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C. Polcino Milies

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C. Polcino Milies

Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 20.570 Ag. Iguatemi
01000 - São Paulo S.P.
Brasil

ABSTRACT:- H. J. Zassenhaus has conjectured that every torsion unit in an integral group ring of a finite group is rationally conjugate to a trivial one. In this paper, we survey the known results regarding this conjecture; namely, that it holds when G is a nilpotent class 2 group or a split metacyclic group $G = A \rtimes B$ with some restrictions on the order of A and B.

1. INTRODUCTION

Let G be a finite group. We denote by $\mathbb{Z}G$ the group ring of G over the ring $\mathbb{Z}$ of rational integers and by $U(\mathbb{Z}G)$ the group of units of $\mathbb{Z}G$.

Let $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ denote the augmentation function. The set

$$V(\mathbb{Z}G) = \{\alpha \in U(\mathbb{Z}G) \mid \varepsilon(\alpha) = 1\}$$

is called the group of *normalized units* of $\mathbb{Z}G$. It is easily seen that

$$U(\mathbb{Z}G) = \{\pm 1\} \times V(\mathbb{Z}G)$$

For a given group X, we shall denote by TX the set of *torsion elements* of X, i.e. the set of elements of finite order in X.

G. Higman, in the first classical paper on units in group rings [6] showed, among other things, that if G is abelian, then every torsion unit of $\mathbb{Z}G$ is trivial, i.e. of the form $\pm g$, $g \in G$ or, in other words, that

$$TV(\mathbb{Z}G) = G$$

When G is not abelian, an obvious way to exibit new units of finite order is to consider the conjugates of the trivial ones by elements $u \in U(\mathbb{Z}G)$. Of course, these units would have the same order as the elements in G.

D.S. Berman [2] showed that, in general, if $u \in TV(\mathbb{Z}G)$ then the order of u is a divisor of $|G|$. Also, S.K. Sehgal has shown that if $u \in TV(\mathbb{Z}G)$ is a unit whose order is a power of a rational prime, then there exists an element $g \in supp(u)$ such that $o(u) = o(g)$ (see [15, theorem VI.2.1]).

Hence, one might have a hope that *every* element $u \in TV(\mathbb{Z}G)$ is conjugate to an element in G.

This question was first considered by I. Hughes and K. R. Pearson [7], who atribute it to Prof. H. Zassenhaus and showed that there exist torsion elements in $V(\mathbb{Z}S_3)$ which *are not* conjugate, *in* $\mathbb{Z}S_3$, to a trivial unit. Shortly afterwords, C. Polcino Milies [10] showed that the same happens in $\mathbb{Z}D_4$. Nevertheless, it is easy to show, in both cases, that this is not so if we allow conjugation to take place inside QG, the rational group algebra of the given group.

Finally, H. Zassenhaus stated his conjecture precisely in [18]:

(1.1) Let $u \in TV(\mathbb{Z}G)$. Then, there exists an invertible element $\alpha \in \mathbb{Q}G$ and an element $g \in G$ such that $\alpha^{-1}u\alpha = g$.

In this case, we shall say that u is rationally conjugate to g and write $u \sim g$.

The first positive result in regard to this conjecture is quite recent. It is due to A. K. Bhandari and I.S. Luthar [3] who (82) proved that the conjecture holds when G is a metacyclic group of order pq, where p and q are both primes and q divides (p-1).

2. SOME GENERAL RESULTS

Before proceeding to survey the known results in regard to this conjecture, let us recall several well-known facts.

A. Whitcomb [17] showed that if G is a metabelian group, i.e. if it contains a normal subgroup A such that both A and G/A are abelian, then G is determined by its integral group ring. In other words, if H is another group such that $\mathbb{Z}G \cong \mathbb{Z}H$ then $G \cong H$.

Let us denote by I(A) the kernel of the augmentation $\varepsilon_A: \mathbb{Z}A \to \mathbb{Z}$ and by I(G,A) the kernel of the map $\mathbb{Z}G \to \mathbb{Z}G/A$ induced by the natural homomorphism $G \to G/A$.

It is easily seen that:

(2.1) $$I(G,A) = \mathbb{Z}G.I(A) = \{ \sum_{a\in A} \gamma_a(a-1) \mid \gamma_a \in \mathbb{Z}G \}.$$

The essencial part of whitcomb's argument consists in showing that, for each unit $u \in TV(\mathbb{Z}G)$, there exists a unique element $g \in G$ such that

(2.2) $$u \equiv g \pmod{I(G).I(A)}.$$

This is derived from the following remark which we record for latter use:

Given elements $a,b \in A$ we have that:

$$(a-1)(b-1) = ab-a-b+1 = (ab-1)-(a-1)-(b-1)$$

and hence:

(2.3) $$\begin{cases} (a-1)+(b-1) \equiv (ab-1) & \pmod{I(G)I(A)} \\ -(a-1) \equiv a^{-1} - 1 & \pmod{I(G)IA()} \end{cases}$$

Also, we recall the following result.

<u>(2.4) Proposition</u> (G. Cliff, S.K. Sehal and A. Weiss [4]) - The group $u(1+I(G)I(A))$ is torsion-free.

Hence, we have:

<u>(2.5) Corollary</u> Let $u \in TV(\mathbb{Z}G)$ and $g \in G$ be as in formula (2.2). Then

$$o(u) = o(g)$$

<u>Proof</u> Set $o(u) = n$, $o(g) = m$. Then:

$$u^m \equiv g^m \pmod{I(G)I(A)}$$

Since $g^m = 1$, we have that $u^m \in 1 + I(G)I(A)$ and, because of Proposition (2.4), we must have $u=1$. Hence $n|m$.

In a similar way we also get that $m|n$. ∆

We now turn our attention to split metabelian groups i.e. we shall assume that G is a semidirect product of the form $A \rtimes B$ where A and B are abelian.

Hence, we have a split exact sequence of groups:

$$1 \to A \to G \rightleftarrows B \to 1$$

which, is turn, induces a sequence

$$o \to I(G,A) \to \mathbb{Z}G \rightleftarrows \mathbb{Z}B \to 0$$

Since the sequence splits, the restriction to units $V(\mathbb{Z}G) \to V(\mathbb{Z}B)$ is onto and thus we have another split sequence:

$$1 \to V(1+I(G,A)) \to V(\mathbb{Z}G) \rightleftarrows V(\mathbb{Z}B) \to 1$$

Hence, we see that $V(\mathbb{Z}G)$, is again a semidirect product of groups:

(2.6) $$V(\mathbb{Z}G) = V(1+I(G,A)) \rtimes V(\mathbb{Z}B).$$

We shall now show that the conjecture holds at least for some of the units in $\mathbb{Z}G$. The technique involved goes back to Hilbert's theorem 90. First we need the following lemma.

(2.7) Lemma Let G be a finite group and let A be a normal p-subgroup of G, where p is a rational prime. Then any element of the form $\alpha = t+\delta$, where $t \in \mathbb{Z}$ is not divisible by p and $\delta \in I(G,A)$, is invertible in $\mathbb{Q}G$.

Proof Since $\mathbb{Q}G$ is semisimple, artinian, if α was not invertible it should be a zero divisor. Hence, there would exist an element $\beta \in \mathbb{Z}G$ such that:

$$\beta\alpha = \beta t+\beta\delta = 0$$

thus,

$$\beta t = -\beta\delta.$$

Since $(t,p) = 1$, we can find integers r,s such that $rt+ps = 1$ so, if we work modulo p (i.e. if we go to $\mathbb{Z}_pG$), we obtain:

$$\beta \equiv -r\beta\delta \pmod p.$$

It follows immediately that:

$$\beta \equiv \pm r^n\beta\delta^n \pmod p, \text{ for all } n \in \mathbb{N}.$$

Since $I(G,A)$ is nilpotent in $\mathbb{Z}_pG$, we see that $\beta \equiv 0 \pmod p$ hence

$$\beta = p\beta_1 \quad \text{with} \quad \beta_1 \in \mathbb{Z}G.$$

Then, again we must have that $\beta_1\alpha = 0$ and the same argument above shows that β_1 is of the form $\beta_1 = p\beta_2$, with $\beta_2 \in \mathbb{Z}G$. Inductively, we would prove that all powers of p divide the coefficients of β a contradiction. Δ

(2.8) Theorem (C. Polcino Milies and S.K. Sehgal [11]) Let $G = A \rtimes B$ be such that $(o(u), |A|) = 1$. Then u is rationally conjugate to an element $b \in B$.

Proof We shall proceed by induction on the number of primes dividing $|A|$. So, let us first assume that A is a p-group, for some rational prime p.

Let $u \in V(\mathbb{Z}G)$ be as in the statement of the theorem and write $u=vw$, with $v \in V(1+I(G,A))$ and $w \in V(\mathbb{Z}B)$ and set $t = o(u)$. writing $v = wvw^{-1}$

we have that:

$$u^t = v.v^w \dots .v^{w^{t-1}} .w^t = 1$$

Hence:

$$w^t = 1$$

$$v.v^w. \dots .v^{w^{t-1}} = 1.$$

Now, set:

$$Z = 1+v+v.v^w+\dots+v.v^w. \dots .v^{w^{t-2}}$$

Then

$$wzw^{-1} = 1+v^w+\dots+v^w.v^{w^2} \dots .v^{w^{t-1}}$$

and hence

$$vwzw^{-1} = v+v^w+\dots+v.v^w.\dots.v^{w^{t-1}} = Z .$$

Now, since $v \equiv 1 \pmod{I(G,A)}$, it follows that $z \;\; t \pmod{I(G,A)}$ thus, it exists $\delta \in I(G,A)$ such that $z = t+\delta$ and, since $(t,p) = 1$, the previous lemma shows that z is invertible

Consequently:

$z^{-1}uz = z^{-1}(vw)z = w$, with $w \in TV(\mathbb{Z}B)$.

Since B is abelian, the result by G. Higman mentioned in the introduction shows that $w = b$, for some $b \in B$.

For the induction step, write $A = A_1 \times A_2$, where A_1 is a p-group and A_2 a p'-group for some rational prime p. Then writting $G_2 = \langle A_2,B \rangle$ we have that $G_1 = A_1 \rtimes G_2$.

If $u \in TV(\mathbb{Z}G)$ is such that $(o(u),|A|) = 1$ then also $(o(u),|A_1|)=1$ and we can use the argument above to show that there exists an invertible element $z \in \mathbb{Q}G$ such that $z^{-1}uz = w \in TV(\mathbb{Z}G_2)$. From the induction hypothesis, since $G_2 = A_2 \rtimes B$, w itself is conjugate to an element $g \in G_2$ and the proof is completed. ▲

We still have another information about units in $V(\mathbb{Z}G)$ which is worth mentioning.

<u>(2.9) Proposition</u> If $u \in TV(1+I(G,A))$ then the order of u is a divisor of $|A|$.

<u>Proof</u> Assume $u=1+\delta$ with $\delta \in I(G,A)$. Then u is of the form:

$$u = 1 + \sum_a \gamma_a(a-1) \quad , \quad \gamma_a \in \mathbb{Z}G.$$

Now, each γ_a is of the form:

$$\gamma_a = \sum_{i,j} z_{ij}\, b_i\, a_j = \sum_{i,j} z_{ij}((b_i-1)a_j + a_j).$$

Hence:

$$\gamma_a(a-1) = \sum_{i,j} z_{ij}(b_i-1)(a-1)a_j + \sum_{i,j} z_{ij}(a_j a-1)$$

and thus

$$\gamma_a(a-1) \equiv \sum_{i,j} z_{ij}(a_j a-1) \quad (\text{mod } I(G)I(A))$$

and formulas (2.3) show that

$$\sum_a \gamma_a(a-1) \equiv a_0 - 1 \, (\text{mod}\, I(G)I(A)) \text{ for some } a_0 \in A$$

and, hence:

$$u \equiv a_0 \quad (\text{mod} \quad I(G).I(A)).$$

Now, corollary (2.5) shows that $o(u) = o(a_0)$, which is a divisor of $|A|$ Δ

No other general result is known so far, so we shall have to consider some special cases. However, we shall first show how representations can be used to help to solve this problem.

3. CONNECTION WITH REPRESENTATIONS

First, we shall prove a result that evables us to consider the problem in a bigger field.

(3.1) Lemma Let $k \subset K$ be fields of characteristic 0 and G any finite group. If two given elements $\alpha, \beta \in kG$ are conjugate in KG then they are also conjugate in kG.

Proof Let α, β be given elements in kG and let us first consider the equation

$$\alpha x = x\beta$$

with $x = \sum_i x_i g_i$ where x_i, $1 \leq i \leq n$ are unknowns.

If we write down explicitely both sides of the equation, we shall obtain a linear sistem of the form:

$$MX = 0$$

with $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $M \in k_{n\times n}$, where $n = |G|$.

Since the assumptions of the theorem imply that there exists a non trivial solution in $K_{n\times 1}$ then we must have $\det M = 0$ and hence there exists a non trivial solution also in $k_{n\times 1}$. Actually, we can find t independent vectors $v_1, \ldots, v_t \in k_{t\times 1}$ such that every solution in $k_{n\times 1}$ of the system is of the form:

$$v = k_1 v_1 + \dots + \lambda_t v_t \quad , \quad \lambda_i \in k \ , \quad 1 \le i \le n.$$

We wish to show first that at least for one of these solutions $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ the corresponding element $u = \sum a_i g_i$ is not a zero divisor in kG.

Hence assume that all of them are. Let T be the regular representation of KG. We shall denote by $T(v_i)$, for briefness, the representation of the element corresponding to v_i in kG.

Then:

$$T(v) = \sum_{i=1}^{t} \lambda_i T(v_i)$$

Since we are assuming that T(v) is a zero divisor, for every choice of $\lambda_1, \dots, \lambda_t \in k$ we have:

$$\det\Big(\sum_{i=1}^{t} \lambda_i T(v_i)\Big) = 0$$

Since k is infinite, this means that the polynomial $\det\Big(\sum_{i=1}^{t} X_i T(v_i)\Big)$ in the indeterminates $X_1, \dots, X_t$ is 0. However, we know that there exists a solution of the system, in $K_{n\times 1}$, say $(r_1, \dots, r_t)$ such that the corresponding element is invertible, and thus

$$\det\Big(\sum_{i=1}^{t} r_i T(v_i)\Big) \neq 0$$

a contradiction.

Hence, we must also have a solution $\lambda_1, \dots, \lambda_t \in k$ such that the corresponding element u is not a zero divisor and verifies:

$$\alpha u = u\beta \ .$$

Since kG is artinian, semisimple, an element $u \in kG$ which is not a zero divisor is invertible. Thus $u^{-1}\alpha u = \beta$ and α, β are conjugate in kG, as desired. ▲

As an immediate consequence, it follows:

(3.2) Corollary Let $u \in TV(\mathbb{Z}G)$. To show that u is rationally conjugate to an element $g \in G$ is suffices to show that there exists an element $\alpha \in U(\mathbb{C}G)$ such that $\alpha^{-1} u \alpha = g$.

Hence, we have:

(3.3) Lemma Let $u \in TV(\mathbb{Z}G)$. Then u is rationally conjugate to an element $g \in G$ if and only if for every irreducible complex representation T of G the matrices T(u) and T(g) are conjugate.

Proof From Wedderburn's theorem we know that $\mathbb{C}G$ is of the form:

$$\mathbb{C}G \underset{\phi}{\cong} \bigoplus_{i=1}^{t} \mathbb{C}_{n_i \times n_i}$$

Let T_i be the irreducible representation corresponding to the i-th simple component, and assume that there exists $A_i \in \mathbb{C}_{n_i \times n_i}$ such that $A_i^{-1} T_i(u) A_i = T_i(g)$.

We can assume that the isomorphism ϕ is given by $\phi = \bigoplus_{i=1}^{t} T_i$. Then, if $\alpha \in \mathbb{C}G$ is the element corresponding to $(A_1, \dots, A_t)$ is the isomorphism, we have that $\alpha^{-1} u \alpha = g$.

(3.4) Lemma Let $u \in V(\mathbb{Z}G)$ and $g \in G$ be elements such that $o(u) = o(g)$. If $\chi(u^k) = \chi(g^k)$ for all positive integers k and all complex irreducible characters χ then u is conjugate to g in $\mathbb{C}G$.

Proof If T is the irreducible representation corresponding to the character χ, then:

$$g \to T(g) \quad \text{and} \quad g \to u \to T(u)$$

are two representations of the cyclic group <g> which afford the same character; hence, they are similar and thus $T(g) \sim T(u)$.

Since this happens for all irreducible complex representations, the result follows from lemma (3.3). Δ

4. NILPOTENT CLASS 2 GROUPS

J. Ritter and S.K. Sehgal [13] have shown that the Zassenhaus Conjecture holds for nilpotent class 2 groups. More precisely, their result is as follows:

(4.1) Theorem Let G be a nilpotent class 2 group and set $u \in TV(\mathbb{Z}G)$. Let g be the unique element of G such that $u \equiv g \pmod{I(G)IG')}$. Then u is rationally conjugate to g.

Proof - Let χ be one such irreducible character and let $N \lhd G$ be the kernel of χ. For an element $u \in \mathbb{Z}G$ we shall denote by $\bar{u}$ its image in $\mathbb{Z}G/N$.

Since $\overline{I(G)I(G')} = I(\bar{G})I(\bar{G}')$ we have that

$$\bar{u} \equiv \bar{g} \pmod{I(\bar{G})I(\bar{G}'))}.$$

Write

$$\bar{u} = \sum_{\bar{x} \in \bar{G}} a_{\bar{x}} . \bar{x}$$

If $a_{\bar{x}} \neq 0$ for same $\bar{x}$ in the center of $\bar{G}$, then a well-know theorem of Berman [2] shows that actually $\bar{u} = \bar{x}$. Because of the uniqueness of the correspondence modulo $I(\bar{G})I(\bar{G}')$ we must have:

$$\bar{u} = \bar{x} = \bar{g}$$

hence, obviously $\chi(u) = \chi(g)$ in this case.

Then, we can assume that all elements in the expression of $\bar{u}$ are noncentral.

Notice that also $\bar{g}$ must be non-central, since otherwise we would have

$$\bar{g}^{-1}.\bar{u} \equiv 1 \quad \mathrm{mod}(I(\bar{G})I(\bar{G}')$$

and $\bar{g}^{-1}\bar{u}$ would be of finite order, contradicting (2.4).

Now, T.R. Berger [1] has shown that irreducible complex characters of nilpotent class 2 groups vanish outside the center. Hence we have:

$$\chi(\bar{g}) = 0$$

$$\chi(\bar{u}) = \sum_{\bar{x}\in\bar{G}} a_{\bar{x}}.\chi(\bar{x}) = 0$$

Thus, $\chi(g) = \chi(u)$ also in this case.

Finally, since $u \equiv g$ (mod $I(G)I(G')$) also implies that $u^k \equiv g^k$ (mod $I(G)I(G')$), our proof is complete because of Lemma (3.4). ▲

The result above might suggest that, in the metabelian case,if $u \equiv g$(mod $I(G).I(A)$) and the Zassenhaus conjecture holds, then u is rationally conjugate precisely to g. However, this is not the case.

Actually G. Gliff, S.K. Sehgal and A. Weiss gave in [4] a whole family of ideals I_k such that for each unit $u \in TV(ZG)$ there exists an element $g_k \in G$ such that $u \equiv g_k$ (mod I_k).

In the very same paper [13] , J. Ritter and S.K. Sehgal gave the following example:

Set $D_{10} = \langle a,b \mid a^5 = b^2 = 1 , a^b = a^{-1}\rangle = \langle a\rangle \times \langle b\rangle$.

Then, all the ideals I_k coincide with either $I(G)I(A)$ or $I(A)I(G)$, where $A = \langle a\rangle$.

The element $u = -a^2+a^3+a^4+(-1+a^3)b$ is a unit of order 5 such that:

$$u \equiv a^3 \mod I(A)I(G)$$

$$u = a^2 \mod I(G)I(A).$$

On the other hand, they show that u is rationally conjugate to both a and a^4.

5. METACYCLIC GROUPS

As we mentioned in the introduction, the first positive result on the Zassenhaus conjecture was obtained by A.K. Bhandari and I. S. Luther [3] who proved it for the metacyclic group:

$$G = \langle a,b \mid a^p = b^q = 1,\ b a b^{-1} = a^j \rangle = \langle a \rangle \rtimes \langle b \rangle$$

where p,q are primer, p odd, $q \mid (p-1)$ and $j^f \equiv 1 \pmod p$.

This result has been extended to the following.

<u>Theorem</u> (C. Polcino Milies, J. Ritter, S.K. Sehgal [12]) Let G be the split metacyclic group $G = \langle a \rangle \rtimes \langle b \rangle$ with $(o(a),o(b)) = 1$. Then every unit $u \in TV(\mathbb{Z}G)$ is rationally conjugate to an element $g \in G$.

Actually, this result was achieved in several steps. With the notations of the theorem, we have:

The Zassenhaus conjecture holds if $\langle a \rangle$ is a p-group, $\langle b \rangle$ a p'-group and the action of b on $\langle a \rangle$ is faithful [11].

In [13], the restriction about the faithfullness of the action was removed and it was also shown that the conjecture holds when $o(a) = n$ an odd integer and $o(b) = q$, a prime not dividing n.

The final result was obtained in [12].

To give an idea of the methods involved in the proof, let us consider the first case, where $o(a) = p^m$, $o(b) = s$, $o(p,s) = 1$ and $b a b^{-1} = a^j$ and $o(j) = s$ in $\mathbb{Z}/p^m\mathbb{Z}$.

Given an element $g \in G$ we can write it on the form $g = \alpha\beta$ where $\alpha \in \langle a \rangle$ and $\beta \in \langle b \rangle$.

If $\beta \neq 1$, it is of the form $\beta = b^h$. As before, we see that:
$g^s = \alpha.\alpha^{j^h}. \dots .\alpha^{j^{h(s-1)}}.\beta^s = \alpha^{(1+j^h+\dots+j^{h(s-1)})}$.

And we have:

$$(1-j^h)(1+j^h+\dots+j^{h(s-1)}) = 1-j^{hs} \equiv 0 \pmod{p^m}.$$

If $p \mid (1-j^h)$ we can write $j^h = 1+kp$, hence

$$j^{hp} = 1+k_1p^2$$

and, inductivelly

$$j^{hp^{m-1}} = 1$$

So we would have $s|hp^{m-1}$ and since $(s,p)=1$, we have that $s|h$, a contradiction. Then $p\nmid(1-j^h)$ so

$$1+j^h + \ldots + j^{h(s-1)} \equiv 0 \pmod{p^m}$$

This argument shows that, for any element $g \in G$ we have that either $o(g)|s$ or $o(g)|p^m$.

Because of Corollary (2.5), the same is true for a unit $u \in TV(\mathbb{Z}G)$ and Theorem (2.8) shows that we only need to consider units u such that $o(u)|p^m$.

Notice also that, if $u=v.w$ with $v \in V(1+I(G,\langle a\rangle))$ and $w \in V(\mathbb{Z}\langle b\rangle)$, with $w \neq 1$, the initial part of the argument in theorem (2.8) shows that $(O(u),s) \neq 1$.

Hence, we only have to consider units $u \in V(1+I(G,\langle a\rangle)$ and Proposition (2.9) shows that these are certainly p-elements.

To treat this case we shall use representations. It is shown in Curtis-Reiner, [5, p.336] that the irreducible representations of G are among the ones given by:

$$T_i(a) = \begin{pmatrix} \xi^i & & & \\ & \xi^{ij} & & \\ & & \ddots & \\ & & & \xi^{ij^{s-1}} \end{pmatrix}, \quad T_i(b) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$

where ξ is a p^mth root of unity.

Write u in the form $u = \sum_{k=0}^{s-1} \alpha_k(a)b^k$ where $\alpha_k(a) \in \mathbb{Z}\langle a\rangle$ $1 \leq k \leq s-1$.

If $\phi: \mathbb{Q}(\xi) \to \mathbb{Q}(\xi)$ denotes the automorphism such that $\phi(\xi) = \xi^j$, it is easy to see that:

$$T_i(u) = \begin{bmatrix} \alpha_0(\xi^i) & \alpha_1(\xi^i) \ldots \alpha_{s-1}(\xi^i) \\ \alpha^{\phi}_{s-1}(\xi^i) & \alpha^{\phi}_0(\xi^i) \ldots \alpha^{\phi}_{s-2}(\xi^i) \\ \alpha_1^{\phi^{s-1}}(\xi^i) & \alpha_2^{\phi^{s-1}}(\xi^i) \ldots \alpha_0^{\phi^{s-1}}(\xi^i) \end{bmatrix}$$

Now, we notice that

$$T_i(b)\ T_i(u)\ T_i(b)^{-1} = T_i^{\phi}(u)$$

Since $T_i(u)^{p^m} = 1$, the matrix is diagonalizable and, because of the observation above, we see that if ξ^{r_i} is an eigenvalue of $T_i(u)$, then $\xi^{r_i j}$ is again an eingenvalue. Thus:

$$T_i(u) \sim \begin{bmatrix} \xi^{r_i} & & & \\ & \xi^{r_i j} & & \\ & & \ddots & \\ & & & \xi^{r_i j^{s-1}} \end{bmatrix} = T_i(a^{r_i})$$

The rest of the proof, which is still rather long, consists in showing that actually the element a^{r_i} does not depend an the representation T_i considered.

In the general case additional difficulties arrise, since the reduction to the case where $u \in TV(1+I(G,\langle a\rangle))$ is more delicate and the representations of G, up to rational equivalence, are given by:

$$T_{d,\mu}^{(a)} = \begin{bmatrix} \xi^{d} & & & \\ & \xi^{dj} & & \\ & & \ddots & \\ & & & \xi^{dj^{t_j-1}} \end{bmatrix}_{t_d \times t_d}, \qquad T_{d,\mu}^{(b)} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \eta^{\mu} & & & 0 \end{bmatrix}_{t_d \times t_d}$$

where d runs over the divisors of n, ξ is a fixed n-th root of unity; t_d is the order of j in $\mathbb{Z}/\frac{n}{d}\mathbb{Z}$, η is a primitive t/t_d-th root of unity and $\mu = 0,1,2,\dots,t/t_d-1$.

6. RECENT RESULTS

Recently, some results regarding the Zassenhaus Conjecture have been obtained for some special classes of split metabelian groups, which we wish to mention.

<u>Theorem (S.K. Sehgal and A. Weiss [16])</u> Let $G = A \rtimes B$ be a split metabelian group, where A is an elementary abelian p-group and B is any abelian group. If B acts faithfully irreducibly on A then the Zassenhaus Conjecture holds for $\mathbb{Z}G$.

<u>Theorem (Z. Marciniak, J. Ritter, S.K. Sehgal and A. Weiss [8])</u>

Let $G = A \rtimes B$ be a split metabelian group. Then the Zassenhaus conjecture holds for $\mathbb{Z}G$ in the following two cases:

(i) When A is abelian, B is of prime order q and $q < p$ for every prime p dividing $|A|$.

(ii) When $A = \langle a \rangle$, B is abelian of order m and $m < p$ for every prime p dividing $|A|$.

7. FINAL REMARK

There is anothe conjecture due to H.J. Zassenhaus which is stronger then the one we have considered, namely:

Every finite subgroup of units in $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of G.

If we restrict ourselves to consider only maximal subgroups of $V(\mathbb{Z}G)$ i.e. subgroups $H \subset V(\mathbb{Z}G)$ such that $|H| = |G|$ then some results are known.

For example, in 1969, S.K. Sehgal [14] showed that this is true for nilpotent class 2 groups and in 1976 G. Peterson [9] showed that it is also true for the symmetric groups S_n. In his conference in this same meeting, prof. K.W. Roggenkamp has announced that the result also holds for arbitrary nilpotent groups.

No results are known in regard to the conjecture in its full generality as stated above.

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