Proceedings of the 13th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'25
Budapest, August 25 - 29, 2025

Extremal Problems on Forest Cuts and Acyclic Neighborhoods in Sparse Graphs*

(Extended abstract)

F. Botler† Y. S. Couto† C. G. Fernandes† E. F. de Figueiredo‡
R. Gómez§ V. F. dos Santos‡ C. M. Sato§

Abstract

Chernyshev, Rauch, and Rautenbach proved that every connected graph G on n vertices for which $e(G) < \frac{11}{5}n - \frac{18}{5}$ has a vertex cut that induces a forest, and conjectured that the same remains true if $e(G) < 3n - 6$ edges. We improve their result by proving that every connected graph on n vertices for which $e(G) < \frac{9}{4}n - \frac{15}{4}$ has a vertex cut that induces a forest. We also study weaker versions of the problem that might lead to an improvement on the bound obtained.

1 Introduction

Let G be a connected graph. A set $S \subset V(G)$ is a *vertex cut* if $G - S$ is disconnected. If $|S| = k$, we say S is a *k-vertex cut.* If S is an independent set, we say S is an *independent cut.* Vertex cuts with special properties have been studied in different contexts. Chen and Yu [1] showed that every connected graph with less than $2n-3$ edges has an independent cut, confirming a conjecture due to Caro. Recently, Chernyshev, Rauch, and Rautenbach proposed the following analogue conjecture, replacing independent set by forest [2, Conjecture 1]. A *forest cut* is a vertex cut that induces a forest.

*This work started during the 6th WoPOCA (Workshop Paulista em Otimização, Combinatória e Algoritmos) in Campinas, Brazil. We thank the organizers for the opportunity to start new collaborations and the agencies that helped making the workshop possible: Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq (404315/2023-2) and FAEPEX (2422/23). F. Botler would also like to thank the support of CNPq (304315/2022-2), CAPES (88887.878880/2023-00) and the São Paulo Research Foundation – FAPESP (2024/14906-6). Y.S. Couto would also like to thank the support of FAPESP (2024/18049-0). C.G. Fernandes would also like to thank the support of CNPq (310979/2020-0). V.F. dos Santos would also like to thank the support of CNPq (312069/2021-9, 406036/2021-7 and 404479/2023-5) and FAPEMIG (APQ-01707-21). C.M. Sato would also like to thank the support of CNPq (408180/2023-4). E.F. de Figueiredo would also like to thank the support of CAPES (Finance Code 001).

†Institute of Mathematics and Statistics, University of São Paulo, Brazil. E-mail: {fbotler,cris,yancouto}@ime.usp.br.

‡Department of Computer Science, Federal University of Minas Gerais, Brazil. E-mail: {ederfgd,viniciussantos}@dcc.ufmg.br.

§Center for Mathematics, Computing and Cognition, Federal University of ABC, Brazil. E-mail: {gomez.renzo,c.sato} @ufabc.edu.br.

Conjecture 1 (Chernyshev–Rauch–Rautenbach, 2024). *If G is a connected graph on n vertices with no forest cut, then $e(G) \geq 3n - 6$.*

Chernyshev et al. [2] also showed that Conjecture 1 holds for some classes of graphs. For instance, they showed that a graph G with n vertices has a forest cut if (i) G is a planar graph that is not triangulated; (ii) G has a universal vertex and $e(G) < 3n - 6$; or (iii) G is connected and $e(G) < \frac{11}{5}n - \frac{18}{5}$.

We say a graph is *k-cyclic* if every vertex set of size at most k is dominating or has a cycle in its neighborhood. Note that any (forest) cut disconnects the graph into at least two components, which are not dominating sets, and one of these components has less than $n/2$ vertices. So, Conjecture 1 claims that any $(\frac{n-1}{2})$-cyclic graph has at least $3n - 6$ edges. Moreover, any 2-vertex cut is trivially a forest, so Chernyshev et al. [2] noted that finding good lower bounds for the number of edges on 1-cyclic 3-connected graphs would imply a result towards Conjecture 1, and stated the following.

Conjecture 2 (Chernyshev–Rauch–Rautenbach, 2024). *If G is a 3-connected graph on n vertices such that there is a cycle in the neighborhood of every vertex, then $e(G) \geq \frac{7}{3}n - \frac{7}{3}$.*

The conjecture addresses a proper subclass of 1-cyclic graphs as it requires cycles in the neighborhood of universal vertices. However, it is functionally the same as for 1-cyclic graphs, as even Conjecture 1 holds for graphs with universal vertices [2]. In this paper, we improve the bound from [2] towards Conjecture 1, disprove Conjecture 2, and present lower bounds on the number of edges for 3-connected graphs to be 1-cyclic and 2-cyclic.

Theorem 3. *Let G be a graph on n vertices. Then the following hold. (a) If G is connected and has no forest cut, then $e(G) \geq \frac{9}{4}n - \frac{15}{4}$; (b) If G is 3-connected, 1-cyclic, and $n \geq 6$, then $e(G) \geq \frac{15}{8}n$; (c) If G is 3-connected, 2-cyclic, and $n \geq 6$, then $e(G) \geq 2n$.*

The $n \geq 6$ is necessary in Theorem 3(b) and 3(c) as K_5 minus an edge is 3-connected and 2-cyclic (hence also 1-cyclic), has five vertices and nine edges, but $9 < \frac{15}{8} \cdot 5 = \frac{75}{8} < 10$.

Remark 4. *There are infinite families of (a) 3-connected 1-cyclic graphs on n vertices with exactly $\frac{15n}{8}$ edges and no universal vertices; (b) 4-connected 1-cyclic graphs on n vertices with exactly $2n$ edges; (c) 3-connected 2-cyclic graphs on n vertices with exactly $\frac{9}{4}n$ edges; (d) 4-connected 2-cyclic graphs on n vertices with exactly $\frac{7}{3}n$ edges.*

Remark 4(a) disproves Conjecture 2, proving that Theorem 3(b) is asymptotically tight. For Theorem 3(c), we present a 3-connected 2-cyclic graph and a 4-connected 2-cyclic graph, both with 6 vertices and 12 edges, and, based on Remark 4(d), we pose the following conjecture that would imply an improvement on Theorem 3(a), towards Conjecture 1.

Conjecture 5. *If G is a 4-connected 2-cyclic graph on $n \geq 9$ vertices, then $e(G) \geq \frac{7}{3}n$.*

In Section 2, we prove Theorem 3(a). In Section 3, we prove Theorem 3(b)-(c), and Remark 4. A recent independent work by Li, Tang, and Zhan [3] contains results similar to the ones on 1-cyclic graphs in Section 3. Due to space constraints, we omit a few proofs.

2 Avoiding forest cuts

Chernyshev et al. [2] proved that a connected graph on n vertices with no forest cut must have at least $\frac{11n}{5} - \frac{18}{5}$ edges. For that, they studied properties its counterexamples with a minimum number of vertices. Such properties are in fact shared with a minimum counterexample to Theorem 3(a) and Conjecture 1. To help the exposition, we state a conjecture parameterized by a number α with $2 \le \alpha \le 3$.

Conjecture 6 (α-FC Conjecture)**.** If G is a connected graph on n vertices with no forest cut, then $e(G) \ge \alpha(n-3)+3$.

Note that Theorem 3(a) is the same as the $\frac{9}{4}$-FC Conjecture, Chernyshev et al. [2] proved the $\frac{11}{5}$-FC Conjecture and Conjecture 1 is the same as the 3-FC Conjecture. For $2 \le \alpha \le 3$, a minimum counterexample to the α-FC Conjecture is a graph G on n vertices with no forest cut, $e(G) < \alpha(n-3)+3$ and n as small as possible. The following lemma is used in the proof of Theorem 3(a).

Lemma 7. *Let G be a minimum counterexample to the α-FC Conjecture, for $2 \le \alpha \le 3$. Then (a) G is 4-connected and has at least 8 vertices; (b) no degree-4 vertex in G has a C_4 in its neighborhood; and (c) no two degree-4 vertices are in the same K_4 in G.*

Lemma 7(a) was adapted from the proof of Claim 1 in Chernyshev et al. [2]. They [2, Claim 2] also proved that, in a minimum counterexample to Conjecture 1, every degree-4 vertex has at most two neighbors of degree 4. Lemma 7(b) and 7(c) are strengthenings of this statement. Lemma 7(b) implies that every degree-4 vertex in a minimum counterexample to the α-FC Conjecture lies in a K_4, and we deduce the following from Lemma 7(c).

Corollary 8. *Let G be a minimum counterexample to the α-FC Conjecture, for $2 \le \alpha \le 3$. Then the following hold: (a) every degree-4 vertex in G has at most one degree-4 neighbor; and (b) each vertex with degree at least 5 in G has at least two neighbors of degree at least 5.*

Corollary 8(b) is also a strengthening of a result of Chernyshev et al. [2, Claim 3]. We conclude this section with the proof of Theorem 3(a).

Proof of Theorem 3(a). Suppose G is a minimum counterexample to Theorem 3(a), and hence to the $\frac{9}{4}$-FC Conjecture. Let n be the number of vertices of G, and n_i be the number of degree-i vertices in G. By Lemma 7(a), G is 4-connected and $n = \sum_{i=4}^{n-1} n_i \ge 8$. Let F_4 be the set of edges joining degree-4 vertices to vertices with degree at least 5. By Corollary 8(a), we have that $|F_4| \ge 3n_4$. By Corollary 8(b), each degree-j vertex in G with $j \ge 5$ contributes with at most $j-2$ edges to F_4, and hence $|F_4| \le \sum_{j=5}^{n-1}(j-2)n_j$. Now, since $j-2 \le 6j-27$ for $j \ge 5$, we have $3n_4 \le \sum_{j=5}^{n-1}(j-2)n_j \le \sum_{j=5}^{n-1}(6j-27)n_j = 6(2e(G)-4n_4) - 27(n-n_4) = 12e(G) + 3n_4 - 27n$, so $e(G) \ge 9n/4$, a contradiction. □

3 Bounds for 1-cyclic and 2-cyclic graphs

First, we present a family of counterexamples to Conjecture 2 and prove Remark 4(a). Take any 3-connected 3-regular graph (see [4]) with k vertices and replace each vertex with a K_4, connecting each of its neighbors to a distinct vertex in the K_4 and leaving only one vertex of each K_4 with degree 3 (see, e.g., Figure 1). We obtain a 3-connected graph G with precisely $n = 4k$ vertices and $m = \frac{3k}{2} + 6k = \frac{15}{8}n$ edges. Moreover, G is 1-cyclic because each of its vertices is in a K_4.

Remark 4(a) shows that Theorem 3(b) is tight. We denote by $K_s^{\triangle}$ the graph obtained from K_3 by adding s new vertices adjacent to the three vertices of the K_3. The proof of Theorem 3(b) uses the following lemma, whose proof we omit.

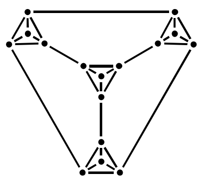

Figure 1: A counterexample to Conjecture 2 built from K_4.

Lemma 9. *If G is a 3-connected 1-cyclic graph on $n \geq 5$ vertices. Then the following hold: (a) every degree-3 vertex has no degree-3 neighbor; and (b) either G is isomorphic to $K_{n-3}^{\triangle}$ or every vertex of G has at least three neighbors of degree at least 4.*

Proof of Theorem 3(b). Let G be a 3-connected 1-cyclic graph on $n \geq 6$ vertices, and n_i be the number of degree-i vertices in G. By Lemma 9(b), either G is isomorphic to $K_{n-3}^{\triangle}$ or every vertex of G has at least three neighbors of degree at least 4. In the former case, as desired, $e(G) = 3n - 6 > \frac{15}{8}n$ as $n \geq 6$. In the latter case, as $4j - 15 \geq j - 3$ for $j \geq 4$, we have $3n_3 \leq \sum_{j=4}^{n-1}(j-3)n_j \leq \sum_{j=4}^{n-1}(4j-15)n_j = 8e(G) - 15n + 3n_3$, i.e., $e(G) \geq \frac{15}{8}n$. □

Note that if we pick an arbitrary 4-connected 4-regular graph and replace each of its vertices by a K_4, leaving all vertices of each K_4 with degree 4, then the graph obtained is 4-connected, 4-regular, and 1-cyclic. Therefore, the lower bound $e(G) \geq 2n$ is best possible for 4-connected 1-cyclic graphs, and proves Remark 4(b). Now, we prove a lower bound on the number of edges for a 3-connected graph to be 2-cyclic. Specifically, we prove Theorem 3(c). We start by proving some properties of 3-connected 2-cyclic graphs.

Lemma 10. *Let G be a 3-connected 2-cyclic graph on $n \geq 6$ vertices. Then every degree-3 vertex has at least two neighbors of degree at least 5.*

Proof. Let v be a degree-3 vertex in G, and x, y, and z be its neighbors. By Lemma 9(a), these three vertices have degree at least 4, and they form a triangle, because $n \geq 5$ and G is 1-cyclic. Suppose, for a contradiction, that x and y have degree 4. Then the neighborhood $N(\{v, x\}) = \{y, z, w\}$, where w is the other neighbor of x. As $n \geq 6$ and G is 2-cyclic, y, x, w form a triangle, and w is also the other neighbor of y. But then $N(\{x, y\}) = \{v, z, w\}$, which must form a cycle because $n \geq 6$. However there is no edge vw, a contradiction. □

Proof of Theorem 3(c). Let n_i be the number of degree-i vertices in G and F be the set of edges joining degree-3 vertices to vertices with degree at least 5. By Lemma 10, we have that $|F| \geq 2n_3$. By Lemma 9(b), either G is isomorphic to $K_{n-3}^{\triangle}$ or every vertex of G has at least three neighbors of degree at least 4. In the former case, G has $3n - 6 \geq 2n$ edges as $n \geq 6$. In the latter case, each degree-j vertex for $j \geq 5$ contributes with at most $j - 3$ edges to F, so $|F| \leq \sum_{j=5}^{n-1}(j-3)n_j$. As $2j - 8 \geq j - 3$ for $j \geq 5$, we have $2n_3 \leq |F| \leq \sum_{j=5}^{n-1}(j-3)n_j \leq \sum_{j=5}^{n-1}(2j-8)n_j = 4e(G) - 8n + 2n_3$, i.e., $e(G) \geq 2n$. □

In Figure 2, on the left, we show two tight examples for Theorem 3(c): the graph $K_3^{\triangle}$, which is 3-connected, and the octahedral graph, which is 4-connected. The third graph in Figure 2 has 9 vertices and 20 edges. Consider the construction illustrated in Figure 1, starting from a 3-connected 3-regular graph on k vertices. If we replace each vertex by an octahedral graph instead of a K_4, we end up with a 3-connected 2-cyclic graph on $6k$ vertices and $\frac{3}{2}k + 12k = \frac{27}{2}k = \frac{9}{4}n$ edges, which proves Remark 4(c). As far as we know, it may hold that $m \geq \frac{9}{4}n$ for the graphs addressed by Theorem 3(c) if $n \geq 10$. The requirement $n \geq 10$ is necessary to exclude the third graph in Figure 2, because $\frac{20}{9} < \frac{9}{4}$.

The lower bound on the number of edges in a 4-connected 2-cyclic graph might be larger. Take a 4-connected 4-regular graph on k vertices, and replace each of its vertices by an octahedral graph, leaving precisely four vertices of each octahedral graph with degree 5. The graph obtained is 4-connected, 2-cyclic, has $6k$ vertices and $m = 2k + 12k = 14k = \frac{7}{3}n$ edges. This proves Remark 4(d), which shows that Conjecture 5 is tight. In Figure 2, on the right, we show a 4-connected 2-cyclic graph on 7 vertices and 16 edges, and two 4-connected 2-cyclic graphs with 8 vertices and 18 edges. Since $\frac{16}{7}$ and $\frac{18}{8}$ are less than $\frac{7}{3}$, these examples justify the condition $n \geq 9$ in Conjecture 5.

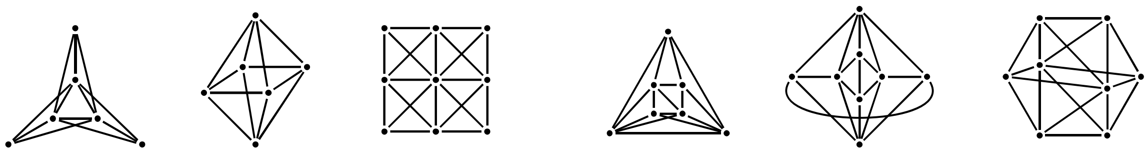

Figure 2: Left: Three 3-connected 2-cyclic graphs, two with 6 vertices and 12 edges and one with 9 vertices and 20 edges. Right: Three 4-connected 2-cyclic graphs, one with 7 vertices and 16 edges, and two with 8 vertices and 18 edges.

4 Final remarks

Several questions remain open. Of course it would be nice to settle Conjecture 1, or to obtain an improvement on Theorem 3(a). Proving Conjecture 5 or finding a family of 4-connected 2-cyclic graphs on n vertices with less than $\frac{7}{3}n$ edges would also be interesting.

The study of k-cyclic graphs with k more than 2 seems to be a possible way to achieve better results towards Conjecture 1. Our exposition points out that we barely use the forest cut requirement for sets larger than 2 in the current results.

References

[1] Guantao Chen and Xingxing Yu. A note on fragile graphs. *Discrete mathematics*, 249(1-3):41–43, 2002.

[2] Vsevolod Chernyshev, Johannes Rauch, and Dieter Rautenbach. Forest cuts in sparse graphs. *Available at arXiv:2409.17724*, 2024.

[3] Chengli Li, Yurui Tang, and Xingzhi Zhan. The minimum size of a 3-connected locally nonforesty graph. *Available at arXiv:2410.23702*, 2024.

[4] Nicholas C. Wormald. The asymptotic connectivity of labelled regular graphs. *Journal of Combinatorial Theory, Series B*, 31(2):156–167, 1981.