ON A CONJECTURE OF ZASSENHAUS IN AN ALTERNATIVE SETTING

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ABSTRACT. H. J. Zassenhaus has suggested that every finite subgroup of normalized units in the integral group ring of a finite group G is conjugate to a subgroup of G. While this conjecture has not been settled for group rings, a reformulation within the context of alternative loop rings has recently been established and is discussed here.

1. Introduction. Let $\mathbf{Z}G$ denote the group ring of a finite group G over the integers. What role does G play in determining the torsion units of $\mathbf{Z}G$? H. J. Zassenhaus has conjectured that every normalized unit in $\mathbf{Z}G$ is a conjugate of an element of G via a unit in the rational group algebra $\mathbf{Q}G$. Several years ago the authors established a variation of this result for alternative loop rings. Specifically, we proved

Theorem 1.1. [5, 3] *Let r be a normalized torsion unit in the integral alternative loop ring $\mathbf{Z}L$ of a finite loop L which is not a group. Then there exist units $\gamma_1, \gamma_2 \in \mathbf{Q}L$ and $\ell \in L$ such that $\gamma_2^{-1}(\gamma_1^{-1} r \gamma_1)\gamma_2 = \ell$.*

A second, far stronger conjecture of Zassenhaus says that every finite subgroup of normalized units of $\mathbf{Z}G$ is conjugate to a subgroup of G, via a unit of $\mathbf{Q}G$. Recently, the authors have shown that here too, with some modification, the conjecture is true for alternative loop rings. We have established, and it is our intent to discuss here, the following theorem.

Theorem 1.2. *If H is a finite subloop of normalized units in an alternative loop ring $\mathbf{Z}L$ which is not associative, then H is isomorphic to a subloop of L. Moreover, there exist units $\gamma_1, \gamma_2, \ldots, \gamma_k$ of $\mathbf{Q}L$ such that*

$$\gamma_k^{-1}(\ldots(\gamma_2^{-1}(\gamma_1^{-1} H \gamma_1)\gamma_2)\ldots)\gamma_k \subseteq L. \tag{1}$$

1991 *Mathematics Subject Classification.* Primary 17D05; Secondary 16S34, 20N05.

This research was supported by the Instituto de Matemática e Estatística, Universidade de São Paulo, by FAPESP and CNPq. of Brasil (Proc. No. 93/4440-0 and 501253/91-2, respectively) and by the Natural Sciences and Engineering Research Council of Canada, Grant No. 0GP0009087.

Corollary 1.3. *If H is a finite subloop of normalized units in $\mathbf{Z}L$, then $|H| \le |L|$. If $|H| = |L|$, then H is isomorphic to L and there exist units $\gamma_1, \dots, \gamma_k \in \mathbf{Q}L$ such that $L = \gamma_k^{-1}(\dots(\gamma_2^{-1}(\gamma_1^{-1}H\gamma_1)\gamma_2)\dots)\gamma_k$.*

2. Definitions and Terminology. An *alternative* ring is one which satisfies the left and right alternative laws

$$x(xy) = x^2y \qquad \text{and} \qquad (yx)x = yx^2.$$

Such rings are nearly associative; for example, the subring generated by any two elements is always associative and also, if three elements of an alternative ring associate in some order, then they too generate an associative subring. Composition algebras provide examples of alternative rings, and examples of particular relevance to this work. A *composition algebra* is an algebra A with 1 over a field F on which there is a multiplicative nondegenerate quadratic form $q\colon A \to F$. By *multiplicative*, we mean that $q(xy) = q(x)q(y)$ for all $x, y \in A$. Since alternative rings satisfy the three *Moufang identities*

$$((xy)x)z = x(y(xz)), \qquad ((xy)z)y = x(y(zy)) \quad \text{and} \quad (xy)(zx) = (x(yz))x,$$

the loop of units in an alternative ring is necessarily a *Moufang loop* which is, by definition, a loop in which any of these (equivalent) identities is valid. In particular, if RL is an alternative loop ring for some commutative and associative ring R and loop L, then L is a Moufang loop (of a very special nature). We refer the reader to the book by Zhevlakov, Slin'ko and Shestakov [8] for a treatment of alternative rings and composition algebras, to Pflugfelder's text [7] for an introduction to Moufang loops, and to two papers in the literature [2, 1] for the basic properties of loops which have alternative loop rings.

In an alternative loop ring RL, the map $\epsilon\colon RL \to R$ which sends $\sum \alpha_\ell \ell$ to $\sum \alpha_\ell$ is a homomorphism. Thus $\epsilon(\gamma^{-1}\ell\gamma) = \epsilon(\ell) = 1$ for any $\ell \in L$ and unit γ. So, if we hope to establish (1), it is clear that we must restrict ourselves to subloops H whose elements are *normalized* in the sense that $\epsilon(h) = 1$ for all $h \in H$.

3. Sketch of the Proof of Theorem 1.2. Let H be a finite subloop of normalized units in an alternative loop ring $\mathbf{Z}L$. In a proof of the isomorphism theorem for alternative loop rings over $\mathbf{Z}$ [4], the authors constructed a map ρ from the normalized torsion units of $\mathbf{Z}L$ to L. Restricted to H, this map turns out to be a one-to-one homomorphism, thus H is isomorphic to a subloop of L. What is more difficult is to establish an equation of the form (1). For this, we show that there exist units γ_i in the rational loop algebra $\mathbf{Q}L$ such that

$$\gamma_k^{-1}(\ldots(\gamma_2^{-1}(\gamma_1^{-1}\alpha\gamma_1)\gamma_2)\ldots)\gamma_k = \rho(\alpha). \tag{2}$$

for all $\alpha \in H$. We consider the three possibilities: (i) H is an abelian group, (ii) H is a nonabelian group, and (iii) H is a Moufang loop which is not associative.

If H is an abelian group, we show that H is generated by a single element and elements in the centre of $\mathbf{Z}L$ and then appeal to Theorem 1.1. Cases (ii) and (iii) are more delicate. Here we use the fact that the alternative loop algebra $\mathbf{Q}L$ is the direct sum of simple alternative algebras and observe that it is sufficient to establish (2) in each simple component. Since the associative components are fields, as we are able to show, and since the ℓ of Theorem 1.1 is, in fact, $\rho(r)$, each element of H and its image under ρ have equal images in the associative components. Thus we need consider only those components which are not associative.

Now an alternative loop algebra has an involution (anti-automorphism of period 2) $\alpha \mapsto \alpha^*$ such that $\alpha\alpha^*$ is central for any α. From this, one can show that $n(\alpha) = \alpha\alpha^*$ induces a multiplicative quadratic form on each simple component of $\mathbf{Q}L$. This form turns out to be non-degenerate, so each simple component is a composition algebra. Let A be such a component and $\pi\colon \mathbf{Q}L \to A$ be the natural projection. Let $L_0 = \rho(H)$.

If H is not associative, we show that each of $\pi(H)$ and $\pi(L_0)$ generate A and that ρ induces an automorphism of A. By a theorem of Jacobson [6], any automorphism of a composition algebra is the product of *reflections* (maps of the form $x \mapsto \gamma^{-1}x\gamma$), so we have the result in case (iii).

The most difficult case is that in which H is a nonabelian group. Here we let x and y be noncommuting elements of L_0 and let B be the subalgebra of $\mathbf{Q}L$ generated by x and y. Similarly, we consider the subalgebra $\hat{B}$ of $\mathbf{Q}L$ generated by $\hat{x}$ and $\hat{y}$, the preimages under ρ of x and y respectively. It can be shown that $\rho\colon H \to L_0$ extends to a ring isomorphism $\hat{B} \to B$ which induces a ring isomorphism $\pi(\hat{B}) \to \pi(B)$. We show that this map extends to an automorphism of the composition algebra A which, by the theorem of Jacobson already mentioned, is the product of reflections. Thus the proof is complete.

REFERENCES

1. Orin Chein and Edgar G. Goodaire, *Loops whose loop rings are alternative*, Comm. Algebra 14 (1986), no. 2, 293–310.
2. Edgar G. Goodaire, *Alternative loop rings*, Publ. Math. Debrecen 30 (1983), 31–38.
3. Edgar G. Goodaire and César Polcino Milies, *Isomorphisms and units in alternative loop rings*, C. R. Math. Rep. Acad. Sci. Canada IX (1987), no. 5, 259–263.
4. ———, *Isomorphisms of integral alternative loop rings*, Rend. Circ. Mat. Palermo XXXVII (1988), 126–135.
5. ———, *Torsion units in alternative loop rings*, Proc. Amer. Math. Soc. 107 (1989), 7–15.
6. N. Jacobson, *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo VII (1958), 55–80.
7. H. O. Pflugfelder, *Quasigroups and loops: Introduction*, Heldermann Verlag, Berlin, 1990.
8. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings that are nearly associative*, Academic Press, New York, 1982, Translated by Harry F. Smith.

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Received March 29, 1994