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Separating the Edges of a Graph by Cycles and by Subdivisions of K_4

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ABSTRACT

A *separating system* of a graph G is a family $\mathcal{S}$ of subgraphs of G for which the following holds: for all distinct edges e and f of G, there exists an element in $\mathcal{S}$ that contains e but not f. Recently, it has been shown that every graph of order n admits a separating system consisting of $19n$ paths, improving the previous almost linear bound of $O(n \log^\star n)$, and settling conjectures posed by Balogh, Csaba, Martin, and Pluhár and by Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan. We investigate a natural generalization of these results to subdivisions of cliques, showing that every graph admits both a separating system consisting of $41n$ edges and cycles and a separating system consisting of $82n$ edges and subdivisions of K_4.

1 | Introduction

Consider a family $\mathcal{S}$ of subsets of a set X. Given $e, f \in X$, we say that an element $S \in \mathcal{S}$ *separates* e from f if $e \in S$ and $f \notin S$. Furthermore, we say that $\mathcal{S}$ *weakly separates* X if for every pair of elements $e, f \in X$ there is an element $S \in \mathcal{S}$ that either separates e from f or that separates f from e, and we say that $\mathcal{S}$ *strongly separates* X if for every pair of elements $e, f \in X$ there is an element $S \in \mathcal{S}$ that separates e from f, and an element that separates f from e. Such a family $\mathcal{S}$ is then called a (weak or strong, respectively) *separating system* of X. The study of separating systems dates back to the 1960s [1–3] and gained substantial attention in the last years (see, e.g., [4–6]). Our focus is a variation posed in 2013 by Katona (see [7, 8]) in which one seeks to separate the *edge* set of a given graph by a small collection of paths. In particular, special attention has been given to the case of complete graphs [9–11].

For our purposes, a *separating path system* (resp., *separating cycle system*) of a graph G is a collection $\mathcal{S}$ of paths (resp., edges and cycles) in G such that for all distinct edges $(e, f) \in E(G) \times E(G)$ there exists $S \in \mathcal{S}$ that separates e from f. Note that this definition fits into the category of *strong* separating systems. Also note that, in the definition of cycle separating system, one is allowed to use isolated edges. This is required to separate graphs that contain bridges (i.e., edges that do not lie in any cycle). In particular, each bridge must appear as a single edge in such a system, and a cycle separating system of a tree is precisely the collection of its edges. This edge requirement also occurs in similar problems, such as cycle decomposition [12, 13] and covering by subdivisions of cliques [14] (see Theorem 6).

Inspired by Katona's question, Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [8] conjectured that every graph on n vertices admits a weak separating system size $O(n)$ consisting solely of paths while verifying it for a number of cases. Their conjecture was later strengthened by Balogh, Csaba, Martin, and Pluhár [7].

Conjecture 1 (Balogh et al. [7] and Falgas-Ravry et al. [8]). *Every graph of order n admits a path separating system of size $O(n)$.*

In 2023, Bonamy, Dross, Skokan, and the authors confirmed Conjecture 1, proving that every graph on n vertices admits a separating path system of size at most $19n$ [15]. This improved the almost linear bound of $O(n \log^{\star} n)$ found by Letzter in 2022 [16]. It is then natural to ask whether every graph also admits a separating cycle system of size $O(n)$. This question was independently posed by Girão and Pavez-Signé,[1] and in this paper, we answer it affirmatively (see Section 3).

Theorem 2. *Every graph on n vertices admits a separating cycle system of size $41n$.*

In fact, we are interested in a more general setting in which elements of the separating system are either edges or subdivisions of a given graph, which we now make precise. We say that H^* is a *subdivision* of H if H can be obtained from H^* by repeatedly deleting a vertex of degree 2 and adding a new edge joining its neighbors. Let H and G be graphs, and let $\mathcal{F}$ be a family of subgraphs of G such that every element of $\mathcal{F}$ is either an edge or a subdivision of H. We say that $\mathcal{F}$ is an *H-cover* of G if $E(G) = \bigcup_{H' \in \mathcal{F}} E(H')$, and we say that $\mathcal{F}$ is an *H-separating system* of G if for all distinct edges $(e, f) \in E(G) \times E(G)$ there is an element $H' \in \mathcal{F}$ that separates e from f. It is not hard to see that every H-separating system of G is an H-cover of G and that a separating path (resp., cycle) system is a K_2-separating (resp., K_3-separating) system. Therefore, the results mentioned above say that every graph on n vertices admits a K_2- and a K_3-separating system of size $O(n)$. Motivated by Conjecture 1, we propose the following more general edge separation conjecture.

Conjecture 3. *For every graph H, there is a constant $C = C(H)$ for which every graph on n vertices admits an H-separating system of size at most $C \cdot n$.*

It is not hard to check that to solve Conjecture 3, it suffices to verify it in the case H is a complete graph. The main result of this paper is that Conjecture 3 holds in the case $H = K_4$ (see Section 2).

Theorem 4. *Every graph on n vertices admits a K_4-separating system of size $82n$.*

Since every subdivision of K_4 can be covered by two cycles, Theorem 4 implies a version of Theorem 2 in which the constant 41 is replaced by 164.

The strategy presented here is similar to the strategy in [15]. Namely, we reduce the main problem to the case of graphs containing a certain spanning subdivision of a clique. Next, we define a linear number of special matchings that separate the edges outside this structure and cover each such matching by a suitable subdivision. We emphasize that we make no effort to reduce the multiplicative constant.

Path separation versus K_4 separation: In [15], the authors reduce the problem to the case of graphs containing a Hamilton path $P = v_1 \cdots v_n$. They consider $5n$ matchings $M_k = \{v_i v_j \in E(G): i + j = k, i < j\}$ and $N_k = \{v_i v_j \in E(G) : i + 2j = k, i < j\}$ and show that each such matching M can be covered by a path P_M with $M \subseteq E(P_M) \subseteq E(P \cup M)$. The argument is completed using a linear path-covering result.

In contrast, here the reduction is to graphs containing a subdivision K of K_4 with the property that K contains a Hamilton cycle $C = v_1 \cdots v_n v_1$. We consider $3n$ matchings $M_k = \{v_i v_j \in E(G) : j - i = k, i < j\}$ and $N_k = \{v_i v_j \in E(G): j - 2i = k, i < j\}$. While some of these matchings cannot be covered by a single subdivision (see Figure 1), we can show that a bounded number of subdivisions suffice. The argument is completed using a linear clique-subdivision–covering result.

Pósa rotation–extension: We use the following standard notation. Given a graph G and a set $S \subseteq V(G)$, we denote by $N_G(S)$ the set of vertices *not* in S adjacent in G to some vertex in S. We omit subscripts when clear from the context.

Given a graph G and vertices u, v in G, let $P = u \cdots v$ be a path from u to v. If $x \in V(P)$ is a neighbor of u in G and x^- is the vertex preceding x in P, then $P' = P - xx^- + ux$ is a path in G for which $V(P') = V(P)$. We say that P' *has been obtained from* P by an *elementary exchange* fixing v (see Figure 2). A path obtained from P by a (possibly empty) sequence of elementary exchanges fixing v is said to be a path *derived* from P. The set of end vertices of paths derived from P distinct from v is denoted by $S_v(P)$. Since all paths derived from P have the same vertex set as P, we have $S_v(P) \subseteq V(P)$. The following lemma arises when P is a longest path ending at v (for a proof, see also [15]).

Lemma 5 (Brandt et al. [17]). *Let $P = u \cdots v$ be a longest path of a graph G and let $S = S_v(P)$. Then $|N_G(S)| \leq 2|S|$.*

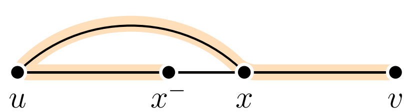

FIGURE 2 | A path (highlighted) obtained by an elementary exchange fixing v.

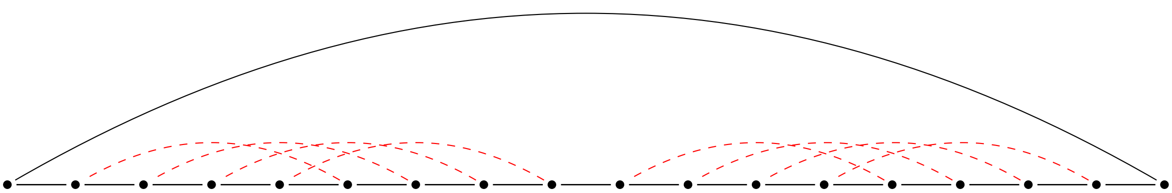

FIGURE 1 | A graph consisting of a cycle together with a matching $M \subseteq \{u_i u_j : j - i = 4\}$ (in dashed red) having no cycle that contains all edges of M.

2 | K_4-Separating Systems

In this section, we verify Conjecture 3 when $H = K_4$. Our argument requires a result stating that every n-vertex graph admits a K_4-cover of size $O(n)$. Although we can prove such a statement simultaneously to the separation result, we obtain a better leading constant by using the following result due to Jørgensen and Pyber [14]. We stress that Jørgensen and Pyber obtained a more general result, proving that every n-vertex graph admits a K_t-cover of size $O_t(n)$, but for our purposes, the following sharper estimate for K_4-covers suffices.

Theorem 6 (Jørgensen and Pyber [14]). *Every graph on n vertices admits a K_4-cover of size at most $2n - 3$.*

The proof of Theorem 4 is divided into four parts. First we apply Lemma 5 to reduce the problem to the case where the studied graph contains a special spanning subdivision of K_4; then, we define the M_k's and N_k's and show how to partition them into matchings to avoid the problem illustrated in Figure 1; the third step is to consider the union of each matching and the spanning subdivision of K_4 and show that it contains another subdivision of K_4 covering the matching, and we finally argue that the obtained collection is the desired K_4-separating system.

Let H_1 and H_2 be (not necessarily edge-disjoint) subgraphs of a graph G, and consider a family $\mathcal{S}$ of subgraphs of G. We say that *$\mathcal{S}$ separates H_1 from H_2* if for all distinct edges $(e, f) \in E(H_1) \times E(H_2)$, there is $S \in \mathcal{S}$ that separates e from f. Therefore, $\mathcal{S}$ is a separating system of G if $\mathcal{S}$ separates G from itself.

Proof of Theorem 4. We proceed by induction on n. Let G be a graph with n vertices. If G is empty, the result trivially holds. If not, we consider a longest path $P = u \cdots v$ of G and put $S = S_v(P)$ as in Lemma 5. Now, let w be the vertex that is closest to v in P and has a neighbor u' in S, and let $P' = u' \cdots v$ be a path obtained from P by elementary exchanges fixing v so that P' starts with u'. Then $C = (P' + u'w) \setminus E(P[w, v])$ is an edge or a cycle that contains $S \cup N(S)$ (C is an edge when $S = \{u\}$ and $|N(S)| = 1$).

Let H be the subgraph of G induced by the edges incident to vertices of S. Note that $V(H) = S \cup N(S)$ and hence $V(H) \subseteq V(C)$. Moreover, by Lemma 5, we have $|N(S)| \leq 2|S|$, and hence $h = |V(H)| \leq 3|S|$. Let $G' = G \setminus S$. Note that, since G is not empty, S is not empty. By the induction hypothesis, there is a K_4-separating system $\mathcal{S}'$ of G' of size at most $82(n - |S|)$. Note that $\mathcal{S}'$ separates from H since $\mathcal{S}'$ covers G'. In what follows, we construct a set $\mathcal{S}$ of edges and subdivisions of K_4 that separates H from G. Moreover, we obtain $\mathcal{S}$ so that $|\mathcal{S}| \leq 82|S|$, and hence $\mathcal{S}' \cup \mathcal{S}$ is a K_4-separating system of G with cardinality at most $82(n - |S|) + 82|S| = 82n$ as desired.

Now, we show that either $e(H) \leq 82|S|$ or H contains a subdivision of K_4 that contains C. For that, let us write $C = u_1 \cdots u_m u_1$. We remark that C may contain vertices in $V(G) \setminus V(H)$. Let $v_1, \ldots, v_h$ be the vertices of H in the order that they appear in C. Formally, for $i \in [h] = \{1, \ldots, h\}$, we define $\sigma(i)$ to be the index of the ith vertex $u_{\sigma(i)}$ of H in C, and set $v_i = u_{\sigma(i)}$. In what follows, each edge $v_i v_j$ is written so that $i < j$. We say that two edges $v_i v_j$ and $v_{i'} v_{j'}$ of $H \setminus E(C)$ *cross* if $v_{i'}$ and $v_{j'}$ lie in distinct components of $C \setminus \{v_i, v_j\}$. If no two edges in $H \setminus E(C)$ cross, then H is an outerplanar graph, and hence $e(H) \leq 2h - 3$ (note that $h \geq 2$). In this case, we set $\mathcal{S}$ as the set of subgraphs each consisting of a single edge in H, and we are done because $|\mathcal{S}| = |e(H)| \leq 2h - 3 \leq 6|S| \leq 82|S|$.

Therefore, we may assume that there are crossing edges e, e' in $H \setminus E(C)$. Note that in this case $K = C + e + e'$ is a subdivision of K_4. Let $K_S = E(K) \cap E(H)$ be the set of edges of K having at least one vertex in S (see Figure 3), and let $\mathcal{K}_S$ be the set of subgraphs each consisting of a single edge in K_S. Observe that $|\mathcal{K}_S| = e(K_S) \leq 2|S| + 2$. By Theorem 6, there is a K_4-cover D' of $H' = H \setminus E(K_S)$ of size at most $2h - 3 \leq 6|S|$. Note that $\mathcal{K}_S \cup D'$ has size at most $2|S| + 2 + 6|S| \leq 10|S|$ (using that $2 \leq 2|S|$) and separates (i) H from G', (ii) H' from K, and (iii) K_S from G.

It remains to create a set of at most $24h \leq 72|S|$ edges and subdivisions of K_4 that separates H' from itself (see Figure 3). To obtain the final elements of the separating system, we define special matchings.

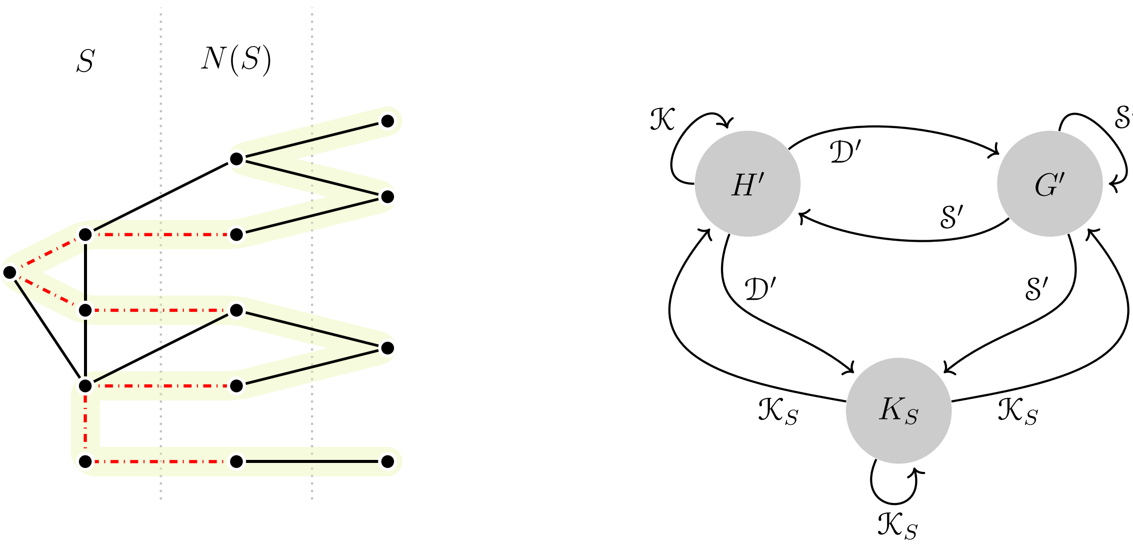

FIGURE 3 | (Left) A set S in a Hamiltonian graph and its neighborhood $N(S)$; part of a subdivision of K_4 is highlighted, and dashed red edges are the edges in K_S. (Right) Families that separate subgraphs of G, where $A \xrightarrow{\alpha} B$ indicates that α separates A from B (e.g., $\mathcal{S}'$ separates G' from H', and also from K_S and from G' itself).

Easy matchings: A *slicing* $A_1, ..., A_s$ of $[a_1, h]$ is a sequence of intervals which together partition $[a_1, h]$. We always assume that intervals are ordered consecutively, that is, so that each $x \in A_i$ precedes all $y \in \bigcup_{j>i} A_j$. Given an interval $A \subseteq [h]$, we say that $v_i v_j$ *starts* in A if $i \in A$ and that $v_i v_j$ *ends* in A if $j \in A$. We say that a matching $M \subseteq E(H')$ is *easy* if there is $a_1 \in [h]$, a slicing $A_1, ..., A_s$ of $[a_1, h]$ and $t \in \{0, 1\}$ such that the following hold.

a. for each $v_i v_j \in M$ there is $r \in [s-1]$ such that $r \equiv t \pmod 2$, $v_i \in A_r$ and $j \in A_{r+1}$,
b. the edges starting in A_r are pairwise crossing for all $r \in [s]$, and
c. for each $r \in [s]$, the number of edges starting in A_r is either zero or odd.

As we shall see, (a)–(c) suffice to overcome our main obstruction when covering matchings by K_4 subdivisions.

Now, for $k \in [h-1]$, let $M_k = \{v_i v_j \in E(H') : j - i = k\}$, and for $k \in [-h+2, h-2]$, let $N_k = \{v_i v_j \in E(H') : j - 2i = k\}$. It is not hard to check that M_k and N_k are linear forests, and each edge of H' is in precisely one M_k and precisely one N_k. This defines at most $3h$ linear forests. In what follows, we partition each M_k and N_k into four easy matchings, yielding $12h$ easy matchings. First, in items (i) and (i′) below, we partition, respectively, each M_k and N_k into two matchings satisfying (a) and (b).

i. Let $k \in [h-1]$. We partition M_k as follows. Let $A_1, ..., A_{\lceil n/k \rceil}$ be a slicing of $[h]$ into sets of size k and at most one set $A_{\lceil n/k \rceil}$ of size at most k. Now, given $t \in \{0, 1\}$, we let $\mathcal{A}_t = \bigcup_{r \equiv t \pmod 2} A_r$, and then put $M_{k,t} = \{v_i v_j \in M_k : i \in \mathcal{A}_t\}$. So, for example, $M_{k,1}$ consists of the edges of M_k that start in "odd" intervals of $[h]$. If $i \in A_r$, then $j \in A_{r+1}$ because $j - i = k = |A_r|$ (see Figure 4). This implies that if $v_i v_j \in M_{k,t}$, then $i \in \mathcal{A}_t$ while $j \in \mathcal{A}_{1-t}$. In particular, $M_{k,t}$ is a matching. Therefore, the matchings $M_{k,t}$ satisfy (a). Now, suppose that $v_i v_j, v_{i'} v_{j'} \in M_{k,t}$ are such that $i, i' \in A_r$ for some r, and suppose, without loss of generality, that $i < i'$. Then $j = k + i < k + i' = j'$, and hence $v_i v_j$ and $v_{i'} v_{j'}$ cross. Therefore, the matchings $M_{k,t}$ satisfy (b) as desired. This step defined at most $2h$ matchings.

i′. Let $k \in [-h+2, h-2]$. We partition N_k as follows. We consider two cases. If $k \geq 0$, then let $a_1 = 1$; otherwise let $a_1 = 1 - k \geq 2$. Now, for each $r \geq 1$ let $a_{r+1} = 2a_r + k$. Since $a_1 > -k$, we can prove by induction that $a_{r+1} > a_r > -k$ for all r. Also, let d be the minimum integer for which $h < a_{d+1}$. We partition $[a_1, h]$ into sets $A_1, ..., A_d$ so that $A_r = [a_r, a_{r+1} - 1]$ for $r \in [d-1]$, and $A_d = [a_d, h]$ (see Figure 5).

Note that every edge of N_k starts in $\bigcup_{j \in [d]} A_d$. Indeed, let $v_i v_j \in N_k$. Then $j = k + 2i$ and we claim that $i \geq a_1$. This is clear if $a_1 = 1$. Otherwise, $a_1 = 1 - k$, but since $j > i$, we must have $k + 2i \geq i + 1$, and hence $i \geq 1 - k = a_1$. Moreover, if $v_i v_j$ starts in A_r, then $v_i v_j$ ends in A_{r+1} since

$$\begin{aligned} a_{r+1} = k + 2a_r \leq k + 2i \leq k + 2(a_{r+1} - 1) \\ < k + 2a_{r+1} = a_{r+2}. \end{aligned}$$

Given $t \in \{0, 1\}$, we let $\mathcal{A}_t = \bigcup_{r \equiv t \pmod 2} A_r$, and put $N_{k,t} = \{v_i v_j \in N_k : i \in \mathcal{A}_t\}$. As in the preceding case, we can prove that the $N_{k,t}$ are matchings that satisfy both (a) and (b) as desired. This step defined at most $4h$ matchings.

ii. Finally, we partition each $M_{k,t}$ and $N_{k,t}$ obtained in steps (i) and (i′) into matchings satisfying (c). It is not hard to see that properties (a) and (b) are hereditary; that is, if a matching M satisfies (a) and (b), then any subset of M satisfies (a) and (b). Now, for each $k \in [h]$ and each $t \in \{0, 1\}$ we partition $M_{k,t}$ into sets $M_{k,t,0}$ and $M_{k,t,1}$ so that $M_{k,t,0}$ and $M_{k,t,1}$ have either zero or an odd number of edges starting in each A_r. This works because every even-sized matching can be partitioned into two odd-sized matchings and every odd-sized matching can be partitioned into an odd-sized matching and an empty matching (see Figure 6). We obtain $N_{k,t,0}$ and $N_{k,t,1}$ analogously. This step defined at most $12h$ matchings.

Let

$$\begin{aligned} \mathcal{M} &= \{M_{k,t,\rho} : k \in [h], t, \rho \in \{0, 1\}\}, \\ \mathcal{N} &= \{N_{k,t,\rho} : k \in [-h+2, h-2], t, \rho \in \{0, 1\}\} \end{aligned}$$

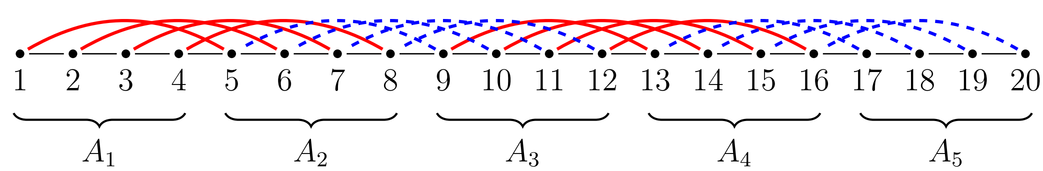

FIGURE 4 | A partition of M_4 into $M_{4,0}$ and $M_{4,1}$ in, respectively, dashed blue and solid red.

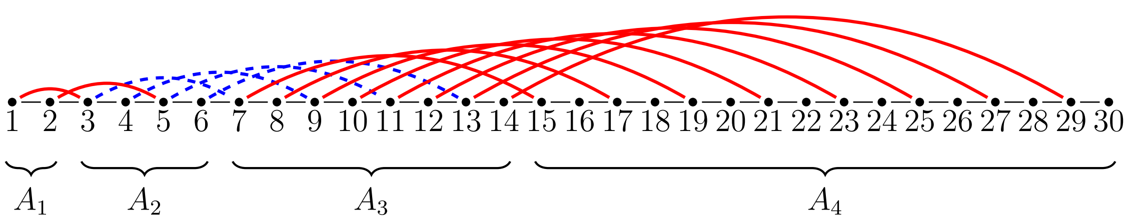

FIGURE 5 | A partition of N_1 into $N_{1,0}$ and $N_{1,1}$ in, respectively, dashed blue and solid red.

and note that

$$\bigcup_{M \in \mathcal{M}} M = E(H') = \bigcup_{M \in \mathcal{N}} M.$$

Building subdivisions of K_4*:* In what follows, given $M \in \mathcal{M} \cup \mathcal{N}$ we show that $K \cup M$ contains either one subdivision of K_4 that contains M or one edge and a subdivision of K_4 that together cover M. Let $A_1, \ldots, A_d$ be the slicing defined in (i) or (i′) when selecting edges for M. This construction is divided into two cases. We say that M is *elementary* if for every $r \in [d]$, there is at most one edge of M that starts in A_r; otherwise, we say that M is *jumbled*. By construction, if M is elementary, then no two edges of M cross. Recall that e and e' are edges not in H' that we use to complete C to a subdivision K of K_4.

Elementary: Suppose M is elementary. We divide this proof into two cases, depending on whether M contains an edge crossing e or e'. Let w_1, w_2, w_3, w_4 be the vertices of e and e' in the order that they appear in C, and assume, without loss of generality, that $e = w_1w_3$ and $e' = w_2w_4$.

Case 1. Suppose that no edge of M crosses e or e'. Since no two edges of M cross, there is at most one edge, say $f = v_iv_j$, of M that starts before w_1 and ends after w_4, in which case we pick f as a single edge. In either case, we cover M (or $M \backslash \{f\}$) by a subdivision K_M of K_4 obtained from K by "taking shortcuts": For every edge $v_iv_j \in M$ (or $v_iv_j \in M \backslash \{f\}$), we replace the path $C[v_i, v_j]$ by the edge v_iv_j.

Case 2. Suppose that M contains some edge f that crosses e or e'. Assume without loss of generality that f and e cross. Note, in particular, that by the construction of M, no edge of M can start before w_1 and also end after w_4. Since no two edges of M cross, there is at most one other edge, say f', of M that crosses e, in which case we pick f' as a single edge. Then $K' = C + f + e$ is a subdivision of K_4, and we obtain a subdivision K_M of K_4 from K' analogously to Case 1: for every edge $v_iv_j \in M \backslash \{f\}$ (or $v_iv_j \in M \backslash \{f, f'\}$), we replace the path $C[v_i, v_j]$ by v_iv_j.

This concludes the analysis when M is elementary.

Jumbled: In this case, we do not use e and e'. The idea here is to modify C into a cycle $C' \subseteq C \cup M$ that contains M and then restore some of the edges removed from C to obtain a subdivision of K_4. Recall that M is easy, and hence there exist $a_1, A_1, \ldots, A_d$, and t such that (a)–(c) hold.

Modifying C*:* Let A_r be an interval of $[h]$ for which the set M_r of edges of M that start in A_r (and end in A_{r+1}) is nonempty. Then, by (a), no edge of M starts in A_{r+1}. Let Q_r be the shortest subpath of C that contains the vertices v_i with $i \in A_r \cup A_{r+1}$, and let $R_r = Q_r \cup M_r$. We show that there is a path $Q'_r \subseteq R_r$ that contains M_r and has the same end vertices as Q_r. Then C' is obtained from C by replacing each Q_r by Q'_r.

The existence of Q'_r can easily be shown by induction. Alternatively, one can use the following construction. Let $s = |M_r|$ and let $w_1, \ldots, w_{2s}$ be the vertices of Q_r incident to edges of M_r in the order that they appear in Q_r. Note that w_1 and w_{2s} are the end vertices of Q_r and that s is odd by (c). By (b), $M_r = \{w_iw_{i+s} : i \leq s\}$. Now, let Q'_r be the graph obtained from R_r by removing the edges in $Q_r[w_i, w_{i+1}]$ for odd i with $i \leq 2s - 1$ and removing isolated vertices (see Figure 7). Note that R_r is a subcubic graph whose vertices with degree 3 are precisely $w_2, \ldots, w_{2s-1}$ and that, to obtain Q'_r, we remove from R_r precisely one edge incident to each w_i. Thus, $\Delta(Q'_r) \leq 2$. We claim that Q'_r is connected. For that, we prove that Q'_r contains a path joining w_i to w_{i+1} for each $i \in [2s - 1]$. This is clear if i is even, because the segment $Q_r[w_iw_{i+1}]$ has not been removed. For odd i, on the other hand, the segment $Q_r[w_{i+s}w_{i+s+1}]$ has not been removed because s is odd (and consequently $i + s$ is even), and thus $w_iw_{i+s}Q_r[w_{i+s}w_{i+s+1}]w_{i+1}$ is the desired a path in Q'_r. Moreover, the end vertices of Q'_r are w_1 and w_{2s} and $M_r \subseteq E(Q'_r)$. This concludes the construction of Q'_r.

Turning C' *into a subdivision of* K_4*:* To turn C into a subdivision of K_4 we only need to restore two of the removed $Q_r[w_i, w_{i+1}]$ for some r. More precisely, since M is jumbled, there is an interval A_r in which at least two edges of M start. Let $M_r \subseteq M$ be the set of edges that start in A_r. Let $s = |M_r|$ and let $w_1, \ldots, w_{2s}$ be the vertices of P incident to edges of M_r as above. Recall that the removed subpaths were $Q_r[w_i, w_{i+1}]$ for odd i. Moreover, recall that s is odd by (c) and thus $s \geq 3$. Therefore, $Q_r[w_1, w_2]$ and $Q_r[w_{s+2}, w_{s+3}]$ were removed. Let $K_M = C' \cup Q_r[w_1, w_2] \cup Q_r[w_{s+2}, w_{s+3}]$. We

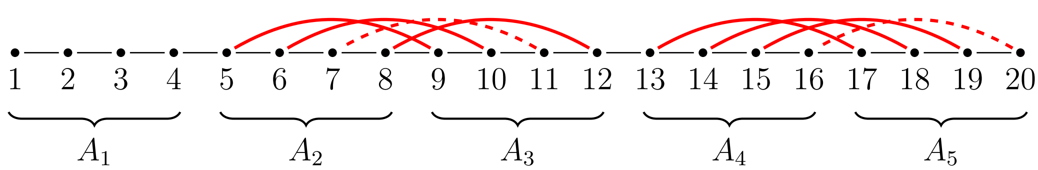

FIGURE 6 | A partition of $M_{4,0}$ into $M_{4,0,0}$ and $M_{4,0,1}$ in, respectively, solid and dashed lines.

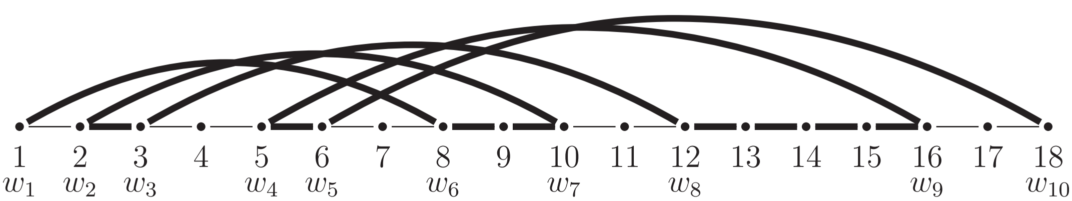

FIGURE 7 | Thick lines illustrate the path Q'_r that contains $M_r \subseteq N_{6,t,r}$ for some t and r.

claim that K_M is a subdivision of K_4. Indeed, by construction of C', the order in which the vertices $w_1, \ldots, w_{2s}$ appear is

$$w_1, w_{s+1}, w_{s+2}, w_2, w_3, w_{s+3}, \ldots, w_i, w_{i+s}, w_{i+s+1}, w_{i+1}, \ldots, w_{2s}.$$

Let $p(w_i)$ be the position of w_i in the order above. In particular, we have $p(w_1) = 1, p(w_{s+2}) = 3, p(w_2) = 4$, and $p(w_{s+3}) = 6$. Then, $p(w_1) < p(w_{s+2}) < p(w_2) < p(w_{s+3})$, and hence K_M is a subdivision of K_4. This step creates a single subdivision of K_4 for each M.

Now, by applying the construction above on each M in $\mathcal{M} \cup \mathcal{N}$, we obtain a set $\mathcal{K}$ of at most $24h$ edges or subdivisions of K_4. We claim that $\mathcal{K}$ separates each pair of edges in H'. Indeed, let $v_i v_j$ and $v_{i'} v_{j'}$ be edges of H'. Each such edge belongs to exactly one $M \in \mathcal{M}$ and exactly one $N \in \mathcal{N}$. If they belong to different elements of $\mathcal{M}$, then they belong to different elements of $\mathcal{K}$ and are separated, because $\mathcal{M}$ partitions $E(H')$. The same argument works for $\mathcal{N}$. Therefore, we may assume that $i - j = i' - j'$ and $i - 2j = i' - 2j'$. This immediately yields $j = j'$ and $i = i'$, so $v_i v_j = v_{i'} v_{j'}$. Therefore, any two *distinct* edges in H' are separated by $\mathcal{K}$, as desired. This concludes the proof. □

3 | Separating into Cycles

First, note that K_4 can be covered by two cycles (indeed, suppose $V(K_4) = [4]$ and note that the cycles 12,341 and 12,431 cover it). Hence, every subdivision of K_4 can be covered by two cycles. This means that any K_4-separating system $\mathcal{S}$ yields a K_3-separating system of size at most $2|\mathcal{S}|$. And thus, by Theorem 4, every graph on n vertices admits a K_3-separating system of size at most $164n$, which verifies Conjecture 3 when $H = K_3$. A slightly better result, Theorem 2 (any graph on n vertices can be separated by at most $41n$ edges and cycles), can be obtained using the following theorem of Pyber [18].

Theorem 7 (Pyber [18]). *Every graph G contains $|V(G)| - 1$ cycles and edges covering $E(G)$.*

The proof Theorem 2 follows that of Theorem 4. However, after partitioning the M_k's and N_k's into $12h$ special matchings, there is no need to consider elementary and jumbled cases, since there is no need to find crossing edges (one proceeds directly to the "Modifying C" part of the argument, in which we obtain a cycle C'). Of course, one should also skip the step for "Turning C' into a subdivision of K_4".

4 | Conclusion

In this paper, we give the first steps toward showing that every graph admits a linearly sized separating system formed by edges and subdivisions of K_t, for fixed t. Perhaps these techniques can be fitted to find subdivisions of larger cliques. If a suitable subdivision of K_t ($t > 4$)—namely, a subdivision containing a Hamilton cycle—can be shown to exist for larger cliques, then the tricks used in our proof might still be useful to produce a K_t-separating system.

Conjecture 3 could be weakened by allowing a wider class of structures. For example, one could seek a separating system consisting of graphs that yield some fixed graph H through a series of edge contractions or consisting of immersions of H (for more about immersions, see, e.g., [19]).

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Data Availability Statement

The authors have nothing to report.

Endnotes

[1] Personal communication.

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