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THE MEASURABILITY OF PETTIS INTEGRABLE FUNCTIONS AND APPLICATIONS

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The Measurability of Pettis Integrable Functions and Applications

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In this paper we study the class of Banach spaces X for which every Pettis integrable map $f : [0,1] \longrightarrow X$ is measurable, where $[0,1]$ is the real unit interval with Lebesque measure λ. Although we assume familiarity with the notions of measurability of functions in a Banach space ([Y]), as well as with Pettis and Riemann integration ([DU],[P]), we recall in section 1, for the reader's convenience, the basic concepts of Pettis integration over probability spaces with values in a Banach space, together with some of the fundamental results of this theory. One can find a wealth of information on this topic in [DU] and [T]. We also note that, for [0,1] with the Lebesgue measure, every Riemann integrable function is Pettis integrable.

The basic categories of spaces we will be interested in are singled out in :

Definition 1 *: A Banach space X is said to be*

i) **Pm** *when all bounded Pettis integrable maps $f : [0,1] \longrightarrow X$ are measurable. Otherwise, it is said to be* **nPm**.

ii) **Rm** *when all Riemann integrable functions $f : [0,1] \longrightarrow X$ are measurable. Otherwise, it is said to be* **nRm**.

X is said to have **property P** *if for each bounded Pettis integrable map $f : [0,1] \longrightarrow X$, with null Pettis integral over* **all** *measurable $A \subseteq [0, 1]$, we have $f = 0$ ae.*

X is said to have **property R** *if for each Riemann integrable map $f : [0, 1] \longrightarrow X$ such that $\forall\, t \in [0,1]$ $\int_{[0,t]} f \, \mathrm{dt} = 0$, we have $f = 0$ λ - ae.*

A compact Hausdorff space is said to be Pm or to have property P if C(K) has the corresponding property.

In section 2 we gather some of the fundamental, but mostly elementary, relations between these concepts and give examples of Pettis and Riemann integrable maps which are not measurable.

We verify that every Pm space has property P and observe that the converse is false.

There are several examples of nPm (not Pm) spaces : among them, $l_p(I)$, with $|I| \geq 2^{\aleph_0}$, for $1 < p \leq \infty$.

On the other hand, the study of the Pm property for the case $p = 1$ involves a combinatorial problem related to intersections of finite sets, indexed by a subset $A \subseteq [0,1]$ with $\lambda^*A > 0$. This lead us to state a measure theoretic analogue of the Erdös-Rado theorem, whose proof is the subject of section 3.

As an application of this property and assuming the continuum hypothesis (CH), we prove in section 4 that the operation of l_1-sums preserves the property of being Pm. For spaces of type $[l_1(I) \oplus [\Sigma_{\alpha \in A} L_1[0,1]]_1]_1$, it's possible to show, without CH, that they are Pm.

In section 5 we discuss abstract L_p-spaces, with $1 < p < \infty$. Assuming CH, and using methods analogous to those in [MR], it is shown that such spaces are Pm iff they are separable.

Since $C(K)^*$ is an abstract L_1-space for every compact Hausdorff topological space K, we apply the results of section 4 to show that $C(K)^*$ is Pm iff K is measure separable.

In section 6, a sufficient condition, indepedent of CH, for X to be a Pm space is presented. If X has property P and the **separable projection property** (every separable subspace of X is contained in a complemented separable subspace of X) then X is Pm.

In section 7 we prove that nonseparable WCG Banach spaces are not Pm. This allows us to conclude that if K is a measure separable compact then $C(K)^*$ doesn't contain any subspace isomorphic to a nonseparable WCG space.

In sections 8 and 9, we study the Pm property in C(K) spaces. It's shown that if K is dyadic then C(K) is Pm. We also show that there are non dyadic compacts with this property. In fact, more is true : if K is a separable compact and X a Banach space, both Pm, so is C(K, X). Conditions are also presented for C(K) to be a Pm space when K is an arbitrary product of compacts.

In section 10 we discuss, using Martin's axiom, the independence of some of the statements proved with aid of the continuum hypothesis.

A comment on notation and terminology. Unless express mention to the contrary, all vector spaces will be over the reals and all normed spaces complete (Banach); all compact spaces shall be Hausdorff and all measures will be positive and finite.

We use standard notation for duals. Thus X^* is the dual of X, with its elements denoted by x^*. As usual, the norm in X^* is given by $\|x^*\| = \sup_{\|x\| \leq 1} |x^*(x)|$.

For $A \subseteq X$, $\overline{A}$ denotes the closure of A in the norm topology, while $\overline{A}^w$ is the closure of A in the weak topology.

CH is the continuum hypothesis, $2^{\aleph_0} = \aleph_1$. We may use c as shorthand for $2^{\aleph_0}$.

If A is a set, $|A|$ is its cardinality. We write $\triangle$ for the symmetric difference of sets ($A \triangle B = (A - B) \cup (B - A)$), as well as for the end of a proof.

If X is a vector space and $K \subseteq X$, span K is the subspace generated by K.

1 Preliminaries

Let X be a Banach space, X^* the dual of X and $(S, \mathcal{B}, \mu)$ a complete probability space.

Definition 2 : *A function $f : S \longrightarrow X$ is called*

a) **measurable** *if there is a sequence $S \xrightarrow{f_n} X$ of simple functions such that $f_n \longrightarrow f$ μ-ae.*

b) **weakly (or scalarly) measurable** *if, for each $x^* \in X^*$, $x^* \circ f : S \longrightarrow \mathbb{R}$ is measurable.*

c) **scalarly L_1** *if $x^* \circ f \in L_1(\mu)$, for all $x^* \in X^*$.*

The connection between measurability and weak measurability is described by the following theorem of Pettis.

Theorem 1 ([T], pg. 33) : *For a function $f : S \longrightarrow X$, are equivalent:*

a) f is measurable;

b) f is scalarly measurable and f has almost separable range (that is, there exists $A \in \mathcal{B}$ such that $\mu A = 1$ and $f(A) \subseteq X$ is separable). $\triangle$

In particular, if X is separable, the notions of measurability and weak measurability coincide.

Definition 3 : *Let $f : S \longrightarrow X$ be a scalarly L_1 function. We say that f is* **Pettis integrable** *if, for each $E \in \mathcal{B}$, there is $x_E \in X$ such that*

$$x^*(x_E) = \int_E f \, d\mu, \quad \forall \, x^* \in X^*.$$

We denote the element x_E of X by $\int_E f \, d\mu$.

Observe that a Pettis integrable function doesn't need to be measurable (as the Bochner integrable ones) but only scalarly measurable (and scalarly L_1).

Example 1 : Recall that if I is any set, $c_0(I)$ is the Banach space whose underlying set is given by

$$c_0(I) = \{ x = (x_i) \in \mathbb{R}^I : x \text{ is bounded and for each } \epsilon > 0, \text{ there is a finite set } F_\epsilon \subseteq I \text{ such that } \{t : |x_t| > \epsilon\} \subseteq F_\epsilon \},$$

with the norm $\|x\|_\infty = \sup \{|x_t| : t \in I\}$.

Let [0,1] provided with the Lebesgue measure and define

$$f : [0,1] \longrightarrow c_0([0,1]), \quad \text{by } f(t) = e_t$$

where

$$e_t(t') = \begin{cases} 1 & \text{if } t' = t \\ 0 & \text{if } t \neq t' \end{cases}$$

We have $c_0([0,1])^* = l_1([0,1])$. Then, given $x^* \in c_0([0,1])$, there is $a_t \in l_1([0,1])$ such that $x^*(e_t) = a_t$, $\forall\, t \in [0,1]$. Moreover, the set $\{t \in [0,1] : a_t \neq 0\}$ is countable, and so we may write it as $\{t_1, \ldots, t_n, \ldots\}$. Thus,

$$x^* \circ f(t) = \begin{cases} a_{t_n} & \text{if } t = t_n \text{ for some } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and we conclude that $x^* \circ f \neq 0$ only in a countable subset of [0,1], i.e., $x^* \circ f = 0$ λ ae. It's simple to verify that $x_E = 0$ for all $E \in \Sigma$; thus f is Pettis integrable.

But f is not measurable because it doesn't have almost separable range : given a measurable A with $\mu A = 1$, we have $|A| = |\mathbb{R}|$ and for distinct t, t' $\in$A, $\|e_t - e_{t'}\|_\infty = 1$. **This shows that $c_0([0, 1])$ is not Pm and does not have property P.**

Example 2 : When $f : S \longrightarrow X$ is Pettis integrable, we have $x^* \circ f : S \longrightarrow \mathbb{R}$ integrable (in the classical sense), for all $x^* \in X^*$. But f need not be bounded. As an example take $f : [0, 1] \longrightarrow c_0\,([0, 1])$ given by

$$f(t) = \begin{cases} (1/t)e_t & \text{if } t \neq 0 \\ e_0 & \text{if } t = 0 \end{cases}$$

The map f is scalarly measurable and Pettis integrable, with $x_E = 0$, for all measurable E. But f is neither bounded nor bounded almost everywhere; moreover, f is not measurable and, consequently, not Bochner integrable. The above example illustrates that there are Pettis integrable functions that are not Bochner integrable. We register that all Bochner integrable maps are Pettis integrable, with the same integral over all measurable sets.

We assume that the reader is familiar with the basic results of Pettis integration, as presented for instance in [DU] or [T]. In particular, for each measurable E, the map $f \mapsto \int_E f\, d\mu$ is linear and preserved by bounded linear operations $T : X \longrightarrow Y$, that is, $\int_E T \circ f\, d\mu = T(\int_E f\, d\mu)$.

It's well known that a Pettis integrable $f : S \longrightarrow X$ originates a map from $\mathcal{B} \longrightarrow X$, given by $E \mapsto \int_E f\, d\mu$, which is a completely additive X valued vector measure with weakly compact range. Furthermore, this vector measure is μ continuous : for all $E \in \mathcal{B}$, $\mu\, E = 0$ implies $F(E) = 0$.

For perfect measure spaces, there is an important theorem due to Stegall.

Definition 4 *: A finite measure space $(S, \mathcal{B}, \mu)$ is* **perfect** *if, for each measurable map $h : S \longrightarrow \mathbb{R}$, for each set $E \subseteq \mathbb{R}$, if $h^{-1}(E) \in \mathcal{B}$, there is a Borel set $C \subseteq E$ such that $\mu h^{-1}(C) = \mu h^{-1}(E)$. All Radon measure spaces are perfect.*

Theorem 2 (Stegall, [T], pg.47) *: Let $f : S \longrightarrow X$ be a Pettis integrable function. If $(S, \mathcal{B}, \mu)$ is a perfect measure space, then $\{\int_E f \, d\mu : E \in \mathcal{B}\}$ is relatively compact (and so separable) in X.* $\triangle$

Corollary 1 ([T], pg.46ff) *: If $f : [0,1] \longrightarrow X$ is Pettis integrable then $\overline{\{\int_E f \, d\mu : E \in \mathcal{B}\}}$ is norm compact.* $\triangle$

The next Lemma is a straightforward application of the Hahn-Banach theorem.

Lemma 1 *: Suppose that $f : S \longrightarrow X$ is Pettis integrable and let A, B be measurable sets. Then, $\mu(A \triangle B) = 0$ implies $\int_A f \, d\mu = \int_B f \, d\mu$.* $\triangle$

Observation 1 : Let $(S, \mathcal{B}, \nu)$ be a finite measure space.

Given A, B $\in \mathcal{B}$, define A to be equivalent to B (A $\sim$ B) if $\nu(A \triangle B) = 0$, where $A \triangle B = (A - B) \cup (B - A)$; $\sim$ is an equivalence relation. Let $\tilde{\mathcal{B}} = \mathcal{B}/\sim$ be the quotient algebra and for each $A \in \mathcal{B}$, write $\tilde{A} = \{B \in \mathcal{B} : A \sim B\}$; then $\tilde{\mathcal{B}} = \{\tilde{A} : A \in \mathcal{B}\}$.

Observe that A $\sim$ A' and B $\sim$ B' implies $\nu(A \triangle B) = \nu(A' \triangle B')$. Thus, the function

$$\tilde{\nu} : \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \longrightarrow [0, \infty), \quad \text{given by} \quad \tilde{\nu}(\tilde{A}, \tilde{B}) = \nu(A \triangle B)$$

is well defined, where A and B are representatives of $\tilde{A}$ and $\tilde{B}$, respectively; $\tilde{\nu}$ defines a metric on $\tilde{\mathcal{B}}$.

A measure ν is said to be **separable (complete)** and $(S, \mathcal{B}, \nu)$ to be **separable in measure (resp., complete)** when $(\tilde{\mathcal{B}}, \tilde{\nu})$ is a separable (resp., complete) metric space. It can be shown that the following are equivalent : (See [L], page 121 and [DS], page 169).

(i) ν is a separable measure ; (ii) $L_1(\nu)$ is a separable Banach space ;

(iii) $L_p(\nu)$ is a separable Banach space, for $1 \leq p < \infty$.

According to the notation set down above, if $\mu(A \triangle B) = 0$, for A, B $\in \mathcal{B}$, then $\tilde{A} = \tilde{B}$ in $\tilde{\mathcal{B}}$. Moreover, for $f : S \longrightarrow X$ Pettis integrable, $x_A = x_B$ (Lemma 1). Therefore, we can consider the function $\tilde{f} : (\tilde{\mathcal{B}}, \tilde{\mu}) \longrightarrow X$ defined by $\tilde{f}(\tilde{E}) = \int_E f \, d\mu$ where $E \in \mathcal{B}$ is a representative of $\tilde{E} \in \tilde{\mathcal{B}}$.

Proposition 1 *: If $f : S \longrightarrow X$ is a bounded Pettis integrable function then*

$$\tilde{f} : (\tilde{\mathcal{B}}, \tilde{\mu}) \longrightarrow X, \text{ defined by } \tilde{f}\,(\tilde{E}) = \int_E f \, d\mu$$

is $\tilde{\mu}$ - $\|.\|$ uniformly continuous : for each $\varepsilon > 0$, there is $\delta > 0$ such that, if $E, E' \in \mathcal{B}$ and $\mu(E \triangle E') < \delta$ then $\|\int_E f \, d\mu - \int_{E'} f \, d\mu\| < \epsilon$.

Proof : By hypothesis, there is M > 0 such that $\|f(t)\| \leq M$, $\forall$ t $\in$ S. Given E, E' $\in \mathcal{B}$ and $x^* \in X^*$ we have :

$$\begin{aligned}|x^*(x_E) - x^*(x_{E'})| &= |\int_E x^* \circ f \, d\mu - \int_{E'} x^* \circ f \, d\mu| \\ &= |\int_{E-E\cap E'} x^* \circ f \, d\mu - \int_{E'-E\cap E'} x^* \circ f \, d\mu| \\ &\leq \int_{E-E\cap E'} |x^* \circ f| \, d\mu + \int_{E'-E\cap E} |x^* \circ f| \, d\mu \\ &= \int_{E\Delta E'} |x^* \circ f| \, d\mu \leq M \, \|x^*\| \, \mu(E \, \Delta \, E').\end{aligned}$$

So, $\|x_E - x_{E'}\| \leq M \, \tilde{\mu}(\tilde{E}, \tilde{E'})$, showing that $\tilde{f}$ is uniformly continuous. Δ

In [P], by a more elaborate argument, Pettis showed that $\tilde{f}$ is absolutely continuous, even when f is not bounded.

When the measure space is [0,1] with Lebesque measure we have

Corollary 2 : *Let $f : S \longrightarrow X$ be a bounded and Pettis integrable function. If we define*

$$\tilde{f} : [0,1] \longrightarrow X, \text{ by } \tilde{f}(t) = \int_{[0,t]} f \, d\mu = x_{[0,t]},$$

then $\tilde{f}$ is uniformly continuous. Δ

Lemma 2 : *Let $(S, \mathcal{B}, \mu)$ be a separable measure space and $(D_n)_{n\geq 1} \subseteq \mathcal{B}$ be such that $(\widetilde{D_n})_{n\geq 1}$ is dense in $\tilde{\mathcal{B}}$.*

a) If $f : S \longrightarrow X$ is a bounded Pettis integrable map and Y is a closed subspace of X such that $\int_{D_n} f \, d\mu \in Y$, $\forall \, n \geq 1$, then $\int_E f \, d\mu \in Y$, $\forall \, E \in \mathcal{B}$.

b) If $f, g : S \longrightarrow X$ are bounded and Pettis integrable functions and $\int_{D_n} f \, d\mu = \int_{D_n} g \, d\mu$, $\forall \, n \geq 1$, then $\int_E f \, d\mu = \int_E g \, d\mu$, $\forall \, E \in \mathcal{B}$.

c) Let $f, g : [0,1] \longrightarrow X$ Pettis integrable maps. If $\int_{[0,t]} f \, d\mu = \int_{[0,t]} g \, d\mu$, $\forall \, t \in [0,1]$, then f and g have the same Pettis integral : $\int_E f \, d\mu = \int_E g \, d\mu$, $\forall \, E \in \Sigma$.

Proof : Item (a) is a direct consequence of Proposition 1.

b) It's sufficient to prove that if $g : S \longrightarrow X$ is bounded, Pettis integrable and such that $\int_{D_n} g \, d\mu = 0$, $\forall$ n $\geq$ 1, then $\int_E g \, d\mu = 0$, $\forall$ E $\in \mathcal{B}$. Fix $\varepsilon > 0$ and E $\in \mathcal{B}$. By Proposition 1 we can select $\delta > 0$ such that

$$\mu(E \, \Delta \, E') < \delta \implies \|\int_E g \, d\mu - \int_{E'} g \, d\mu\| < \varepsilon.$$

Moreover, there is n $\geq$ 1 such that $\mu(E \, \Delta \, D_n) < \delta$. Then,

$$\left\|\int_{D_n} g \, d\mu - \int_E g \, d\mu\right\| < \varepsilon.$$

Since $\int_{D_n} g \, d\mu = 0$, we get $\|\int_E g \, d\mu\| < \varepsilon$,

c) It's sufficient to note that if $s \leq r$ are rationals in [0, 1] then

$$\int_{[s,r]} f \, d\mu = \int_{[0,r]} f \, d\mu - \int_{[0,s]} f \, d\mu = \int_{[s,r]} g \, d\mu,$$

and that the family of intervals with rational endpoints is measure dense in the Lebesque measurable sets. $\triangle$

Before showing that Riemann integrable maps are Pettis integrable we set down

Definition 5 : *(i) A partition P of the interval [a, b] $\subseteq \mathbb{R}$ is a finite sequence $a = t_0 < t_1 < \ldots < t_n = b$. We set $\Delta t_i = t_i - t_{i-1}$, $1 \leq i \leq n$. Further, $\Delta P = max\{\Delta t_i : 1 \leq i \leq n\}$ is the* **diameter** *of P.*

(ii) Let $f : [a, b] \longrightarrow X$ be a function and $P = \{t_0, \ldots, t_n\}$ a partition of [a, b]. An element $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_n) \in \prod_{i=1}^n [t_{i-1}, t_i]$ is called a **sample** *in the partition P. We set $S(f, P, \vec{\xi}) = \sum_{i=1}^n f(\xi_i)\Delta t_i$.*

(iii) $f : [a, b] \longrightarrow X$ is Riemann integrable iff there is $x \in X$ such that $lim_{\Delta P \to 0} S(f, P, \vec{\xi}) = x$, that is :

for each $\varepsilon > 0$, there is $\delta > 0$ such that, for each partition P of [a, b], if $\Delta P < \delta$,

then $\|S(f, P, \vec{\xi}) - x\| < \varepsilon$, for all samples $\vec{\xi}$ in P.

This unique element x is called **the Riemann integral of f** *and will be denoted by* $\int_{[a,b]} f \, dt$. **The use of the variable t will always indicate that we are considering the Riemann integral**. *Typically, in what follows, we will have [a, b] = [0,1].*

The definition in (iii) is equivallent to the following : there is $x \in X$ such that

$\forall \varepsilon > 0 \; \exists n_0 \geq 1$ such that, if P is a partition of [a, b] in subintervals of lenght δ_1,

with $\delta_1 < 1/n_0$, then $\|S(f, P, \vec{\xi}) - x\| < \varepsilon$, for all samples $\vec{\xi}$ in P.

The map in Example 1 is Riemann integrable with null integral everywhere.

Proposition 2 : *A Riemann integrable function $f : [0,1] \longrightarrow X$ is Pettis integrable, with the same integral over every subinterval of [0,1].*

Proof : For $E \in \Sigma$, we must find x_E such that $x^*(x_E) = \int_E x^* \circ f \, d\lambda$, $\forall x^* \in X^*$. (1)

For $E = [0,t]$ we take $x_E = \int_{[0,t]} f \, dt$. Then, for $0 \leq t_1 < t_2 \leq 1$, if $E = [t_1, t_2]$ or (t_1, t_2) or $[t_1, t_2)$ or $(t_1, t_2]$, we have $x_E = \int_E f \, dt$. In all these cases, $x^*(x_E) = \int_E x^* \circ f \, d\lambda$,

$\forall\, x^* \in X^*$. Further, equality (1) is still true if E is a finite union of mutually disjoint intervals.

If $E \in \Sigma$ any measurable set, there is, given $\varepsilon > 0$, a finite union V of open intervals such that $\lambda(E \triangle V) < \varepsilon$. Suppose $V = \bigcup \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$ and set $I_j = (a_j, b_j) - \bigcup_{i=1}^{j-1} (a_i, b_i),\ \ 1 \le j \le r$.

We have $V = \bigcup_{j=1}^{r} I_j$ and $I_j \cap I_{j'} = \emptyset$ if $j \neq j'$; so, we can suppose V is a finite union of mutually disjoint intervals (now, not necessarily open).

For each $n \ge 1$, choose a finite union E_n of mutually disjoint intervals I_j^n, $1 \le j \le k_n$, such that $\lambda(E \triangle E_n) < 1/n$. We know that we have $x_{E_n} = \sum_{j=1}^{k_n} \int_{I_j^n} f\, dt$.

Fact : $(x_{E_n})_{n \ge 1}$ is a Cauchy sequence in X.

In effect, given $x^* \in X^*$ and $n, m \ge 1$, $n \ne m$, we have :

$$\begin{aligned}
|x^*(x_{E_n} - x_{E_m})| &= |x^*(x_{E_n}) - x^*(x_{E_m})| \\
&= \left|\int_{E_n} x^* \circ f\, d\lambda - \int_{E_m} x^* \circ f\, d\lambda\right| \\
&\le \int_{E_n - E_m} |x^* \circ f|\, d\lambda + \int_{E_m - E_n} |x^* \circ f|\, d\lambda \\
&= \int_{E_n \triangle E_m} |x^* \circ f|\, d\lambda \le \|x^*\|\, M\, \lambda(E_n \triangle E_m),
\end{aligned}$$

where $M > 0$ is such that $\|f(t)\| \le M$, $\forall\, t \in [0,1]$.

We also have $\lambda(E_n \triangle E_m) \le \lambda(E_n \triangle E) + \lambda(E_m \triangle E)$. Thus,

$$|x^*\,(x_{E_n} - x_{E_m})| \le \|x^*\|\, M\, (\, 1/n + 1/m\,),\ \forall\, m, n \ge 1.$$

Given $\varepsilon > 0$, choose $n_0 > (2M)/\varepsilon$. Then, for $m, n \ge n_0$,

$\|x_{E_n} - x_{E_m}\| = \sup_{\|x^*\| \le 1} |x^*(x_{E_m} - x_{E_n}| \le M(1/n + 1/m) \le (2/n_0)\, M < \varepsilon$,

proving the Fact.

Let $x_E \in X$ be the limit of the sequence x_{E_n} in X. We have to verify that $x^*(x_E) = \int_E x^* \circ f\, d\mu$, $\forall\, x^* \in X^*$.

Since $x_{E_n} \longrightarrow x_E$ in norm, for each $x^* \in X^*$, $\int_{E_n} x^* \circ f\, d\mu \longrightarrow x^*(x_E)$ in $\mathbb{R}$. (2)

We also have, $\left|\int_E x^* \circ f\, d\mu - \int_{E_n} x^* \circ f\, d\mu\right| \le \|x^*\|\, M\, \mu(E_n \triangle E) < \|x^*\|\, M\, 1/n$ and so, $\int_{E_n} f\, d\lambda \longrightarrow \int_E f\, d\lambda$ in $\mathbb{R}$. (3)

From (2) and (3) we get $x^*(x_E) = \int_E x^* \circ f\, d\lambda$, ending the proof. $\triangle$

Observation 2 : We can derive more from the above result : if $f : [0,1] \longrightarrow X$ is a bounded function such that, for each $t \in [0,1]$, there is $x_{[0,t]} \in X$

with $x^*(x_{[0,t]}) = \int_{[0,t]} x^* \circ f \, d\mu, \forall x^* \in X^*$, then f is Pettis integrable.

2 Pm spaces and property P

In this section we discuss the relations between the concepts set down in Definition 1 of the Introduction. We start with

Proposition 1 : *Let X and Y be Banach spaces.*

i) If X is separable then X is Pm (and Rm).

ii) If X is Pm (Rm) and Y is a closed subspace of X then Y is Pm (resp., Rm).

iii) If X is Pm (Rm) and $T : X \longrightarrow Y$ is a linear isomorphism of X onto Y then Y is Pm (resp., Rm).

Proof : (i) is a consequence of Pettis' theorem, caracterizing measurable functions, while (ii) is clear. Item (iii) is a consequence of the fact that a linear isomorphism preserves measurability. $\triangle$

From Proposition 1.2 we get

Proposition 2 : *All Pm spaces are Rm.* $\triangle$

Problem 1 : Is the converse of Proposition 1 true ?

A Pettis integrable $S \xrightarrow{f} X$ is a **null function** if $\int_E f \, d\mu = 0$, for all measurable $E \subseteq S$. Clearly, f is a null function iff $x^* \circ f = 0$ a.e., $\forall x^* \in X^*$. When $S = [0,1]$, we have that f is a null function iff for all $t \in [0,1]$ and $x^* \in X^*$, $\int_{[0,t]} x^* \circ f \, d\mu = 0$.

Observe that the function $f : [0,1] \longrightarrow c_0([0,1])$, $f(t) = e_t$, has null Pettis integral but is not zero almost everywhere in [0, 1]. Thus there is an important distinction between a null function and one which is zero a.e. In a Pm space, both concepts of course coincide.

With respect to property P, note that Observation 1.2 yields that X has P when, for each bounded Pettis integrable function $f : [0,1] \longrightarrow X$, if $\int_{[0,t]} f \, d\mu = 0$, $\forall t \in [0,1]$, then $f = 0$ λ - ae.

We have already noted in Example 1.1 that $c_0([0,1])$ does not have property P. It's quite clear that we have

Lemma 1 : *Property P is inherited by subspaces and preserved by linear isomorphism.* $\triangle$

We shall have more to say about property P latter on. For the moment, we prove

Proposition 3 : *All Pm spaces have property P.*

Proof : Let be f : [0,1] $\longrightarrow$ X be a bounded, Pettis integrable function with $\int_E f\, d\mu = 0, \forall E \in \Sigma$. Since X is Pm, f is measurable, and so $\|f\|$ is measurable. Moreover, since f is bounded, $\|f\|$ is Lebesgue integrable. Consequently, f is Bochner integrable and its Bochner integral is zero over every measurable E. But this implies that $f = 0$ λ - ae. $\triangle$

There are, of course, analogous results for Rm spaces.

Proposition 4 *([DI]) : A Rm space has property R.* $\triangle$

There are, however, spaces with R and P which are not Pm or Rm. An example is $l_\infty(\mathbb{N})$ (see example 4 below).

Example 1 : The spaces $l_p(I)$, $1 < p < \infty$, with $|I| \geq 2^{\aleph_0}$, do not have property R and thus, do not have P and are neither Pm nor Rm. To see this, let $t \mapsto \eta_t$ be an injective map from [0,1] into I. Define f : [0,1] $\longrightarrow l_p(I)$ by $f(t) = e_{\eta_t}$ where

$$e_{\eta_t}(i) = \begin{cases} 1 & \text{if } i = \eta_t \\ 0 & \text{otherwise} \end{cases}$$

It can be shown that f is Riemann integrable, with $\int_{[0,t]} f\, dt = 0, \forall t \in [0,1]$. It's clear that f is distinct from zero at all $t \in [0, 1]$. In fact, if t, t' are distinct points in [0,1], then $\|f(t) - f(t')\|_p = 2^{1/p}$, and the range of f cannot be almost separable.

Example 2 : $c_0(I)$, $|I| \geq 2^{\aleph_0}$, is nPm (and nRm), because $c_0(I)$ contains a subspace isomorphic to $c_0([0,1])$.

Example 3 : $L_\infty([0,1])$ is nRm (and so, nPm). To see this, define f : [0,1] $\longrightarrow L_\infty([0,1])$ by

$$f(t) = \chi_{[0,t]} = \text{characteristic map of the interval } [0,t].$$

We will show that $\int_{[0,1]} f\, dt = g$, where $g(t) = 1 - t$, $\forall t \in [0,1]$ (i.e., f is Riemann integrable).

Let $P = \{0 = t_0 < t_1 < \ldots t_n = 1\}$ be a partition of [0,1]. For each i, $1 \leq i \leq n$, choose $\xi_i \in [t_{i-1}, t_i]$. Then, $(\Sigma_{i=1}^n f(\xi_i) \triangle t_i - g)(t) = \Sigma_{i=1}^n \chi_{[0,\xi_i]}(t) \triangle t_i - g(t)$.

For $t \neq 0$, choose j such that $t_{j-1} < t \leq t_j$. Then,

$|\Sigma_{i=1}^n \chi_{[0,\xi_i]}(t) \Delta t_i - g(t)| =$

$$= \begin{cases} |\Sigma_{i=j}^n \Delta t_i - (1-t)| = t - \Sigma_{i=1}^{j-1} \Delta t_i \leq \Delta t_j \leq \Delta P & \text{if } t \in [t_{j-1}, \xi_j] \\ |\Sigma_{i=j+1}^n \Delta t_i - (1-t)| = t - \Sigma_{i=1}^{j} \Delta t_j \leq \Delta P & \text{if } t \notin [t_{j-1}, \xi_j] \end{cases}$$

So, $\|\Sigma_{i=1}^{n} f(\xi_i) \triangle t_i - g\|_\infty \leq \triangle P$, proving that f is Riemann (and Pettis) integrable. But f does not have almost separable range and so cannot be measurable : for distinct t, t' in [0,1], $\|f(t) - f(t')\|_\infty = 1$.

Example 4 : $l_\infty(\mathbb{N})$ is nRm (so, it's nPm). Although it's possible to show this directly, it's sufficient to remember Pelczynski's theorem : $l_\infty(\mathbb{N})$ is isomorphic to $L_\infty([0,1])$.

Note that $l_\infty(I)$ is nRm (and nPm), if $|I| \geq \aleph_0$, since it contains $l_\infty(\mathbb{N})$ isometrically.

Example 5 : Let be $(X_i)_{i\in I}$ a family of non zero spaces with $|I| \geq 2^{\aleph_0}$. Then, the l_p sum $X = (\oplus X_i)_p$, $1 < p \leq \infty$ or $p = 0$, is not Pm (Rm).

Recall that, for $1 < p < \infty$, $X = \{x = (x_i)_{i\in I} : x_i \in X_i, \forall\, i \in I \text{ and } \sum_{i\in I} \|x_i\|^p < \infty\}$, with $\|x\|_p = (\sum_{i\in I} \|x_i\|^p)^{1/p}$.

For $p = \infty$, $X = \{x = (x_i)_{i\in I} : \sup_{i\in I}\|x_i\| < \infty\}$, with $\|x\|_\infty = \sup_{i\in I} \|x_i\|$; the case $p = 0$ was described in Example 1.1.

In each case it's possible to construct a linear isometry of $l_p(I)$ or $(c_0(I))$ into X and so it inherits from the classical sequence spaces the fact that it does not have R, P, Rm or Pm.

For $p = 1$, the situation is quite different : in the presence of CH, the l_1 - sum of Pm spaces is Pm . The next two sections shall be devoted to this result.

We end this section with

Proposition 5 : *A Pm space that contains an isomorphic copy of $c_0(\mathbb{N})$ is not a dual.*

Proof : By Pelczynski's theorem ([Pe]), if a dual space contains $c_0(\mathbb{N})$ then it contains $l_\infty(\mathbb{N})$. This is not possible because X is Pm and $l_\infty(\mathbb{N})$ is not. $\triangle$

3 A measure theoretic Erdös-Rado theorem

To simplify exposition we set down

Definition 1 : *A family $\{S_i : i \in I\}$ of sets is* **quasi-disjoint** *if there is J such that $S_i \cap S_{i'} = J$, for all distinct i, i' $\in$ J. In particular, a family of pairwise disjoint sets is quasi-disjoint.*

We may phrase the well known Erdös - Rado theorem for finite sets as

Theorem 1 (CH) : (Erdös-Rado, see [C], page 5) : *Let $\{S_\xi : \xi \in A\}$ be a family of finite sets, with $|A| = c$. Then there is $B \subseteq A$ such that $|B| = c$ and $\{S_\xi : \xi \in B\}$ is quasi-disjoint.*

We wish to establish the following measure theoretic analogue of Theorem 1, where λ^* denotes Lebesque **outer measure**.

Theorem 2 (CH) : *Let A be a subset of [0,1] with $\lambda^* A > 0$ and $\{S_t : t \in A\}$ a family of finite sets. Then there is $B \subseteq A$, with $\lambda^* B > 0$, such that $\{S_t : t \in B\}$ is quasi-disjoint.*

Before the proof of Theorem 2, we need some preparatory steps.

Observation 1 : If $C \subseteq [0,1]$ is such that $\lambda^*C = \delta > 0$, then

$\lambda^*C = \inf \{\Sigma\, \lambda I_n : C \subseteq \bigcup I_n$ and every I_n is an interval$\}$

$= \inf \{\Sigma\, \lambda I_n : C \subseteq \bigcup I_n$ and every I_n is an interval with rational endpoints$\}$.

The set $\{J : J \subseteq [0,1]$ and J is an interval with rational endpoints$\}$ is countable. So, if we assume CH, the collection

$$\mathcal{I}_\delta = \{(I_n)_{n \geq 1} : \Sigma\, \lambda I_n < \delta \text{ and every } I_n \text{ is an interval with rational endpoints}\}$$

has cardinal $\aleph_1$; we **fix** a bijection $h : [1, \aleph_1) \longrightarrow \mathcal{I}_\delta$, $h(\alpha) = (I_n^\alpha)_{n \geq 1}$.

Proposition 1 (CH)) *: Let A be a subset of [0,1] with $\lambda^*A > 0$ and $\{S_t : t \in A\}$ be a family of singletons. Then there is $B \subseteq A$ such that $\lambda^*B > 0$ and $\{S_t : t \in B\}$ is quasi-disjoint. Thus, one of the following two possibilities occurs :*

(i) there is an element a such that $S_t = \{a\}$, $\forall\, t \in B$; **or**

(ii) the sets S_t, $t \in B$, are pairwise disjoint.

Proof : The proof involves a diagonal argument and transfinite induction. Let $\delta = \lambda^*A$.

Suppose $\mathcal{A} = \{a_t : t \in A\}$; for $x \in \mathcal{A}$, set $A_x = \{t \in A : S_t = \{x\}\}$.

If, for some $x \in \mathcal{A}$, $\lambda^*A_x > 0$ then the proposition is proved : it's sufficient to take $B = A_x$ and $S_t = \{x\}$, $t \in B$.

Thus, we may assume that for each $x \in \mathcal{A}$, $\lambda^*A_x = 0$.

Let $(I_n^\alpha)_{n \geq 1}$, $\alpha < \aleph_1$ be the enumeration of $\mathcal{I}_\delta$ constructed in observation 1.

We claim that, by transfinite induction on $\aleph_1$, we may choose a sequence $\{t_\alpha : \alpha < \aleph_1\}$ such that if ν, β are distinct countable ordinals then $t_\nu \neq t_\beta$ and $S_{t_\nu} \cap S_{t_\beta} = \emptyset$.

For the first step in the induction, since $\lambda^*A = \delta > 0$ and $\Sigma\lambda I_n^1 < \delta$ there is $t_1 \in A$ such that $t_1 \notin \cup\, I_n^1$. Note that $t_1 \in A_{x_1}$, where $S_{t_1} = \{x_1\}$.

Having constructed t_η, $\eta < \alpha$, and recalling that $\bigcup_{\eta<\alpha} A_{x_\eta}$ has measure zero, because it's a countable union of sets measure zero, we may choose $t_\alpha \in A$ - $(\bigcup_{\eta<\alpha} A_{x_\eta} \cup \bigcup I_n^\alpha)$.

It's straightforward that the sequence $B = \{t_\alpha : \alpha < \aleph_1\}$ has the claimed properties. It remains to verify that $\lambda^*B > 0$.

If $\lambda^*B = 0$ then there is a covering (I_n) of B by intervals with rational endpoints such that $\Sigma\, \lambda I_n < \delta/2$. By observation 1, we can find $\alpha < \aleph_1$ such that $(I_n) = (I_n^\alpha)$. Thus, $B \subseteq \bigcup I_n^\alpha$. But this is impossible since $t_\alpha \in B$ was chosen outside $\bigcup I_n^\alpha$. $\triangle$

Observation 2 : The proof of Proposition 1 actually shows that if there is no subset of C of A of positive outer measure, such that $S_t = S_{t'}$, for all t, t' in C, **then** there is $B \subseteq A$ such that $\lambda^*B = \lambda^*A$ and $S_t \cap S_{t'} = \emptyset$, for all distinct t, t' in B.

Proof of Theorem 2

We begin by generalizing Proposition 1.

Fact : Let A be a subset of [0,1] with $\lambda^*A > 0$ and $\{S_t : t \in B\}$ a family of finite sets, each of them with $k \geq 0$ elements. Then there is $B \subseteq A$ such that $\lambda^*B > 0$ and $\{S_t : t \in B\}$ is quasi-disjoint.

Proof : We proceed by induction on $k \geq 0$, the case $k = 0$ being trivial and that in which $k = 1$ having been taken care of by Proposition 1. So, suppose the statement holds for each $j \leq k$; we are going to verify it for $k + 1$. Let $\lambda^*A = \delta > 0$ and define $\mathcal{A} = \bigcup_{t \in A} S_t$.

If $u \subseteq \mathcal{A}$, set $A_u = \{t \in A : u \subseteq S_t\}$. Note that if we write $S_t = \{t_i : 1 \leq i \leq k + 1\}$ then $t \in \bigcup_{i=1}^{k+1} A_{\{t_i\}}$. We shall use the enumeration of $\mathcal{I}_\delta$ described in Observation 1.

Case 1 : There is $u \subseteq \mathcal{A}$, $1 \leq |u| \leq k + 1$ such that $\lambda^*A_u > 0$.

Then, $\{S_t - u : t \in A_u\}$ is a family of sets with $k + 1 - |u| \leq k$ elements such that $\lambda^*A_u > 0$. By induction, there is $B \subseteq A_u$ with $\lambda^*B > 0$, such that $\{S_t - u : t \in B\}$ is quasi-disjoint. It's clear that $\{S_t : t \in B\}$ has the same property.

Case 2 : For all $u \subseteq \mathcal{A}$ with $|u| \leq k + 1$, $\lambda^*A_u = 0$.

Here we proceed as in the proof Proposition 1. By transfinite induction on $\alpha \in \aleph_1$, it's possible to construct a sequence $B = \{t_\alpha : \alpha < \aleph_1\}$ of elements of A such that

$$t_\alpha \in A - (\bigcup I_n^\alpha \cup \bigcup_{\beta<\alpha} (A_1^\beta \cup \ldots \cup A_{k+1}^\beta)),$$

where $S_{t_\beta} = \{x_i^\beta : 1 \leq i \leq k + 1\}$ and $A^\beta j = \{t \in A : x_j^\beta \in S_{t_\lambda}\} = A_{\{x_j^\beta\}}$.

The inductive step comes, as in the proof Proposition 1, from the fact that $\lambda^*A^\beta j = 0$ for all $\beta < \aleph_1$ and $j \leq k + 1$, by the hypothesis assumed in Case 2. Moreover, the constructed sequence satisfies

$$\beta < \eta \implies t_\eta \notin A_1^\beta \cup \ldots \cup A_{k+1}^\beta \implies S_{t_\beta} \cap S_{t_\eta} = \emptyset,$$

and $\{S_t : t \in B\}$ is a disjoint family of sets.

The same diagonal argument used in the proof of Proposition 1 will show that $\lambda^*B > 0$, establishing the Fact.

To finish the proof write $A = \bigcup_{k \geq 1} A_k$, where $A_k = \{t \in A : S_t \text{ has cardinal } k\}$. Since A has strictly positive outer measure, the same must be true of at least one A_k. The desired conclusion follows from an application of the Fact to A_k. $\triangle$

Observation 3 : An analysis of the preceding proof will show that the statement holds for separable regular Borel measures. This of course might also be obtained as a Corollary of the above result, using the well known Caratheodory classification of such spaces. It would be interesting to find other classes of measure spaces for which this generalization of the Erdös-Rado theorem holds true.

It's clear that Theorem 1 is in fact a consequence of Theorem 2.

4 l_1 sums of Pm spaces

Given a family $(X_i)_{i\in I}$ of Banach spaces, their l_1 sum will be indicated by $[\oplus_{i\in I}X_i]_1$; the definition of this space is recalled in Example 2.5.

Theorem 1 (CH) *: The operation of taking l_1 sum preserves property P.*

Proof : Let $f : [0,1] \longrightarrow X$ be a bounded Pettis integrable function, and suppose $\int_{[0,t]} f\, d\lambda = 0$, $\forall$ t $\in$ [0,1]. We reason by contradiction to show that $f = 0$ λ-ae.

If f is not zero λ-ae, there are $\varepsilon > 0$ and $A \subseteq [0,1]$, with $\lambda^* A > 0$, such that $\|f(t)\| > \varepsilon$, $\forall$ t $\in$ A.

For t $\in$ [0,1], set $f(t) = (f_i(t))_{i\in I}$, where $f_i(t) \in X_i$, $\forall$ i $\in$ I. Moreover, for each t $\in$ A, $f(t)$ is summable : $\exists$ a finite $J_t \subseteq I$ such that

$$(1) \qquad \Sigma_{i\in J_t} \|f_i(t)\| > \varepsilon/2 \quad \text{and} \quad \Sigma_{i\notin J_t} \|f_i(t)\| < \varepsilon/3.$$

Observe that $\{J_t : t \in A\}$ is a family of finite sets and $\lambda^* A > 0$. By Theorem 3.2, there is $B \subseteq A$, with $\lambda^* B > 0$, and a set J such that for distinct t, t' $\in$ B, $J_t \cap J_{t'} = J$.

We have two cases to consider :

Case 1 : $J = \emptyset$.

The dual of X is $[\oplus_{i\in I}X_i^*]_\infty$; and so we may write $x^* = (x_i^*)_{i\in I}$, with $x_i^* \in X_i^*$, $\sup_{i\in I} \|x_i^*\| < \infty$ and $x^*((x_i)_{i\in I}) = \Sigma_{i\in I} x_i^*(x_i)$.

Consider $x^* = (x_i^*)_{i\in I} \in X^*$ given by :

- if $i \notin \bigcup\{J_t : t \in B\}$, set $x_i^* = 0$;
- if $i \in J_t$, for some t $\in$ B, choose x_i^* with $\|x_i^*\| = 1$ and $x_i^*(f_i(t)) = \|f_i(t)\|$.

Put $L = \bigcup \{J_{t'} : t' \in B\}$ and, for t $\in$ B, $L_t = \bigcup \{J_{t'} : t' \in B, t' \neq t\}$. Note that for t $\in$ B, we have $x^* \circ f(t) = \Sigma_{i\in I}\, x_i^*(f_i(t)) = \Sigma_{i\in L}\, x_i^*(f_i(t))$, since $x_i^* = 0$, for $i \notin L$.

Moreover,

$$(2) \qquad x^* \circ f(t) = \Sigma_{i\in J_t} x_i^*(f_i(t)) + \Sigma_{i\in L_t} x_i^*(f_i(t)) = \Sigma_{i\in J_t} \|f_i(t)\| + \Sigma_{i\in L_t} x_i^*(f_i(t)).$$

But $|\Sigma_{i\in L_t}\, x_i^*(f_i(t))| \leq \Sigma_{i\in L_t} \|x_i^*\|\, \|f_i(t)\| \leq \Sigma_{i\notin J_t} \|f_i(t)\| < \varepsilon/3.$ (3)

It follows from equations (1) - (3) that $x^* \circ f(t) > \varepsilon/2 - \varepsilon/3 = \varepsilon/6$, $\forall$ t $\in$ B.

Consequently, $\lambda(\{t : x^* \circ f(t) > 0\}) > 0$. But this is absurd since f has null Pettis integral and so $x^* \circ f$ has be zero almost everywhere.

Case 2 : $J = \{i_1, \ldots, i_r\}$.

For each $j \leq r$, consider the projection $\pi_{i_j} : X \longrightarrow X_{i_j}$, $\pi_{i_j}(x) = x_{i_j}$ and define $f_{i_j} = \pi_{i_j} \circ f : [0,1] \longrightarrow X_{i_j}$.

Since $\pi_{i_j}\left(\int_{[0,t]} f\,d\lambda\right) = \int_{[0,t]} \pi_{i_j} \circ f\,d\lambda = \int_{[0,t]} f_{i_j}\,d\lambda$, and $\int_{[0,t]} f\,d\lambda = 0$, $\forall$ t $\in$ [0,1], it follows that $\int_{[0,t]} f_{i_j}\,d\lambda = 0$, $\forall$ t $\in$ [0,1], $1 \leq j \leq r$.

So, $f_{i_1}, f_{i_2}, \ldots f_{i_r}$ are functions with null Pettis integral. By hypothesis, the spaces X_{i_j} have property P and so there is $C_j \subseteq [0,1]$ with $\lambda^* C_j = 0$ and $f_{i_j}(t) = 0$, $\forall$ t $\notin C_j$ and $j \leq r$. Set $C_0 = B - C_1 \cup C_2 \cup \ldots \cup C_r$. Then we have :

(1) $\lambda^* C_0 > 0$;

(2) $t \in C_0 \implies \|f(t)\| > \varepsilon$ and $f_{i_j}(t) = 0$, $1 \leq j \leq r$.

Since $J = \{i_1, \ldots, i_r\}$, we have that

$$t \in C_0 \implies \Sigma_{i \in J_t - J} \|f_i(t)\| > \varepsilon/2 \quad \text{and} \quad \Sigma_{i \notin J_t - J} \|f_i(t)\| < \varepsilon/3.$$

Observe that, for t $\in$ B, the sets J_t - J are pairwise disjoint; and we profit from this fact to define the following element $x^* = (x_i)_{i \in I}$ of X^* :

– if $i \notin \bigcup_{t \in C_0} (J_t - J)$, then $x_i^* = 0$;

– if $i \in J_t - J$, $t \in C_0$, choose $x_i^* \in X^*$ such that $\|x_i^*\| = 1$ and $x_i^*(f_i(t)) = \|f_i(t)\|$.

Let $W = \bigcup \{J_{t'} - J : t' \in C_0\}$ and, for $t \in C_0$, $W_t = \bigcup \{J_{t'} - J : t' \in C_0, t' \neq t\}$.

For $t \in C_0$,

$$x^* \circ f(t) = \Sigma_{i \in I}\, x_i^*(f_i(t)) = \Sigma_{i \in W}\, x_i^*(f_i(t)) = \Sigma_{i \in J_t - J}\, x_i^*(f_i(t)) + \Sigma_{i \in W_t}\, x_i^*(f_i(t)).$$

But $|\Sigma_{i \in W_t} x_i^*(f_i(t))| \leq \Sigma_{i \in W_t} \|x_i^*\|\,\|f_i(t)\| \leq \varepsilon/3$, and just as in Case 1, it follows that $x^* \circ f(t) > \varepsilon/6$, $\forall$ t $\in C_0$.

Thus, $\lambda(\{t : x^* \circ f(t) > 0\}) > 0$, a contradiction, because we have $x^* \circ f = 0$ λ-ae.

From cases 1 and 2 we conclude that $f = 0$ λ - ae, as desired. $\triangle$

We fix some notation that will be useful below. Let $(X_i)_{i \in I}$ be a family of Banach spaces and $X = [\oplus_{i \in I} X_i]_1$ their l_1 sum.

For each $J \subseteq I$, let $X_J = [\oplus_{i \in J} X_i]_1$ and $\pi_J : X \longrightarrow X_J$ be the map that forgets the ccordinates outside J : $\pi_J((x_i)) = (x_i)_{i \in J}$. If $J = \{i\}$ we write X_i instead of $X_{\{i\}}$.

Define $i_J : X_J \longrightarrow X$, $i_J(x_J) = (x_J, 0)$ and identify X_J with its image in X, $i_J(X_J)$.

Proposition 1 : *Let X be the l_1 sum of Pm spaces X_i, $i \in I$. Let $f : [0,1] \longrightarrow X$ be a bounded Pettis integrable function. Then there are bounded and Pettis integrable functions $f_1, f_2 : [0,1] \longrightarrow X$ such that :*

(i) $f = f_1 + f_2$; *(ii) f_1 is measurable;*

(iii) f_2 has null Pettis integral : $\int_{[0,t]} f_2\,d\lambda = 0$, $\forall$ t $\in$ [0,1].

Proof : Let $\{r_n : n \geq 1\}$ be an enumeration of the rational numbers in [0,1]. For each $n \geq 1$, let $x_n = \int_{[0,r_n]} f\,d\lambda \in X$. Since X is an l_1 sum of the X_i, there is a countable subset

$J_n \subseteq I$ such that $\{i : x_n(i) \neq 0\} \subseteq J_n$. Let $J = \bigcup_{n\geq 1} J_n \subseteq I$; J is countable and we have that $\int_{[0,r_n]} f\, d\lambda \in X_J$, $\forall\, n \geq 1$. By Lemma 1.2.a) we have $\int_{[0,t]} f\, d\lambda \in X_J$, $\forall\, t \in [0,1]$.

Define $f_1 = i_J \circ \pi_J \circ f$ and $f_2 = f - f_1$. It's clear that f_1, f_2 are bounded and Pettis integrable. Moreover, $\int_{[0,r_n]} f_1\, d\lambda = i_J \circ \pi_J \left(\int_{[0,r_n]} f\, d\lambda\right) = \int_{[0,r_n]} f\, d\lambda$, $\forall\, n \geq 1$.

Thus, $\int_{[0,t]} f_1\, d\lambda = \int_{[0,t]} f\, d\lambda$, $\forall\, t \in [0,1]$ and it follows that f and f_1 have the same Pettis integral over all measurable sets (Lemma 1.1.c).

Since X_i is Pm, for each $i \in J$, there is $C_i \subseteq [0,1]$, $\lambda^* C_i = 0$, such that $(\pi_i \circ f)([0,1] - C_i)$ is separable. Set $C = \bigcup_{i\in J} C_i$. Then $\lambda^* C = 0$ and $(\pi_J \circ f)([0,1] - C)$ is also separable. Since i_J is an isometry into X, f_1 has almost separable range and so must be measurable.

Finally, $\int_{[0,t]} f_2\, d\lambda = \int_{[0,t]} f\, d\lambda - \int_{[0,t]} f_1\, d\lambda$, and so $\int_{[0,t]} f_2\, d\lambda = 0$, $\forall\, t \in [0,1]$. $\triangle$

Observation 1 : Proposition 1 holds true if we replace $([0,1], \Sigma, \lambda)$ by any complete measure separable space $(S, \mathcal{B}, \mu)$: If X is the l_1 sum of a family of Pm spaces, any bounded Pettis integrable $f : S \longrightarrow X$ can be written as $f = f_1 + f_2$, where $f_1, f_2 : S \longrightarrow X$ are such that f_1 is measurable and f_2 has null Pettis integral.

Theorem 2 (CH) *: The operation of taking l_1 sums preserves the property of being Pm.*

Proof : Let $f : [0,1] \longrightarrow X$ be a bounded Pettis integrable function. By Proposition 1, there are bounded Pettis integrable $f_1, f_2 : [0,1] \longrightarrow X$, such that :

(i) $f = f_1 + f_2$; (ii) f_1 is measurable; (iii) f_2 has null Pettis integral.

Since all the components of X are Pm, they have property P. By Theorem 1, X also has this property and so f_2 must be zero λ- ae in [0,1]. Thus, $f = f_1$ λ- ae and consequently also measurable. $\triangle$

Corollary 1 *: Any l_1 sum of separable spaces is Pm. In particular, $\forall$ sets Γ and A, $[l_1(\Gamma) \oplus [\Sigma_{\alpha\in A} L_1([0,1])]_1]_1$ is Pm.* $\triangle$

Observation 2 : A remark due to M. Ignez S. V. Diniz implies that spaces of the form $[l_1(\Gamma) \oplus [\Sigma_{\alpha\in A} L_1[0,1]]_1]_1$ are Pm, independently of CH. We will comment on this at the end of section 8.

5 The L_p spaces and the Pm property

We refer the reader to [L], chapter 5, for a detailed account of abstract L_p spaces. For his convenience we transcribe here the results that are relevant to our discussion.

(1) If X is an abstract L_p space, $1 \leq p < \infty$, then X is isometric to $L_p(\nu)$, for some measure ν.

(2) If X is an abstract L_p space, $1 \leq p < \infty$, then X is isometric to

$[l_p(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A} L_p[0,1]^{m_\alpha}]_p]_p$, for some set Γ and some set of cardinal numbers $m_\alpha \geq \aleph_0$.

(3) For $\eta \geq \aleph_0$ and $1 \leq p < \infty$, $L_p([0,1]^\eta)$ has a subspace isomorphic to $l_2(\eta)$.

(4) If $L_p(\nu)$ is separable then $L_p(\nu)$ is isometric to one of the following spaces :

(i) $l_p(\Gamma)$, where $|\Gamma| \leq \aleph_0$; (ii) $L_p([0,1])$; (iii) $[l_p(\Gamma) \oplus L_p([0,1])]_p$, where $|\Gamma| \leq \aleph_0$.

In particular, if $|A| \leq \aleph_0$ and $m_\alpha \leq \aleph_0$, $\forall \alpha \in A$, then $[\oplus\Sigma_{\alpha\in A} L_p([0,1]^{m_\alpha})]_p$ is isometric to $L_p([0,1])$.

(5) Let (Z, ξ, δ) be a finite measure space. Then $L_1(\delta)$ is isometric to one of the following spaces :

(i) $[\oplus_{k\in\mathbb{N}} L_1([0,1]^{m_k})]_1$, with $m_k \geq \aleph_0$, $\forall\, k \geq 1$; (ii) $l_1(\Gamma)$, with $|\Gamma| \leq \aleph_0$;

(iii) $[l_1(\Gamma) \oplus [\oplus\Sigma_{k\in\mathbb{N}} L_1([0,1]^{m_k})]_1]_1$, with $|\Gamma| \leq \aleph_0$, and $m_k \geq \aleph_0$, $\forall\, k \geq 1$.

Theorem 1 (CH) *: Let be $1 < p < \infty$ and X an abstract L_p space. Are equivalent :*

(i) X is Pm;

(ii) X is isometric to $l_p(\Gamma)$ or $[l_p(\Gamma) \oplus L_p[0,1]]_p$ or $L_p[0,1]$, where $|\Gamma| \leq \aleph_0$.

Proof : (i) $\Rightarrow$ (ii) : We can ssume that X is of the form $[l_p(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A} L_p([0,1]^{m_\alpha})]_p]_p$, where $m_\alpha \geq \aleph_0$, $\forall \alpha \in A$.

If $|\Gamma| > \aleph_0$ then, by CH, $|\Gamma| \geq c$. By example 2.1, $l_p(\Gamma)$ is nPm. This is impossible since $l_p(\Gamma)$ is a closed subspace of X and X is Pm. Thus, $|\Gamma| \leq \aleph_0$.

If $m_\alpha > \aleph_0$, for some $\alpha \in A$, then $m_\alpha \geq c$. By result (3) above, $L_p([0,1]^{m_\alpha})$ contains an isomorphic copy of $l_2(I)$, with $|I| = m_\alpha \geq c$. Again this is impossible because $l_2(I)$ is nPm and X is Pm. Therefore, $m_\alpha \leq \aleph_0$, $\forall \alpha \in A$, and so $L_p([0,1]^{m_\alpha})$ is isometric to $L_p([0,1])$.

Thus, X is a space of the type $[l_p(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A} L_p[0,1]]_p]_p$.

Now we show that $[\oplus\Sigma_{\alpha\in A} L_p([0,1])]_p$ contains an isometric copy of $l_p(A)$.

For each $\alpha \in A$, choose $e_\alpha \in L_p([0,1])$, with $\|e_\alpha\|_p = 1$. For $x = (x_\alpha)_{\alpha\in A} \in l_p(A)$, consider $Tx = (x_\alpha e_\alpha)_{\alpha\in A}$. Since $\|Tx\|_p{}^p = \Sigma_{\alpha\in A} \|x_\alpha e_\alpha\|^p = \Sigma_{\alpha\in A} |x_\alpha|^p = \|x\|_p{}^p$, T is a linear isometry from $l_p(A)$ to $[\oplus\Sigma_{\alpha\in A} L_p([0,1])]_p$.

If $|A| > \aleph_0$ then $l_p(A)$ is nPm, and X would be nPm. Thus, $|A| \leq \aleph_0$ and, by the result (4) mentioned above X is of one of the forms asserted in (ii).

For (ii) $\Rightarrow$ (i), it's sufficient to observe that all spaces mentioned in (ii) are separable and consequently Pm. $\triangle$

Corollary 1 (CH) *: For $1 < p < \infty$, an abstract* L_p *space is Pm iff it is separable.* $\triangle$

We now turn to the case $p = 1$.

Theorem 2 (CH) : *Let X be an abstract L_1 space. Are equivalent :*

(i) X is Pm ;

(ii) There are sets Γ and A such that X is isometric to $l_1(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A} L_1[0,1]]_1]_1$;

(iii) Let $(Z,\ \xi,\ \delta)$ be a finite measure space. If $L_1(\delta)$ is isomorphic to a subspace of X then $L_1(\delta)$ is separable.

Proof : (i) $\Rightarrow$ (ii) : We may assume that X is $[l_1(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A}\, \mathrm{L}_1([0,1]^{\mathrm{m}_\alpha})]_1]_1$, with $\mathrm{m}_\alpha \geq \aleph_0$. Moreover, $\mathrm{m}_\alpha \leq \aleph_0$ by the same argument used in the proof of Theorem 1. By (4), $\mathrm{L}_1([0,1]^{\mathrm{m}_\alpha})$ is isometric to $L_1([0,1])$, and so, X is isometric to $[l_1(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A}\, \mathrm{L}_1([0,1])]_1]_1$.

(i) $\Rightarrow$ (iii) : We have that $\mathrm{L}_1(\delta)$ is isomorphic to $[l_1(\Gamma) \oplus [\oplus\Sigma_{k\geq 1}\, \mathrm{L}_1([0,1]^{\mathrm{m_k}})]_1]_1$, where $|\Gamma| \leq \aleph_0$ and $\mathrm{m_k} \leq \aleph_0$, $\forall$ k ≥ 1. Since X is Pm, we have $\mathrm{m_k} \leq \aleph_0$, $\forall$ k ≥ 1 and again by (4), $[\oplus\Sigma_{k\leq 1}\, \mathrm{L}_1([0,1]^{\mathrm{m_k}})]_1$ is isometric to $\mathrm{L}_1([0,1])$. Thus, $\mathrm{L}_1(\delta)$ is isomorphic to $[l_1(\Gamma)\oplus L_1([0,1])]_1$ or $l_1(\Gamma)$ (with $|\Gamma| \leq \aleph_0$) or $\mathrm{L}_1([0,1])$, and therefore separable.

(ii) $\Rightarrow$ (i) is an immediate consequence of Theorem 4.2.

(iii) $\Rightarrow$ (ii) : We can suppose X is of the form $[l_1(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A}\, \mathrm{L}_1([0,1]^{\mathrm{m}_\alpha})]_1]_1$, because X is an abstract L_1 space. Since $[0,1]^{\mathrm{m}_\alpha}$ is a finite measure space and $\mathrm{L}_1([0,1]^{\mathrm{m}_\alpha})$ is isometric to a closed subspace of X, it follows from (iii) that $\mathrm{L}_1([0,1]^{\mathrm{m}_\alpha})$ is separable; but this is possible only if $\mathrm{m}_\alpha \leq \aleph_0$ in which case $\mathrm{L}_1([0,1]^{\mathrm{m}_\alpha})$ is isometric to $\mathrm{L}_1([0,1])$. It's now clear that (ii) must hold. $\triangle$

Observation 1 : In [MR] an analogous classification of the abstract L_p spaces, $1 \leq \mathrm{p} < \infty$, is given with respect to property Rm (assuming CH), as follows :

Theorem (CH) : Let be X an abstract L_p space, $1 \leq \mathrm{p} < \infty$.

(a) For $1 < \mathrm{p} < \infty$, X is Rm iff X is isometric to $l_p(\Gamma)$ or $[l_p(\Gamma) \oplus \mathrm{L_p}([0,1])]_\mathrm{p}$ or $\mathrm{L_p}([0,1])$, where $|\gamma| \leq \aleph_0$. Thus, X is Rm iff it is separable.

(b) For p = 1, are equivalent :

(1) X is Rm;

(2) Let $(\mathrm{Z},\ \xi,\ \delta)$ be a finite measure space. If $\mathrm{L}_1(\delta)$ is isomorphic to a closed subspace of X then $\mathrm{L}_1(\delta)$ is separable, that is, it is isometric to $l_1(\Gamma)$, $\mathrm{L}_1([0,1])$ or $[l_1(\Gamma) \oplus \mathrm{L}_1([0,1])]_1$, where Γ is countable;

(3) There are sets Γ and A such that X is isometric to $[l_1(\Gamma) \oplus [\oplus\Sigma_{\alpha\in A}\, \mathrm{L}_1([0,1])]_1]_1$.

Theorems 1 and 2 together with the result mentioned in Observation 1 yield an affirmative answer to Problem 2.1 in the class of $\mathrm{L_p}$ spaces:

Corollary 2 (CH) : *For $1 \leq p < \infty$, an abstract L_p space is Pm iff it is Rm.* $\triangle$

We now apply our results to duals of spaces of continuous real function on compact spaces. If K is a compact space, C(K) is the Banach space of continuous real functions defined on K with the sup norm, $\|f\|_\infty = \sup_{\mathrm{t}\in\mathrm{K}}|f(\mathrm{t})|$. It's known that $\mathrm{C(K)}^*$ is an abstract L_1 space.

Definition 1 *: A compact space K is* **measure separable** *if every regular Borel measure on K is separable.*

Examples of measure separable compacts are the compact metric spaces, compact dispersed spaces and the one point compactification of the disjoint union of measure separable compacts.

The following result describes the dual of measurable separable compacts :

Theorem ([L]) : Let K be a measure separable compact.

a) If K is a not dispersed then $C(K)^* = [l_1(K) \oplus [\oplus\Sigma_{i\in I} L_1([0,1])]_1]_1$, where $|I| \geq \aleph_0$.

b) If K is dispersed then $C(K)^* = l_1(K)$.

An immediate consequence of Theorems 4.1, Theorem 2 and the preceding results is

Corollary 3 *(CH) : If K is a compact, $C(K)^*$ is Pm iff K is measure separable.* $\triangle$

Corollary 4 *(CH) : If K is measure separable then $C(K)^*$ does not contain a subspace isomorphic to a non separable Hilbert space.*

Proof : It's sufficient to remember that a non separable Hilbert space is isomorphic to $l_2(\Gamma)$, for some uncountable set Γ. Since we are assuming CH, it is nPm, and so cannot be a subspace of a Pm space. $\triangle$

Observation 2 : In section 7 we will show that in fact if K is measure separable then $C(K)^*$ cannot contain a subspace isomorphic to a non separable WCG space.

6 A sufficient condition for the Pm property

We have already observed that a Pm space has property P. The converse is false : $l_\infty(\mathbb{N})$ has P but is not Pm. In this section we discuss a condition, the separable projection property (spp), which, together with P, guarantees that a space is Pm.

Definition 1 *: A Banach space X has the* **separable projection property (spp)** *when each separable subspace of X is contained in a separable* **complemented** *subspace of X.*

Equivalently, if Y is a separable subspace of X, there is a separable $Z \subseteq X$ such that $Y \subseteq Z$ and Z is the image of a continuous linear projection defined in X.

Example 1. $X = C([0, \aleph_1])$ has the spp, where $[0, \aleph_1]$ is the usual compact order space. To see this, let Y be a separable subspace of X and $D = \{f_n : n \leq 1\}$ a countable dense subset of Y. For $n \geq 1$, let $\alpha_n < \aleph_1$ be an ordinal such that f_n is constant for all $\beta \geq \alpha_n$. Set $\alpha = \sup\{\alpha_n : n \geq 1\}$; clearly α is countable.

Let $Z = \{g \in X : g$ is constant for all $\beta \geq \alpha\}$. It's clear that Z is a closed subspace of X, containing Y. It's easily verified that Z is separable.

To see that Z is complemented in X, define, for $f \in X$, $\tilde{f} : [0, \aleph_1] \longrightarrow \mathbb{R}$ by

$$\tilde{f}(\beta) = \begin{cases} f(\beta) & \text{if } \beta \le \alpha \\ f(\alpha) & \text{if } \beta \ge \alpha \end{cases}$$

Since $(\alpha, \aleph_1]$ is clopen in $[0, \aleph_1]$, $\tilde{f}$ is a continuous function. Moreover, $\tilde{f}$ is constant for $\beta \ge \alpha$, and so it is in Z. It's readily verified that $f \mapsto \tilde{f}$ is a linear continuous projection from X onto Z. This shows that X has the spp.

Example 2. $l_p(\mathrm{I})$, $1 \le p < \infty$, and $c_0(\mathrm{I})$ both have the spp. In fact, any l_p sum of separable spaces has spp, if $1 \le p < \infty$. It's a consequence of the long chain of projections that are a decomposition of the identity in a WCG space, that all such spaces have the separable projection property (see [Li]).

Theorem 1 : *A Banach space with the spp and property P is Pm.*

Proof : Let $f : [0,1] \longrightarrow X$ be a bounded Pettis integrable function. Then, the map $\tilde{f} : [0,1] \longrightarrow X$, given by $\tilde{f}(t) = \int_{[0,t]} f \, d\lambda = x_t$ is continuous (Corollary 1.1). Set $Y = \operatorname{span}\{x_r : r \in [0,1] \cap Q\} \subseteq X$; clearly, Y is separable in X. Since X has spp, there is Z, a closed separable complemented subspace of X, with $Y \subseteq Z$. Let $p : X \longrightarrow Z$ be the projection of X onto Z.

Set $f_1 = p \circ f$. Then f_1 is bounded and Pettis integrable. Moreover, since $f_1([0,1]) \subseteq Z$, f_1 has separable range and so must be measurable. Since all x_t can be approximated by a sequence of x_n's, we conclude that $x_t \in Z$, $\forall\, t \in [0,1]$. Thus, $p(x_t) = x_t$, $\forall\, t \in [0,1]$ and we get

$$\int_{[0,t]} f_1 \, d\lambda = \int_{[0,t]} p \circ f \, d\lambda = p\left(\int_{[0,t]} f \, d\lambda\right) = p(x_t) = x_t,$$

showing that the Pettis integrals of f and f_1 are the same.

Now consider $f_2 = f - f_1$; it is bounded, Pettis integrable and $\int_{[0,t]} f_2 \, d\lambda = 0$, $\forall\, t \in [0,1]$. Since X has property P, it follows that $f_2 = 0$ λ-ae. It's then clear that $f = f_1$ λ-ae is measurable, proving that X is Pm. △

With essentially the same proof one can show

Theorem 2 : *Let $(S, \mathcal{B}, \nu)$ be a complete separable measure space and X a Banach space with the spp and such that :*

(*) If $S \xrightarrow{f} X$ is bounded, Pettis integrable and $\int_E f \, d\nu = 0$, $\forall\, E \in \mathcal{B}$, then $f = 0$ ν-ae.

Then, all bounded Pettis integrable X valued maps defined in S are measurable. △

Corollary 1 : *If X has a countable total subset $\Omega \subseteq X^*$, then X has property P. If X also has the spp, then it is Pm.*

Proof : It's enough to show that X has property P. Let f : [0,1] $\longrightarrow$ X be a bounded null function; then $x^*(x_t) = \int_{[0,t]} x^* \circ f \, d\lambda = 0$, $\forall\, x^* \in X^*$ and $t \in [0,1]$. In particular, for each $x^* \in \Omega$, we may choose $A_{x^*} \subseteq [0,1]$ with $\lambda A_{x^*} = 0$ and $x^* \circ f = 0$, $\forall\, t \notin A_{x^*}$.

Set $A = \bigcup_{x^* \in \Omega} A_{x^*}$. Since Ω is countable, $\lambda A = 0$; further, $t \notin A$ implies $x^* \circ f(t) = 0$, $\forall\, x^* \in \Omega$. Since Ω is total in X^*, this yields $f(t) = 0$ if $t \notin A$, that is, $f = 0$ λ-ae. $\triangle$

Observation 1. The proof of Theorem 1 does not depend on CH. In section 10 we shall use Theorem 1, Martin's Axiom and the negation of CH to prove the independence of certain statements concerning the separability of Pm spaces.

7 WCG spaces and the Pm property

Our aim here is to prove that if we assume CH, nonseparable WCG spaces are not Pm. A good reference on WCG is [Li].

Definition 1 : *A Banach space X is WCG if there is a weakly compact $K \subseteq X$, such that $X = \overline{span\, K}$.*

Recall our conventions about closures : $\overline{*}$ is norm closure while $\overline{*}^w$ is closure in the weak topology.

Example 1. : a) Separable spaces; b) $c_0(I)$;

c) Any reflexive space, in particular $L_p(\nu)$, $1 < p < \infty$;

(d) $L_1(\nu)$ is WCG iff ν is a σ - finite measure.

(e) Every complemented subspace of a WCG space is WCG.

(f) C(K), where K is the one point compactification of a discrete space.

Recall that a compact space K is an Eberlein compact if it is homeomorphic to a weakly compact subset in some Banach space. In [Li] it is shown that a compact space K is an Eberlein compact iff C(K) is WCG.

We have already mentioned in Example 6.2 that all WCG spaces have the separable projection property.

Let be X a WCG space. Then there is U weakly compact in X such that $X = \overline{\text{span}\, U}$. Set $K = \overline{co(U \cup -U)}$. Clearly K is convex and by the Krein-Smulian theorem it is also weakly compact. We write B_r for the open ball of radius r in X.

Lemma 1 : *For K as defined above*

a) K is absolutely convex.

b) If $B = span\, K$, then 0 is an interior point of B and $B = \bigcup_{n \geq 1} nK$.

c) For each x^* in X^* we have $\|x^*\| = sup\, \{|x^*(y)| : y \in B \text{ and } \|y\| \leq 1\}$

Thus, if $K_n = B_1 \cap nK$, $n \leq 1$, we have $\|x^*\| = sup\, \{x^*_{|K_n} : n \geq 1\}$.

Proof : a) Since K is convex and $0 \in K$, it's enough to show that $x \in K$ implies $-x \in K$. If $x \in U \subseteq K$ then $-x \in U$, and there is nothing to do.

Suppose $x \in co(U \cup -U)$. Then there are $x_1, \ldots, x_n \in U \cup -U$, and $\alpha_1, \ldots, \alpha_n \geq 0$, such that $\Sigma\ \alpha_k = 1$ and $x = \Sigma\ \alpha_k x_k$. But then $-x = \Sigma\ \alpha_k(-x_k)$, showing that $-x \in K$.

Now assume that $x \in \overline{co(U \cup -U)}$; then there is a sequence (y_n) in $co(U \cup -U)$ such that y_n (norm) converges to x. But then $-x$ is the limit of $-y_n \in K$ and so $-x \in K$.

b) Given $x \in B$, we have $x = \sum_{i=1}^{n} \alpha_k x_k$, with the α_k in $\mathbb{R}$ and the x_k in K. For $k \leq n$, consider

$$y_k = \begin{cases} x_j & \text{if } \alpha_j > 0 \\ -x_j & \text{otherwise} \end{cases}$$

Of course $y_1, \ldots, y_n \in K$ (K is absolutely convex) and we have that $x = \Sigma\ |\alpha_k| y_k$. Let $\alpha = \Sigma\ |\alpha_k|$; we may as well suppose that $\alpha \neq 0$. It's straightforward to verify that x/α is in K. Thus $\beta\ (x/\alpha) \in K$ and we may conclude that $\beta\, x \in K$, $\forall \beta \in [0, 1/\alpha]$. This shows that 0 is internal to B.

If $x \in B$, since 0 is an internal point of B, there is $\alpha > 0$ such that $\alpha x \in K$, i.e., $x \in (1/\alpha)K$. Now choose $n \geq 1/\alpha$ to get that $x \in nK$. Since it's clear that $nK \subseteq B$, for all $n \geq 1$, we conclude $B = \bigcup \{nK : n \geq 1\}$.

c) Let $\|x^*\| = a > 0$ and fix $\varepsilon > 0$. Choose $x \in X$ with $\|x\| = 1$ and $|x^*(x)| > a - \varepsilon/2$. Since $x \in \overline{B}$, there is $(y_n) \subseteq B$, with $\|y_n\| \leq 1$, such that y_n (norm) converges to x. Since $|x^*(y_n)| \longrightarrow |x^*(x)|$ and $|x^*(x)| > a - \varepsilon/2$, there is $n \geq 1$ such that $|x^*(y_n)| > a - \varepsilon$. Now (c) follows immediately. $\triangle$

Observe that for each $n \geq 1$ the sets K_n in item (c) of Lemma 1 are convex, weakly compact and $\overline{\bigcup K_n} = B_1$.

The next result, that appears in [Hg] with a different proof, was obtained independently by the authors. We denote by dens X the **density** of the space X. The symbol ω denotes the cardinal number of $\mathbb{N}$.

Theorem 1 *: If X is a WCG space, then dens $X^* \leq (\text{dens}\, X)^{\omega}$.*

Proof : By Lemma 1, we can suppose that $X = \overline{\text{span}\, K}$, where K is absolutely convex and weakly compact. For each $n \geq 1$, let $K_n = B_1 \cap nK$ be the weakly compact sets defined in Lemma 1.(c). Consider the map

$$T : X^* \longrightarrow Z = [\oplus_{n \geq 1} C(K_n)]_\infty, \text{ defined by } T(x^*) = (x^*_{|K_n}).$$

T is obviously linear and since $\|x^*\| = \sup\ \{\|x^*_{|K_n}\| : n \geq 1\}$, it is also an isometry onto its range. Thus, dens $X^* \leq$ dens Z. But dens $Z = \prod_{n \geq 1}$ dens $C(K_n)$ and so

$$(*) \qquad \text{dens } X^* \leq \prod_{n \geq 1} \text{dens } C(K_n).$$

Let $\{x_i : i \in I\}$ be a dense subset of B_1, with $|I| = \text{dens } X$ and select $(x_i^*)_{i \in I}$ in X^* such that $\|x_i^*\| = 1$ and $x_i^*(x_i) = \|x_i\|$, $\forall\, i \in I$. Then $\{x_i^* : i \in I\}$ separates points in B_1 and so separates points in K_n, $\forall\, n \geq 1$. Now the Stone-Weirstrass theorem yields $\text{dens } C(K_n) \leq |I| = \text{dens } X$, $\forall\, n \geq 1$; thus, $\prod_{n \geq 1} \text{dens } C(K_n) \leq (\text{dens } X)^{\omega}$ and the result follows from (*). $\triangle$

Proposition 1 *: If X is a WCG space with dens $X = \alpha \geq c$ then it has a closed WCG subspace Y such that dens $Y = c$.*

Proof : Let K be a convex weakly compact subset of X such that such that X is the closure of span K; dens $X = \alpha$ implies that dens $K = \alpha$ (in the $\|.\|$ topology).

Fact : If $\alpha \geq c$ there are $\delta > 0$ and $D \subseteq K$ such that $|D| = c$ and satisfying $\forall\, x, x' \in D, x \neq x' \Rightarrow \|x - x'\| \geq \delta$.

We first make the following observation : let γ be a strictly positive real and define $\mathcal{A} = \{C \subseteq K : \forall\, x, x' \in C, x \neq x' \Rightarrow \|x - x'\| \geq \gamma\}$. If $\mathcal{A} \neq \emptyset$, we can order $\mathcal{A}$ by inclusion and an application of Zorn's lemma will yield a **maximal element** of $\mathcal{A}$.

Choose $n \geq 1$ such that $1/n <$ diam K. Note that for every $m \geq n$ we have that $\mathcal{A}_m = \{C \subseteq K : \forall\, x, x' \in C, x \neq x' \Rightarrow \|x - x'\| \geq 1/m\}$ is not empty. By the observation in the preceding paragraph, there is $A_m \subseteq K$ such that A_m is maximal in $\mathcal{A}_m$.

We claim that $K = \overline{\bigcup_{m \geq n} A_m}$. For suppose we could find $x \in K$ and $k \geq n$ such that $B_{1/k} \cap (\bigcup_{m \geq n} A_m) = \emptyset$, where $B_{1/k}$ is the open ball of radius $1/k$ in X. In particular, there is no $y \in A_k$ such that $\|x - y\| \leq 1/k$ and so $A_k \cup \{x\}$ is an element of $\mathcal{A}_k$ properly containing A_k, an impossibility since this set is maximal in $\mathcal{A}_k$.

Now dens $K = \alpha$ forces $|\bigcup_{m \geq n} A_n| \geq \alpha$, and so $|A_k| \geq \alpha$, for some $k \geq n$. To finish the proof of the Fact, just choose $D \subseteq A_k$, with $|D| = c$.

A moment of thought will convince the reader that D is closed and that dens $D = c$. Thus, dens $co(D) = c$; furthermore $K_1 = \overline{co(D)}^{w}$ is weakly compact and has density c. The desired WCG of density c is then $\overline{\text{span } K_1}$ and the proof is complete. $\triangle$

Definition 2 *: If X is a Banach space, define* **dens* X** *as the least cardinal γ such that X has a total subset of cardinal γ.*

Proposition 2 ([Li]) *: If X is WCG then dens* X = dens X.* $\triangle$

Proposition 3 *: In a WCG space X, any family of closed hyperplanes of cardinality strictly less than dens X has non trivial intersection.*

Proof : Suppose $\alpha <$ dens X and H_λ, $\lambda < \alpha$, is a family of closed hyperplanes in X. For each λ, let x^*_λ be the continuous norm 1 linear functional associated to H_λ. It's easily seen that $\bigcap H_\lambda = \emptyset$ implies that $\{x^*_\lambda : \lambda < \alpha\}$ is total in X. Consequently, dens* $X \leq \alpha$, which is impossible by Proposition 2. $\triangle$

Theorem 2 (CH) *: A WCG space is Pm iff it is separable.*

Proof : We have only to verify that if X is not separable then it is not Pm. By CH, dens X $\geq$ c; since the property of being Pm is inherited by subspaces, if it is shown that WCG subspaces of density c are not Pm, the same will be true of X. On the other hand, Proposition 1 guarantees that X has a WCG subspace of density c and so cannot be Pm.

We may therefore assume that dens X = c. Since dens* X $\leq$ (densX)$^{\omega}$, it follows from Theorem 1 that dens X* = c. Let $\{x^*_\alpha : \alpha < c\}$ be a norm dense subset in X*. Using Proposition 3 and transfinite induction we can define a sequence $\{x_\alpha : \alpha < c\} \subseteq X$ such that, for α, $\beta < c$, we have

$$\|x_\alpha\| = 1 \quad \text{and} \quad \alpha < \beta \implies x^*_\beta(x_\alpha) = 0.$$

Fix a bijection h : [0,1] $\longrightarrow$ [0, c), h(t) = α_t from [0, 1] to the set of ordinals strictly less than c. Now define $f : [0,1] \longrightarrow X$ by $f(t) = x_{\alpha_t}$. Since $\|x_{\alpha_t}\| = 1$, f is bounded in X and in fact Im $f \subseteq B_1$. Clearly, $f(t) \neq 0$, $\forall t \in [0,1]$.

We contend that $x^* \circ f = 0$ λ-ae, $\forall\, x^* \in X^*$, that is, f is a null function. To see this, fix s $\in$ [0,1]; then for each t $\in$ [0,1], $x^*_{\alpha_s}(f(t)) = x^*_{\alpha_s}(x_{\alpha_t})$, Consequently,

$$\alpha_t > \alpha_s \implies x^*_{\alpha_s}(x_{\alpha_t}) = 0.$$

Thus, $\{t \in [0,1] : x^*_{\alpha_s} \circ f(t) \neq 0\} = \{t \in [0,1] : \alpha_t \leq \alpha_s\}$; since $\alpha_s < \aleph_1 = c$, it follows that this set is countable and so has Lebesque measure zero. This shows that $x^*_{\alpha_s} \circ f = 0$ λ-ae, as desired.

Now let $x^* \in X^*$. Recalling that $\{x^*_\alpha : \alpha < c\}$ is norm dense in X*, for each n $\geq$ 1, there is $\alpha_n < c$ such that $\|x^* - x^*_{\alpha_n}\| < 1/n$; let $A_n = \{t \in [0,1] : x^*_{\alpha_n}(x_{\alpha_t}) = 0\}$. Since $x^*_{\alpha_n} \circ f = 0$ λ-ae, $\lambda A_n = 1$, $\forall$ n $\geq$ 1.

Moreover, it's clear that $\lambda(\bigcap_{i=1}^{n} A_i) = 1$ for all n and so ([0,1] has finite measure), $\lambda(\bigcap_{n\geq 1} A_n = \inf \{\lambda(\bigcap_{i=1}^{n} A_i) : n \leq 1\} = 1$. Set $A = \bigcap_{n\geq 1} A_n$; we have $\lambda A = 1$ and for t $\in$ A,

$$|x^*(x_{\alpha_t})| \leq |x^*(x_{\alpha_t}) - x^*_{\alpha_n}(x_{\alpha_t})| + |x^*_{\alpha_n}(x_{\alpha_t})| \leq \|x^* - x^*_{\alpha_n}\| + |x^*_{\alpha_n}(x_{\alpha_t})| \leq 1/n,$$

for all n $\geq$ 1. It follows immediately that $x^* \circ f = 0$ λ-ae. Since x* is arbitrary in X*, f is Pettis integrable with zero integral over all measurable sets in [0,1]. This shows that X is not Pm, ending the proof. $\triangle$

If K is an Eberlein compact, C(K) is separable iff K is metrizable. Thus

Corollary 1 (CH) *: An Eberlein compact is Pm iff it is metrizable.* $\triangle$

From Corollary 5.3 and Theorem 2 we get

Corollary 2 (CH) *: If K is a measure separable compact, $C(K)^*$ does not contain a subspace isomorphic to a nonseparable WCG space.* $\triangle$

Observation : In the presence of Martin's Axiom and the negation of CH it's possible to exibit non separable WCG spaces which are Pm. But we also show that even then WCG spaces with density $\geq 2^{\aleph_0}$ are not Pm. This will be discussed in section 10.

8 The Pm property for compact spaces

Let $(K_i)_{i \in I}$ be a family of compact spaces and $K = \prod_{i \in I} K_i$ be their product, with the product topology. For $x \in K$, let x_j be its j^{th} coordinate in K_j. This notation will remain fixed throughout this section.

Definition 1 : *(i) A function $g \in C(K)$* **depends on a subset A $\subseteq$ I of coordinates** *whenever the following condition is satisfied:*

$$[dep]: \quad \text{if } x,\ y \in K \text{ are such that } x_j = y_j,\ \forall\, j \in A, \text{ then } g(x) = g(y).$$

(ii) A function $f : [0,1] \longrightarrow C(K)$ is said **to depend on a set Γ of coordinates** *if, for all $t \in [0,1]$, $f(t)$ depends only on the coordinates in Γ.*

Observation 1. If $f \in C(K_i)$, $i \in I$, we can 'lift' f to $C(K)$ by defining $f(x) = f(x_i)$. Observe that f depends only on i. If F is any finite subset of I, the elements of the subalgebra of C(K) generated by the 'lifting' of the maps in $C(K_i)$, $i \in F$, depend only on the coordinates in F.

Let $\mathcal{A}$ be the collection of maps in C(K) that depend only on a finite set of coordinates in I. Then $\mathcal{A}$ is an algebra wich contains the constant functions and separates the points of K. By the Stone - Weirstrass theorem, $\mathcal{A}$ is dense in C(K); so, each function in C(K), can be uniformly approximated by a sequence each term of which depends only on a finite number of coordinates. It follows that **every function in C(K) depends only on a countable set of coordinates.**

For $J \subseteq I$, set $K_J = \prod_{i \in J} K_i$ and $K_{I-J} = \prod_{i \in I-J} K_i$. We identify K with $K_J \times K_{I-J}$ **that is, x with (x_J, x_{I-J}).**

We have natural projections $p_J : K \longrightarrow K_J$, given by $p_J(x) = (x_j)_{j \in J}$. Whenever convenient we write x_J for $p_J(x)$. Thus in this notation, $Id_K = p_J \oplus p_{I-J}$.

We shall need the following simple result.

Lemma 1 : *With notation as above*

(a) (i) The map p_J is continuous and onto.

(ii) The function $\alpha_J : C(K_J) \longrightarrow C(K)$, given by $\alpha_J(g) = g \circ p_J$ is linear and an isometry onto its range.

(b) Let y be any element of $\prod_{i \in I-J} K_i$. Then

(i) The map $K_J \xrightarrow{\varphi_J} K$, $\varphi_J(x) = (x, y)$, *is continuous and injective.*

(ii) The map $C(K) \xrightarrow{\pi_J} C(K_J)$, $\pi_J(g) = g \circ \varphi_J$ *is linear, continuous and onto.* △

Theorem 1 : *Let be (S, B, ν) a complete separable measure space and $f : S \longrightarrow C(K)$ a bounded and Pettis integrable function, where $K = \prod_{i\in I} K_i$. Then there are functions $f_1, f_2 : S \longrightarrow C(K)$ such that :*

(i) $f = f_1 + f_2$; *(ii)* f_1 *and* f_2 *are bounded and Pettis integrable;*

(iii) $\int_E f_1 \,\mathrm{d}\nu = \int_E f \,\mathrm{d}\nu, \forall\, E \in B$;

(iv) $\overline{\mathrm{span}\, f_1(S)}$ *is isometric to a subspace of $C(K_J)$, where $J \subseteq I$ is a countable set;*

(v) $\int_E f_2 \,\mathrm{d}\nu = 0, \forall\, E \in B$.

Proof : Let $(\mathrm{D_n})_{n\geq 1} \subseteq \mathcal{B}$ be a sequence of measurable sets such that $\{\widetilde{\mathrm{D_n}} : \mathrm{n} \geq 1\}$ is dense in $\widetilde{\mathcal{B}}$ (notation as in Observation 1.1). Let $h_n = \int_{\mathrm{D_n}} f \,\mathrm{d}\nu$, $\mathrm{n} \geq 1$. Then $h_n \in \mathrm{C(K)}$ and it depends only on a countable set $\mathrm{J_n}$ of coordinates in I. Set $\mathrm{J} = \bigcup_{n\geq 1} \mathrm{J_n}$.

With notation as in Lemma 1, define $f_1 = \alpha_\mathrm{J} \circ \pi_\mathrm{J} \circ f : \mathrm{S} \longrightarrow \mathrm{C(K)}$. Obviously f_1 is bounded and Pettis integrable. Moreover, for each $\mathrm{n} \geq 1$,

$$\int_{\mathrm{D_n}} f_1 \,\mathrm{d}\nu = \int_{\mathrm{D_n}} \alpha_\mathrm{J} \circ \pi_\mathrm{J} \circ f \,\mathrm{d}\nu = \alpha_\mathrm{J} \circ \pi_\mathrm{J} \int_{\mathrm{D_n}} f \,\mathrm{d}\nu = \alpha_\mathrm{J} \circ \pi_\mathrm{J}(h_\mathrm{n}),$$

Note that $\alpha_\mathrm{J} \circ \pi_\mathrm{J}(h_\mathrm{n}) = \pi_\mathrm{J}(h_\mathrm{n}) \circ \mathrm{p_J} = h_\mathrm{n} \circ \varphi_\mathrm{J} \circ \mathrm{p_J}$. We claim that $h_\mathrm{n} \circ \varphi_\mathrm{J} \circ \mathrm{p_J} = h_n$. This will then immediately yield $\int_{\mathrm{D_n}} f_1 \,\mathrm{d}\nu = \int_{\mathrm{D_n}} f \,\mathrm{d}\nu$.

In effect, if $\mathrm{x} = (\mathrm{x_J}, \mathrm{x_{I-J}}) \in \mathrm{K}$, then

$$h_\mathrm{n} \circ \varphi_\mathrm{J} \circ \mathrm{p_J}(\mathrm{x}) = h_\mathrm{n} \circ \varphi_\mathrm{J}(\mathrm{x_J}) = h_\mathrm{n}(\mathrm{x_J}, \mathrm{y}) = = h_\mathrm{n}(\mathrm{x_J}, \mathrm{x_{I-J}}) = h_n(\mathrm{x}),$$

because h_n depends only on the coordinates which are in J and $\mathrm{y} \in \prod_{i\in I-J} \mathrm{K_i}$ is the element used in the definition of α_J, as in Lemma 1.(b).

Since $\int_{\mathrm{D_n}} f_1 \,\mathrm{d}\nu = \int_{\mathrm{D_n}} f \,\mathrm{d}\nu, \forall\, \mathrm{n} \geq 1$, Lemma 1.2 yields $\int_\mathrm{E} f_1 \,\mathrm{d}\nu = \int_\mathrm{E} f \,\mathrm{d}\nu, \forall\, \mathrm{E} \in \mathrm{B}$.

Moreover, $\overline{\mathrm{span}\, f_1(\mathrm{S})}$ is isometric to a subspace of $\mathrm{C(K_J)}$ since all functions in $f_1(\mathrm{S})$ depend only on the coordinates in J. If we define $f_2 = f - f_1$, then f_2 is bounded, Pettis integrable and $\int_\mathrm{E} f_2 \,\mathrm{d}\nu = 0, \forall\, \mathrm{E} \in \mathrm{B}$. △

The above result applies in particular to [0,1] with Lebesque measure. Recall that a compact K is said to have property P if C(K) has P.

Proposition 1 : *For compact spaces, the property of being Pm as well as property P are preserved by continuous image.*

Proof : If $\pi : T \longrightarrow T'$ is a continuous onto map of compact spaces, it induces a linear isometry $\pi^* : C(T) \longrightarrow C(K)$, $f \mapsto \pi \circ f$. The conclusion now follows from the fact that properties P and Pm are inherited by subspaces. $\triangle$

Proposition 2 : *All compact separable spaces have property P.*

Proof : Suppose T is a separable compact and $D = \{d_n : n \geq 1\}$ a countable dense subset of T. For each $n \geq 1$, let $d_n^* : C(K) \longrightarrow \mathbb{R}$ be the linear continuous functionals 'evaluation at d_n', $d_n^*(f) = f(d_n)$.

It's clear that $\Omega = \{d_n^* : n \geq 1\}$ is total in C(T) and so the statement follows from Corollary 6.1. $\triangle$

Proposition 3 : *Any product of separable compact spaces has property P.*

Proof : Let $f : [0,1] \longrightarrow C(K)$ be bounded function such that $\int_{[0,t]} f \, d\mu = 0$, $\forall\, t \in [0,1]$. For each $t \in [0,1]$, the element $f(t)$ of C(K) depends only on a countable set Γ_t of coordinates. Set $\Gamma = \bigcup_{t\in[0,1]} \Gamma_t$. Then $|\Gamma| \leq c$ and f depends only on the coordinates in Γ. For $y \in \prod_{i\in I-\Gamma} K_i$, let $\varphi_\Gamma : K_\Gamma \longrightarrow K$, $\varphi_\Gamma(x) = (x, y)$, and $\pi_\Gamma : C(K) \longrightarrow C(K_\Gamma)$, $\pi_\Gamma(g) = g \circ \varphi_\Gamma$, be the maps defined in Lemma 1.(b).

Consider the function $\pi_\Gamma \circ f : [0,1] \longrightarrow C(K_\Gamma)$. Since $|\Gamma| \leq c$ and K_i is a separable space, K_Γ is a separable compact space. Thus, it has property P. But the fact that f has null Pettis integral implies that the same is true of $\pi_\Gamma \circ f$ and so we conclude that $\pi_\Gamma \circ f = 0$ λ-ae. Let $A \subseteq [0,1]$ be such that $\lambda A = 0$ and $\pi_\Gamma \circ f(t) = 0$ if $t \notin A$.

To finish the proof observe that if $t \notin A$, then for all $x \in K$,

$$f(t)(x) = f(x_\Gamma, x_{I-\Gamma}) = f(x_\Gamma, y) = 0,$$

because for all $t \in [0,1]$, f(t) depends only on the coordinates in Γ. $\triangle$

Recall that a compact space is said to be **dyadic** if it is the continuous image of the product $2^\eta = \{0,1\}^\eta$, for some cardinal η.

Corollary 1 : *A continuous image of a product of separable spaces has property P. In particular, all dyadic compacts have property P.* $\triangle$

As the last step to one of the main result of this section we prove

Proposition 4 : *For all cardinals η, $C(2^\eta)$ has the separable projection property (spp).*

Proof : Let $Y \subseteq C(2^\eta)$ a separable subspace of $C(2^\eta)$ and $D = \{g_n : n \geq 1\}$ a countable dense subset of Y. Every g_n depends on a countable set $J_n \subseteq \eta$ of coordinates. Let $J = \bigcup_{n\geq 1} J_n$.

Observe that all $g \in Y$ depend only on the coordinates in J because there is a subsequence of the g_n converging to g in the sup norm.

We write 0 for the sequence in a product of $2 = \{0, 1\}$ whose coordinates are all equal to zero.

Let $Z = \{h \in C(2^\eta) : h(x) = h(x_J, 0), \forall x \in 2^\eta\}$. Of course $Y \subseteq Z$ and Z is a closed subspace of $C(2^\eta)$. For $g \in C(2^\eta)$, define $\tilde{g} : 2^\eta \longrightarrow \mathbb{R}$ by $\tilde{g}(x) = g(x_J, 0)$. Clearly $\tilde{g}$ is continuous and in Z.

If we define $p : C(2^\eta) \longrightarrow Z$ by $p(g) = \tilde{g}$, then p is a linear projection from 2^η onto Z, and so Z is a closed complemented subspace of $C(2^\eta)$. Moreover, Z is isometric to $C(2^J)$, and since J is countable, $C(2^J)$ is separable. This completes the proof. $\triangle$

Theorem 6.1 and the results of this section yield

Theorem 2 : *All dyadic compacts are Pm spaces.* $\triangle$

Under CH, nonseparable WCG spaces are nPm. We get Hagler's theorem ([Hg]) :

Corollary 2 (CH) : *If K is a dyadic compact, C(K) has no subspace isomorphic to a non separable WCG space.* $\triangle$

Observation 3 : In [DI] it's shown (by an entirely distinct method) that a dyadic compact is Rm. This property is a consequence of the fact these spaces are Pm.

Observation 4 : M. Ignez S. V. Diniz has remarked that every space X of the form $[l_1(I) \oplus [\Sigma_{\alpha \in A} L_1[0,1]]_1]_1$ is isomorphic to a closed subspace of C(T), for some dyadic compact T. This obtained by observing that the unit ball of the components of X with the weak* topology are either homeomorphic to a product of copies of [0,1] or are metrizable. Thus, X is isometric to a subspace of C(T) where T is a product of dyadics and consequently dyadic. It follows from Theorem 2 that X is Pm. This proof is independent of CH.

We now our attention to providing an example of a Pm compact of a quite distinct nature. We will show that there is a separable, measure separable Pm compact K such that C(K) is not a subspace of C(D) for any dyadic D. In particular, K is not dyadic. To provide this example, we shall make use of a construction due to Talagrand and presented in [T], pages 199 ff. We ask the reader to consult [T] for the details omitted below.

Let $L = [-1,1]^\omega$ and, for each $n \in \omega$, let $\delta_n : L \longrightarrow [-1, 1]$ be the projection on the n-coordenate. Recalling that L is a compact metric space, we can enumerate $(\lambda_\gamma)_{\gamma<\omega_1}$ the Borel regular measures defined on L. Let $(D_\alpha)_{\alpha<\omega_1}$ be the family of infinite subsets of $\mathbb{N}$.

We begin with the following result, whose proof may be found in [T] :

Lemma 2 : *There is a collection $(A_\alpha)_{\alpha<\omega_1}$ of infinite subsets of $\mathbb{N}$ verifying the following conditions :*

i) $\beta < \alpha \implies A_\alpha$ - A_β *is a finite set ;*

ii) For every subset $D \in \mathbb{N}$ there is $\alpha < \omega_1$ such that either A_α - D or $A_\alpha \cup D$ is a finite set ;

iii) For every $\gamma < \omega_1$, if $\alpha = \gamma + 1$, then one of the following holds :

a) $\lambda_\gamma(\{x \in L : \exists \lim_{A_\alpha} \delta_n(x)\}) = 1$ **or**

b) For $B = A_\gamma - A_\alpha$, $\lambda_\gamma(\{x \in L : \lim_B \delta_n(x)$ *does not exists*$\}) > 0$. $\triangle$

Let $\mathcal{A} = \{A \subseteq \mathbb{N} : A \text{ is a finite set}\} \bigcup \{A_\alpha : \alpha < \omega_1\}$. Let H be the Boolean algebra generated by $\mathcal{A}$. It's clear that the elements of H can be written as $(A_{1,1} \cap \ldots \cap A_{1,r_1}) \bigcup \ldots \bigcup (A_{s,1} \cap \ldots \cap A_{s,r_s})$, where $A_{i,j}$ or $A^c_{i,j}$ are in $\mathcal{A}$.

Let K = S(H), be the Stone space of H. For A $\in$ H, put $S_A = \{ \mathcal{F} \in K : A \in \mathcal{F}\}$. It is known that K is a compact Hausdorff space and that $\{ S_A : A \in H \}$ is a basis clopens in K. Since the finite sets are in H, it's not difficult to verify that K is separable, with the countable collection of principal ultrafilters $\mathcal{F}(n) = \{C \in H : n \in C\}$ as an open dense set in K.

In [T], Talagrand proves that every scalarly measurable function f : [0,1] $\longrightarrow$ C(K) is strongly measurable. Thus, **K is a separable Pm compact space.**

The following result shows that C(K) provides an example of a nonseparable Banach space such that both it and its dual are Pm. It will in fact imply that C(K) is not isomorphic to a subspace of C(D) for any dyadic space D. In particular, K is not dyadic. We just register that it is possible to prove directly from the construction that K is not dyadic.

Theorem 3 *: K is measurable separable.*

Proof : Let μ be a regular positive probability Borel measure on K.

We must prove that there is a countable set $\{ Z_n : n \in \mathbb{N}\} \subseteq \{$ borel sets of K $\}$ such that for every Borel set Z and $\varepsilon > 0$, there is n $\in \mathbb{N}$ such that $\mu(Z \triangle Z_n) < \varepsilon$.

It is sufficient to consider the case that μ is a measure without atoms, that is, in which the measure of finite sets is zero.

The sets $\mathcal{G} = \{\mu S_{A_\alpha} : \alpha < \omega_1\}$ and $\mathcal{H} = \{\mu S^c_{A_\alpha} : \alpha < \omega_1\}$ are subsets of [0,1], and so there is a sequence $(\alpha_n)_{n \geq 1}$ such that $\{\mu S_{A_{\alpha_n}} : n \geq 1\}$ and $\{\mu S^c_{A_{\alpha_n}} : n \geq 1\}$ are dense respectively in $\mathcal{G}$ and $\mathcal{H}$. Put $\overline{\alpha} = \sup \{\alpha_n : n \geq 1\}$. Then $\overline{\alpha} < \omega_1$ and we have $\{\mu S_{A_\alpha} : \alpha \leq \overline{\alpha} \}$ and $\{\mu S^c_{A_\alpha} : \alpha \leq \overline{\alpha} \}$ countable and dense in $\mathcal{G}$ and $\mathcal{H}$, respectively.

We first prove the folowing Facts, where $\varepsilon > 0$ is a real number :

1. $\forall \alpha < \omega_1, \forall \varepsilon > 0$ there is $\delta \leq \overline{\alpha}$ such that $\mu(S_{A_\alpha} \triangle S_{A_\delta}) < \varepsilon$.

If $\alpha \leq \overline{\alpha}$, it is sufficient to take $\delta = \alpha$. Suppose $\alpha > \overline{\alpha}$; then we may choose $\delta \leq \overline{\alpha}$ such that $|\mu S_{A_\alpha} - \mu S_{A_\delta}| < \varepsilon$. We have $S_{A_\alpha} \triangle S_{A_\delta} = (S_{A_\alpha - A_\delta}) \cup (S_{A_\delta - A_\alpha})$ which implies $\mu(S_{A_\alpha} \triangle S_{A_\delta}) = \mu(S_{A_\alpha - A_\delta}) + \mu(S_{A_\delta - A_\alpha})$.

Since $\delta < \alpha$ implies that A_α - A_δ is a finite set, we get $\mu(S_{A_\alpha}$ - $S_{A_\delta}) = 0$ and $\mu(S_{A_\alpha} \triangle S_{A_\delta}) = \mu(S_{A_\delta}$ - $S_{A_\alpha})$.

Therefore, from $A_\alpha = (A_\alpha$ - $A_\delta) \cup (A_\alpha \cap A_\delta)$ and $\mu(S_{A_\alpha - A_\delta}) = 0$ we get $\mu S_{A_\alpha} = \mu S_{A_\alpha - A_\delta}$.

Thus, $|\mu S_{A_\alpha} - \mu S_{A_\delta}| < \varepsilon$ implies $|\mu S_{A_\alpha \cap A_\delta} - \mu S_{A_\delta}| < \varepsilon$, which yields $\mu S_{A_\delta - A_\alpha} < \varepsilon$ and it follows that $\mu(S_{A_\alpha} \triangle S_{A_\delta}) < \varepsilon$, proving Fact 1.

2. $\forall \alpha < \omega_1$, $\exists\, \delta \leq \overline{\alpha}$ such that $\mu(S^c_{A_\alpha} \triangle S^c_{A_\delta}) < \varepsilon$.

3. Suppose $R = A_1 \cap \ldots \cap A_n$ and $Q = B_1 \cap \ldots \cap B_n$, with A_i, $B_i \in H$, are such that $\mu(S_{A_i} \triangle S_{B_i}) < \varepsilon/n$, $1 \leq i \leq n$. Then, $\mu(S_R \triangle S_Q) < \varepsilon$.

4. Suppose $R_1, \ldots, R_t, Q_1, \ldots, Q_t$ are elements of H such that $\mu(S_{R_i} \triangle S_{Q_i}) < \varepsilon/t$, $1 \leq i \leq t$. Then $\mu(S_{R_1 \cup \ldots \cup R_t} \triangle S_{Q_1 \cup \ldots \cup Q_t}) < \varepsilon$.

The proofs of items 2 - 4 are straightforward calculations.

Set $W = \{A_\alpha : \alpha \leq \overline{\alpha}\} \cup \{A^c_\alpha : \alpha \leq \overline{\alpha}\} \cup \{U : U$ is a finite or cofinite set in $\mathbb{N}\}$ and $\mathcal{E} = \{\bigcap Z : Z \subseteq W, Z$ finite $\}$.

Then $\mathcal{D} = \{S_{Q_1 \cup \ldots \cup Q_t} : Q_i \in \mathcal{E},\ 1 \leq i \leq t\}$ is countable. We will prove that $\mathcal{D}$ is dense in the measure μ.

Let Y be a Borel set in K and $\varepsilon > 0$. We may select a closed F and an open set Θ such that $F \subseteq Y \subseteq \Theta$ and $\mu(\Theta - F) < \varepsilon/2$.

For each $x \in F$, choose $V_x \in H$ such that $x \in S_{V_x} \subseteq \Theta$. Since F is compact, there are $x_1, \ldots, x_k \in F$ such that $F \subseteq \bigcup_{i=1}^k S_{V_{x_i}} \subseteq \Theta$. We have $Y - \bigcup_{i=1}^k S_{V_{x_i}} \subseteq Y - F$ and $\bigcup_{i=1}^k S_{V_{x_i}} - Y \subseteq \Theta - Y$; thus $\mu(Y \triangle \bigcup_{i=1}^k S_{V_{x_i}}) < \varepsilon/2$.

It's straightforward to see that we may write $\bigcup_{i=1}^k V_{x_i} = R_1 \cup \ldots \cup R_t$, where each R_i is an intersection of elements from $\mathcal{A}$ or whose complement is in $\mathcal{A}$, that is

$$R_i = A^i_1 \cap \ldots \cap A^i_{r_i},\ 1 \leq i \leq t, \qquad (*)$$

where each A^i_j is finite, cofinite, a member of the sequence constructed in Lemma 2 or the complement of one such element.

For each $i \leq t$, $1 \leq j \leq r_i$ (as in (*)), if $A^i_j = A_\alpha$ (or A^c_α), with $\alpha < \omega_1$, by Fact 1 we may select $\delta_j \leq \overline{\alpha}$ such that $\mu(S_{A^i_j} \triangle S_{A_{\delta_j}}) < \varepsilon/2tr_i$ (or $\mu(S_{A^i_j} \triangle S_{A^c_{\delta_j}}) < \varepsilon/2tr_i$).

Let $B^i_j = A_{\delta_j}$ (or $A^c_{\delta_j}$). If A^i_j is a finite or cofinite set, put $B^i_j = A^i_j$.

Let $Q_i = B^i_1 \cap \ldots \cap B^i_{r_i}$, $1 \leq i \leq t$. Then $S_{Q_1 \cup \ldots \cup Q_t} \in \mathcal{D}$, and since (by Fact 3) $\mu(S_{R_i} \triangle S_{Q_i}) < \varepsilon/2t$, it follows (from Fact 4) that $\mu(S_{R_1 \cup \ldots \cup R_t} \triangle S_{Q_1 \cup \ldots \cup Q_t}) < \varepsilon/2$.

Finally,

$$\mu(Y \triangle S_{Q_1 \cup \ldots \cup Q_t}) \leq \mu(Y \triangle S_{R_1 \cup \ldots \cup R_t}) + \mu(S_{R_1 \cup \ldots \cup R_t} \triangle S_{Q_1 \cup \ldots \cup Q_t}) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

ending the proof. $\triangle$

The preceding discussion and Corollary 5.3 yield

Theorem 4 : *Both C(K) and $C(K)^*$ are Pm spaces.* $\triangle$

Corollary 3 : *If D is a dyadic, C(K) is not isomorphic to a closed subspace of C(D).*

Proof : We recall here a theorem of Hagler ([.]) :

Let X be a Banach space and D a dyadic compact with X isomorphic to a subspace of C(D). Let δ be a regular cardinal number. Are equivalent :

(i) the dimension of X is $\geq \delta$; (ii) X^* contains a subspace isomorphic to $C(2^\delta)^*$.

In our case, the dimension of C(K) is $\geq$ c. If C(K) were isomorphic to a subspace of C(D), $C(K)^*$ would contain subspace isomorphic to $C(2^c)^*$. Note that $C(2^c)^*$ is not Pm since 2^c is not measure separable (just consider the usual Haar measure on 2^c). But this is impossible because the dual of C(K) is Pm. $\triangle$

9 The spaces C(K, X) and the Pm property

Given a compact K and a Banach space X, we denote by C(K, X) the Banach space of the continuous functions $g : K \longrightarrow X$, with the sup norm $\|g\| = \sup_{p\in K} \|g(t)\|$.

In this section we show that if K is a separable Pm compact and X is Pm then C(K, X) is Pm.

For $L \subseteq C(K, X)$ and $p \in K$, we set $L(p) = \{f(p) : f \in L\} \subseteq X$.

The next result is useful in identifying separable subsets of C(K, X).

Proposition 1 : *For a subset L of C(K, X), are equivalent :*

(1) L is separable ;

(2) There is a countable $D \subseteq C(K, X)$ such that :

(a) $\forall\, p \in K,\ L(p) \subseteq \overline{D(p)}$.

(b) $\forall\, p, q \in K\ \ \{[\forall\, d \in D\ (d(p) = d(q))] \Longrightarrow [\forall\, \alpha \in L\ (\alpha(p) = \alpha(q))]\}$.

(3) There is a countable set $W \subseteq C(K)$ and a separable $Y \subseteq X$ such that

(a) $\forall\, \alpha \in L$, $Im\, \alpha \subseteq Y$;

(b) $\forall\, p, q \in K\ \ \{[\forall\, \sigma \in W\ (\sigma(p) = \sigma(q))] \Longrightarrow [\forall\, \alpha \in L\ (\alpha(p) = \alpha(q))]\}$.

(4) There is a compact metric space K_m, a closed subspace $Y \subseteq X$, a surjective continuous function $\pi : K \longrightarrow K_m$ and an injective continuous function $\gamma : L \longrightarrow C(K_m, Y)$ such that $\pi_* \circ \gamma = i_L$, where $\pi_* : C(K_m, Y) \longrightarrow C(K, X)$ is the isometry induced by π and i_L is the canonical injection of L in C(K , X).

Proof : (1) $\Longrightarrow$ (2) : Since $L \subseteq C(K, X)$ is separable, there is a countable $D \subseteq L$ such that $\overline{D} = L$. For $p \in K$, consider $L(p) = \{f(p) : f \in L\}$ and $D(p) = \{f(p) : f \in D\}$.

First notice that $L(p) = \overline{D(p)}$. To see this, let $f(p) \in L(p)$ and $\varepsilon > 0$; since D is dense in L, there is $g \in D$ such $\|f - g\| < \varepsilon$. In particular, $\|f(p) - g(p)\| < \varepsilon$ and so $f(p) \in \overline{D(p)}$.

Now suppose $p, q \in K$ are such that $d(p) = d(q)$, $\forall\, d \in D$. If $\alpha \in L$ and $\varepsilon > 0$, there is $d \in D$ such that $\|\alpha - d\| < \varepsilon/2$, and so

$$\|\alpha(p) - d(p)\| < \varepsilon/2 \quad \text{and} \quad \|\alpha(q) - d(q)\| < \varepsilon/2.$$

Thus, $\|\alpha(p) - \alpha(q)\| < \varepsilon$; since ε is arbitrary we conclude that $\alpha(p) = \alpha(q)$.

$(2) \Longrightarrow (3)$: Let $D = \{g_n : n \geq 1\}$; for each $n \geq 1$, the image of g_n is separable because it's a compact in the metric space X. Let be $Y = \overline{\text{span} \cup g_n(K)}$. Then Y is a separable closed subspace of X. Let $\{y_n : \|y_n\| = 1, n \geq 1\}$ be a countable dense subset of the set of norm 1 vectors in Y.

For each $n \geq 1$, choose $x_n^* \in X^*$ such that $x_n^*(y_n) = 1 = \|x_n^*\|$. It's readily verified that $\{x_n^* : n \geq 1\}$ is total in Y, that is, if $y \in Y$ and $x_n^*(y) = 0$, $\forall\, n \geq 1$, then $y = 0$.

Define $W = \{x_m^* \circ g_n : n, m \geq 1\}$.

(a) : Fix $\alpha \in L$; given $\varepsilon > 0$, there is $n \geq 1$ such that $\|\alpha - g_n\| < \varepsilon$. Thus, $\|\alpha(p) - g_n(p)\| < \varepsilon$, $\forall\, p \in K$. Recalling that $\operatorname{Im} g_n \subseteq Y$, this reasoning shows that

$$\forall\, \varepsilon > 0\ \forall\, p \in K\ \exists\, y \in Y \text{ such that } \|\alpha(p) - y\| < \varepsilon.$$

Since Y is closed in X we get that $\alpha(p) \in Y$, $\forall\, p \in K$.

(b) : Let $p, q \in K$ be such that $\sigma(p) = \sigma(q)$, $\forall \sigma \in W$. Then,

$$x_m^* \circ g_n(p) = x_m^* \circ g_n(q), \quad \forall\, m \geq 1.$$

Since $\{x_m^* : m \geq 1\}$ is total in Y, $g_n(p) = g_n(q)$, $\forall\, n \geq 1$, which, by 2.(b), implies $\alpha(p) = \alpha(q)$, $\forall \alpha \in L$.

$(3) \Longrightarrow (4)$: Let $W = \{\sigma_n : n \geq 1\}$ be the countable set satisfying the conditions in (3). Define the following equivalence relation $\approx$ on K

$$p \approx q \quad \text{iff} \quad \sigma_n(p) = \sigma_n(q),\ \forall\, n \geq 1.$$

Let $K_m = K/\approx$, considered as a topological space with the quotient topology. Then the function $\pi : K \longrightarrow K_m$, given by $\pi(p) = p/\approx$ (the class of p under $\approx$), is continuous and onto and so K_m is compact. We must verify that the topology on K_m is Hausdorff. The following facts are easily established :

i) If $U \subseteq \mathbb{R}$ is an open subset then $\sigma^{-1}(U)$ is invariant by $\approx$, for all $\sigma \in W$, i.e. : if $p \in \sigma^{-1}(U)$ and $p \approx q$, then $q \in \sigma^{-1}(U)$.

ii) If p and q are not equivalent under $\approx$ then there are open subsets $U, V \subseteq \mathbb{R}$, and $\sigma_n \in W$ such that $U \cap V = \emptyset$, $\sigma_n(p) \in U$ and $\sigma_n(q) \in V$.

It follows from (i) that if $U \subseteq \mathbb{R}$ is open and $\sigma \in W$, then $\pi^{-1}(\pi(\sigma^{-1}(U))) = \sigma^{-1}(U)$. Since K_m has the quotient topology, we get that $\pi(\sigma^{-1}(U))$ is open in K_m, for all $\sigma \in W$ and all open $U \subseteq \mathbb{R}$.

Now, to separate the points of K_m proceeds as follows : if $p/\approx \neq q/\approx$, using (ii) choose opens U, V in $\mathbb{R}$ and $n \geq 1$ such that $\sigma_n(p) \in U$, $\sigma_n(q) \in V$ and $U \cap V = \emptyset$. By (i), $p/\approx \subseteq \sigma^{-1}(U)$ and $q/\approx \subseteq \sigma^{-1}(V)$, $\sigma^{-1}(U) \cap \sigma^{-1}(V) = \emptyset$ and both these inverse images are open and invariant. The projections of these inverse images on K_m will then be non empty, disjoint neighbourhoods of $p/\approx$ and $q/\approx$, proving that K_m is Hausdorff.

Our next step is to verify that $C(K_m)$ is separable which will then imply that K_m is metrizable. For each $n \geq 1$, define $\widetilde{\sigma_n} : K_m \longrightarrow \mathbb{R}$ by $\widetilde{\sigma_n}(p/\approx) = \sigma_n(p)$. $\widetilde{\sigma_n}$ is well defined and, since $\widetilde{\sigma_n} \circ \pi = \sigma_n$, it follows that it is continuous. A moment of thought will convince the reader that this countable family of continuous maps separates the points of K_m and the Stone-Weierstrass Theorem then guarantees the separability of $C(K_m)$.

Now, For every function $\alpha \in L$, define $\tilde{\alpha} : K_m \longrightarrow X$ by $\tilde{\alpha}(p/\approx) = \alpha(p)$. It's straightforward to check that $\tilde{\alpha}$ is well defined and continuous because $\tilde{\alpha} \circ \pi = \alpha$.

Let $\gamma : L \longrightarrow C(K_m, Y)$ be given by $\gamma(\alpha) = \tilde{\alpha}$; then γ is linear, continuous and injective. Moreover, if π^* is the isometry from $C(K_m, Y) \longrightarrow C(K, X)$ induced by π ($\pi^*(h) = h \circ \pi$), it's clear that $\pi^* \circ \gamma = i_L$, the canonical immersion of L in C(K, X).

(4) $\Longrightarrow$ (1) : Since $\pi^* \circ \gamma = i_L$ and γ is injective, L is a subspace of $C(K_m, Y)$.

On the other hand, Y is separable and so by a result of Banach ([Ba]), isometric to a closed subspace of C([0,1]). From this we obtain that $C(K_m, Y)$ is a subspace of $C(K_m, C([0,1]))$, which is isomorphic to $C(K_m \times [0,1])$. But $K_m \times [0,1]$ is metrizable which in turn implies that $C(K_m, Y)$ is separable. Thus, the same must be true of its subspace L, concluding the proof. $\triangle$

Theorem 1 : *If K is a separable Pm compact and X a Pm Banach space, C(K, X) is Pm.*

Proof : Let $D = \{d_n \ ; \ n \geq 1\}$ be a dense subset of K and $f : [0,1] \longrightarrow C(K, X)$ a bounded Pettis integrable function.

For each $n \geq 1$, we define $d_n^* : [0,1] \longrightarrow X$ by $d_n^*(t) = f(t)(d_n)$. It is easy to see that d_n^* is bounded and Pettis integrable because it is the composition of f and the operation of calculation at d_n.

Since X is Pm, **there is** $C \subseteq [0,1]$, $\lambda C = 1$, such that $\bigcup_{n\geq 1} d_n^*(C)$ is separable. If $Y = \overline{\operatorname{span} \cup_{n\geq 1} d_n^*(C)}$, we have that Y is separable. Consequently, proceding just as in the proof of (2) implies (3) in Proposition 1, there is a countable family of continuous linear functionals in X^* which is total in Y.

For each $n \geq 1$, consider the function $h_n : [0,1] \longrightarrow C(K)$ given by $h_n(t) = x_n^* \circ f(t)$. Each h_n is bounded and Pettis integrable; since K is Pm, **there is B** $\subseteq [0,1]$, $\lambda B = 1$, such that $Z = \overline{\operatorname{span} \cup_{n\geq 1} h_n(B)} \subseteq C(K)$ is separable. Let $W = \{\sigma_n : n \geq 1\}$ be a dense set in Z.

Now consider $A = B \cap C \subseteq [0,1]$, B and C as above. Then $\lambda A = 1$ and we will show that $L = f(A)$ is separable, using the conditions in Proposition 1.3.

(a) It's clear that D being dense in K, Y being closed and that $\forall$ n and $t \in A$ $f(t)(d_n) = d_n^*(t) \in Y$, forces $f(t) \subseteq Y$, for all t in A.

b) Let be $p, q \in K$ such that $\sigma_n(p) = \sigma_n(q)$, $\forall\, n \geq 1$. Then $x_n^* \circ f(t)(p) = x_n^* \circ f(t)(q)$, $\forall\, n \geq 1$. Since $\{x_n^* : n \geq 1\}$ is total in Y, we get $f(t)(p) = f(t)(q)$, $\forall\, t \in A$.

This ends the proof. $\triangle$

Corollary 1 : *a) If $K_1, \ldots , K_n$ are separable Pm compacts, then $K_1 \times \ldots \times K_n$ is Pm.*

b) If K is a Pm compact and X a separable Banach space then C(K, X) is Pm.

c) If K_1 and K_2 are Pm compact spaces and K_1 is separable then the tensor product $C(K_1) \otimes C(K_2)$ is Pm.

Proof : a) It follows from Theorem 1 that $C(K_1, C(K_2))$ is Pm; but this space is isometric to $C(K_1 \times K_2)$. Thus, $K_1 \times K_2$ is Pm. Induction will complete the proof of (a).

b) By Banach's theorem, X is isometric to a closed subspace of $C([0,1])$ and so $C(K, X)$ is isometric to a subspace of $C(K, C([0,1]))$. This space is isomorphic to $C(K \times [0,1])$, in turn isomorphic to $C([0,1], C(K))$, which is Pm by Theorem 1.

c) It'sufficient to remember ([S], pg.357) that $C(K_1) \otimes C(K_2)$ is isometric to $C(K_1, C(K_2))$ and then apply Theorem 1. $\triangle$

Theorem 2 *Let be $(K_n)_{n \geq 1}$ a family of compacts such that, every finite product of the K_n's is Pm. Then $K = \prod_{n \geq 1} K_n$, as well as every continuous image of K, is Pm.*

Proof : Notation will be as in Lemma 1, except that we write p_j, α_j, φ_j and π_j for the maps corresponding to the finite subset $J = \{1, \ldots, j\}$. For the definition of all φ_j and π_j, we fix $y = (y_n)$ in K. By Proposition 8.1, it's sufficient to verify that K is Pm.

Let $f : [0,1] \longrightarrow C(K)$ be a bounded Pettis integrable function. For $j \geq 1$, consider $h_j = \alpha_j \circ \pi_j \circ f$. Then h_j is Pettis integrable and $h_j(t)((x_n)) = f(t)(x_1, \ldots, x_j, (y_i)_{i>j})$; thus Im h_j is isometric to a subspace of $C(K_1 \times \ldots \times K_j)$. By hypothesis, $K_1 \times \ldots \times K_j$ is Pm and so there is $A_j \subseteq [0,1]$ such that $\lambda A_j = 1$ and $h_j(A_j)$ is separable.

Set $A = \bigcap_{j \geq 1} A_j$. Then $\lambda A = 1$ and $Z = \overline{\cup_{j \geq 1} h_j(A)} \subseteq C(K)$ is separable. Let $D = \{g_n : n \geq 1\}$ be a dense subset of Z. We will prove that $f(A) \subseteq Z$ and consequently, it must be separable. The measurability of f is then clear.

Fix $t \in A$ and $\varepsilon > 0$. Then there is $g \in C(K)$, depending only on the coordinates below some $j \geq 1$, such that $\|f(t) - g\| < \varepsilon/3$.

In particular, we have $|f(t)(x_1, \ldots, x_j, (y_i)_{i>j}) - g(x_1, \ldots, x_j, (y_i)_{i>j})| < \varepsilon/3$, $\forall\, (x_i) \in \prod_{i \leq j} K_i$; thus, one has $\|h_j(t) - g\| < \varepsilon/3$.

On the other hand, since $h_j(t) \in Z$, there is $g_r \in D$ such that $\|h_j(t) - g_r\| < \varepsilon/3$.

These inequalities clearly imply $\|f(t) - g_r\| < \varepsilon$. Since Z is closed and ε arbitrary, we conclude $f(A) \subseteq Z$, ending the proof. $\triangle$

Corollary 2 *: Let $(K_i)_{i \in I}$ be a family of compact and $K = \prod_{i \in I} K_i$. Are equivalent :*

(1) K is Pm ; *(2) (a) $\forall\, F \subseteq I$, F finite, $\prod_{i \in F} K_i$ is Pm ;*

(b) K has property P.

Proof : (1) implies (2) is a direct consequence of Proposition 8.1.

(2) $\Longrightarrow$ (1) : By Theorem 8.1 we can write a bounded Pettis integrable f as $f = f_1 + f_2$ where these maps have the properties described in its statement. Since K has property P and f_2 is a null function (Theorem 8.1 item (v)), we have that $f = f_1$ λ-ae.

On the other hand, by item (iv) in the same result, f_1 (and thus f) can be considered as a map into a **countable** product of components of K. An application of Theorem 2 yields the measurability of f_1 and consequently, that of f. $\triangle$

Proposition 8.3, Corollary 1 and the preceding result give

Corollary 3 : *Any product of separable Pm compacts is Pm.*

10 Some aspects of Pm theory under $\neg$CH + MA

In the previous sections we used the continuum hypothesis (CH) to obtain a number of results. Here, the independence of some of these statements from ZFC will be discussed, using Martin's axiom (MA) and the negation of CH. A convenient reference for Martin's axiom is [K]. A formulation of MA can be stated as

MA : In a CCC compact Hausdorff space, the union of a family of meager sets with cardinality $\alpha < 2^{\aleph_0}$ is a meager set.

It is known that there are models of ZFC set theory (Zermelo - Fraenkel with Axiom of Choice) satifying MA in which CH is false. The following consequence of Martin's axiom will be important in what follows.

Theorem 1 *(see [K]) : (MA) If η is a cardinal number such that $\eta < 2^{\aleph_0}$ then the union of η subsets of $\mathbb{R}$ of Lebesgue measure zero has measure zero.* $\triangle$

In section 6 we proved that if X is a Banach space which have spp and the null Pettis integral property then X is Pm. The proof did not use CH and so this result holds true under $\neg$CH + MA. Thus, with a proof analogous to that of Corollary 6.1 we can show

Theorem 2 : *Let X be a Banach space and η an infinite cardinal such that the union of η subsets in [0,1] of Lebesgue measure zero has measure zero. If X has a total subset $\Omega \subseteq X^*$ with $|\Omega| \leq \eta$, then X has property P. If X also has the spp, then X is Pm.* $\triangle$

Consequently,

Corollary 1 *($\neg$CH + MA) : A Banach space with density $\eta < 2^{\aleph_0}$ has property P. If it also has the spp, then it is Pm.*

Proof : It is enough to recall that a Banach space with density $\eta < 2^{\aleph_0}$ has a total subset $\Omega \subseteq X^*$ such that $|\Omega| = \eta$. By MA, the union of η subsets of measure zero in [0,1] has measure zero and thus we can apply Theorem 2 to conclude. $\triangle$

In section 7, it was proved that CH implied that a WCG space is Pm iff it is separable. In section 5 it was shown that, under CH, an abstract L_p space, $1 < p < \infty$, is Pm iff it is separable. Furthermore, we had observed that the continuum hypothesis implied that a non CCC compact space did not have property P. These statements are all independent of ZFC since we can apply Corollary 1 to get

Corollary 2 *($\neg$CH + MA) : For $\aleph_1 \leq \eta < 2^{\aleph_0}$, $l_p(\eta)$, $1 \leq p < \infty$, $c_0(\eta)$ are Pm and $C([0, \eta])$ has property P. Moreover, $C([0, \omega_1])$ is Pm.* $\triangle$

Nevertheless, the following result shows that the class of nonseparable Pm WCG spaces in a model of $\neg$ CH + MA is not 'very large'. Notation is as set down in section 7.

Proposition 1 *($\neg$CH + MA) : If X is WCG and dens $X \geq c$ then X is not Pm.*

Proof : By Proposition 7.1, we may assume that dens X = c. Thus dens*X = c and therefore, $c = \text{dens } X = \text{dens}^* X \leq \text{dens } X^* \leq (\text{dens}X)^\omega = c$, and we get dens $X^* = c$.

Let $\{x^*_\lambda : \lambda < c\}$ be a dense subset in X^*, and $H_\lambda = (x^*_\lambda)^{-1}(0)$, $\forall\ \lambda < c$.

Just as in the proof of Proposition 7.3, for every $\alpha < c$, we can find $x_\alpha \in \bigcap_{\lambda<\alpha} H_\lambda$, with $\|x_\alpha\| = 1$.

Fix a bijection $h : [0,1] \longrightarrow [0, c)$, $t \mapsto \lambda_t$, and consider $f : [0,1] \longrightarrow X$, given by $f(t) = x_{\lambda_t}$. f is bounded, in fact $\|f(t)\| = 1$, $\forall\ t \in [0,1]$. Moreover, given $t_0 \in [0,1]$, we have $\lambda_t > \lambda_{t_0} \Longrightarrow x_{\lambda_t} \in H_{\lambda_{t_0}} \Longrightarrow x^*_{\lambda_{t_0}}(x_{\lambda_t}) = 0 \Longrightarrow x^*_{\lambda_{t_0}} \circ f(t) = 0$.

Since $|\{t : \lambda_t \leq \lambda_{t_0}\}| \leq |\lambda_{t_0}| < c$, by MA, $\mu\{t : \lambda_t \leq \lambda_{t_0}\} = 0$. To conclude, one can now proceed as in the proof of Theorem 7.2. $\triangle$

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