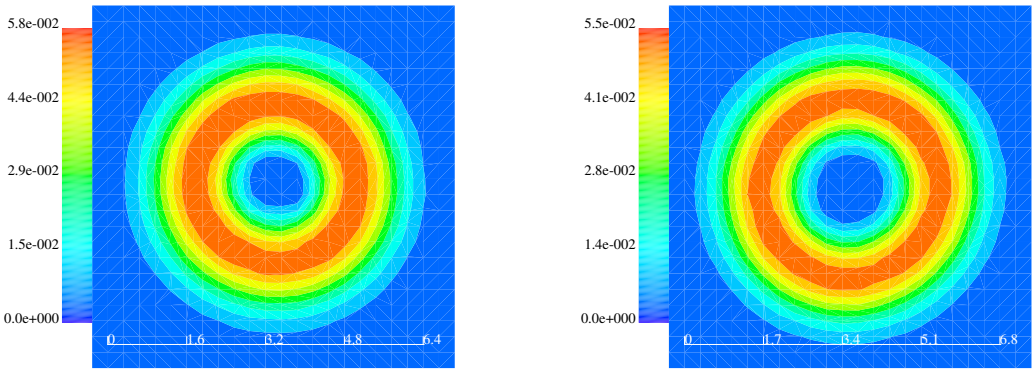



**Figure 7.** The energy density per unit length of $V_\delta = \frac{\mu^2}{2}(1-n_3)(1+n_3)^\delta$ for $\delta = 0.25$ (*left*) and $\delta = 0.5$ (*right*). The results for $\delta = 0.0, 1.0$ are already presented in Fig.4.

Note that for $\epsilon, \delta = 0$ the potentials become the *old*-BS and $\epsilon, \delta = 1$ they are the *new*-BS. The results are stimulating. In the case of $V_\epsilon$, the solutions always break the axial symmetry except only for $\epsilon = 1.0$ (Fig.6). On the other hand, in the $V_\delta$, the solutions always keep the axial symmetry except only for $\delta = 0.0$ (Fig.7). Thus we confirmed the criterion for the mechanism of the symmetry breaking: if a potential has the vacuum value at the point antipodal to the true vacuum, the solution always exhibits the axial symmetry, and if there is no another zero except for the vacuum, the solution deforms.

We summarize our results: for $n = 1$, the potential for the integrable sector is the *old*-BS type and the solution naturally has the axial symmetry. For $n \geq 2$ the potential for the integrable sector is always the *new*-BS type, and in terms of the above simulational study, the solutions always should be axially symmetric. As a result, our prescription of the ansatz (20) is valid for all topological charges.

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