Multiple solutions for an elliptic system near resonance

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Abstract

We consider an elliptic system with linear part depending on two parameters and a sublinear perturbation. We obtain the existence of at least two solutions when the linear part is near resonance. The system is associated to a strongly indefinite functional and the solutions are obtained through saddle point theorem and Galerkin approximation.

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1 Introduction

In this paper we consider the following system of elliptic equations:

$$\begin{cases} -\Delta u = au + bv \pm (f_1(x,v) + h_1(x)) & \text{in } \Omega, \\ -\Delta v = bu + av \pm (f_2(x,u) + h_2(x)) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $h_1, h_2 \in L^2(\Omega)$ and f_1, f_2 are sublinear nonlinearities, in particular, they satisfy

$$\begin{array}{r} f_i : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function and there exist constants } S > 0 \\ \text{and } q \in (1,2), \text{ such that } |f_i(x,t)| \leq S(1 + |t|^{q-1}), \text{ for } i = 1,2. \end{array} \tag{1.2}$$

We will also assume one of the following two sets of hypotheses on the functions $f_{1,2}$ and $h_{1,2}$:

$$\lim_{t \to \pm\infty} f_i(x,t) = \pm\infty, \text{ uniformly with respect to } x \in \Omega,\ i = 1,2, \tag{$\mathbf{f}$}$$

or

$$\begin{cases} (i) & \lim_{|t|\to\infty} F_i(x,t) = +\infty, \text{ uniformly with respect to } x \in \Omega,\ i = 1,2, \\ (ii) & \int_\Omega h_1\phi + h_2\psi = 0, \quad \text{for every } (\phi,\psi) \in Z, \end{cases} \tag{$\mathbf{F}$}$$

where $F_i(x,t) = \int_0^t f_i(x,s)ds$ and Z is a space which will be defined in the statements of the theorems.

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We will refer to problem (1.1) as (1.1+) or (1.1−), depending on the sign preceding the two nonlinearities. Our purpose is to obtain a multiplicity result when the linear part of (1.1) is near resonance (this situation is often called "almost-resonance" in literature).

Throughout the paper we denote by $\sigma(-\Delta)$ the spectrum of the Laplacian in $H_0^1(\Omega)$, that is the set of the eigenvalues λ_k where $0<\lambda_1<\lambda_2\leq\lambda_3\leq ... \leq\lambda_k\leq ...$, and by ϕ_k $(k=1,2,..)$ the corresponding eigenfunctions, which will be taken orthogonal and normalized with $\|\phi_k\|_{H_0^1}=1$. Finally, we denote by H_{λ_i} the eigenspace corresponding to the eigenvalue λ_i.

The main motivation for this paper is [dPM08], where a similar result was proven in the scalar case, in particular, it was considered the problem

$$\begin{cases} -\Delta u=\lambda u\pm f(x,u)+h(x) & \text{in} \quad \Omega\,, \\ u=0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.3}$$

(again, we will refer to this problem as (1.3+) or (1.3−), based on the sign before the nonlinearity f). The main result stated that given an eigenvalue λ_k and $h\in L^2(\Omega)$, if f satisfies conditions analogous to (1.2) and (**f**) (or (**F**)), then

a) there exists $\varepsilon_0>0$, such that for $\lambda\in(\lambda_k-\varepsilon_0,\lambda_k)$ there exist two solutions of (1.3+);

b) there exists $\varepsilon_1>0$, such that for $\lambda\in(\lambda_k,\lambda_k+\varepsilon_1)$ there exist two solutions of (1.3−).

As remarked in [dPM08], the possibility to find two solutions is suggested by the following observation: for $\lambda\notin\sigma(-\Delta)$ there always exists a solution of problem (1.3), which (for $\lambda>\lambda_1$) may be obtained through the saddle point theorem, however, in the saddle geometry, the dimension of the negative space is different if we are above or below the eigenvalue λ_k.
This means that the geometry of the quadratic part of the functional changes when λ passes from below to above an eigenvalue λ_k, and then it is possible to have both saddle point geometries at the same time, if λ is near to λ_k and the perturbation f makes the functional change sign in the eigenspace H_{λ_k}. In order to achieve this, the term $\pm f$ needs to have the appropriate sing (different if we are above or below to λ_k) and to be "large enough", in the sense of the hypotheses (**f**) or (**F**).

In the case of system (1.1), the non-resonance condition reads $a\pm b\notin\sigma(-\Delta)$. Our aim is to guarantee the existence of at least two solutions when $a+b$ or $a-b$ are near to an eigenvalue of the Laplacian. We will first obtain a result in the case where one of the values $a\pm b$ is near resonance, while the other one is at a certain distance from the spectrum $\sigma(-\Delta)$ (see the Theorems 1.1-1.2). Then we will consider the case in which both values are near resonance (see the Theorems 1.3-1.4).

The following theorem guarantees the existence of two distinct solutions when $a+b$ is almost-resonant, while $a-b$ is far from the spectrum $\sigma(-\Delta)$.

Theorem 1.1. *Let λ_k be an eigenvalue of $(-\Delta)$, λ_l the first eigenvalue above $a-b$ and*

$$Z=span\,\{\,(\phi,\phi):\ \phi\in H_{\lambda_k}\}\,.$$

*Suppose $h_1,h_2\in L^2(\Omega)$, f_1,f_2 satisfy hypotheses (1.2) and (**f**) (or (**F**)). Then*

(a) given $\delta>0$, there exists $\varepsilon_0>0$ such that, if $dist(a-b,\sigma(-\Delta))>\delta$ and $a+b\in(\lambda_k-\varepsilon_0,\lambda_k)$, then problem (1.1+) has two distinct solutions,

(b) *given* $\delta > 0$, *there exists* $\varepsilon_1 > 0$ *such that, if* $dist(a-b, \sigma(-\Delta)) > \delta$ *and* $a+b \in (\lambda_k, \lambda_k+\varepsilon_1)$, *then problem (1.1−) has two distinct solutions.*

As a consequence of Theorem 1.1, it is simple to prove the corresponding result in the case where $a - b$ is almost-resonant, while $a + b$ is far from $\sigma(-\Delta)$.

Theorem 1.2. *Let* λ_k *be an eigenvalue of* $(-\Delta)$, λ_l *the first eigenvalue above* $a + b$ *and*

$$Z = span\,\{\,(\phi, -\phi) : \ \phi \in H_{\lambda_k}\}\,.$$

Suppose $h_1, h_2 \in L^2(\Omega)$, f_1, f_2 *satisfy hypotheses (1.2) and* (***f***) *(or* (***F***)*). Then*

(c) *given* $\delta > 0$, *there exists* $\varepsilon_0 > 0$ *such that, if* $dist(a+b, \sigma(-\Delta)) > \delta$ *and* $a-b \in (\lambda_k-\varepsilon_0, \lambda_k)$, *then problem (1.1−) has two distinct solutions,*

(d) *given* $\delta > 0$, *there exists* $\varepsilon_1 > 0$ *such that, if* $dist(a+b, \sigma(-\Delta)) > \delta$ *and* $a-b \in (\lambda_k, \lambda_k+\varepsilon_1)$, *then problem (1.1+) has two distinct solutions.*

By comparing item (a) of Theorem 1.1 with item (d) of Theorem 1.2, we observe that problem (1.1+) has two solutions when $a + b$ is almost-resonant from below of the eigenvalue, and when $a - b$ is almost-resonant from above of the eigenvalue. In the following theorem we show the existence of two solutions also when both resonances happen at the same time.

Theorem 1.3. *Let* λ_k *and* λ_l *be two eigenvalues (not necessarily distinct) of* $(-\Delta)$, *and*

$$Z = span\,\{\,(\phi, \phi) : \ \phi \in H_{\lambda_k}, \quad (\phi, -\phi) : \ \phi \in H_{\lambda_l}\}\,.$$

Suppose $h_1, h_2 \in L^2(\Omega)$, f_1, f_2 *satisfy hypotheses (1.2) and* (***f***) *(or* (***F***)*). Then*

(e) *there exists* $\varepsilon_2 > 0$ *such that, if* $a - b \in (\lambda_l, \lambda_l + \varepsilon_2)$ *and* $a + b \in (\lambda_k - \varepsilon_2, \lambda_k)$, *then problem (1.1+) has two distinct solutions.*

The case when problem (1.1−) has, at the same time, $a + b$ almost-resonant from above of the eigenvalue and $a - b$ almost-resonant from below of the eigenvalue, follows again easily from Theorem 1.3.

Theorem 1.4. *Let* λ_k *and* λ_l *be two eigenvalues (not necessarily distinct) of* $(-\Delta)$, *and*

$$Z = span\,\{\,(\phi, -\phi) : \ \phi \in H_{\lambda_k}, \quad (\phi, \phi) : \ \phi \in H_{\lambda_l}\}\,.$$

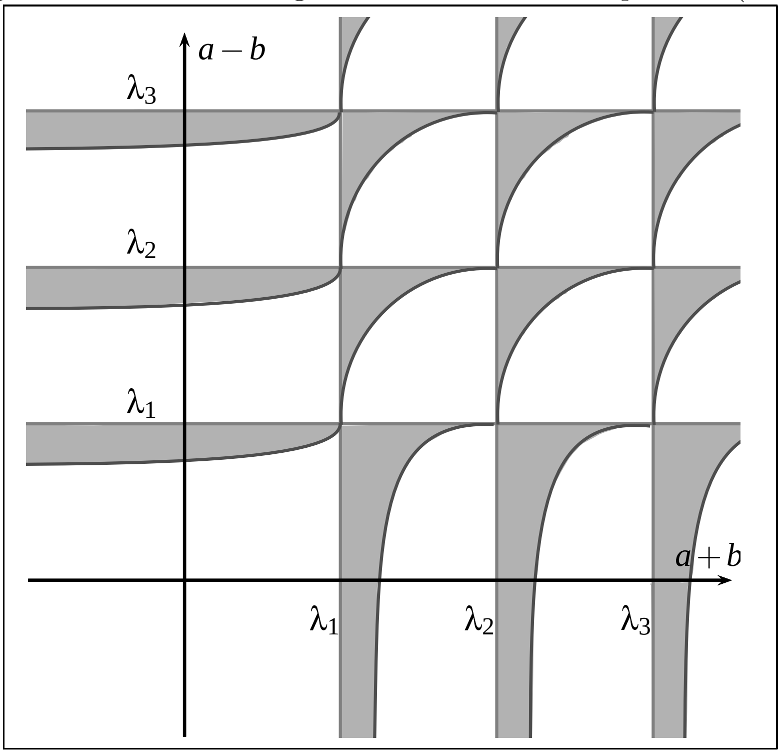
Suppose $h_1, h_2 \in L^2(\Omega)$, f_1, f_2 *satisfy hypotheses (1.2) and* (***f***) *(or* (***F***)*). Then*

(f) *there exists* $\varepsilon_2 > 0$ *such that, if* $a - b \in (\lambda_k - \varepsilon_2, \lambda_k)$ *and* $a + b \in (\lambda_l, \lambda_l + \varepsilon_2)$, *then problem (1.1−) has two distinct solutions.*

Joining the theorems from 1.1 to 1.4, it is possible to infer a qualitative sketch of the regions, in the plane $(a + b, a - b)$, where one can guarantee the existence of at least two solutions for problem (1.1−): see Figure 1. The sketch for problem (1.1+) would be analogous, with the axes switched.

The paper is structured as follows: in section 2 we discuss the techniques we use and the bibliography related to our problem, in section 3 we set the notation and we prove some preliminary lemmas, while in section 4 we will prove the main results.

Figure 1: Sketch of the regions of two solutions for problem (1.1−)



2 Techniques, bibliography and remarks

The result in [dPM08] was obtained by finding two saddle point geometries, once with a linking of order $k-1$, and another time with a linking of order $k+m-1$, where m is the multiplicity of the eigenvalue λ_k); then one proved that these solutions were distinct since they lay at different levels.

In the case of our system, we will always deal with a functional (see the definition in (3.1)) which is strongly indefinite, in the sense that there exist two infinite dimensional subspaces of its space of definition, such that the principal part of the functional is unbounded from above in one and from below in the other (see the estimates in Lemma 4.1). This implies that the standard linking theorems are no more available in order to find critical points. Some of the techniques used in approaching this kind of problems may be seen in [BR79, dFF94, HvdV93, dFdOR04, Mas06, Mas07]. Here, we will use an approximation technique (Galerkin procedure), namely, we will solve finite dimensional problems, where we can use the standard linking theorems, then take limit on the dimension of such problems and prove that the limit is actually the critical point we were looking for.

It is interesting to point out that, in [dPM08], the authors had to consider different kinds of saddle point geometries when the parameter λ in (1.3) was approaching the eigenvalue from below or from above. This was due to the fact that the situation was asymmetric in the sense that the resonant subspace was part of the (finite-dimensional) negative subspace of the quadratic part of the functional in one case, while it was in the (infinite-dimensional) positive one in the other case. Since in this paper we use the Galerkin technique in order to deal with the lack of compactness in both subspaces, there is no more this difference and then the saddle point geometries turn out to be the same in the two cases (compare section 4.1 with section 4.3).

Multiplicity results like those in [dPM08] were known for the first eigenvalue and were studied by many authors since the work of Mawhin and Schmitt [MS90], where the problem in dimension one is considered, using bifurcation from infinity and degree theory; we cite [BL89, LR90], which also consider the one dimensional case, and [CMN92, CdF93], which deal with the higher dimension problems; these works are all based on bifurcation theory. Results for higher eigenvalues were obtained in [LR90], again using bifurcation from infinity and degree theory, but only for the one dimensional case and making use of the fact that in this case all the eigenvalues are simple. In [RS97, MRS97], the same kind of problems were analyzed from a variational point of view and at least three solutions were found when approaching the first eigenvalue from below and from above, under conditions which are basically our set of hypotheses (**F**). The variational approach was later exploited in [MP02] to obtain a similar result for the p-Laplacian operator (see also [DNM01]).

After [dPM08], some improvements were obtained in [KT11b], in the sense of considering a somewhat weaker condition on the nonlinearity f and the forcing term h: instead of the condition $\int_\Omega hu = 0$ (which corresponds to (**F**-*ii*) here), a Landesman-Lazer type condition is imposed, which in some cases allows the functions h to have a bounded (instead of necessarily zero) component in the almost-resonant eigenspace.

Regarding almost-resonant problems in the case of systems, we cite [ST10, OT09, AS12, KT11a]. The first three consider variational systems in gradient form, which means that the principal part of the functional associated to the problem has a finite dimensional negative space, so that many techniques used in the scalar case can be used directly in this case.
On the other hand, in [KT11a] the authors consider a system which is different from problem (1.1) but, like in our case, the associated functional is strongly indefinite. They obtain two solutions near resonance under a condition similar to the one they used in [KT11b], by using generalized linking theorems for the case of strongly indefinite functionals (see [Rab86, MMP94]). We remark that they consider a single variable parameter μ which takes the system near resonance, while here we study the behavior of our system (1.1) with respect to both parameters a, b (see also Remark 2.2, where we discuss why we have only two parameters in system (1.1)). It is also worth saying that in their approach they still need two different proofs for the case above and below the eigenvalue, like in [dPM08], which we could avoid in this paper, as explained above.

We end this section with some remarks on our hypotheses and some model functions.

Remark 2.1. Hypothesis (**f**) is stronger than (**F**-*i*), but it does not require to impose the additional "nonresonance" condition on $h_{1,2}$ contained in (**F**-*ii*). For this reason we consider both sets of hypotheses. Observe that our hypotheses only deal with the asymptotical behavior of f: no condition in the origin is required for our multiplicity result.
Model functions for hypothesis (**f**) could be

$$f_i(x,u) = |u|^{q-2}u\,, \qquad \text{or} \qquad f_i(x,u) = \ln(1+|u|)\arctan(u)\,.$$

For the case (**F**), one could consider, for example,

$$f_i(x,u) = \arctan(u) \qquad \text{or} \qquad f_i(x,u) = \begin{cases} u & \text{if } |u| \le 1\,, \\ \frac{1}{u} & \text{if } |u| > 1\,. \end{cases}$$

Observe that the last example would fit in the hypotheses used in [KT11b, KT11a], but not even a null forcing term would satisfy their Landesman-Lazer type condition, while it satisfies our ($\mathbf{F}$-ii).

Remark 2.2. We observe that one could think of a more general system than (1.1), having four different coefficient for the linear part:

$$\begin{cases} -\Delta u = au + cv \pm (f_1(x,v) + h_1(x)) & \text{in } \Omega, \\ -\Delta v = du + ev \pm (f_2(x,u) + h_2(x)) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

However, since we aim to use variational methods, we are forced to take $a = e$ in (2.1). Moreover, we need the eigenvalues of the matrix of the coefficients to be real, and for this we impose $cd \geq 0$. If we also suppose $cd \neq 0$ then we may always transform (2.1) into (1.1) with $b = \sqrt{cd}$, by the change of unknowns $(U,V) = (\sqrt{d/c}\,u, v)$, which preserves the hypotheses ($\mathbf{f}$) and ($\mathbf{F}$) (see for example in [Mas07]).

As a consequence, if $a = e$ and $cd > 0$, it is no loss of generality, to consider the system (2.1) in the form (1.1).
The case $c = d = 0$ has no interest in this paper since it would never satisfy the hypotheses of our Theorems 1.1 to 1.4.

3 Notation and preliminary lemmas

Throughout the paper we will use the notation $H = H_0^1(\Omega)$, $E = H \times H$ and we will use the following norms: if $u \in H$ and $\mathbf{u} = (u,v) \in E$, then

$$\begin{cases} \|u\|_{L^2} = \sqrt{\int_\Omega u^2}\,, & \|\mathbf{u}\|_{[L^2]^2} = \sqrt{\|u\|_{L^2}^2 + \|v\|_{L^2}^2}\,, \\ \|u\|_H = \sqrt{\int_\Omega |\nabla u|^2}\,, & \|\mathbf{u}\|_E = \sqrt{\|u\|_H^2 + \|v\|_H^2}\,. \end{cases}$$

The internal products in $L^2(\Omega) \times L^2(\Omega)$ and in E, associated to the above norms, will be denoted by $\langle \cdot, \cdot \rangle_{[L^2]^2}$ and $\langle \cdot, \cdot \rangle_E$, respectively. Observe that, by Poincaré inequality, $\|u\|_{L^2} \leq S\,\|u\|_H$ for some positive constant S, but we will assume throughout the paper that $S = 1$ in order to simplify the estimates.

We will define the applications $\mathcal{F} : E \to \mathbb{R}$ and $\mathcal{H} : E \to \mathbb{R}$, given by

$$\mathcal{F}(u,v) = \int_\Omega F_1(x,v) + \int_\Omega F_2(x,u), \qquad \mathcal{H}(u,v) = \int_\Omega h_1 v + \int_\Omega h_2 u.$$

Moreover, we define the bilinear form $B : E \times E \to \mathbb{R}$:

$$B((u,v),(\phi,\psi)) = \int_\Omega \nabla u \nabla \psi + \int_\Omega \nabla v \nabla \phi - a \int_\Omega (u\psi + v\phi) - b \int_\Omega (u\phi + v\psi).$$

We will consider the functionals $J^\pm : E \to \mathbb{R}$, associated to the problems (1.1$\pm$), given by

$$J^\pm(u,v) = \pm \frac{1}{2} B((u,v),(u,v)) - \mathcal{F}(u,v) - \mathcal{H}(u,v)\,, \tag{3.1}$$

actually, one can prove in a standard way that $J^\pm$ are $\mathcal{C}^1$ functionals and their critical points are solutions for the problems (1.1±), respectively.
Denoting $\mathbf{u} = (u, v)$, $\boldsymbol{\phi} = (\phi, \psi)$ and $\overline{\boldsymbol{\phi}} = (\psi, \phi)$, we have that

$$B(\mathbf{u}, \boldsymbol{\phi}) = \langle \mathbf{u}, \overline{\boldsymbol{\phi}} \rangle_E - a \langle \mathbf{u}, \overline{\boldsymbol{\phi}} \rangle_{[L^2]^2} - b \langle \mathbf{u}, \boldsymbol{\phi} \rangle_{[L^2]^2} \,, \tag{3.2}$$

$$J^\pm(\mathbf{u}) = \pm 1/2 B(\mathbf{u}, \mathbf{u}) - (\mathcal{F}(\mathbf{u}) + \mathcal{H}(\mathbf{u})) \,, \tag{3.3}$$

$$(J^\pm)'(\mathbf{u})[\boldsymbol{\phi}] = \pm B(\mathbf{u}, \boldsymbol{\phi}) - \big(\mathcal{F}'(\mathbf{u})[\boldsymbol{\phi}] + \mathcal{H}'(\mathbf{u})[\boldsymbol{\phi}]\big) \,. \tag{3.4}$$

We collect in the following lemma several estimates that will be used in the rest of the paper. We will denote by C, C_1, $C_2, \ldots$ positive constants whose value is not important and which may be different from line to line.

Lemma 3.1. *Given f_1, f_2 satisfying (1.2) and $h_1, h_2 \in L^2(\Omega)$, there exist constants S_0 and H such that*

$$|\mathcal{H}(\mathbf{u})| \le H \, \|\mathbf{u}\|_{[L^2]^2} \,, \qquad |\mathcal{H}'(\mathbf{u})[\boldsymbol{\phi}]| \le H \, \|\boldsymbol{\phi}\|_{[L^2]^2} \,, \tag{3.5}$$

$$|\mathcal{F}(\mathbf{u})| \le S_0 \left(1 + \|\mathbf{u}\|^q_{[L^2]^2}\right) \,, \qquad \left|\mathcal{F}'(\mathbf{u})[\boldsymbol{\phi}]\right| \le S_0 \left(1 + \|\mathbf{u}\|^{q-1}_{[L^2]^2}\right) \|\boldsymbol{\phi}\|_{[L^2]^2} \,, \tag{3.6}$$

for every $\mathbf{u} = (u, v) \in E$ and $\boldsymbol{\phi} = (\phi, \psi) \in E$.

Proof. These estimates are rather standard. By Hölder inequality,

$$\left| \int_\Omega h_1 v \right| \le \|h_1\|_{L^2} \|v\|_{L^2} \quad \text{and} \quad \left| \int_\Omega h_2 u \right| \le \|h_2\|_{L^2} \|u\|_{L^2} \,. \tag{3.7}$$

Then for a suitable $H > 0$, depending on h_1 and h_2, we get

$$|\mathcal{H}(\mathbf{u})| \le \frac{H}{2} \|u\|_{L^2} + \frac{H}{2} \|v\|_{L^2} \le H \, \|\mathbf{u}\|_{[L^2]^2} \,. \tag{3.8}$$

The second estimate in (3.5) follows immediately, since $\mathcal{H}'(\mathbf{u})[\boldsymbol{\phi}] = \mathcal{H}(\boldsymbol{\phi})$.

Now, by (1.2), we have

$$\left| \int_\Omega F_1(x, v) \right| \le \int_\Omega C \, (1 + |v|^q) = C \left(|\Omega| + \|v\|^q_{L^q} \right) \le C_1 \left(1 + \|v\|^q_{L^2} \right) \tag{3.9}$$

and an analogous estimate holds for $\int_\Omega F_2(x, u)$. Then

$$|\mathcal{F}(\mathbf{u})| \le C_1 (1 + \|v\|^q_{L^2}) + C_1 (1 + \|u\|^q_{L^2}) \le 2 C_1 (1 + \|\mathbf{u}\|^q_{[L^2]^2}), \tag{3.10}$$

where we used that $\|u\|^r_{L^2} + \|v\|^r_{L^2} \le 2 \max\left(\|u\|^r_{L^2}, \|v\|^r_{L^2}\right) \le 2 \, \|\mathbf{u}\|^r_{[L^2]^2}$. On the other hand,

$$\left|\mathcal{F}'(\mathbf{u})[\boldsymbol{\phi}]\right| = \left| \int_\Omega f_1(x, v) \psi + f_2(x, u) \phi \right| \le \left| \int_\Omega f_1(x, v) \psi \right| + \left| \int_\Omega f_2(x, u) \phi \right| \,,$$

where $\left|\int_\Omega f_1(x, v)\psi\right| \le \int_\Omega S(1 + |v|^{q-1})|\psi|$. Using Hölder inequality with $r = \frac{2}{3-q} < 2$, we get

$$\left| \int_\Omega f_1(x, v) \psi \right| \le C \left(\|\psi\|_{L^1} + \left(\int_\Omega |v|^{q-1 \frac{2}{q-1}} \right)^{\frac{q-1}{2}} \|\psi\|_{L^r} \right) \le C_1 \left(\|\psi\|_{L^2} + \|v\|^{q-1}_{L^2} \|\psi\|_{L^2} \right); \tag{3.11}$$

again, an analogous estimate holds for $\left|\int_\Omega f_2(x,u)\phi\right|$ and then

$$\left|\mathcal{F}'(\mathbf{u})[\boldsymbol{\phi}]\right| \le C_1(1+\|v\|_{L^2}^{q-1})\,\|\psi\|_{L^2} + C_1(1+\|u\|_{L^2}^{q-1})\,\|\phi\|_{L^2} \le C_2\left(1+\|\mathbf{u}\|_{[L^2]^2}^{q-1}\right)\|\boldsymbol{\phi}\|_{[L^2]^2}\,.$$

□

In order to study the functional J^+, we define, as in [Mas07], an orthogonal basis for E which diagonalizes B: let us consider the eigenvalue problem to find $\mu \in \mathbb{R}$ and $(u,v) \in E$ such that

$$B((u,v),(\phi,\psi)) = \mu\langle (u,v),(\phi,\psi)\rangle_E,\ \ \forall(\phi,\psi)\in E. \tag{3.12}$$

We write $u = \sum_{j\in\mathbb{N}} c_j\phi_j$ and $v = \sum_{j\in\mathbb{N}} d_j\phi_j$. By using $(\phi,\psi) = (0,\phi_i)$ and $(\phi,\psi) = (\phi_i,0)$ in (3.12), since the eigenfunctions ϕ_i are mutually orthogonal, we get that (3.12) is equivalent to

$$\begin{pmatrix} a-\lambda_i & \mu\lambda_i+b \\ \mu\lambda_i+b & a-\lambda_i \end{pmatrix}\begin{pmatrix} c_i \\ d_i \end{pmatrix} = 0,\ \ \forall i\in\mathbb{N}, \tag{3.13}$$

which has nontrivial solutions when μ is such that the determinant of the above matrix is zero for some $i \in \mathbb{N}$, that is, $(a-\lambda_i)^2 - (\mu\lambda_i+b)^2 = 0$.

In view of the above computations, we define the two sequences of eigenvalues

$$\mu_{\pm i} = \frac{-b \pm (\lambda_i - a)}{\lambda_i},\ \ i\in\mathbb{N}$$

and the corresponding eigenfunctions (normalized in E)

$$\boldsymbol{\psi}_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2}},\ \ i\in\mathbb{N}.$$

As a consequence we have

$$\|\boldsymbol{\psi}_i\|_E = 1,\ \ \langle \boldsymbol{\psi}_i,\boldsymbol{\psi}_j\rangle_E = \delta_{i,j},\ \ B(\boldsymbol{\psi}_i,\boldsymbol{\psi}_j) = \mu_i\delta_{i,j},\ \ \langle \boldsymbol{\psi}_i,\boldsymbol{\psi}_j\rangle_{[L^2]^2} = \lambda_{|i|}^{-1}\delta_{i,j}, \tag{3.14}$$

for $i,j \in \mathbb{Z}_0 := \mathbb{Z}\setminus\{0\}$. Moreover, if we write $\mathbf{u} = \sum_{i\in\mathbb{Z}^0} c_i\boldsymbol{\psi}_i$, then

$$\|\mathbf{u}\|_E^2 = \sum_{i\in\mathbb{Z}^0} c_i^2,\ \ B(\mathbf{u},\mathbf{u}) = \sum_{i\in\mathbb{Z}^0}\mu_i c_i^2,\ \ \|\mathbf{u}\|_{[L^2]^2}^2 = \sum_{i\in\mathbb{Z}^0}\lambda_{|i|}^{-1}c_i^2. \tag{3.15}$$

4 Proof of the main results

In this section we will give the proofs of the main theorems, in particular, we give in full details the proof of point (a) in Theorem 1.1. We find two saddle point geometries in section 4.1, from which we obtain (in section 4.2) two sequences of critical points of finite dimensional approximations of the functional J^+ and we show, in section 4.4, how to obtain two distinct solutions of problem (1.1) from these sequences. In section 4.3 we explain what has to be adapted when dealing with the case (b) of Theorem 1.1, while Theorem 1.2 is proved at the end of section 4.4. Finally, in section 4.5, we prove the Theorems 1.3 and 1.4. Section 4.6, is devoted to the proof of two technical lemmas (a kind of PS condition) used in the proofs of the main results.

4.1 Two saddle point geometries when $a+b$ is almost-resonant from below

In this section we build the saddle point geometries that will provide critical points for the functional J^+, when $a+b$ is sufficiently near to λ_k from below and $a-b$ is far from $\sigma(-\Delta)$.

Let us fix λ_k (of multiplicity $m \in \mathbb{N}$, in particular let $\lambda_k = \lambda_{k+m-1}$) and λ_l, the eigenvalues stated in Theorem 1.1. We define the subspaces of E

$$\begin{cases} V = \overline{span\{\boldsymbol{\psi}_i : i \in \mathbb{Z}_0,\ \mu_i < 0 \text{ and } \mu_i \neq \mu_k\}}, \\ Z = \overline{span\{\boldsymbol{\psi}_i : i \in \mathbb{Z}_0,\ \mu_i = \mu_k\}}, \\ W = \overline{span\{\boldsymbol{\psi}_i : i \in \mathbb{Z}_0,\ \mu_i > 0 \text{ and } \mu_i \neq \mu_k\}}, \end{cases} \tag{4.1}$$

and we denote by B_V, B_{VZ}, B_W and B_{ZW} the unitary closed balls, with respect to the norm $\|\cdot\|_E$, in the spaces V, $V \oplus Z$, W and $Z \oplus W$ respectively, and by S_V, S_{VZ}, S_W and S_{ZW} their relative boundaries. We also define the sets of indexes

$$\mathbb{Z}_V = \{i \in \mathbb{Z}_0 : \mu_i < 0 \text{ and } \mu_i \neq \mu_k\},$$

$$\mathbb{Z}_Z = \{i \in Z_0 : \mu_i = \mu_k\},$$

$$\mathbb{Z}_W = \{i \in \mathbb{Z}_0 : \mu_i > 0 \text{ and } \mu_i \neq \mu_k\}.$$

Observe that if $a+b \in (\lambda_k - \varepsilon, \lambda_k)$ for some $\varepsilon > 0$, then

$$0 < \mu_k = \frac{\lambda_k - (a+b)}{\lambda_k} < \frac{\varepsilon}{\lambda_k}. \tag{4.2}$$

We start by proving a lemma which provides estimates is the subspaces V, Z and W.

Lemma 4.1. *Suppose $a \pm b \notin \sigma(-\Delta)$ and fix λ_k being the first eigenvalue above $a+b$, of multiplicity m, and λ_l the first eigenvalue above $a-b$.*

If $dist(a-b, \sigma(-\Delta)) > \delta > 0$, then there exists a constant $K_{a+b,\delta} > 0$, depending on the sum $a+b$ and on δ, such that

$$B(\mathbf{u},\mathbf{u}) \leq -K_{a+b,\delta}\,\|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in V, \tag{4.3}$$
$$B(\mathbf{u},\mathbf{u}) \geq K_{a+b,\delta}\,\|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in Z \oplus W. \tag{4.4}$$

Moreover, if $a+b$ is near enough to λ_k (in particular, if $a+b > 0$ and $dist(a+b, \sigma(-\Delta) \setminus \{\lambda_k\}) > \alpha > 0$), then there exists $G_{\alpha,\delta} > 0$, depending on α and δ, such that

$$B(\mathbf{u},\mathbf{u}) \leq -G_{\alpha,\delta}\,\|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in V, \tag{4.5}$$
$$B(\mathbf{u},\mathbf{u}) \geq G_{\alpha,\delta}\,\|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in W. \tag{4.6}$$

Remark 4.2. *As will become clear from the proof below, the constants $K_{a+b,\delta}$ and $G_{\alpha,\delta}$ also depend on k and l, but we do not include this dependence in the notation since we are considering them fixed along the proofs.*

Proof of Lemma 4.1. By hypothesis, λ_k is the first eigenvalue above $a+b$, then

$$a+b \in (\lambda_{k-1}, \lambda_k), \text{ if } k \geq 2 \qquad \text{or} \qquad a+b < \lambda_1. \tag{4.7}$$

If $k \geq 2$, then the sequence $\left\{\mu_i = 1 - \frac{a+b}{\lambda_i}\right\}_{i\in\mathbb{N}}$ is increasing, with $\mu_{k-1} < 0$ and $\mu_k > 0$, thus

$$\min_{i\in\mathbb{N}} |\mu_i| = \min\{\mu_k, -\mu_{k-1}\} = \min\left\{1 - \frac{a+b}{\lambda_k}, \frac{a+b}{\lambda_{k-1}} - 1\right\} > 0\,.$$

If $a + b < \lambda_1$, then

$$\inf_{i\in\mathbb{N}} |\mu_i| \geq \min\left\{1, 1 - \frac{a+b}{\lambda_1}\right\} > 0\,.$$

In both cases, there exists a constant $K_{a+b} > 0$, depending on $a + b$ (and on k), such that

$$|\mu_i| \geq K_{a+b}, \qquad \text{for every } i \in \mathbb{N}. \tag{4.8}$$

In the same way, since λ_l is the first eigenvalue above $a - b$ and $dist(a - b, \sigma(-\Delta)) > \delta > 0$, we have

$$a - b \in (\lambda_{l-1} + \delta, \lambda_l - \delta), \text{ if } l \geq 2 \qquad \text{or} \qquad a - b < \lambda_1 - \delta. \tag{4.9}$$

If $l \geq 2$, the sequence $\left\{\mu_{-i} = -1 + \frac{a-b}{\lambda_i}\right\}_{i\in\mathbb{N}}$ is decreasing, with $\mu_{-(l-1)} > 0$ and $\mu_{-l} < 0$, then

$$\min_{i\in\mathbb{N}} |\mu_{-i}| = \min\left\{\mu_{-(l-1)}, -\mu_{-l}\right\} = \min\left\{-1 + \frac{a-b}{\lambda_{l-1}}, -\frac{a-b}{\lambda_l} + 1\right\} \geq \frac{\delta}{\lambda_l} > 0\,.$$

If $a - b < \lambda_1 - \delta$, then

$$\inf_{i\in\mathbb{N}} |\mu_{-i}| \geq \min\left\{1, 1 - \frac{a-b}{\lambda_1}\right\} \geq \min\left\{1, \frac{\delta}{\lambda_1}\right\} > 0\,.$$

In both cases, there exists a constant $K_\delta > 0$, depending on δ (and on l), such that

$$|\mu_{-i}| \geq K_\delta, \qquad \text{for every } i \in \mathbb{N}. \tag{4.10}$$

From (4.8) and (4.10) we conclude that there exists a constant $K_{a+b,\delta} > 0$, depending on $a + b$ and δ, such that

$$|\mu_{\pm i}| \geq K_{a+b,\delta}, \qquad \text{for every } i \in \mathbb{N}. \tag{4.11}$$

If $\mathbf{u} \in V$, we can write $\mathbf{u} = \sum_{j\in\mathbb{Z}_V} c_j \boldsymbol{\psi}_j$, but $\mu_j \leq -K_{a+b,\delta}$ for $j \in \mathbb{Z}_V$, implying, by (3.15), that $B(\mathbf{u},\mathbf{u}) = \sum_{j\in\mathbb{Z}_V} \mu_j c_j^2 \leq -K_{a+b,\delta} \|\mathbf{u}\|_E^2$, which proves (4.3).

In the same way, if $\mathbf{u} = \sum_{j\in\mathbb{Z}_Z\cup\mathbb{Z}_W} c_j \boldsymbol{\psi}_j \in Z \oplus W$, since $\mu_j \geq K_{a+b,\delta}$ for every $j \in \mathbb{Z}_Z \cup \mathbb{Z}_W$ (actually $\mu_k > 0$), we get $B(\mathbf{u},\mathbf{u}) = \sum_{j\in\mathbb{Z}_Z\cup\mathbb{Z}_W} \mu_j c_j^2 \geq K_{a+b,\delta} \|\mathbf{u}\|_E^2$, proving (4.4).

In order to obtain (4.5) and (4.6) we observe that if $a+b > 0$ and $dist(a+b, \sigma(-\Delta)\setminus\{\lambda_k\}) > \alpha > 0$ we have, if $k \geq 2$,

$$\min_{i\in\mathbb{N}\setminus\{k,\ldots,k+m-1\}} |\mu_i| = \min\{\mu_{k+m}, -\mu_{k-1}\} = \min\left\{1 - \frac{a+b}{\lambda_{k+m}}, \frac{a+b}{\lambda_{k-1}} - 1\right\} > \min\left\{\frac{\alpha}{\lambda_{k+m}}, \frac{\alpha}{\lambda_{k-1}}\right\} = \frac{\alpha}{\lambda_{k+m}} > 0\,. \tag{4.12}$$

If $k = 1$, one gets

$$\min_{i\in\mathbb{N}\setminus\{1\}} |\mu_i| = 1 - \frac{a+b}{\lambda_2} \geq \frac{\alpha}{\lambda_2} > 0\,.$$

In both cases, we get that $|\mu_i|$ is bounded from below, uniformly with respect to $a+b$, for $i \in \mathbb{N}\setminus\{k,\ldots,k+m-1\}$. Then there exists $G_{\alpha,\delta} > 0$, depending on α and δ (and on k,l), such that

$$|\mu_i| \geq G_{\alpha,\delta}, \qquad \text{for every } i \in \mathbb{Z}_0 \setminus \{k,\ldots,k+m-1\}, \tag{4.13}$$

from which it follows that $B(\mathbf{u},\mathbf{u}) \leq -G_{\alpha,\delta}\,\|\mathbf{u}\|_E^2$ for every $\mathbf{u} \in V$ and that $B(\mathbf{u},\mathbf{u}) \geq G_{\alpha,\delta}\,\|\mathbf{u}\|_E^2$ for every $\mathbf{u} \in W$. □

Remark 4.3. *By comparing the estimates (4.10) and (4.13) with (4.2), we observe that, once that α and δ have been fixed, for ε small enough, $\mu_i = \mu_k$ only for $i = k,..,k+m-1$, then the Z defined in (4.1) is exactly the same as the one defined in the statement of Theorem 1.1.*

Lemma 4.1 is what we need in order to obtain the two saddle point geometries described in the following propositions.

Proposition 4.4. *Under the hypotheses of Theorem 1.1, if dist$(a-b,\sigma(-\Delta)) > \delta > 0$, suppose $a+b \notin \sigma(-\Delta)$ and λ_k is the first eigenvalue above $a+b$, then there exist $D_{a+b,\delta} \in \mathbb{R}$ and $\rho_{a+b,\delta} > 0$, such that*

$$J^+(\mathbf{u}) \geq D_{a+b,\delta}, \qquad \forall\, \mathbf{u} \in Z \oplus W, \tag{4.14}$$

$$J^+(\mathbf{u}) < D_{a+b,\delta}, \qquad \forall\, \mathbf{u} \in \rho S_V,\ \rho \geq \rho_{a+b,\delta}. \tag{4.15}$$

Proof. Let $\mathbf{u} \in Z \oplus W$. By (4.4) and (3.5-3.6), we have

$$J^+(\mathbf{u}) \geq K_{a+b,\delta}\,\|\mathbf{u}\|_E^2 - S_0\left(1 + \|\mathbf{u}\|_E^q\right) - H\,\|\mathbf{u}\|_E\,. \tag{4.16}$$

Since $K_{a+b,\delta} > 0$ and $q \in (1,2)$, the function of $\|\mathbf{u}\|_E$ above is bounded from below, then there exists $D_{a+b,\delta} \in \mathbb{R}$ satisfying (4.14).

On the other hand, if $\mathbf{u} \in V$, by (4.3) and (3.5-3.6), we get

$$J^+(\mathbf{u}) \leq -K_{a+b,\delta}\,\|\mathbf{u}\|_E^2 + S_0\left(1 + \|\mathbf{u}\|_E^q\right) + H\,\|\mathbf{u}\|_E\ , \tag{4.17}$$

and then the above function goes to $-\infty$ when $\|\mathbf{u}\|_E \to +\infty$, implying that there exists $\rho_{a+b,\delta} > 0$ satisfying (4.15). □

Remark 4.5. *We observe that the hypotheses **(f)** and **(F)** have not been used in the above proof, actually, they are required only in the next proposition.*

Proposition 4.6. *Under the hypotheses of Theorem 1.1, if dist$(a-b,\sigma(-\Delta)) > \delta > 0$ and λ_k is the first eigenvalue above $a+b$, then there exists $\varepsilon_0 > 0$, depending on δ, such that for $a+b \in (\lambda_k - \varepsilon_0, \lambda_k)$, there exist E_δ, $D_{a+b,\delta} \in \mathbb{R}$, $\rho_{a+b,\delta} > R_\delta > 0$, such that (4.14), (4.15) hold and*

$$J^+(\mathbf{u}) \geq E_\delta, \qquad \forall\, \mathbf{u} \in W, \tag{4.18}$$

$$J^+(\mathbf{u}) < E_\delta - 1, \qquad \forall\, \mathbf{u} \in R_\delta S_{VZ}, \tag{4.19}$$

$$J^+(\mathbf{u}) < E_\delta - 1, \qquad \forall\, \mathbf{u} \in V \text{ with } \|\mathbf{u}\|_E > R_\delta. \tag{4.20}$$

Proof. Set $\alpha = \frac{1}{2}dist(\lambda_k, \sigma(-\Delta) \setminus \{\lambda_k\})$ and let $a + b \in (\lambda_k - \varepsilon_0, \lambda_k)$, where initially we only impose $\varepsilon_0 < \alpha$, which implies $dist(a + b, \sigma(-\Delta) \setminus \{\lambda_k\}) > \alpha$.

- Let $\mathbf{u} \in W$. By (4.6) and (3.5-3.6), we have

$$J^+(\mathbf{u}) \geq G_{\alpha,\delta} \|\mathbf{u}\|_E^2 - S_0 \left(1 + \|\mathbf{u}\|_E^q\right) - H \|\mathbf{u}\|_E . \tag{4.21}$$

  Since $G_{\alpha,\delta} > 0$ and $q \in (1, 2)$, there exists $E_\delta \in \mathbb{R}$, satisfying (4.18). We observe that the constant E_δ depends also on α, which we omit in the notation since at this point we may consider it fixed.

- Let now $\mathbf{u} \in V$. By (4.5) and (3.5-3.6), we have

$$J^+(\mathbf{u}) \leq -G_{\alpha,\delta} \|\mathbf{u}\|_E^2 + S_0 \left(1 + \|\mathbf{u}\|_E^q\right) + H \|\mathbf{u}\|_E . \tag{4.22}$$

  The estimate (4.20) will hold true provided $R_\delta > \widetilde{R}$, where $\widetilde{R} > 0$ may be chosen depending only on δ and α, but independent from a, b.

- In order to prove that there exists $R_\delta > \widetilde{R}$ satisfying (4.19) for some $\varepsilon_0 \in (0, \alpha)$, assume for sake of contradiction that, for any sequences $\varepsilon_j \to 0^+$ and $\{R_j\}$, where $R_j > \widetilde{R}$ for every $j \in \mathbb{N}$, there exist $\mathbf{u}_j \in R_j S_{VZ}$ and $a_j, b_j \in \mathbb{R}$ such that $a_j + b_j \in (\lambda_k - \varepsilon_j, \lambda_k)$, $dist(a_j - b_j, \sigma(-\Delta)) > \delta$ and

$$J_j^+(\mathbf{u}_j) \geq E_\delta - 1 .$$

  Here and below we denote by J_j^+ (resp. B_j) the functional J^+ (resp. the form B) computed with $a = a_j$ and $b = b_j$. Moreover $\left\{\mu_i^j\right\}_{i\in\mathbb{Z}_0}$ will be the eigenvalues of B_j, while the corresponding eigenfunctions do not depend on j.

  Without loss of generality, we may assume that $R_j \to \infty$ and $\varepsilon_j R_j^2 \to 0$. We write $\mathbf{u}_j = \mathbf{v}_j + \mathbf{z}_j$, where $\mathbf{v}_j \in V$ and $\mathbf{z}_j \in Z$, for every $j \in \mathbb{N}$.

  By (4.2), the k-th eigenvalue of the form B_j satisfies $0 < \mu_k^j < \frac{\varepsilon_j}{\lambda_k}$, then $B_j(\mathbf{z}_j, \mathbf{z}_j) \leq \frac{\varepsilon_j}{\lambda_k} \|\mathbf{z}_j\|_E^2$. Using also (4.5), we have

$$B_j(\mathbf{u}_j, \mathbf{u}_j) = B_j(\mathbf{z}_j, \mathbf{z}_j) + B_j(\mathbf{v}_j, \mathbf{v}_j) \leq \frac{\varepsilon_j}{\lambda_k} \|\mathbf{z}_j\|_E^2 - G_{\alpha,\delta} \|\mathbf{v}_j\|_E^2 , \tag{4.23}$$

  and then, for every $j \in \mathbb{N}$, it follows that

$$E_\delta - 1 \leq J_j^+(\mathbf{u}_j) \leq \frac{\varepsilon_j}{\lambda_k} \|\mathbf{z}_j\|_E^2 - G_{\alpha,\delta} \|\mathbf{v}_j\|_E^2 - \mathcal{F}(\mathbf{u}_j) - \mathcal{H}(\mathbf{u}_j) . \tag{4.24}$$

  Dividing equation (4.24) by R_j^2 and reordering, we get

$$\frac{G_{\alpha,\delta}}{R_j^2} \|\mathbf{v}_j\|_E^2 \leq \frac{\varepsilon_j}{\lambda_k R_j^2} \|\mathbf{z}_j\|_E^2 - \frac{E_\delta - 1}{R_j^2} - \frac{\mathcal{F}(\mathbf{u}_j) + \mathcal{H}(\mathbf{u}_j)}{R_j^2} . \tag{4.25}$$

  We note that $\frac{|\mathcal{F}(\mathbf{u}_j)+\mathcal{H}(\mathbf{u}_j)|}{R_j^2} \leq \frac{S_0\left(1+\|\mathbf{u}_j\|_{[L^2]^2}^q\right)+H\|\mathbf{u}_j\|_{[L^2]^2}}{R_j^2} \leq \frac{C\left(1+R_j+R_j^q\right)}{R_j^2} \to 0$ when $j \to \infty$, since $q \in (1, 2)$. Moreover $\frac{E_\delta - 1}{R_j^2} \to 0$ and $\frac{\varepsilon_j}{\lambda_k R_j^2} \|\mathbf{z}_j\|_E^2 \leq \frac{\varepsilon_j}{\lambda_k} \to 0$. Thus it follows that

$$\frac{\|\mathbf{v}_j\|_E}{R_j} \to 0 , \tag{4.26}$$

and since $\|\mathbf{v}_j\|_E^2 = R_j^2 - \|\mathbf{z}_j\|_E^2$, we also get

$$\frac{\|\mathbf{z}_j\|_E}{R_j} \to 1\,. \tag{4.27}$$

Now, for every $j \in \mathbb{N}$, we define $\widehat{\mathbf{z}}_j = \frac{\mathbf{z}_j}{R_j}$, $\widehat{\mathbf{v}}_j = \frac{\mathbf{v}_j}{R_j}$ and $\widehat{\mathbf{u}}_j = \widehat{\mathbf{z}}_j + \widehat{\mathbf{v}}_j$. Then, by (4.26) and (4.27), there exists $\widehat{\mathbf{z}}_0 \in Z$ with $\|\widehat{\mathbf{z}}_0\|_E = 1$, such that (up to a subsequence) $\widehat{\mathbf{z}}_j \to \widehat{\mathbf{z}}_0$ in E and uniformly (since the dimension of Z is finite) and $\widehat{\mathbf{v}}_j \to 0$ in E, when $j \to \infty$. We denote by $P_1\mathbf{u}$ and $P_2\mathbf{u}$ the components of a vector $\mathbf{u} \in E$, that is, $\mathbf{u} = (P_1\mathbf{u}, P_2\mathbf{u})$. Since $\|\widehat{\mathbf{z}}_0\|_E = 1$, it follows that at least one of the components, $P_1\widehat{\mathbf{u}}_j$ or $P_2\widehat{\mathbf{u}}_j$, does not converge to zero in E. Without loss of generality, we suppose that $P_1\widehat{\mathbf{u}}_j$ does not converge to zero in E. Then we have

$$\begin{cases} P_1\widehat{\mathbf{z}}_j \xrightarrow{j\to\infty} P_1\widehat{\mathbf{z}}_0 \neq 0, & \text{uniformly in } \Omega, \\ P_1\widehat{\mathbf{v}}_j \xrightarrow{j\to\infty} 0, & \text{in } L^2(\Omega). \end{cases} \tag{4.28}$$

As a consequence, there exist $\chi > 0$ and $j_0 \in \mathbb{N}$ such that, for $j > j_0$, the sets $\Omega_j := \{x \in \Omega : \ |P_1\widehat{\mathbf{u}}_j(x)| > \chi\}$ satisfy $|\Omega_j| > \chi$. Actually, by (4.28), there exists $\chi > 0$ such that, for j large enough, there exist sets $\widetilde{\Omega}_j$ with $|\widetilde{\Omega}_j| > \chi$ such that $|P_1\widehat{\mathbf{z}}_j(x)| > 2\chi$ and $|P_1\widehat{\mathbf{v}}_j(x)| < \chi$, a.e. in $\widetilde{\Omega}_j$, so that $|P_1\widehat{\mathbf{u}}_j(x)| > \chi$ a.e. in $\widetilde{\Omega}_j$.

At this point we need to consider separately the hypotheses (**f**) and (**F**).

If hypotheses (1.2) and (**f**) hold, then given $M > 0$, there exists $C_M \in \mathbb{R}$ such that $F_{1,2}(x,t) \geq M|t| - C_M$. Then, by taking $M = \frac{1+H}{\chi^2}$, we obtain, for $j > j_0$, the estimate

$$\int_\Omega F_2(x, R_j(P_1\widehat{\mathbf{u}}_j)) \geq M R_j \int_\Omega |P_1\widehat{\mathbf{u}}_j| - D_M \geq M R_j \chi^2 - D_M, \tag{4.29}$$

where $D_M = \int_\Omega C_M$. Moreover, since $F_1(x, P_2\widehat{\mathbf{u}}_j) \geq -C_M$,

$$\int_\Omega F_1(x, R_j(P_2\widehat{\mathbf{u}}_j)) \geq -D_M. \tag{4.30}$$

It follows from the last two estimates and from (3.5), that

$$\mathcal{F}(R_j\widehat{\mathbf{u}}_j) + \mathcal{H}(R_j\widehat{\mathbf{u}}_j) \geq M R_j \chi^2 - 2D_M - R_j H = R_j - 2D_M. \tag{4.31}$$

Inserting (4.31) in (4.24), we get

$$E_\delta - 1 \leq \frac{\varepsilon_j}{\lambda_k} \|R_j\widehat{\mathbf{z}}_j\|_E^2 - G_{\alpha,\delta} \|R_j\widehat{\mathbf{v}}_j\|_E^2 - R_j + 2D_M\,, \tag{4.32}$$

and then

$$G_{\alpha,\delta} R_j^2 \|\widehat{\mathbf{v}}_j\|_E^2 + R_j \leq \frac{\varepsilon_j R_j^2}{\lambda_k} + 2D_M - E_\delta + 1\,, \tag{4.33}$$

which gives rise to a contradiction, since the right hand side of the inequality is bounded, while the left hand side tends to $+\infty$, when $j \to \infty$.

We conclude that there exist $R_\delta > \widetilde{R}$ and $\varepsilon_0 \in (0, \alpha)$ which satisfy (4.19).

We consider now the case in which hypotheses (1.2) and (**F**) hold. We first show that

$$\lim_{R\to\infty} \inf_{j>j_0} \int_\Omega F_2(x, RP_1(\widehat{\mathbf{u}}_j)) = +\infty. \tag{4.34}$$

In fact, we show that given $M > 0$, there exists R_0 large enough, such that

$$\int_\Omega F_2(x, RP_1(\widehat{\mathbf{u}}_j)) \geq M, \qquad \forall j \geq j_0 \text{ and } R \geq R_0. \tag{4.35}$$

First observe that there exists $C_F > 0$, such that

$$F_i(x,t) \geq -C_F, \text{ for every } (x,t) \in \Omega \times \mathbb{R} \text{ and } i = 1, 2\,;$$

actually, by (**F**-i) there exists $t_0 > 0$ such that $F_i(x,t) \geq 0$ for $|t| > t_0$ and by (1.2) $F_i(x,t) \geq -S(t_0 + t_0^q/q)$ for $|t| \leq t_0$.

Then we define $M_1 = \frac{M+|\Omega|C_F}{\chi}$. By point (**F**-$i$), there exists $s_0 > 0$ such that,

$$F_2(x,t) > M_1, \qquad \forall |t| > s_0. \tag{4.36}$$

Also, for every $R > \frac{s_0}{\chi}$, we have $\Omega_j \subseteq \{x \in \Omega : \; |RP_1(\widehat{\mathbf{u}}_j(x))| > s_0\}$. Then, since $|\Omega_j| > \chi$, it follows that

$$\int_\Omega F_2(x, RP_1(\widehat{\mathbf{u}}_j)) \geq \int_{\Omega_j} F_2(x, RP_1(\widehat{\mathbf{u}}_j)) + \int_{\Omega\setminus\Omega_j} -C_F \geq$$
$$\geq \int_{\Omega_j} M_1 - C_F|\Omega| \geq M_1\chi - C_F|\Omega| = M. \tag{4.37}$$

Thus, (4.34) is satisfied, since $\int_\Omega F_1(x, RP_2(\widehat{\mathbf{u}}_j))$ is bounded from below; it follows that

$$\mathcal{F}(\mathbf{u}_j) = \mathcal{F}(R_j\widehat{\mathbf{u}}_j) \to +\infty, \quad \text{when} \quad j \to +\infty. \tag{4.38}$$

Moreover, by Remark 4.3 and (**F**-ii), we have that $\mathcal{H}(\mathbf{u}_j) = \mathcal{H}(\mathbf{v}_j)$, then we may estimate

$$G_{\alpha,\delta}\,\|\mathbf{v}_j\|_E^2 + \mathcal{H}(\mathbf{u}_j) = G_{\alpha,\delta}\,\|\mathbf{v}_j\|_E^2 + \mathcal{H}(\mathbf{v}_j) \geq G_{\alpha,\delta}\,\|\mathbf{v}_j\|_E^2 - H\,\|\mathbf{v}_j\|_E\,. \tag{4.39}$$

As a consequence, there exists $\delta_2 > 0$ such that $G_{\alpha,\delta}\,\|\mathbf{v}_j\|_E^2 + \mathcal{H}(\mathbf{u}_j) \geq -\delta_2$.

Then, from (4.24), we get

$$\mathcal{F}(\mathbf{u}_j) \leq \frac{\varepsilon_j\,\|\mathbf{z}_j\|_E^2}{\lambda_k} - E_\delta + 1 + \delta_2, \tag{4.40}$$

which gives rise to a contradiction, as in the previous case, since the right hand side is bounded, while the left hand side tends to $+\infty$.

We conclude again that there exist $R_\delta > \widetilde{R}$ and $\varepsilon_0 \in (0,\alpha)$ which satisfy (4.19).

The constants in the statement of the proposition can be obtained as follows: we first get $E_\delta \in \mathbb{R}$ satisfying (4.18), then we obtain $\varepsilon_0 > 0$ and $R_\delta > 0$ satisfying (4.19) and (4.20). These estimates hold uniformly for $a+b \in (\lambda_k - \varepsilon_0, \lambda_k)$. Then, fixing $a+b$ in this interval, we can obtain a constant $D_{a+b,\delta} \in \mathbb{R}$ satisfying (4.14). Finally, we get $\rho_{a+b,\delta} > 0$ satisfying (4.15); observe that we may also choose $\rho_{a+b,\delta} > R_\delta$, as desired. $\square$

Remark 4.7. *Proposition 4.4 holds for any fixed value of $a+b$ in the interval $(\lambda_{k-1}, \lambda_k)$. This geometry will produce a solution of problem (1.1+). On the other hand, the geometry from Proposition 4.6 holds for $a+b$ near to the eigenvalue λ_k, from below. This other geometry will also give a solution of problem (1.1+). As a result, for such values of $a+b$, both geometries hold, so that we will have two solutions of problem (1.1+). However, we will still need to prove that they are not the same.*

4.2 Obtaining two sequences of critical points

As observed in the introduction, the functional J^+ defined in (3.1) is strongly indefinite. In order to overcome this difficulty, we will define a sequence of finite dimensional problems, where we can apply the classical saddle point theorem, and then we will take limit to obtain the solutions of problem (1.1).

We define, for every $n > k+m$,

$$E_n = span[\boldsymbol{\psi}_{-n}, \dots, \boldsymbol{\psi}_n] \subseteq E,$$

$$V_n = V \cap E_n \qquad \text{and} \qquad W_n = W \cap E_n.$$

By Remark 4.3, we may assume that $Z \subseteq E_n$ for every $n > k+m$. We will denote by B^n_V, B^n_{VZ} and B^n_{ZW} the closed unitary balls, with respect to the norm of E, in the subspaces V_n, $V_n \oplus Z$ and $Z \oplus W_n$ respectively, and by S^n_V, S^n_{VZ} and S^n_{ZW} their relative boundaries. Finally, let J^+_n be the functional J^+ restricted to the subspace E_n.

Under the hypotheses of Theorem 1.1, the claims in Proposition 4.6 hold. Then, fixed a, b such that $dist(a-b, \sigma(-\Delta)) > \delta > 0$ and $a+b \in (\lambda_k - \varepsilon_0, \lambda_k)$, the functional J^+_n satisfies the same estimates in the subespaces V_n, Z and W_n.
Moreover, the following lemma (whose proof is given in section 4.6) implies that the functionals J^+_n satisfy the PS condition.

Lemma 4.8. *Suppose $a \pm b \notin \sigma(-\Delta)$, f_1, f_2 satisfy (1.2) and $h_1, h_2 \in L^2(\Omega)$. Fix $n > k+m$ and suppose $\{\mathbf{u}_i\} \subset E_n$ is a sequence such that*

$$\left|(J^{\pm}_n)'(\mathbf{u}_i)[\boldsymbol{\phi}]\right| \leq \varepsilon_i \left\|\boldsymbol{\phi}\right\|_E, \qquad \text{for every } \boldsymbol{\phi} \in E_n, \tag{4.41}$$

where $\varepsilon_i \to 0$. Then $\{\mathbf{u}_i\}$ admits a subsequence which converges in E_n.

In these conditions, for every $n > k+m$, we can apply the saddle point theorem two times: in fact, equations (4.14) and (4.15) define a saddle point geometry between the subspaces V_n and $Z \oplus W_n$, while equations (4.18) and (4.19) define a second saddle point geometry, between the subspaces $V_n \oplus Z$ and W_n. As a consequence, we obtain the existence of the critical points $\mathbf{u}^+_n, \mathbf{v}^+_n \in E_n$ of the functionals J^+_n, at the critical levels

$$c^+_n = \inf_{\gamma \in \Gamma^n_V} \sup_{\mathbf{u} \in \rho_{a+b,\delta} B^n_V} J^+_n(\gamma(\mathbf{u})) \geq D_{a+b,\delta}, \tag{4.42}$$

$$d^+_n = \inf_{\gamma \in \Gamma^n_{VZ}} \sup_{\mathbf{v} \in R_\delta B^n_{VZ}} J^+_n(\gamma(\mathbf{v})) \geq E_\delta, \tag{4.43}$$

where

$$\Gamma^n_V = \left\{\gamma \in \mathcal{C}(\rho_{a+b,\delta} B^n_V, E_n) : \ \gamma(\mathbf{u}) = \mathbf{u} \text{ if } \left\|\mathbf{u}\right\|_E = \rho_{a+b,\delta}\right\}, \tag{4.44}$$

$$\Gamma^n_{VZ} = \left\{\gamma \in \mathcal{C}(R_\delta B^n_{VZ}, E_n) : \ \gamma(\mathbf{v}) = \mathbf{v} \text{ if } \left\|\mathbf{v}\right\|_E = R_\delta\right\}. \tag{4.45}$$

Since $\mathbf{u}_n^+, \mathbf{v}_n^+$ are just critical points of the functional J^+ restricted to certain subspaces, they are not solutions of our problem. The solutions that we will find, in section 4.4, will be critical points of J^+ whose critical levels will be the limit of subsequences of c_n^+ and d_n^+. Thus, we need estimates on c_n^+, d_n^+ that guarantee the existence of convergent subsequences and also that the limit levels are distinct. Such estimates are given in the next lemma.

Lemma 4.9. *Under the conditions of Proposition 4.6, fixed a, b such that $dist(a-b, \sigma(-\Delta)) > \delta > 0$ and $a+b \in (\lambda_k - \varepsilon_0, \lambda_k)$, there exists $T_\delta > 0$ such that $c_n^+ \in [D_{a+b,\delta}, E_\delta - 1]$ and $d_n^+ \in [E_\delta, T_\delta]$, for every $n > k+m$.*

Proof. The estimates from below come from (4.42-4.43), as a direct consequence of the saddle point theorem.

In order to obtain the estimates from above, we first note that it is possible to build a map $\gamma_0 \in \Gamma_V^n$ such that $\gamma_0(\rho_{a+b,\delta} B_V^n)$ is the union of the annulus $\{\mathbf{u} \in V_n : \|\mathbf{u}\|_E \in [R_\delta, \rho_{a+b,\delta}]\}$ with a hemisphere contained in $R_\delta S_{VZ}^n$, having $R_\delta S_V^n$ as boundary: for instance, we may take

$$\gamma_0(p) = \begin{cases} p\,, & \text{if } R_\delta \le \|p\|_E \le \rho_{a+b,\delta}\,, \\ p + \sqrt{(R_\delta)^2 - \|p\|_E^2}\ \boldsymbol{\psi}_k\,, & \text{if } \|p\|_E \le R_\delta\,. \end{cases}$$

Then, by (4.19) and (4.20), we get $\sup\limits_{\mathbf{u} \in \rho_{a+b,\delta} B_V^n} J_n^+(\gamma_0(\mathbf{u})) < E_\delta - 1$, and then $c_n^+ < E_\delta - 1$.

On the other hand, since the identity belongs to Γ_{VZ}^n, one has $d_n^+ \le \sup\limits_{\mathbf{v} \in R_\delta B_{VZ}^n} J^+(\mathbf{v})$.

If $\mathbf{u} \in V \oplus Z$, let $\mathbf{z} \in Z$ and $\mathbf{v} \in V$ be such that $\mathbf{u} = \mathbf{z} + \mathbf{v}$. Since $a+b \in (\lambda_k - \varepsilon_0, \lambda_k)$, one has

$$-K_{a+b,\delta} < 0 < \mu_k \le \frac{\varepsilon_0}{\lambda_k}\,. \tag{4.46}$$

Then by (4.46), (4.3) and (3.5-3.6), we have

$$\begin{aligned} J^+(\mathbf{u}) &\le \frac{\varepsilon_0}{\lambda_k} \|\mathbf{z}\|_E^2 - K_{a+b,\delta} \|\mathbf{v}\|_E^2 + S_0(1 + \|\mathbf{u}\|_E^q) + H \|\mathbf{u}\|_E \\ &\le \frac{\varepsilon_0}{\lambda_k} \|\mathbf{u}\|_E^2 + S_0(1 + \|\mathbf{u}\|_E^q) + H \|\mathbf{u}\|_E\,, \end{aligned}$$

implying that J^+ is bounded from above (uniformly with respect to n), in the bounded sets $R_\delta B_{VZ}^n$. As a consequence there exist $T_\delta > 0$ which bounds d_n^+ from above. $\square$

4.3 The saddle point geometries when $a+b$ is almost-resonant from above

In this section we will show how to obtain critical levels analogous to (4.42) and (4.43), in the case (b) of Theorem 1.1, by adapting the arguments used in the previous sections for the case (a).

We consider the functional J^- defined in (3.1), which is the same as the functional J^+, except for the fact that it has the form $(-B)$ in the place of the original form B. Thus, we define the two sequences of eigenvalues of $(-B)$:

$$\widetilde{\mu}_{\pm i} = -\mu_{\pm i} = -\frac{-b \pm (\lambda_i - a)}{\lambda_i}\,, i \in \mathbb{N}\,,$$

which correspond to the same eigenfunctions $\boldsymbol{\psi}_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2}}$, $i \in \mathbb{N}$.
Then we define the subspaces $\widetilde{V}, \widetilde{Z}$ and $\widetilde{W}$ in the same manner as in (4.1), but using the eigenvalues $\widetilde{\mu}_i : \ i \in \mathbb{Z}_0$ in the place of $\mu_i : \ i \in \mathbb{Z}_0$.

We observe that if $a+b \in (\lambda_k, \lambda_k + \varepsilon)$, for some $\varepsilon > 0$, then

$$0 < \widetilde{\mu}_k = \frac{(a+b) - \lambda_k}{\lambda_k} < \frac{\varepsilon}{\lambda_k}\,, \tag{4.47}$$

analogous to (4.2).

We only need to show that Lemma 4.1 holds for $(-B)$ with the new definition of the subspaces: the rest of the arguments will follow as in section 4.1.

Lemma 4.10. *Suppose $a \pm b \notin \sigma(-\Delta)$ and fix λ_k being the largest eigenvalue below $a+b$, of multiplicity m, and λ_l the first eigenvalue above de $a-b$.*

If $dist(a-b, \sigma(-\Delta)) > \delta > 0$, then there exists a constant $K_{a+b,\delta} > 0$, depending on the sum $a+b$ and on δ, such that

$$-B(\mathbf{u},\mathbf{u}) \le -K_{a+b,\delta}\, \|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in \widetilde{V}\,, \tag{4.48}$$

$$-B(\mathbf{u},\mathbf{u}) \ge K_{a+b,\delta}\, \|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in \widetilde{Z} \oplus \widetilde{W}\,. \tag{4.49}$$

Moreover, if $a+b$ is near enough to λ_k (in particular, if $dist(a+b, \sigma(-\Delta) \setminus \{\lambda_k\}) > \alpha > 0$), then there exists $G_{\alpha,\delta} > 0$, depending on α and δ, such that

$$-B(\mathbf{u},\mathbf{u}) \le -G_{\alpha,\delta}\, \|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in \widetilde{V}\,, \tag{4.50}$$

$$-B(\mathbf{u},\mathbf{u}) \ge G_{\alpha,\delta}\, \|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in \widetilde{W}\,. \tag{4.51}$$

Proof. The proof of this lemma follows as in Lemma 4.1.

In fact, since we are considering the eigenvalues of the form $(-B)$, the sequence $\widetilde{\mu}_i$, with $i \in \mathbb{N}$, is decreasing while the sequence $\widetilde{\mu}_{-i}$, with $i \in \mathbb{N}$, is increasing. However, one can prove as before that an estimate like (4.11) holds for these new eigenvalues. Then (4.48) and (4.49) hold true, by the definition of $\widetilde{V}, \widetilde{Z}$ and $\widetilde{W}$.

Considering $a+b$ near enough to λ_k, we also get, as in the proof of Lemma 4.1, the existence of the constant $G_{\alpha,\delta}$ satisfying (4.50) and (4.51). ☐

Reasoning as in Remark 4.3, we observe that for ε small enough, $\widetilde{Z}$ is exactly the same as the Z defined in the statement of Theorem 1.1.
Thus, Lemma 4.10 and estimate (4.47) allow us to show an analogous of the Propositions 4.4 and 4.6 for the functional J^-, with the new subspaces $\widetilde{V}, \widetilde{Z}$ and $\widetilde{W}$.

Proposition 4.11. *Under the hypotheses of Theorem 1.1, if $dist(a-b, \sigma(-\Delta)) > \delta > 0$, suppose $a+b \notin \sigma(-\Delta)$ and λ_k is the largest eigenvalue below $a+b$, then there exist $D_{a+b,\delta} \in \mathbb{R}$ and $\rho_{a+b,\delta} > 0$, such that J^- satisfies (4.14) and (4.15) in the subspaces $\widetilde{V}, \widetilde{Z}$ and $\widetilde{W}$.*

Proposition 4.12. *Under the hypotheses of Theorem 1.1, if $dist(a-b, \sigma(-\Delta)) > \delta > 0$ and λ_k is the largest eigenvalue below $a+b$, then there exists $\varepsilon_1 > 0$, depending on δ, such that for $a+b \in (\lambda_k, \lambda_k + \varepsilon_1)$, there exist E_δ, $D_{a+b,\delta} \in \mathbb{R}$, $\rho_{a+b,\delta} > R_\delta > 0$, such that J^- satisfies (4.14), (4.15), (4.18), (4.19) and (4.20) in the subspaces $\widetilde{V}, \widetilde{Z}$ and $\widetilde{W}$.*

The Propositions 4.11 and 4.12 are enough to guarantee, as in section 4.2, the existence of two sequences of critical points $\{\mathbf{u}_n^-\}$ and $\{\mathbf{v}_n^-\}$, of the functionals $J_n^- = J^-|_{E_n}$, at critical levels c_n^-, d_n^-, analogous to (4.42) and (4.43).

4.4 Existence of solutions in the limit

In this section we conclude the proof of Theorem 1.1, by showing that, up to subsequences, the sequences of critical points $\{\mathbf{u}_n^+\}$ and $\{\mathbf{v}_n^+\}$ obtained in section 4.1 converge to distinct solutions of (1.1+), while $\{\mathbf{u}_n^-\}$ and $\{\mathbf{v}_n^-\}$, from section 4.3, converge to distinct solutions of (1.1−). At the end of the section we also prove Theorem 1.2.

We first state a lemma, which will be proven in section 4.6.

Lemma 4.13. *Suppose $a \pm b \notin \sigma(-\Delta)$, f_1, f_2 satisfy (1.2) and $h_1, h_2 \in L^2(\Omega)$. Suppose $\{\mathbf{u}_i\} \subset E$ is a sequence such that $\mathbf{u}_i \in E_i$, for every $i \in \mathbb{N}$, and*

$$\left|(J_i^{\pm})'(\mathbf{u}_i)[\boldsymbol{\phi}]\right| \le \varepsilon_i \,\|\boldsymbol{\phi}\|_E, \qquad \text{for every } \boldsymbol{\phi} \in E_i, \tag{4.52}$$

where $\varepsilon_i \to 0$. Then $\{\mathbf{u}_i\}$ is bounded in E.

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We will only prove item (a), since item (b) follows by the same argument when applied to the sequences obtained in section 4.3.

First, we note that, by Lemma 4.9, there exist $c^+ \in [D_{a+b,\delta}, E_\delta - 1]$ and $d^+ \in [E_\delta, T_\delta]$ such that, passing to a subsequence, $c_n^+ \to c^+$ and $d_n^+ \to d^+$ for $n \to \infty$.

We claim that there exists $\mathbf{u}^+ \in E$, critical point of the functional J^+, such that $J^+(\mathbf{u}^+) = c^+$. By the saddle point theorem, we know that, for every $n > k + m$,

$$J_n^+(\mathbf{u}_n^+) = c_n^+, \tag{4.53}$$

$$(J_n^+)'(\mathbf{u}_n^+)[\boldsymbol{\phi}] = 0, \qquad \text{for every } \boldsymbol{\phi} \in E_n. \tag{4.54}$$

As a consequence, we may apply Lemma 4.13, to deduce that the sequence $\{\mathbf{u}_n^+\}$ is bounded in E and then there exists $\mathbf{u}^+ \in E$ such that, passing to a further subsequence,

$$\begin{cases} \mathbf{u}_n^+ = (u_n, v_n) \rightharpoonup \mathbf{u}^+ = (u, v) & \text{in } E, \\ \mathbf{u}_n^+ = (u_n, v_n) \to \mathbf{u}^+ = (u, v) & \text{in } L^2 \times L^2. \end{cases} \tag{4.55}$$

Let $h > k + m$. By testing (4.54) with $(0, \psi) \in E_h$ and $(\phi, 0) \in E_h$, we get, for $n > h$,

$$\begin{cases} \displaystyle\int_\Omega \nabla u_n \nabla \psi - a \int_\Omega u_n \psi - b \int_\Omega v_n \psi - \int_\Omega f_1(x, v_n)\psi - \int_\Omega h_1 \psi = 0, \\ \displaystyle\int_\Omega \nabla v_n \nabla \phi - a \int_\Omega v_n \phi - b \int_\Omega u_n \phi - \int_\Omega f_2(x, u_n)\phi - \int_\Omega h_2 \phi = 0. \end{cases} \tag{4.56}$$

By (4.55) and the continuity of the Nemytskii operators associated to f_1 and f_2 (see for example in [dF89]), we have, in the limit,

$$\begin{cases} \displaystyle\int_\Omega \nabla u \nabla \psi - a \int_\Omega u \psi - b \int_\Omega v \psi - \int_\Omega f_1(x, v)\psi - \int_\Omega h_1 \psi = 0, \\ \displaystyle\int_\Omega \nabla v \nabla \phi - a \int_\Omega v \phi - b \int_\Omega u \phi - \int_\Omega f_2(x, u)\phi - \int_\Omega h_2 \phi = 0, \end{cases} \tag{4.57}$$

that is, $(J^+)'(\mathbf{u}^+)[\boldsymbol{\phi}] = 0$, for every $\boldsymbol{\phi} = (\phi, \psi) \in E_h$. Since $\bigcup_{h \in \mathbb{N}} E_h$ is dense in E, it follows that $(J^+)'(\mathbf{u}^+) = 0$, and then $\mathbf{u}^+ = (u, v)$ is a critical point of the functional J^+.

We show now that $\mathbf{u}_n^+ \to \mathbf{u}^+$, strongly in E, which implies that $J^+(\mathbf{u}^+) = c^+$.
Let $P_n : H \to span\{\phi_1, \dots, \phi_n\}$ be the orthogonal projection. Then $P_n u \to u$, $P_n v \to v$ in H and since $(v_n - P_n v, u_n - P_n u) \in E_n$, we can use it as a test function in in (4.54) to obtain

$$(J_n^+)'(u_n, v_n)[v_n - P_n v, u_n - P_n u] = 0, \qquad \text{for every } n > k + m. \tag{4.58}$$

Since $u_n - P_n u,\ v_n - P_n v \to 0$ in L^2 and $u_n,\ v_n$ are bounded in L^2, by using (1.2), we get the following convergences:

$$\begin{cases} a\displaystyle\int_\Omega v_n(v_n - P_n v) + \int_\Omega u_n(u_n - P_n u) + b\int_\Omega u_n(v_n - P_n v) + \int_\Omega v_n(u_n - P_n u) \to 0, \\ \displaystyle\int_\Omega f_1(x, v_n)(u_n - P_n u) + \int_\Omega f_2(x, u_n)(v_n - P_n v) \to 0, \\ \displaystyle\int_\Omega h_1(u_n - P_n u) + \int_\Omega h_2(v_n - P_n v) \to 0. \end{cases} \tag{4.59}$$

Then it follows by (4.58) that

$$\int_\Omega \nabla v_n \nabla(v_n - P_n v) + \int_\Omega \nabla u_n \nabla(u_n - P_n u) \to 0, \tag{4.60}$$

which implies

$$\int_\Omega |\nabla v_n|^2 - \int_\Omega \nabla v_n \nabla v + \int_\Omega |\nabla u_n|^2 - \int_\Omega \nabla u_n \nabla u + \int_\Omega \nabla v_n \nabla(v - P_n v) + \int_\Omega \nabla u_n \nabla(u - P_n u) \to 0. \tag{4.61}$$

The last two terms in (4.61) go to zero because $(P_n v, P_n u) \to (v, u)$ in E and $(v_n,\ u_n)$ is bounded in E. It follows that

$$\int_\Omega |\nabla v_n|^2 - \int_\Omega \nabla v_n \nabla v + \int_\Omega |\nabla u_n|^2 - \int_\Omega \nabla u_n \nabla u \to 0,$$

that is, $\|(u_n, v_n)\|_E \to \|(u, v)\|_E$ and then $(u_n, v_n) \to (u, v)$ strongly in E.

We then conclude that $J^+(\mathbf{u}_n^+) \to J^+(\mathbf{u}^+)$ and then $J^+(\mathbf{u}^+) = c^+$.

By the same argument, there exists $\mathbf{v}^+ \in E$, critical point of J^+, such that $\mathbf{v}_n^+ \to \mathbf{v}^+$ in E and $J^+(\mathbf{v}^+) = d^+$. Moreover, $\mathbf{u}^+ \neq \mathbf{v}^+$ since $J(\mathbf{u}^+) = c^+ \leq E_\delta - 1$ and $J(\mathbf{v}^+) = d^+ \geq E_\delta$.

We have then proved point (a) of Theorem 1.1. $\square$

Theorem 1.2 follows now easily by a change of variables.

Proof of Theorem 1.2. This result is a direct consequence of Theorem 1.1: let (u, v) be one of the solutions of (1.1+) and define $(u, \overline{v}) = (u, -v)$. Then

$$\begin{cases} -\Delta u = au - b\overline{v} - [\,-f_1(x, -\overline{v}) - h_1(x)\,] & \text{in } \Omega, \\ -\Delta \overline{v} = -bu + a\overline{v} - [\,f_2(x, u) + h_2(x)\,] & \text{in } \Omega, \end{cases} \tag{4.62}$$

that is, $(u, \overline{v})$ is a solution of problem (1.1−) with the new coefficients $\widetilde{a} = a$ and $\widetilde{b} = -b$, the new nonlinearities $\widetilde{f_1}(x, v) = -f_1(x, -v)$, $\widetilde{f_2}(x, u) = f_2(x, u)$, which still satisfy (1.2) and (**f**) (resp. (**F**-*i*)), and the new $L^2(\Omega)$ functions $\widetilde{h}_1 = -h_1$, $\widetilde{h}_2 = h_2$.

If we are considering hypothesis (**F**), the condition $\int_\Omega h_1\phi + h_2\psi = 0$ for $(\phi, \psi) \in Z$ which appears in Theorem 1.1, becomes $\int_\Omega \widetilde{h}_1\phi + \widetilde{h}_2\psi = 0$ for $(\phi, -\psi) \in Z$, as in Theorem 1.2.

The proof of item (*d*) follows in the same way from item (*b*) of Theorem 1.1. $\square$

4.5 Double almost-resonance

In this section we outline the proof of the multiplicity results stated in the Theorems 1.3-1.4, which deal with two cases of double almost-resonance, that is, when the values $a+b$ and $a-b$ are near to eigenvalues of the Laplacian operator, at the same time.
The proof will follow the same lines as that of Theorem 1.1 item (a).

Let us fix λ_k (of multiplicity $m \in \mathbb{N}$, in particular $\lambda_k = \lambda_{k+m-1}$) and λ_l (of multiplicity $t \in \mathbb{N}$, in particular $\lambda_l = \lambda_{l+t-1}$) the eigenvalues stated in Theorem 1.3.

Observe that if $a-b \in (\lambda_l, \lambda_l + \varepsilon)$ and $a+b \in (\lambda_k - \varepsilon, \lambda_k)$, for some $\varepsilon > 0$, then $\mu_k = 1 - \frac{a+b}{\lambda_k}$ and $\mu_{-l} = \frac{a-b}{\lambda_l} - 1$ satisfy

$$0 < \mu_k, \mu_{-l} < \frac{\varepsilon}{\min\{\lambda_k, \lambda_l\}}. \tag{4.63}$$

In order to reproduce once more the geometry of Proposition 4.6, we define the spaces V, Z, W in the following way:

$$\begin{cases} V = \overline{span\{\boldsymbol{\psi}_i : \, i \in \mathbb{Z}_0, \, \mu_i < 0, \, \mu_i \neq \mu_{-l} \text{ and } \mu_i \neq \mu_k\}}, \\ Z = \overline{span\{\boldsymbol{\psi}_i : \, i \in \mathbb{Z}_0, \, \mu_i = \mu_k \text{ or } \mu_i = \mu_{-l}\}}, \\ W = \overline{span\{\boldsymbol{\psi}_i : \, i \in \mathbb{Z}_0, \, \mu_i > 0, \, \mu_i \neq \mu_{-l} \text{ and } \mu_i \neq \mu_k\}}. \end{cases} \tag{4.64}$$

We describe below how to build the geometry which guarantees the existence of critical levels analogous to (4.42) and (4.43), stressing the main differences in the proofs.
We start with a lemma, similar to Lemma 4.1, which gives estimates in the subspaces V, Z and W.

Lemma 4.14. *Suppose $a \pm b \notin \sigma(-\Delta)$ and fix λ_k being the first eigenvalue above $a+b$, of multiplicity m, and λ_l the largest eigenvalue below $a-b$, of multiplicity t. Then there exists a constant $\beta_{a,b} > 0$, depending on a and b, such that*

$$B(\mathbf{u}, \mathbf{u}) \leq -\beta_{a,b} \|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in V, \tag{4.65}$$

$$B(\mathbf{u}, \mathbf{u}) \geq \beta_{a,b} \|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in Z \oplus W. \tag{4.66}$$

Moreover, if $a+b > 0$ is near enough to λ_k and $a-b$ to λ_l (in particular, if $dist(a+b, \sigma(-\Delta) \setminus \{\lambda_k\}) > \alpha > 0$ and $dist(a-b, \sigma(-\Delta) \setminus \{\lambda_l\}) > \alpha > 0$), then there exists $\eta_\alpha > 0$, depending on α, such that

$$B(\mathbf{u}, \mathbf{u}) \leq -\eta_\alpha \|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in V, \tag{4.67}$$

$$B(\mathbf{u}, \mathbf{u}) \geq \eta_\alpha \|\mathbf{u}\|_E^2, \qquad \forall\, \mathbf{u} \in W. \tag{4.68}$$

Remark 4.15. *As observed in Remark 4.2, about Lemma 4.1, here the constants $\beta_{a,b}$ and η_α also depend on k and l, which we are considering fixed along the proofs.*

Proof of Lemma 4.14. The proof of this lemma follows by the same argument as that of Lemma 4.1, however, here $a-b$ is not bounded away from the spectrum $\sigma(-\Delta)$, as a consequence, the constant in the first two estimates depends also on the difference $a-b$. In particular, we can set $\beta_{a,b} = \inf\{|\mu_i|, \, i \in \mathbb{Z}_0\}$ (the ínfimum is positive since $\mu_i \to \pm 1$ when $i \to \pm\infty$) and $\eta_\alpha = \{\inf |\mu_i|, \, i \in \mathbb{Z}_V \cup \mathbb{Z}_W\} \geq \frac{\alpha}{\max\{\lambda_{k+m}, \lambda_{l+t}\}}$. □

Again, for ε small enough, the space Z defined in (4.64) is exactly the same as the Z defined in the statement of Theorem 1.3.
Thus, in view of Lemma 4.14, we obtain the geometries in the propositions below.

Proposition 4.16. *Under the hypotheses of Theorem 1.3, suppose $a \pm b \notin \sigma(-\Delta)$, λ_k is the first eigenvalue above $a+b$ and λ_l is the first below $a-b$, then there exist $\Lambda_{a,b} \in \mathbb{R}$ and $\omega_{a,b} > 0$, such that*

$$J^+(\mathbf{u}) \geq \Lambda_{a,b}, \qquad \forall \mathbf{u} \in Z \oplus W, \tag{4.69}$$

$$J^+(\mathbf{u}) < \Lambda_{a,b}, \qquad \forall \mathbf{u} \in \omega S_V,\ \omega \geq \omega_{a,b}. \tag{4.70}$$

Proof. The existence of $\Lambda_{a,b} \in \mathbb{R}$ satisfying (4.69), follows from (4.66) as in Proposition 4.4, while the existence of $\omega_{a,b} > 0$ satisfying (4.70), follows from (4.65). □

Proposition 4.17. *Under the hypotheses of Theorem 1.3, suppose λ_k is the first eigenvalue above a+b and λ_l is the first below a−b. Then there exists $\varepsilon_2 > 0$, such that for $a+b \in (\lambda_k - \varepsilon_2, \lambda_k)$ and $a-b \in (\lambda_l, \lambda_l + \varepsilon_2)$, there exist Θ, $\Lambda_{a,b} \in \mathbb{R}$, $\omega_{a,b} > r_0 > 0$, such that (4.69), (4.70) hold and*

$$J^+(\mathbf{u}) \geq \Theta, \qquad \forall \mathbf{u} \in W, \tag{4.71}$$

$$J^+(\mathbf{u}) < \Theta - 1, \qquad \forall \mathbf{u} \in r_0 S_{VZ}, \tag{4.72}$$

$$J^+(\mathbf{u}) < \Theta - 1, \qquad \forall \mathbf{u} \in V \text{ with } \|\mathbf{u}\|_E > r_0. \tag{4.73}$$

Proof. Set

$$\alpha = \frac{1}{2} \min \{ \mathit{dist}(\lambda_k, \sigma(-\Delta) \setminus \{\lambda_k\}),\ \mathit{dist}(\lambda_l, \sigma(-\Delta) \setminus \{\lambda_l\}) \}$$

and let $a+b \in (\lambda_k - \varepsilon_2, \lambda_k)$ and $a-b \in (\lambda_l, \lambda_l + \varepsilon_2)$, where we initially only impose $\varepsilon_2 < \alpha$, which implies $\mathit{dist}(a+b, \sigma(-\Delta) \setminus \{\lambda_k\}) > \alpha$ and $\mathit{dist}(a-b, \sigma(-\Delta) \setminus \{\lambda_l\}) > \alpha$.

- The existence of Θ satisfying (4.71) follows directly from (4.68) as in Proposition 4.6. Since we fixed α we omit its dependence in the notation for the constant Θ.

- Let $\mathbf{u} \in V$. By (4.67) and (3.5-3.6),

  $$J^+(\mathbf{u}) \leq -\eta_\alpha \|\mathbf{u}\|_E^2 + S_0(1 + \|\mathbf{u}\|_E^q) + H \|\mathbf{u}\|_E. \tag{4.74}$$

  Since $\eta_\alpha > 0$ and $q \in (1,2)$ the estimate (4.73) will be satisfied provided $r_0 > \widetilde{r}$, where $\widetilde{r} > 0$ can be chosen depending only on α, not on a or b.

- In order to prove that there exists $r_0 > \widetilde{r}$ satisfying (4.72) for some $\varepsilon_2 \in (0, \alpha)$, reasoning as in Proposition 4.6, we suppose by sake of contradiction that for any sequences $\varepsilon_j \to 0^+$ and $\{r_j\}$, with $r_j > \widetilde{r}$ for every $j \in \mathbb{N}$, there exist $\mathbf{u}_j \in r_j S_{VZ}$ and $a_j, b_j \in \mathbb{R}$ such that $a_j + b_j \in (\lambda_k - \varepsilon_j, \lambda_k)$, $a_j - b_j \in (\lambda_l, \lambda_l + \varepsilon_j)$ and $J_j^+(\mathbf{u}_j) \geq \Theta - 1$.

  Without loss of generality we may assume that $r_j \to \infty$ and $\varepsilon_j r_j^2 \to 0$. We write $\mathbf{u}_j = \mathbf{v}_j + \mathbf{z}_j$, where $\mathbf{v}_j \in V$ and $\mathbf{z}_j \in Z$, for every $j \in \mathbb{N}$. By (4.63) and (4.67), we have

  $$B_j(\mathbf{u}_j, \mathbf{u}_j) = B_j(\mathbf{v}_j, \mathbf{v}_j) + B_j(\mathbf{z}_j, \mathbf{z}_j) \leq \frac{\varepsilon_j}{\min\{\lambda_l, \lambda_k\}} \|\mathbf{z}_j\|_E^2 - \eta_\alpha \|\mathbf{u}\|_E^2. \tag{4.75}$$

Then, for every $j \in \mathbb{N}$, it follows

$$\Theta - 1 \leq J^{+}(\mathbf{u}_j) \leq \frac{\varepsilon_j}{\min\{\lambda_l, \lambda_k\}} \|\mathbf{z}_j\|_E^2 - \eta_\alpha \|\mathbf{u}\|_E^2 - \mathcal{F}(\mathbf{u}_j) - \mathcal{H}(\mathbf{u}_j). \tag{4.76}$$

Now, proceeding as in the proof of Proposition 4.6, we reach a contradiction analogous to (4.33) or to (4.40).

The constants in the statement of the proposition can be obtained as follows: we first get $\Theta \in \mathbb{R}$ satisfying (4.71), then we obtain $\varepsilon_2 > 0$ and $r_0 > 0$ satisfying (4.72) and (4.73). These estimates hold uniformly for $a + b \in (\lambda_k - \varepsilon_2, \lambda_k)$ and $a - b \in (\lambda_l, \lambda_l + \varepsilon_2)$. Then, fixing a, b respecting these conditions, we can obtain a constant $\Lambda_{a,b} \in \mathbb{R}$ satisfying (4.69). Finally, we get $\omega_{a,b} > 0$ satisfying (4.70); observe that we may also choose $\omega_{a,b} > r_0$, as desired. □

With the geometries obtained above and the estimate (4.63) in the place of (4.46), by repeating the same arguments from sections 4.2 and 4.4, we conclude that the functional J^+ has two distinct critical points in E, corresponding to solutions of problem (1.1+), and then proving Theorem 1.3.

Theorem 1.4 follows from Theorem 1.3 by the same argument used to prove Theorem 1.2.

4.6 The PS-conditions

In this section we give the proof of the Lemmas 4.8 and 4.13, which show that the functionals $J_n^{\pm}$ satisfy the PS-condition in the finite dimensional space E_n, for every $n > k + m$, and that the sequences of constrained critical points obtained in the sections 4.2 and 4.3 are bounded.

Proof of Lemma 4.8 and Lemma 4.13. The two lemmas have different statements but the proof in quite similar.

By the hypothesis $a \pm b \notin \sigma(-\Delta)$, the eigenvalues $\mu_i : \ i \in \mathbb{Z}_0$ are all different from zero, then we may divide the space E in the two orthogonal components

$$E^- = \overline{span\{\boldsymbol{\psi}_i : \, i \in \mathbb{Z}_0, \, \mu_i < 0\}} \qquad \textit{and} \qquad E^+ = \overline{span\{\boldsymbol{\psi}_i : \, i \in \mathbb{Z}_0, \, \mu_i > 0\}}\,,$$

and for every $j > k + m$, we set $E_j^+ = E^+ \cap E_j$ and $E_j^- = E^- \cap E_j$. Since $\mu_i \to \pm 1$ when $i \to \pm\infty$, we may set $\xi := \inf\{|\mu_i| : \ i \in \mathbb{Z}_0\} > 0$ and we have

$$\begin{cases} B(\mathbf{u}, \mathbf{u}) \leq -\xi \|\mathbf{u}\|_E^2\,, & \text{for every } \mathbf{u} \in E^-, \\ B(\mathbf{u}, \mathbf{u}) \geq \xi \|\mathbf{u}\|_E^2\,, & \text{for every } \mathbf{u} \in E^+. \end{cases} \tag{4.77}$$

Obviously ξ depends on a, b, but this in of no importance in this proof.

We write $\mathbf{u}_i = \mathbf{z}_i + \mathbf{p}_i$, where $\mathbf{z}_i \in E^-$ and $\mathbf{p}_i \in E^+$. We will show that $\mathbf{z}_i$ and $\mathbf{p}_i$ are bounded in E.

Actually, by testing (4.41) (resp. (4.52)) with $\boldsymbol{\phi} = \mathbf{z}_i$ we get

$$\left| B(\mathbf{u}_i, \mathbf{z}_i) \mp \mathcal{F}'(\mathbf{u}_i)[\mathbf{z}_i] \mp \mathcal{H}'(\mathbf{u}_i)[\mathbf{z}_i] \right| \leq \varepsilon_i \|\mathbf{z}_i\|_E\,. \tag{4.78}$$

By (3.14), we have $B(\mathbf{u}_i, \mathbf{z}_i) = B(\mathbf{z}_i, \mathbf{z}_i)$, then

$$-\varepsilon_i \|\mathbf{z}_i\|_E \leq B(\mathbf{z}_i, \mathbf{z}_i) + S_0 \left(1 + \|\mathbf{u}_i\|_E^{q-1}\right) \|\mathbf{z}_i\|_E + H \|\mathbf{z}_i\|_E\,. \tag{4.79}$$

Using (4.77), it follows that

$$-\varepsilon_i \|\mathbf{z}_i\|_E \leq -\xi \|\mathbf{z}_i\|_E^2 + S_0 \left(1 + \|\mathbf{u}_i\|_E^{q-1}\right) \|\mathbf{z}_i\|_E + H \|\mathbf{z}_i\|_E\,; \tag{4.80}$$

dividing by $\|\mathbf{z}_i\|_E$ and reordering, we get

$$\xi \|\mathbf{z}_i\|_E \leq \varepsilon_i + S_0 \|\mathbf{u}_i\|_E^{q-1} + (S_0 + H). \tag{4.81}$$

On the other hand, by testing (4.41) (resp. (4.52)) with $\boldsymbol{\phi} = \mathbf{p}_i$ and reasoning as above, we get

$$B(\mathbf{p}_i, \mathbf{p}_i) - S_0 \left(1 + \|\mathbf{u}_i\|_E^{q-1}\right) \|\mathbf{p}_i\|_E - H \|\mathbf{p}_i\|_E \leq \varepsilon_i \|\mathbf{p}_i\|_E\,; \tag{4.82}$$

by using (4.77), reordering and dividing by $\|\mathbf{p}_i\|_E$, it follows that

$$\xi \|\mathbf{p}_i\|_E \leq \varepsilon_i + S_0 \|\mathbf{u}_i\|_E^{q-1} + (S_0 + H). \tag{4.83}$$

From (4.81) and (4.83), we get

$$\xi \|\mathbf{u}_i\|_E \leq 2\varepsilon_i + 2S_0 \|\mathbf{u}_i\|_E^{q-1} + 2(S_0 + H), \tag{4.84}$$

implying that $\{\mathbf{u}_i\}$ is bounded in E, since $q - 1 < 1$.

This concludes the proof of Lemma 4.13. In the case of Lemma 4.8, since $\{\mathbf{u}_i\} \subseteq E_n$, which is finite dimensional, it follows that it has a convergent subsequence. □

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