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Active redundancy allocation for a k-out-of-n:F system of dependent components

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Abstract

We characterize active redundancy through compensator transform and use the reverse rule of order 2 (RR_2) property between compensator processes to investigate the problem of where to allocate a spare in a k-out-of-n:F system of dependent components through active redundancy.

Keywords: Reliability; Martingales; Marked point process; k-out-of-n:F system; Active redundancy; Minimal standby redundancy; Reverse rule of order 2

1. Introduction

The problem of where to allocate a redundant component in a system in order to stochastically increase the system lifetime is important in reliability theory. For instance see Boland et al. (1992), Singh and Misra (1994), Meng (1996), Prasad et al. (1999), Kuo and Prasad (2000), Kuo et al. (2001), Bueno (2005a), among others.

There are two common types of redundancy that are used, namely active redundancy, which stochastically lead to consider the maximum of random variables, and standby redundancy, which stochastically lead to consider the convolution of random variables. The problem of where to allocate a spare component, is addressed in Boland et al. (1992) for either active or standby redundancy, in a k-out-of-n:G system of independent components in order to stochastically increase the system reliability. For a k-out-of-n:G system, Boland et al. (1992) consider stochastic ordering to shows that for active redundancy it

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is stochastically optimal always to allocate a spare to the weakest component. For standby redundancy and under likelihood ratio ordering Boland et al. (1992) gives sufficient conditions to ensure that in a series system the allocation should go to the weakest component while in a parallel system it should go to the strongest.

Singh and Misra (1994) consider the same problem with another criterion of optimality and proves that if the lifetime of the k-out-of-n system resulting from an active redundancy operation of component i is denoted by $\tau_k^i = \tau_k(S_1, \dots, S_{i-1}, S_i \vee S, S_{i+1}, \dots, S_n)$, then

$$P(\tau_k^i > \tau_k^j) \geqslant P(\tau_k^j > \tau_k^i),$$

provided the lifetime of component j is stochastically larger than that of component i, i.e. with respect to the above criterion, it is preferable to allocate the active redundant component to the stochastically weakest component for stochastically ordered component lifetimes. The problem of allocating a standby redundant component either, in a series and parallel systems, is also considered, comparing the probabilities that one system has a longer lifetime than the other.

Meng (1996) uses the concept of permutation equivalency (Boland et al., 1989) to shows that if two components in a coherent system are permutation equivalent, then allocating a spare to the weaker position as an active redundant optimally improve the system lifetime.

Clearly, in a k-out-of-n system all components are permutation equivalent and the sufficient condition of Boland et al. (1992) is a consequence of this result.

Bueno (2005a) defined a particular form of standby redundancy which was called minimal standby redundancy. Intuitively, the minimal standby redundancy gives to component i an additional lifetime as it had just before the failure. Bueno (2005a) uses the reverse rule of order 2 (RR_2) property between compensator processes to investigate the problem of where to allocate a spare in a k-out-of-n:F system, for minimal standby redundancy, under dependence conditions. As in Boland et al. (1992), Bueno (2005a) proves that for minimal standby redundancy it is stochastically optimal always to allocate a spare to the weakest component.

In this paper, to consider the problem of where to allocate a spare component for active redundancy under dependence conditions, we use a martingale approach to reliability theory. In Section 2 we give a point process formulation of a coherent system and we get a compensator characterization of active redundancy operation. In Section 3 we use Kwieciński and Szekli (1991) to investigate the best active redundancy allocation in a k-out-of-n:F system in order to increase system reliability. In Section 4 we give a constructive example.

2. Active redundancy under dependence conditions

2.1. The point process formulation of a coherent system

We consider the vector $(S_1, S_2, \dots, S_n)$ of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(S_i \neq S_j) = 1$, for all $i \neq j, i, j$ in $E = \{1, 2, \dots, n\}$. The lifetimes can be dependent but simultaneous failures are ruled out.

The evolution of the components in time define a marked point process given through the part failure times and the corresponding marks.

We denote by $T_1 < T_2 < \cdots < T_n$ the ordered lifetimes $S_1, S_2, \dots, S_n$, as they appear in time, and by $X_i = \{j : T_i = S_j\}$ the corresponding marks.

As a convention we set $T_{n+1} = T_{n+2} = \cdots = \infty$ and $X_{n+1} = X_{n+2} = \cdots = e$ where e is a fictitious mark not in E. Therefore the sequence $(T_n, X_n)_{n \geqslant 1}$ define a marked point process.

The mathematical formulation of our observations is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geqslant 0}$, where

$$\Im_t = \sigma\{1_{\{T_i>s, X_i=j\}}, 0 < s \leqslant t, 1 \leqslant i \leqslant n, j \in E\}.$$

Intuitively, at each instant t the observer knows if the events $\{T_i \leqslant t,\ X_i = j\}$ have either occurred or not and if they have, he knows exactly the value T_i and the mark X_i.

We assume that S_i, $1 \leqslant i \leqslant n$ are totally inaccessible $\Im_t$-stopping time.

An extended and positive random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leqslant t\} \in \Im_t$ for all $t \geqslant 0$; an $\Im_t$-stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n\geqslant 0}$, of $\Im_t$-stopping time, $\tau_n < \tau$, exists such that $\lim_{n\to\infty}\tau_n = \tau$; an $\Im_t$-stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping time σ. For a mathematical basis of stochastic processes applied to reliability theory see the recent book by Aven and Jensen (1999). Also, to simplify the notation, we assume that relations such as $\subset, =, \leqslant, <, \neq$ between random variables and measurable sets, respectively, always hold with probability one, which means that the term $P - a.s.$ is suppressed.

The simple marked point $N^i_j(t) = 1_{\{T_i\leqslant t, X_i=j\}}$ is an $\Im_t$-submartingale and from the Doob Meyer decomposition we know that there exists a unique $\Im_t$-predictable process $(A^i_j(t))_{t\geqslant 0}$, called the $\Im_t$-compensator of $N^i_j(t)$, with $A^i_j(0) = 0$, such that $N^i_j(t) - A^i_j(t)$ is an $\Im_t$-martingale. $A^i_j(t)$ is absolutely continuous by the totally inaccessibility of S_i, $1 \leqslant i \leqslant n$.

As $N^i_j(t)$ can only count on the time interval $(T_{i-1}, T_i]$, the corresponding compensator $A^i_j(t)$ must vanish outside that interval. To count the failure of component j we let $N_j(t) = \sum_{n\geqslant 1} N^n_j(t)$ with $\Im_t$-compensator process $A_j(t) = \sum_{n\geqslant 1} A^n_j(t)$.

The compensator process is expressed in terms of the conditional probability, given the available information. Its generalize the classical notion of hazards. Intuitively, this corresponds to producing whether the failure is going to occur now, on the basis of all observations available up to, but not including, the present.

The $\Im_t$-stopping times S_i are rarely of direct concern in reliability theory. One is more interested in the system lifetime:

$$\tau = \tau(\mathbf{S}) = \min_{1\leqslant j\leqslant r} \max_{i\in K_j} S_i,$$

where K_j, $1 \leqslant j \leqslant r$ are the system minimal cut sets, that is, a minimal set of components whose joint failure causes the system to fail. As $\{\tau > t\} = \{\cap_{1\leqslant j\leqslant r} \cup_{i\in K_j} \{S_i > t\}\}$, τ is also an $\Im_t$-stopping time.

Clearly component i contributes to system failure after its critical level Y_i, that is, the first time from which onwards it's failure lead to system failure (for a rigorous definition of critical level see Arjas, 1981b). It is easy to see that the critical level of a component in series with the system is the initial time 0 (a component is in series with the system when it is a unitary cut set) and the critical level of a component in parallel with a subsystem is the subsystem lifetime.

In a forthcoming work Bueno (2005b) proposes to analyze the system lifetime after its component's critical levels observing the family of σ-algebras $(\Im_{Y_i+t})_{t\geqslant 0}$, where

$$\Im_{Y_i+t} = \{A \in \Im_\infty : A \cap \{Y_i + t \leqslant s\} \in \Im_s,\ s \geqslant 0\},$$

the lifetimes $S_i = ((T_i - Y_i)^+ | \Im_{Y_i}),\ i \in E$ and the corresponding counting process

$$M_i(t) = 1_{\{S_i\leqslant t\}} = E[N_i(Y_i + t) - N_i(Y_i) | \Im_{Y_i}] = E[1_{\{Y_i<T_i\leqslant Y_i+t\}} | \Im_{Y_i}].$$

Follows that $B_i(t) = E[A_i(Y_i + t) - A_J(Y_i) | \Im_{Y_i}]$ is the $\Im_{Y_i} + t$-compensator of $M_i(t)$.

To represent a coherent system as a series structure Bueno (2005b) proves the following theorem:

Theorem 2.1.1. *Under the above notation the system lifetime τ is almost surely equal to the series system lifetime*

$$S = \min_{\{i:Y_i<\infty\}} S_i.$$

The ideas for the above result was provided by Arjas (1981b) which gives a rigorous proof of the following Theorem:

Theorem 2.1.2. *Under the above notation, the* $\Im_t$*-compensator of* $M(t) = 1_{\{\tau \leqslant t\}}$ *is*

$$A(t) = \sum_{j=1}^{n} [A_i(t \wedge \tau) - A_i(Y_i)]^+,$$

where $[a]^+ = \max\{a, 0\}$.

Follows that

$$P(\tau \leqslant t) = E[M(t)] = E[A(t)] = \sum_{j=1}^{n} E[[A_i(t \wedge \tau) - A_i(Y_i)]^+]$$

and we can improve system reliability minimizing the quantities $E[[A_i(t \wedge \tau) - A_i(Y_i)]^+]$ for each i, $1 \leqslant i \leqslant n$.

In this paper we are going to specialize in a k-out-of-n:F system which functions if and only if at least $n - k + 1$ out of the n components functions. We denote the lifetime of a k-out-of-n:F system by $\tau_{k:\mathrm{F}}(\mathbf{S}) = T_k$. Clearly, in a k-out-of-n:F system, the critical level of component i, $1 \leqslant i \leqslant n$ is T_{k-1}, and follows as a Corollary of Theorem 2.1.1:

Corollary 2.1.3. *The* $\Im_t$*-compensator process of a k-out-of-n:F system is*

$$A(t) = \sum_{i=1}^{n} [A_i(t \wedge \tau) - A_i(T_{k-1})]^+.$$

2.2. A compensator characterization of active redundancy

An active redundancy of a component i stochastically lead to the maximum of the lifetime S_i and the spare lifetime S. If S_i and S are independent and identically distributed random variables with distribution function $F_i(t) = 1 - \overline{F}_i(t)$ the resulting lifetime, $S_i \vee S$, from an active redundancy operation of component i has a distribution function

$$P(S_i \vee S \leqslant t) = 1 - P(S_i \vee S > t),$$

where

$$P(S_i \vee S > t) = 2P(S_i > t) - P(S_i > t)^2 = 2\overline{F}_i(t) - \overline{F}_i(t)^2.$$

In the case of independent components, $A_i(t) = -\ln P(S_i > t|\Im_t)$ on $\{t < S_i\}$, is the $\Im_t$-compensator of component i and we can write that on $\{t < S_i \vee S\}$

$$-\ln P(S_i + S > t|\Im_t) = A_i(t) - \ln(2 - \exp[-A_i(t)]),$$

is the $\Im_t$-compensator of $S_i \vee S$. Since that $P(S_i \vee S > t|\Im_t) = \exp[-A_i(t)](2 - \exp[-A_i(t)])$.

Clearly, we are considering

$$\Im_t = \sigma\{1_{\{S_i \vee S \leqslant s\}}, 1_{\{S_j \leqslant s\}}, s \leqslant t, j \neq i, 1 \leqslant j \leqslant n\}.$$

Now, in the case of dependent components, we have to find a compensator approach formulation of the active redundancy operation preserving the above intuition. We note that $A_i(t) - \ln(2 - \exp[-A_i(t)])$ is equal to

$$\int_0^t \mathrm{d}A_(s) - \int_0^t \left(\frac{\exp[-A_i(s)]}{2 - \exp[-A_i(s)]}\right)\mathrm{d}A_i(s) = \int_0^t \left(\frac{2 - 2\exp[-A_i(s)]}{2 - \exp[-A_i(s)]}\right)\mathrm{d}A_i(s).$$

We propose, in the general case, the following active redundant transformation:

$$B_j(t) = \int_0^t \alpha_j(s)\,\mathrm{d}A_j(s) \quad \text{where } \alpha_i(s) = \frac{2-2\exp[-A_i(s)]}{2-\exp[-A_i(s)]} \text{ and } \alpha_j(s) = 1 \text{ for } j \neq i.$$

Therefore, we are looking for a probability measure Q, such that, under Q, $B_j(t)$ becomes the $\Im_t$-compensator of $N_j(t)$ with respect to this modified probability measure.

This follows from Theorems 2.2.1 and 2.2.2 below.

Theorem 2.2.1. *The following process is a local* $\Im_t$*-martingale*:

$$L_i(t) = \left(\frac{2-2\exp[-A_i(S_i)]}{2-\exp[-A_i(S_i)]}\right)^{N_i(t)} (2-\exp[-A_i(t)]).$$

Proof. We consider the $\Im_t$-stopping time defined by

$$U_n = \inf\{t \geqslant 0 : A_i(t) \geqslant n \quad \text{or } B_i(t) \geqslant n\}.$$

Clearly we have $U_n \uparrow \infty$ as $n \uparrow \infty$.

It is sufficient to prove that the process

$$L_i^n(t) = \left(\frac{2-2\exp[-A_i(S_i)]}{2-\exp[-A_i(S_i)]}\right)^{N_i(t\wedge U_n)} (2-\exp[-A_i(t\wedge U_n)])$$

is a bounded $\Im_t$-martingale. Thus the process $L_i(t) = \sum_{n=0}^{\infty} 1_{\{U_n \leqslant t \leqslant U_{n+1}\}} L_i^n(t)$ is a local $\Im_t$-martingale.

For any $\Im_t$-stopping time $U \leqslant U_n$, we can write

$$L^n(U) = 1 - \int_0^U \exp\left[\int_0^s \frac{\exp[-A_i(u)]}{2-\exp[-A_i(u)]}\mathrm{d}A_i(u)\right]\left(\frac{\exp[-A_i(s)]}{2-\exp[-A_i(s)]}\right)\mathrm{d}[N_i(s)-A_i(s)]$$

because if $U < S_i$, we have $N_i(U) = 0$ and

$$\begin{aligned}1 &+ \int_0^U \exp\left[\int_0^s \frac{\exp[-A_i(u)]}{2-\exp[-A_i(u)]}\mathrm{d}A_i(u)\right]\left(\frac{\exp[-A_i(s)]}{2-\exp[-A_i(s)]}\right)\mathrm{d}A_i(s)\\ &= \exp\left[\int_0^U \left(\frac{\exp[-A_i(u)]}{2-\exp[-A_i(u)]}\right)\mathrm{d}A_i(u)\right] = 2-\exp[-A_i(U)] = L^n(U).\end{aligned}$$

If $U \geqslant S_i$

$$\begin{aligned}1 &- \int_0^U \exp\left[\int_0^s \frac{\exp[-A_i(u)]}{2-\exp[-A_i(u)]}\mathrm{d}A_i(u)\right]\left(\frac{\exp[-A_i(s)]}{2-\exp[-A_i(s)]}\right)\mathrm{d}[N_i(s)-A_i(s)]\\ &= \exp\left[\int_0^{S_i} \left(\frac{\exp[-A_i(u)]}{2-\exp[-A_i(u)]}\right)\mathrm{d}A_i(u)\right]\left(\frac{2-2\exp[-A_i(S_i)]}{2-\exp[-A_i(S_i)]}\right) = 2-2\exp[-A_i(S_i)] = L^n(U).\end{aligned}$$

As the integrand $\exp\left[\int_0^s \frac{\exp[-A_i(u)]}{2-\exp[-A_i(u)]}\mathrm{d}A_i(u)\right]\left(\frac{\exp[-A_i(s)]}{2-\exp[-A_i(s)]}\right)$ is an $\Im_t$-predictable process and $N_i(s) - A_i(s)$ is an $\Im_t$-martingale, $L_i^n(t)$ is an $\Im_t$-martingale and $E[L_i^n(U_n)] = 1$. □

Theorem 2.2.2. *Under the probability measure* Q^n *defined by the Radon Nikodym derivative*

$$\frac{\mathrm{d}Q_i^n}{\mathrm{d}P} = L_i^n(U_n),$$

$B_j(t)$ *is the* $\Im_t$*-compensator of* $N_j(t)$.

Proof. From Theorem 2.2.1, $L_i^n(t)$ is a $\Im_t$-martingale with $E[L_i^n(t)] = 1$ and therefore $L_i^n(t)$ can be considered as a density function. We denote Q_i^n the probability measure which is defined by the Radon Nikodym derivative $\frac{dQ_i^n}{dP} = L_i^n(U_n)$.

We consider the stopped sub σ-algebra $F^n = (\Im_{t\wedge U_n})_{t\geqslant 0}$.

For any stopping time $U \leqslant U_n$ and for $j = i$,

$$\begin{aligned} E_{Q_i^n}[B_i(U)] &= E[L_i(U)B_i(U)] = E\left[\int_0^U L_i(s)\,\mathrm{d}B_i(s)\right] \\ &= E\left[\int_0^U (2-\exp[-A_i(s)])\left(\frac{2-2\exp[-A_i(s)]}{2-\exp[-A_i(s)]}\right)\mathrm{d}A_i(s)\right] = E\left[\int_0^U (2-2\exp[-A_i(s)])\mathrm{d}N_i(s)\right] \\ &= E[1_{\{S_i\leqslant U\}}(2-2\exp[-A_i(S_i)])] = E_{Q_i^n}[N_i(U)]. \end{aligned}$$

The second equality above follows by Dellacherie's integration formula. For $j \neq i$

$$\begin{aligned} E_{Q_i^n}[B_j(U)] &= E[L_i(U)A_j(U)] = E\left[\int_0^U L_i(s)\,\mathrm{d}A_j(s)\right] = E\left[\int_0^U L_i(s^-)\,\mathrm{d}A_j(s)\right] = E\left[\int_0^U L_i(s^-)\,\mathrm{d}N_j(s)\right] \\ &= E[1_{\{S_j\leqslant U\}}L_i(S_j)] = E_{Q_i^n}[N_j(U)], \end{aligned}$$

because $A_j(s)$ is an increasing natural process and $L_i(S_j^-) = L_i(S_j)$ where $L(S_j^-)$ is the left limit of $L_i(S_j)$. □

Remark 2.2.3. Bueno (2005a) characterizes the minimal standby redundancy through the compensator transformation given by

$$C_j(t) = \int_0^t \alpha_j(s)dA_j(s) \quad \text{where } \alpha_i(s) = \frac{A_i(s)}{1+A_i(s)} \text{ and } \alpha_j(s) = 1 \text{ for } j \neq i.$$

If we decide to make a redundancy operation in a specific component, we must ask for what type of redundancy, either minimal standby redundancy or active redundancy, we should use in order to stochastically increase the system lifetime. If the components are independent it is easy to see that the minimal standby redundancy lifetime $S_i + S$ is stochastically larger than the lifetime $S_i \vee S$ produced by an active redundancy where S is independent and identically distributed as S_i.

In the dependent case, since that $2 - \exp[x] \leqslant x + 1$ we have that the active redundancy transformation $A_i(t) - \ln(2 - \exp[-A_i(t)])$ is larger than the minimal standby redundancy transformation $A_i(t) - \ln(1 + A_i(t))$.

From Corollary 2.1.3 the $\Im_t$-compensator process of a k-out-of-n:F system is

$$A(t) = \sum_{i=1}^n [A_i(t\wedge\tau) - A_i(T_{k-1})]^+$$

and since that $f(x) = \frac{1+x}{2-\exp[x]}$ is an increasing function of x we have

$$\begin{aligned} C_i(t) - C_i(T_{k-1}) &= A_i(t) - \ln(1 + A_i(t)) - A_i(T_{k-1}) + \ln(1 + A_i(T_{k-1})) \\ &\leqslant A_i(t) - \ln(2 - \exp[-A_i(t)]) - A_i(T_{k-1}) + \ln(2 - \exp[-A_i(T_{k-1})]) = B_i(t) - B_i(T_{k-1}) \end{aligned}$$

and we consider that an active redundancy produces a weaker lifetime than a minimal standby redundancy in the sense that the hazard process for failure of component i under an active redundancy is larger than the hazard process for failure of component i under a minimal standby redundancy.

Therefore, if we decide to make a redundancy operation in a specific component, for example in the weakest component in a k-out-of-n:F system, we should use the minimal standby redundancy in order to stochastically increases the system lifetime.

3. Active redundancy in a k-out-of-n:F system of dependent components

We are concerned with the problem of where to allocate a spare component using active redundancy in a k-out-of-n:F system in order to optimize system reliability improvement. We denote the lifetime of a k-out-of-n:F system by $\tau_{k:\mathrm{F}}(\mathbf{S}) = T_k$ where $\mathbf{S} = (S_1, \ldots, S_n)$ is the random vector of component lifetimes and we denote the system lifetime resulting from an active redundancy operation of component i by $\tau^i_{k:\mathrm{F}} = \tau_{k:\mathrm{F}}(S_1, \ldots, S_{i-1}, S_i \vee S, S_{i+1}, \ldots, S_n)$. We count this system failure through $N^i(t) = 1_{\{\tau^i_{k:\mathrm{F}} \leqslant t\}}$, a counting process with $\mathfrak{I}_t$-compensator $A^i(t)$, $1 \leqslant i \leqslant n$.

We are going to use the following result from Kwieciński and Szekli (1991).

Theorem 3.1 Kwieciński and Szekli (1991). *Consider two point processes N and M with corresponding compensator processes*

$$A_n(t) = A_n(t|t_0, t_1, \ldots, t_{n-1}) \quad \text{on } (T_{n-1}, T_n];$$
$$B_n(t) = B_n(t|s_0, s_1, \ldots, s_{n-1}) \quad \text{on } (S_{n-1}, S_n];$$

which are continuous in t. If $A_n(t) \leqslant B_n(t)$ for all t and $s_0, s_1, \ldots, s_{n-1}$ and $t_0, t_1, \ldots, t_{n-1}$, such that $s_i \leqslant t_i$, $0 \leqslant i \leqslant n-1$, then

$$E[\psi(N(t))] \leqslant E[\psi(M(t))]$$

for all decreasing real and right continuous function with left hand limits ψ, that is, equivalent to $N \leqslant^{\mathrm{st}} M$.

In order to compare system's compensator we recall from the Total Positivity Theory (Karlin, 1968) the definition of reverse rule of order 2 functions. Such definition is used to characterize the family of decreasing monotone likelihood ratio property (Kotz and Johnson, 1985).

Definition 3.2. A bivariate real positive function $K(x, y)$, $-\infty < x < \infty$, $-\infty < y < \infty$ is reverse rule of order 2, (RR_2), if and only if

$$K(x_1, y_2)K(x_2, y_1) \geqslant K(x_1, y_1)K(x_2, y_2)$$

for all $-\infty < x_1 < x_2 < \infty$, $-\infty < y_1 < y_2 < \infty$.

At this point it is important to remember Theorem 2.1.1 on the series representation of a coherent system concluding that we only need to compare the compensator processes after the critical levels of the respective components. Our main result is:

Theorem 3.3. *If the transformation $K(j, t) = 2 - \exp[-A_j(t)]$, $1 \leqslant j \leqslant n$, $t \in [0, \infty)$ is reverse rule of order 2 then $N^1(t) \leqslant^{\mathrm{st}} N^2(t) \leqslant^{\mathrm{st}} \cdots \leqslant^{\mathrm{st}} N^n(t)$.*

Proof. Follows from Theorems 2.2.1 and 2.2.2 that the active redundancy through compensator transform of the component i is given by

$$B_i(t) = \int_0^t \left(\frac{2 - 2\exp[-A_i(s)]}{2 - \exp[-A_i(s)]} \right) \mathrm{d}A_i(s) = A_i(t) - \ln(2 - \exp[-A_i(t)]).$$

From Corollary 2.1.3 we have to compare system's compensators expectation values of the form

$$A^i(t) = \sum_{j=1}^{i-1}[A_j(t) - A_j(T_{k-1})] + A_i(t) - \ln(2 - \exp[-A_i(t)]) - A_i(T_{k-1}) + \ln(2 - \exp[-A_i(T_{k-1})]) + \sum_{j=i+1}^{n}[A_j(t) - A_j(T_{k-1})], \quad 1 \leqslant i, j \leqslant n.$$

It is sufficient to prove for $i = 1$ and $j = 2$.

$$\begin{aligned} A^1(t) &= [A_1(t) - \ln(2 - \exp[-A_1(t)]) - A_1(T_{k-1}) + \ln(2 - \exp[-A_1(T_{k-1})]) + A_2(t) - A_2(T_{k-1}) \\ &\quad + \sum_{i=3}^{n}[A_i(t) - A_i(T_{k-1})] \leqslant A_1(t) - A_1(T_{k-1}) + A_2(t) - \ln(2 - \exp[-A_2(t)]) - A_2(T_{k-1}) \\ &\quad + \ln(2 - \exp[-A_2(T_{k-1})]) + \sum_{i=3}^{n}[A_i(t) - A_i(T_{k-1})] \\ &= A^2(t) - \ln\left(\frac{2 - \exp[-A_1(t)]}{2 - \exp[-A_1(T_{k-1})]}\right) \leqslant -\ln\left(\frac{2 - \exp[-A_2(t)]}{2 - \exp[-A_2(T_{k-1})]}\right) \quad \left(\frac{2 - \exp[-A_1(t)]}{2 - \exp[-A_1(T_{k-1})]}\right) \\ &\geqslant \left(\frac{2 - \exp[-A_2(t)]}{2 - \exp[-A_2(T_{k-1})]}\right). \end{aligned}$$

The results follows from Theorem 3.1. □

Under Theorem 3.3, we understand that it is optimal to perform active redundancy on the weakest component of a k-out-of-n:F system. Since we never claim any relation among components in the system our results are valid for components stochastically dependent.

As $A_i(0) = 0$ for all i, if $2 - \exp[-A_i(t)]$ are RR_2 we have $A_i(t) \geqslant A_j(t)$ for all $i \leqslant j$ and we consider component i weaker than component j in the sense that the hazard process for failure of component i is larger than the hazard process for failure of component j.

In the case of components stochastically independent and under the assumption that $2 - \exp[-A_i(t)]$ are RR_2 we can prove that $A_i(t) \geqslant A_j(t)$ and $S_i \leqslant^{\text{st}} S_j$ for all $i \leqslant j$. We have:

Corollary 3.4. *Let* $\{S_1, \ldots, S_n\}$ *be the stochastically independent components lifetime of a k-out-of-n:F system If* $2 - \exp[-A_i(t)]$ *are* RR_2, *then* $S_1 \leqslant^{\text{st}} S_2 \leqslant^{\text{st}} \cdots \leqslant^{\text{st}} S_n$ *and* $\tau^1_{k:\text{F}} \geqslant^{\text{st}} \tau^2_{k:\text{F}} \geqslant^{\text{st}} \cdots \geqslant^{\text{st}} \tau^n_{k:\text{F}}$ *for* $k = 1, \ldots, n$.

Example 3.5. Let $\{S_1, \ldots, S_n\}$ be the stochastically independent components lifetime of a k-out-of-n:F system. If the ith component lifetime S_i in a k-out-of-n:F system has a Gamma distribution with parameters λ and i, $\lambda > 0$ and $i = 1, \ldots, n$ then $S_1 \leqslant^{\text{st}} S_2 \leqslant^{\text{st}} \cdots \leqslant^{\text{st}} S_n$, and we choose the first ($i = 1$) component. In this case the first component lifetime increases from S_1 to $S_1 \vee S$ where S is the spare lifetime independent and identically distributed as S_1.

4. A constructive example

We propose to find lifetimes S_i^*, $i = 1, \ldots, n$, with $\Im_t$-compensators $A_i^*(t)$ of $1_{\{S_i^* \leqslant t\}}$ such that $2 - \exp[-A_j^*(t)]$ has the RR_2 property. We consider, as in Arjas (1981a), a lifetime S_i (or its distribution) which is increasing failure rate relative to $\Im_t$. However, as S_i is $\Im_t$-measurable, $P(S_i > t | \Im_t) = 1_{\{T_i > t\}}$ and it is not suitable for our proposal.

Then we consider a lifetime which is increasing failure rate relative to $\Im_t^i$, where

$$\Im_t^i = \sigma\{1_{\{S_j > s\}},\ s \leqslant t, j = 1, \ldots, n, j \neq i\},$$

shortly S_i is $(\text{IFR}|\mathfrak{I}_t^i)$, which means that

$$P((S-t)^+ > s|\mathfrak{I}_t^i) \downarrow t.$$

Clearly we also have

$$\overline{F}_i(t) = P(S_i > t|\mathfrak{I}_t^i) \downarrow t.$$

Let $M_i(t)$ be the cadlag version of the counting process

$$M_i(t) = E[1_{\{S_i \leqslant t\}}|\mathfrak{I}_t^i].$$

Follows that $M_i(t)$ is a $\mathfrak{I}_t^i$-submartingale with $\mathfrak{I}_t^i$-compensator $C_i(t)$. From Arjas (1981b), if S_i is $(\text{IFR}|\mathfrak{I}_t^i)$, $C_i(t)$ is a.s. convex on $(0, S_i]$. Now, as $M_i(t) - C_i(t)$ is an $\mathfrak{I}_t^i$-martingale, for $s < t$, we have

$$E[P(S_i \leqslant t|\mathfrak{I}_t^i) - P(S_i \leqslant s|\mathfrak{I}_s^i)|\mathfrak{I}_s^i] = E[C_i(t) - C_i(s)|\mathfrak{I}_s^i]$$

and therefore follows from the monotone convergence theorem that

$$\lim_{t\to s} E[P(S_i \leqslant t|\mathfrak{I}_t^i) - P(S_i \leqslant s|\mathfrak{I}_s^i)|\mathfrak{I}_s^i] = 0,$$

that is,

$$\lim_{t\to s} \int_B [P(S_i \leqslant t|\mathfrak{I}_t^i) - P(S_i \leqslant s|\mathfrak{I}_s^i)|\mathfrak{I}_s^i]\mathrm{d}P = 0$$

for all $B \in \mathfrak{I}_t^i$.

As $P(S_i > t|\mathfrak{I}_t) \downarrow t$, $P(S_i \leqslant t|\mathfrak{I}_t^i)$ is left continuous and therefore $\mathfrak{I}_t^i$-predictable. Follows that $C_i(t) = P(S_i \leqslant t|\mathfrak{I}_t^i)$.

We now turn to the $\mathfrak{I}_t$-compensator of $N_i(t) = 1_{\{S_i \leqslant t\}}$ where

$$\mathfrak{I}_t = \mathfrak{I}_t^i \vee \sigma\{1_{\{S_i > s\}}, |\ s \leqslant t\}.$$

Arjas and Yashin (1988) proves that the $\mathfrak{I}_t$-compensator of $N_i(t)$, $A_i(t)$ is given by

$$A_i(t) = \int_0^t \frac{1_{\{S_i > s\}}\,\mathrm{d}C_i(s)}{\overline{F}_i(s-)} = -\ln(\overline{F}_i(t \wedge S_i)).$$

We are looking for an $\mathfrak{I}_t$-compensator of $N_i(t)$, $A_i^*(t)$, a transformation of $A_i(t)$ such that $2-\exp[-A_j^*(t)])$ has the RR_2 property. As S_i is $(\text{IFR}|\mathfrak{I}_t^i)$, we can conveniently choose $\overline{F}_i(t)$ as a PF_2 function, (a TP_2 function under shift) such that $2-\exp[-A_j^*(t)]$ has the RR_2 property. We propose the compensator transform

$$A_i^*(t) = \int_0^t \frac{1}{2\exp[-A_i(s)]-1}\mathrm{d}A_i(s).$$

Now we let

$$\begin{aligned} L_i(t) &= \left(\frac{1}{2\exp[-A_i(s)]-1}\right)^{N_i(t)} \exp[A_i(t) - A_i^*(t)] \\ &= 1 - \int_0^t \exp\left[-\int_0^s \frac{(2\exp[-A_i(u)]-2)}{(2\exp[-A_i(u)]-1)}\mathrm{d}A_i(u)\right]\left[\frac{(2\exp[-A_i(s)]-2)}{(2\exp[-A_i(s)]-1)}\right]\mathrm{d}[N_i(s) - A_i(s)]. \end{aligned}$$

As $N_i(s) - A_i(s)$ is an $\mathfrak{I}_t$-martingale and the integrand is $\mathfrak{I}_t$-predictable, $L_i(t)$ is a local martingale. We suppose that $L_i(t)$ is uniformly integrable.

However, $E[L_i(t)] = 1$, $L_i(t)$ can be considered as a density function and we can define a measure Q_i by the Radon Nikodyn derivative $\frac{dQ_i}{dP} = L_i(S_i)$. Therefore, applying Girsanov Theorem (Bremaud, 1981) we have that $A_i^*(t)$ is the $\Im_t$-compensator of $N_i(t)$ under the measure Q_i.

Follows that

$$2 - \exp([-A_j^*(t)]) = 2 - \exp\left[\int_0^t \frac{1}{2\exp[-A_i(s)] - 1} \mathrm{d}A_i(s)\right] = \exp[A_i(t)] = \frac{1}{\overline{F}_i(t)},$$

which has the RR_2 property.

We define the component lifetimes S_i^* by

$$Q(S_i^* > t|\Im_t) = \exp[-A_i^*(t)], \quad 1 \leqslant i \leqslant n.$$

To give a practical example we consider the ordered lifetimes with a conditional survival function given by

$$\overline{F}(t_i|t_1, t_2, \ldots t_{i-1}) = \exp\left[-\left(\frac{t_i - \eta_i}{\theta}\right)^\beta + \left(\frac{t_{i-1} - \eta_i}{\theta}\right)^\beta\right]$$

for $\eta_i \vee t_{i-1} < t_i$, where t_i are the ordered observations and density functions

$$\begin{aligned} f(t_1, t_2, \ldots, t_n) &= \prod_{i=1}^n f(t_i|t_1, t_2, \ldots t_{i-1}) \\ &= \left(\frac{\beta}{\theta}\right)\left(\frac{t_1 - \eta_1}{\theta}\right)^{\beta-1} \exp\left[\left(\frac{t_1 - \eta_1}{\theta}\right)^\beta\right] \prod_{i=2}^n \left(\frac{\beta}{\theta}\right)\left(\frac{t_i - \eta_i}{\theta}\right)^{\beta-1} \\ &\quad \times \exp\left[-\left(\frac{t_i - \eta_i}{\theta}\right)^\beta + \left(\frac{t_{i-1} - \eta_i}{\theta}\right)^\beta\right]. \end{aligned}$$

Follows that

$$\mathrm{d}A_i(t|t_1, t_2, \ldots, t_{i-1}) = \frac{f(t|t_1, t_2, \ldots, t_{i-1})}{\overline{F}(|t_1, t_2, \ldots, t_{i-1})} = \left(\frac{\beta}{\theta}\right)\left(\frac{t - \eta_i}{\theta}\right)^{\beta-1}, \quad t_{i-1} \leqslant t < t_i, \ t_0 = 0.$$

In particular we can take $\beta = 2$ in which case S_i is a suitable $(\mathrm{IFR}|\Im_t^i)$ distribution such that $\overline{F}(t_i|t_1, t_2, \ldots, t_{i-1})$ is TP_2 (under shift) on t_i and η_i.

Therefore

$$A_i(t) = \frac{2}{\theta^2}\int_{\eta_i}^t (s - \eta_i)\,\mathrm{d}s = \left(\frac{t - \eta_i}{\theta}\right)^2, \quad t > \eta_i, \ t_{i-1} \leqslant t < t_i, \ t_0 = 0$$

and

$$A_i^*(t) = \int_0^t \frac{1}{2\exp[-A_i(s)] - 1} \mathrm{d}A_i(s) = -\ln\left(2 - \exp\left[\left(\frac{t - \eta_i}{\theta}\right)^2\right]\right)$$

on $\eta_i + 0.83\theta > t \geqslant \eta_i$, $t_{i-1} \leqslant t < t_i$, $t_0 = 0$, $i = 1, \ldots, n$ where it is well defined.

Follows that we can define the component life times S_i^* by

$$Q(S_i^* > t|\Im_t) = 2 - \exp\left[\left(\frac{t - \eta_i}{\theta}\right)^2\right], \quad \eta_i + 0.83\theta > t \geqslant \eta_i, \ i = 1, \ldots, n.$$

Remark 4.1. In Bueno (2004) gives a constructive example for minimal standby redundancy. In this case we must consider $A_i^*(t)$, a transformation of $A_i(t)$, such that $1 + A_i^*(t)$ has the RR_2 property. Bueno (2004) proposes the compensator transform:

$$A_i^*(t) = \int_0^t \exp[A_i(s)]\,\mathrm{d}A_i(s) = \exp[A_i(t)] - 1$$

and we have $1 + A_i^*(t) == \frac{1}{\bar{F}_i(t)}$ which has the RR_2 property.

5. Conclusions

We understand that, the main contribution of this paper is that, in the case of a k-out-of-n:F system with dependent components, we can apply active redundancy as we do in the case of independence. Also, there are cases where, in order to increases system lifetime, its better to apply minimal standby redundancy. It is also important the use of a martingale approach to allocate an active redundant spare in a k-out-of-n:F system of dependent components through compensator transforms.

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