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Symbolic dynamics for large non-uniformly hyperbolic sets of three dimensional flows ☆

Jérôme Buzzi [a], Sylvain Crovisier [a], Yuri Lima [b,*]

[a] *Laboratoire de Mathématiques d'Orsay, CNRS - UMR 8628, Université Paris-Saclay, Orsay 91405, France*
[b] *Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, Cidade Universitária, 05508-090, São Paulo - SP, Brazil*

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We construct symbolic dynamics for three dimensional flows with positive speed. More precisely, for each $\chi > 0$, we code a set of full measure for every invariant probability measure which is χ–hyperbolic. These include all ergodic measures with entropy bigger than χ as well as all hyperbolic periodic orbits of saddle-type with Lyapunov exponent outside of $[-\chi, \chi]$. This contrasts with a previous work of Lima & Sarig which built a coding associated to a given invariant probability measure [28]. As an application, we code homoclinic classes of measures by suspensions of irreducible countable Markov shifts.

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* Corresponding author.
E-mail addresses: jerome.buzzi@universite-paris-saclay.fr (J. Buzzi), sylvain.crovisier@universite-paris-saclay.fr (S. Crovisier), yurilima@gmail.com (Y. Lima).

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Contents

1. Introduction

Let M be a smooth closed three dimensional manifold and X be a $C^{1+\beta}$ vector field on M with $\beta > 0$ which is non-singular, i.e. $X_p \neq 0$ for all $p \in M$. We want to code a "large" subset of M with some non-uniform hyperbolicity for the flow $\varphi = \{\varphi^t\}_{t\in\mathbb{R}}$ generated by X. This subset carries all φ–invariant hyperbolic ergodic probability measures with the following nonuniform hyperbolic property. Let $\chi > 0$.

χ–HYPERBOLIC MEASURE: A φ–invariant probability measure μ on M is *χ–hyperbolic* if μ–a.e. point has one Lyapunov exponent $> \chi$ and one Lyapunov exponent $< -\chi$. The Lyapunov exponent along the flow, which always vanishes, is called trivial.

This defines a rather natural, large, and uncountable class of measures. For instance, by the Ruelle inequality, every φ–invariant ergodic probability measure with metric entropy larger than χ is χ–hyperbolic. Also, every φ–invariant probability measure defined by a closed orbit of saddle type with nontrivial Lyapunov exponents larger than χ in absolute value is χ–hyperbolic. In this paper, for each $\chi > 0$ we construct a symbolic system which lifts all χ–hyperbolic measures.

Main Theorem. *Let X be a non-singular $C^{1+\beta}$ vector field ($\beta > 0$) on a closed 3–manifold M. Given $\chi > 0$, there exist a locally compact topological Markov flow (Σ_r, σ_r) and a map $\pi_r : \Sigma_r \to M$ such that $\pi_r \circ \sigma_r^t = \varphi^t \circ \pi_r$, for all $t \in \mathbb{R}$, and satisfying:*

(1) *The roof function r and the projection π_r are Hölder continuous.*
(2) *$\pi_r[\Sigma_r^\#]$ has full measure for every χ–hyperbolic measure on M.*
(3) *π_r is finite-to-one on $\Sigma_r^\#$, i.e.* $\mathrm{Card}(\{z \in \Sigma_r^\# : \pi_r(z) = x\}) < \infty$, *for all $x \in \pi_r[\Sigma_r^\#]$.*

A more precise version of the Main Theorem is stated in Section 9.1, see Theorem 9.1.

A topological Markov flow is the unit speed vertical flow on a suspension space whose basis is a topological Markov shift and whose roof function is continuous, everywhere positive and uniformly bounded. We can endow (Σ_r, σ_r) with a natural metric, called the *Bowen-Walters metric*, that makes σ_r a continuous flow. It is with respect to this metric that π_r is Hölder continuous. The set $\Sigma_r^\#$ is the *regular* set of (Σ_r, σ_r), consisting of all elements of Σ_r for which the symbolic coordinate has a symbol repeating infinitely often in the future and a symbol repeating infinitely often in the past. See Section 1.2 for the definitions.

The Main Theorem provides a *single* symbolic extension that codes all χ–hyperbolic measures at the same time, and that is finite-to-one almost everywhere. This improves on the result by Lima & Sarig [28], whose codings depend on the choice of a measure (or a countable class of measures). We will mention later the importance of this novelty.

In applications, it is useful to work with *irreducible* Markov shifts since, among other properties, they are topologically transitive and they carry at most one measure of maximal entropy (see Section 1.2.1). This is related to the notion of homoclinically related measures and of *homoclinic classes of measures*, defined in Section 10. In this context, we prove the following theorem.

Theorem 1.1. *In the setting of the Main Theorem, let μ be a hyperbolic ergodic measure. Then Σ_r contains an irreducible component Σ_r' which lifts any χ–hyperbolic ergodic measure ν homoclinically related to μ.*

This implies the following local uniqueness result for measures of maximal entropy.

Corollary 1.2. *In the setting of the Main Theorem, let μ be a hyperbolic ergodic measure. Then there is at most one measure ν which is homoclinically related to μ and maximizes the entropy, i.e. satisfies $h(\varphi, \nu) = \sup\{h(\varphi, \rho) : \rho$ is homoclinically related to $\mu\}$.*

Results about uniqueness of the measure of maximal entropy for flows have been obtained previously under various settings, see for instance [8,24,20,11,21,18,19,31].

The field of symbolic dynamics has been extremely successful in analyzing systems displaying hyperbolic behavior. Its modern history includes (but is not restricted to) the construction of Markov partitions in various uniformly and non-uniformly hyperbolic settings:

- Adler & Weiss for two dimensional hyperbolic toral automorphisms [1].
- Sinaĭ for Anosov diffeomorphisms [37].
- Ratner for Anosov flows [34,33].
- Bowen for Axiom A diffeomorphisms [6,4] and Axiom A flows without fixed points [7].
- Katok for sets approximating hyperbolic measures of diffeomorphisms [23].
- Hofbauer [22] and Buzzi [13,14] for piecewise maps on the interval and beyond.
- Sarig for surface diffeomorphisms [36].
- Lima & Matheus for two dimensional non-uniformly hyperbolic billiards [27].
- Ben Ovadia for diffeomorphisms in any dimension [3].
- Lima & Sarig for three dimensional flows without fixed points [28].

- ◦ Lima for one-dimensional maps [29].
- ◦ Araujo, Lima, Poletti for non-invertible maps with singularities in any dimension [2].

In the first four settings above, that dealt with uniformly hyperbolic systems, the coding is surjective and one-to-one in a large (Baire generic) set. Katok was the first to treat non-uniformly hyperbolic systems [23]. When applied to surface diffeomorphisms, it implies the existence of horseshoes of large (but not necessarily full) topological entropy. Sarig was the first to construct non-uniformly hyperbolic horseshoes of full topological entropy [36]. His work improved Katok's to a great extent, proving that for each $\chi > 0$ there is a symbolic coding with good properties, among them the finiteness-to-one property in the regular set $\Sigma^\#$ (see Section 1.2.1 for the definition of $\Sigma^\#$). It codes all χ–hyperbolic measures *simultaneously*, and it implies many dynamical consequences such as estimates on the number of closed orbits [36], an at most countable set of ergodic measures of maximal entropy [36], ergodic properties of equilibrium measures [35], and the almost Borel structure of surface diffeomorphisms [10]. In recent years, more advances are being obtained, such as the coding of homoclinic classes of measures by irreducible Markov shifts and finiteness/uniqueness of measures of maximal entropy [17], and continuity properties of Lyapunov exponents [16].

The work of Lima & Sarig was the first to construct, for three dimensional flows, horseshoes of full topological entropy [28]. It is not as strong as Sarig's, since it only codes one χ–hyperbolic measure at a time (actually, by an easy adaptation in the proof, it codes countably many such measures). It implies some dynamical consequences, such as estimates on the number of closed orbits [28], the countability on the number of measures of maximal entropy [28], and ergodic properties of equilibrium measures [26]. Unfortunately, their techniques do not seem to extend to, say, the coding of all χ-hyperbolic measures as in the case of diffeomorphisms.

Our Main Theorem identifies a subset of points in M with non-uniform hyperbolicity at least χ possessing local product structure, and constructs a finite-to-one extension of this set by a locally compact topological Markov flow. This set carries all χ–hyperbolic measures. As an application, we code homoclinic classes of measures by irreducible Markov flows.

1.1. Method of proof

We build on the seminal work of Sarig [36] and its extension to flows by Lima & Sarig [28]. Lima & Sarig study a flow by considering the Poincaré return map to a section. This yields a surface map to which they apply a version of Sarig's result. The Poincaré map has singularities, which are controlled at the price of choosing the section in a way that almost all orbits slowly approach the boundary of the section. Here is where their construction becomes specific to a single measure. It is still unknown whether there is a global Poincaré section such that this latter property holds for every χ–hyperbolic measure. We call the presence of boundary the *boundary effect*.

Additionally to the work of Sarig [36] and Lima & Sarig [28], we are also inspired by the remarkable work of Bowen [7]. Bowen's idea to construct Markov partitions for flows is to replace the Poincaré map by good returns (suitable holonomy maps), which are smooth by construction: the artificial singularities of [28] have disappeared. In this way, we proceed as follows:

(1) Construct two global Poincaré sections $\Lambda, \widehat{\Lambda}$ such that $\Lambda \subset \widehat{\Lambda}$. We use Λ as the reference section for our construction, and $\widehat{\Lambda}$ as a security section.
(2) Let $f : \Lambda \to \Lambda$ be the Poincaré return map of Λ (note: f is not the Poincaré return map of $\widehat{\Lambda}$). If μ is χ–hyperbolic and ν is the measure induced on Λ, then ν–almost every $x \in \Lambda$ has a Pesin chart $\Psi_x : [-Q(x), Q(x)]^2 \to \widehat{\Lambda}$ whose size satisfies $\lim \frac{1}{n} \log Q(f^n(x)) = 0$. Note that the center of the chart is in Λ, while the image is on the security section $\widehat{\Lambda}$. Local changes of coordinates by linear maps of norm Q^{-1} allow to conjugate f to a uniformly hyperbolic map.
(3) Introduce ε–double charts $\Psi_x^{p^s,p^u}$, which are versions of Pesin charts that control separately the local stable and local unstable hyperbolicity at x (the parameters p^s/p^u can be seen as choices of sizes of the local stable/unstable manifolds). Define the transition between ε–double charts so that the parameters p^s, p^u are *almost* maximal, given the previous and next charts.
(4) Construct a countable collection $\mathscr{A}$ of ε–double charts that are dense in the space of all ε–double charts. The notion of denseness is defined in terms of finitely many parameters of the ε–double charts. Using pseudo-orbits, shadowing and the graph transform method, the collection $\mathscr{A}$ defines a Markov cover $\mathscr{Z}$. Unfortunately, $\mathscr{Z}$ defines a symbolic coding that is usually infinite-to-one. Fortunately, $\mathscr{Z}$ is locally finite.
(5) $\mathscr{Z}$ satisfies a Markov property: for every $x \in \bigcup_{Z \in \mathscr{Z}} Z$ there is $k > 0$ such that $f^k(x)$ satisfies a Markov property in the stable direction and $\ell > 0$ such that $f^{-\ell}(x)$ satisfies a Markov property in the unstable direction. The values of k, ℓ are uniformly bounded.
(6) The local finiteness of $\mathscr{Z}$ and the uniform bounds on k, ℓ allow to apply a refinement method to obtain a countable Markov partition, which defines a topological Markov flow (Σ_r, σ_r) and a map $\pi_r : \Sigma_r \to M$ satisfying the Main Theorem.

In analogy with Bowen [7], in our case a good return of the center of a chart is a return to Λ. The ideas of [7] are also used in steps (5) and (6).

We use the same method of [36] to obtain step (2). Steps (3) and (4) use ideas of [36,28], but they require *novel* ideas. The main difficulty is the following: there is no canonical way to parse a flow orbit into good returns, hence a single orbit might be cut into different ways. We call this the *parsing problem*. It relates to the *inverse problem*, whose goal is to prove that the parameters of the ε–double charts coding an orbit are defined "up to bounded error". Firstly, since the flow transition times of good returns might belong to a continuum (hence uncountable), our definition of a transition between ε–double charts requires inequalities between the parameters p^s, p^u, see relations (4.1) and (4.2). This contrasts with all previous recent literature, whose definitions of transition require equalities. Secondly, we compare the parameters of an orbit directly with the parameters of the ε–double charts coding it. To do that, we introduce analogues of the parameters p^s, p^u for points of M. Indeed, we introduce continuous and discrete versions of such parameters, see Sections 3.3 and 3.5. The continuous version is intrinsic and only depends on the flow, while the discrete depends on the parsing. The discrete one can more easily be compared with the parameters of the ε–double charts. These new parameters, already used in [27] in a non-essential way, are essential to us.

The definition of transition between ε–double charts introduces new difficulties. Since equalities between the parameters no longer hold, a single orbit can be shadowed by two different sequences of ε–double charts, and the accumulated transition times of the two sequences might differ. To investigate this difference, which we call shear, we first show that parameters at *hyperbolic times* are defined “up to bounded error”, and then prove that between two hyperbolic times the shear is uniformly bounded, regardless the number of hits to $\widehat{\Lambda}$ and Λ, see Section 6.3.

Another difficulty we encounter related to Step (4) above is the *coarse graining*, which consists on selecting a countable collection $\mathscr{A}$ of ε–double charts that are dense in the space of all ε–double charts and such that the pseudo-orbits they generate shadow all relevant orbits. This also relates to the definition of transition between ε–double charts, that has to be loose enough to code all relevant orbits and tight enough to impose that charts parameters are defined “up to bounded error”. To guarantee that the definition is loose enough, the countable collection we consider is much larger than those constructed in the recent literature. Yet, proving that this family is sufficient also requires an analysis at hyperbolic times, where parameters are essentially uniquely defined. We can then define parameters between successive hyperbolic times. See Section 5.

1.2. Preliminaries

1.2.1. Symbolic dynamics

Let $\mathscr{G} = (V, E)$ be an oriented graph, where V, E are the vertex and edge sets. We denote edges by $v \to w$, and assume that V is countable.

Topological Markov shift (TMS): It is a pair (Σ, σ) where

$$\Sigma := \{\mathbb{Z}\text{–indexed paths on } \mathscr{G}\} = \left\{\underline{v} = \{v_n\}_{n\in\mathbb{Z}} \in V^{\mathbb{Z}} : v_n \to v_{n+1}, \forall n \in \mathbb{Z}\right\}$$

is the symbolic space and $\sigma : \Sigma \to \Sigma$, $[\sigma(\underline{v})]_n = v_{n+1}$, is the *left shift*. We endow Σ with the distance $d(\underline{v}, \underline{w}) := \exp[-\inf\{|n| \in \mathbb{Z} : v_n \neq w_n\}]$. The *regular set* of Σ is

$$\Sigma^\# := \left\{\underline{v} \in \Sigma : \exists v, w \in V \text{ s.t. } \begin{array}{l} v_n = v \text{ for infinitely many } n > 0 \\ v_n = w \text{ for infinitely many } n < 0 \end{array}\right\}.$$

We only consider TMS that are *locally compact*, i.e. for all $v \in V$ the number of ingoing edges $u \to v$ and outgoing edges $v \to w$ is finite.

Given (Σ, σ) a TMS, let $r : \Sigma \to (0, +\infty)$ be a continuous function. For $n \geq 0$, let $r_n = r + r \circ \sigma + \cdots + r \circ \sigma^{n-1}$ be n–th *Birkhoff sum* of r, and extend this definition for $n < 0$ in the unique way such that the *cocycle identity* holds: $r_{m+n} = r_m + r_n \circ \sigma^m$, $\forall m, n \in \mathbb{Z}$.

Topological Markov flow (TMF): The TMF defined by (Σ, σ) and the *roof function* r is the pair (Σ_r, σ_r) where $\Sigma_r := \{(\underline{v}, t) : \underline{v} \in \Sigma, 0 \leq t < r(\underline{v})\}$ and $\sigma_r : \Sigma_r \to \Sigma_r$ is the flow on Σ_r given by $\sigma_r^t(\underline{v}, t') = (\sigma^n(\underline{v}), t' + t - r_n(\underline{v}))$, where n is the unique integer such that $r_n(\underline{v}) \leq t' + t < r_{n+1}(\underline{v})$. We endow Σ_r with a natural metric $d_r(\cdot, \cdot)$, called the *Bowen-Walters metric*, such that σ_r is a continuous flow [5,28]. The *regular set* of (Σ_r, σ_r) is $\Sigma_r^\# = \{(\underline{v}, t) \in \Sigma_r : \underline{v} \in \Sigma^\#\}$.

In other words, σ_r is the unit speed vertical flow on Σ_r with the identification $(\underline{v}, r(\underline{v})) \sim (\sigma(\underline{v}), 0)$. The roof functions we will consider will be Hölder continuous. In this case, there exist $\kappa, C > 0$ such that $d_r(\sigma_r^t(z), \sigma_r^t(z')) \leq C d_r(z, z')^\kappa$ for all $|t| \leq 1$ and $z, z' \in \Sigma_r$, see [28, Lemma 5.8].

Irreducible component: If Σ is a countable Markov shift defined by an oriented graph $\mathscr{G} = (V, E)$, its *irreducible components* are the subshifts $\Sigma' \subset \Sigma$ over maximal subsets $V' \subset V$ satisfying the following condition:

$$\forall v, w \in V', \ \exists \underline{v} \in \Sigma \text{ and } n \geq 1 \text{ such that } v_0 = v \text{ and } v_n = w.$$

An irreducible component Σ'_r of a suspended shift Σ_r is a set of elements $(\underline{v}, t) \in \Sigma_r$ with $\underline{v}$ in an irreducible component Σ' of Σ.

1.2.2. Notations

For $a, b, \varepsilon > 0$, we write $a = e^{\pm\varepsilon} b$ when $e^{-\varepsilon} \leq \frac{a}{b} \leq e^{\varepsilon}$. We also write $a \wedge b := \min(a, b)$. We write $\bigsqcup A_n$ to represent the *disjoint union* of sets A_n.

The *Frobenius norm* of a 2×2 matrix is $\left\| \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \right\|_{\text{Frob}} := \sqrt{a^2 + b^2 + c^2 + d^2}$. It is equivalent to the sup norm $\|\cdot\|$, since $\|\cdot\| \leq \|\cdot\|_{\text{Frob}} \leq \sqrt{2}\|\cdot\|$. The *co-norm* of an invertible matrix A is denoted by $m(A) = \|A^{-1}\|^{-1}$. We write $u \asymp v$ if $\lim u/v = 1$.

1.2.3. Metrics

If M is a smooth Riemannian manifold, we denote by d_M the distance induced by the Riemannian metric. The Riemannian metric induces a Riemannian metric $d_{\text{Sas}}(\cdot, \cdot)$ on TM, called the *Sasaki metric*, see e.g. [12, §2]. For nearby small vectors, the Sasaki metric is almost a product metric in the following sense. Given a geodesic γ in M joining y to x, let $P_\gamma : T_y M \to T_x M$ be the parallel transport along γ. If $v \in T_x M$, $w \in T_y M$ then $d_{\text{Sas}}(v, w) \asymp d(x, y) + \|v - P_\gamma w\|$ as $d_{\text{Sas}}(v, w) \to 0$, see e.g. [12, Appendix A]. The rate of convergence depends on the curvature tensor of the metric on M.

Given an open set $U \subset \mathbb{R}^n$ and $h : U \to \mathbb{R}^m$, let $\|h\|_{C^0} := \sup_{x\in U} \|h(x)\|$ denote the C^0 norm of h. For $0 < \beta \le 1$, let $\mathrm{H\ddot{o}l}_\beta(h) := \sup \frac{\|h(x)-h(y)\|}{\|x-y\|^\beta}$ where the supremum ranges over distinct elements $x, y \in U$. Note that $\mathrm{H\ddot{o}l}_1(h)$ is a Lipschitz constant of h, that we will also denote by $\mathrm{Lip}(h)$. If h is differentiable, let $\|h\|_{C^1} := \|h\|_{C^0} + \|dh\|_{C^0}$ denote its C^1 norm, and $\|h\|_{C^{1+\beta}} := \|h\|_{C^1} + \mathrm{H\ddot{o}l}_\beta(dh)$ its $C^{1+\beta}$ norm.

For any x, y close to some point z in a Riemannian manifold M, the parallel transport along the shortest geodesic between x and y induces a linear map $P_{x,y} : T_xM \to T_yM$. To any linear map $A\colon T_xM \to T_yM$, one associates a map $\widetilde{A} := P_{y,z} \circ A \circ P_{z,x}$. By definition, $\widetilde{A}$ depends on z but different basepoints z define a map that differs from $\widetilde{A}$ by pre and post composition with isometries. In particular, $\|\widetilde{A}\|$ does not depend on the choice of z. With this notation, a map $f : U \subset M \to M$ is $C^{1+\beta}$ if it is C^1 and $\exists C > 0$ such that $\|\widetilde{df_x} - \widetilde{df_y}\| \le Cd(x, y)^\beta$ for all nearby $x, y \in U$. In this case, define $\mathrm{H\ddot{o}l}_\beta(df) := \sup \frac{\|\widetilde{df_x}-\widetilde{df_y}\|}{d(x,y)^\beta}$ where the supremum ranges over distinct nearby elements $x, y \in U$.

1.3. Standing assumptions

Let M be a three dimensional closed smooth Riemannian manifold, and let $X : M \to TM$ be a $C^{1+\beta}$ vector field such that $X(x) \ne 0$, $\forall x \in M$, and let $\varphi = (\varphi^t)_{t\in\mathbb{R}}$ be the flow generated by X. We will denote the value of the vector field X at x by either X_x or $X(x)$. Given a set $Y \subset M$ and an interval $I \subset \mathbb{R}$, write $\varphi^I(Y) := \bigcup_{t\in I} \varphi^t(Y)$.

Since obtaining a coding for the flow generated by X is equivalent to obtaining a coding for the flow generated by cX for some $c > 0$, we assume from now on that $\|\nabla X\|_0 \le 1$ (just change X to cX for $c > 0$ small enough).[1] This assumption avoids the introduction of some multiplicative constants. For instance, since an application of the Grönwall inequality implies that $\|d\varphi^t\| \le e^{\|\nabla X\|_0|t|}$ for all $t \in \mathbb{R}$ (see e.g. [25]), we will simply write that $\|d\varphi^t\| \le e^{|t|}$, $\forall t \in \mathbb{R}$. Another consequence is that every Lyapunov exponent of φ has absolute value at most 1, hence we can take $\chi \in (0, 1)$ in the definition of χ-hyperbolicity.

2. Poincaré sections

In this section, we:

(1) Construct two sections $\Lambda, \widehat{\Lambda}$ with good geometrical properties such that $\Lambda \subset \widehat{\Lambda}$, $d(\Lambda, \partial\widehat{\Lambda}) > 0$, and the orbit under φ of every point of M hits Λ after some time $\rho \ll 1$. The section Λ induces a Poincaré return map f and a return time r. We call Λ the *reference section* and $\widehat{\Lambda}$ the *security section*.

[1] The notation ∇X represents the covariant differential, i.e. for each $x \in M$ we have a linear map $\nabla X(x) : T_xM \to T_xM$ defined by $[\nabla X(x)](Y) = \nabla_Y X$.

(2) Introduce the *induced linear Poincaré flow* Φ, which is a flow that describes the local behavior of φ in the complementary direction to X.
(3) Introduce the *holonomy* maps g_x^+, g_x^- for each $x \in \Lambda$, which are local and continuous versions of Poincaré return maps. It is for these maps that we will construct suitable systems of coordinates in Section 3.

2.1. Transverse discs and flow boxes

Let $\rho > 0$.

ρ–TRANSVERSE DISC: An open disc $D \subset M$ is *ρ–transverse* if:

- D is compactly contained in a C^∞ disc of M.
- $\mathrm{diam}(D) < 4\rho$.
- For every $x \in D$, $\angle(X(x), T_x D^\perp) < \rho$.

In other words, a ρ–transverse disc is a small disc that is almost orthogonal to X. It is easy to build ρ–transverse discs. For instance, we know by the tubular neighborhood theorem that φ can be conjugated in local charts to the flow $(x, t_0) \in \mathbb{R}^2 \times \mathbb{R} \mapsto (x, t_0 + t)$. If ρ' is small enough, then the image of $B(0, \rho') \times \{t_0\}$ under the local chart is a ρ–transverse disc.

FLOW BOX: Every ρ–transverse disc D defines a *flow box* $\varphi^{[-4\rho,4\rho]}D$.

The assumption that X does not vanish implies that for all $\rho > 0$ small enough, the map $\Gamma_D\colon (y,t) \in D \times [-4\rho, 4\rho] \mapsto \varphi^t(y)$ is a diffeomorphism onto the flow box $\varphi^{[-4\rho,4\rho]}D$. We denote its inverse by $x \in \varphi^{[-4\rho,4\rho]}D \mapsto (\mathfrak{q}_D(x), \mathfrak{t}_D(x))$, where $\mathfrak{q}_D : \varphi^{[-4\rho,4\rho]}D \to D$ and $\mathfrak{t}_D : \varphi^{[-4\rho,4\rho]}D \to [-4\rho, 4\rho]$.

Lemma 2.1. *There is a $\rho_0 = \rho_0(M, X) > 0$ such that for every ρ_0–transverse discs D, D':*

(1) *The maps $\mathfrak{q}_D, \mathfrak{t}_D$ are $C^{1+\beta}$.*
(2) *The map $\mathfrak{q}_D$ has a Lipchitz constant smaller than 2.*
(3) *If D' intersects the flow box $\varphi^{[-4\rho_0,4\rho_0]}D$, then the restriction to D' of the map $\mathfrak{t}_D$ has a Lipschitz constant smaller than 1.*

Proof. By the implicit function theorem, for any ρ–transverse disc D the chart $\Gamma_D\colon (x,t) \in D \times [-4\rho, 4\rho] \mapsto \varphi^t(x)$ is $C^{1+\beta}$, hence the inverse maps $\mathfrak{q}_D, \mathfrak{t}_D$ are also $C^{1+\beta}$. This proves part (1).

Let us consider the foliation of $\mathbb{R}^3$ whose leaves are the verticals $\{(x,y)\} \times \mathbb{R}$, and let Δ be a section whose tangent spaces $T_x\Delta$ define angles with the horizontal planes smaller than $\gamma > 0$. Then the holonomy along the vertical lines define a projection to Δ which is $(1/\cos\gamma)$–Lipschitz.

Given $e > 0$ arbitrarily small, there exists a covering of M by finitely many charts $\Theta\colon (-a,a)^3 \to M$ which are $(1+e)$-biLipschitz and such that the lifted vector field $\widehat{X} := \Theta^* X$ is tangent to the vertical lines. Choosing e and then $\rho_0 > 0$ small enough, for any ρ_0–transverse disc D, the set $\varphi_{[-4\rho,4\rho]}(D)$ is contained in the image of a chart Θ; moreover $\Theta^{-1}(\Delta)$ is a disc whose tangent spaces define angles with the horizontal planes smaller than $\gamma > 0$. The projection $\mathfrak{q}_D$ to D is conjugated by Θ to the projection by holonomy along vertical lines to the set $\Theta^{-1}(\Delta)$. Consequently, the map $\mathfrak{q}_D$ is $(1+e)^2/\cos\gamma$–Lipschitz, which can be chosen arbitrarily close to 1 if e, γ, and hence ρ_0, are small enough. This proves part (2).

Now consider two ρ_0–transverse discs D, D' such that D' intersects $\varphi^{[-4\rho_0,4\rho_0]}D$. Since ρ_0 is chosen small, both D, D' are contained in the image of a same chart Θ. As before, let $\widehat{X} := \Theta^* X$ be the vector field X lifted in the chart. The discs $\Theta^{-1}(D'), \Theta^{-1}(D)$ are graphs $\{(x, y, \varphi_i(x, y))\}$ over the horizontal hyperplane of $C^{1+\beta}$ maps φ_1 and φ_2 respectively. For $z = \Theta(x, y, \varphi_1(x, y)) \in D'$ which intersects the flow box $\varphi^{[-4\rho_0,4\rho_0]}D$, the projection time to D can be computed in the chart as:

$$\mathfrak{t}_D(z) = \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{1}{\|\widehat{X}(x,y,t)\|} dt.$$

The C^1–norm of $\frac{1}{\|\widehat{X}\|}$ is bounded, independently from the charts Θ. Taking ρ_0 small, the derivatives $D\varphi_i$ and the differences $\varphi_2(x, y) - \varphi_1(x, y)$ are close to 0. Hence the derivative of the map $z \mapsto \mathfrak{t}_D(z)$ for $z \in D'$ is close to 0, proving part (3). □

2.2. Proper sections and Poincaré return maps

We begin with some definitions.

Proper section: A *proper section of size* ρ is a finite union $\Lambda = \bigcup_{i=1}^{\mathfrak{n}} D_i$ of ρ–transverse discs $D_1, \dots, D_{\mathfrak{n}}$ such that:

(1) Cover: $M = \bigcup_{i=1}^{\mathfrak{n}} \varphi^{[0,\rho)} D_i$.
(2) Partial order: For all $i \neq j$, at least one of the sets $\overline{D_i} \cap \varphi^{[0,4\rho]} \overline{D_j}$ or $\overline{D_j} \cap \varphi^{[0,4\rho]} \overline{D_i}$ is empty; in particular $\overline{D_i} \cap \overline{D_j} = \emptyset$.

Define the *return time function* $r_\Lambda : \Lambda \to (0, \rho)$ by $r_\Lambda(x) := \inf\{t > 0 : \varphi^t(x) \in \Lambda\}$.

Poincaré return map: The *Poincaré return map* of a proper section Λ is the map $f_\Lambda : \Lambda \to \Lambda$ defined by $f_\Lambda(x) := \varphi^{r_\Lambda(x)}(x)$.

In the following, we fix $\rho < \min\{0.25, \rho_0\}$ small and consider two proper sections $\Lambda, \widehat{\Lambda}$ of size $\rho/2$ such that $\Lambda \subset \widehat{\Lambda}$ and $d_M(\Lambda, \partial\widehat{\Lambda}) > 0$. We let $d = d_{\widehat{\Lambda}}$ be the metric on $\widehat{\Lambda}$ defined by the induced Riemannian metric on $\widehat{\Lambda}$. For $x \in \widehat{\Lambda}$ and $r > 0$, we write:

- $B(x,r) \subset \widehat{\Lambda}$ for the ball in the distance d with center x and radius r;
- $B_x[r] \subset T_x\widehat{\Lambda}$ for the ball with center 0 and radius r;
- $R[r] := [-r,r]^2 \subset \mathbb{R}^2$.

Since the associated flow boxes are $C^{1+\beta}$, there exists $L > 0$ such that for any transverse disc D_i defining the section Λ, the maps $\mathfrak{q}_{D_i}, \mathfrak{t}_{D_i}$ satisfy:

$$\text{Höl}_\beta(d\mathfrak{q}_{D_i}) < L \text{ and } \text{Höl}_\beta(d\mathfrak{t}_{D_i}) < L.$$

2.3. Exponential maps

Given $x \in \widehat{\Lambda}$, let $\text{inj}(x)$ denote the injectivity radius of $\widehat{\Lambda}$ at x, and let $\exp_x$ be the *exponential map* of $\widehat{\Lambda}$ at x, wherever it can be defined. Below we list the properties of $\exp_x$ that we will use.

Regularity of $\exp_x$: There is $\mathfrak{r} \in (0,\rho)$ such that for every $x \in \Lambda$ the following properties hold on the ball $B_x := B(x, 2\mathfrak{r}) \subset \widehat{\Lambda}$:

(Exp1) If $y \in B_x$ then $\text{inj}(y) \geq 2\mathfrak{r}$, the map $\exp_y^{-1} : B_x \to T_y\widehat{\Lambda}$ is a diffeomorphism onto its image, and for all $v \in T_x\widehat{\Lambda}, w \in T_y\widehat{\Lambda}$ with $\|v\|, \|w\| \leq 2\mathfrak{r}$ it holds

$$\tfrac{1}{2}(d(x,y) + \|v - P_{y,x}w\|) \leq d_{\text{Sas}}(v,w) \leq 2(d(x,y) + \|v - P_{y,x}w\|),$$

where $P_{y,x}$ is the parallel transport along the geodesic joining y to x.

(Exp2) If $y_1, y_2 \in B_x$ then $d(\exp_{y_1} v_1, \exp_{y_2} v_2) \leq 2d_{\text{Sas}}(v_1, v_2)$ for $\|v_1\|$, $\|v_2\| \leq 2\mathfrak{r}$, and $d_{\text{Sas}}(\exp_{y_1}^{-1} z_1, \exp_{y_2}^{-1} z_2) \leq 2[d(y_1,y_2) + d(z_1,z_2)]$ for $z_1, z_2 \in B_x$ whenever the expression makes sense. In particular, $\|d(\exp_x)_v\| \leq 2$ for $\|v\| \leq 2\mathfrak{r}$ and $\|d(\exp_x^{-1})_y\| \leq 2$ for $y \in B_x$.

Conditions (Exp1)–(Exp2) say that the exponential maps and their inverses are well-defined and have Lipschitz constants bounded by 2 in balls of radius $2\mathfrak{r}$. The existence of $\mathfrak{r}$ follows from compactness, since $d_M(\Lambda, \partial\widehat{\Lambda}) > 0$ and $d(\exp_x)_0$ is the identity map.

The next two assumptions describe the regularity of $d\exp_x$. For $x, x' \in \widehat{\Lambda}$, let $\mathscr{L}_{x,x'} := \{A : T_x\widehat{\Lambda} \to T_{x'}\widehat{\Lambda} : A \text{ is linear}\}$ and $\mathscr{L}_x := \mathscr{L}_{x,x}$. In particular, $P_{y,x}$ considered in (Exp1) is in $\mathscr{L}_{y,x}$. Given $y \in B_x, z \in B_{x'}$ and $A \in \mathscr{L}_{y,z}$, let $\widetilde{A} \in \mathscr{L}_{x,x'}$, $\widetilde{A} := P_{z,x'} \circ A \circ P_{x,y}$. The norm $\|\widetilde{A}\|$ does not depend on the choice of x, x'. If $A_i \in \mathscr{L}_{y_i,z_i}$ then $\|\widetilde{A_1} - \widetilde{A_2}\|$ does depend on the choice of x, x', but if we change the basepoints x, x' to w, w' then the respective differences differ by precompositions and postocompositions with norm of the order of the areas of the geodesic triangles formed by x, w, y_i and by x', w', z_i, which will be negligible to our estimates. For $x \in \Lambda$, define the map $\tau = \tau_x : B_x \times B_x \to \mathscr{L}_x$ by $\tau(y,z) = d\widetilde{(\exp_y^{-1})}_z$, where we use the identification $T_v(T_y\widehat{\Lambda}) \cong T_y\widehat{\Lambda}$ for all $v \in T_y\widehat{\Lambda}$.

Regularity of $d\exp_x$: There is $\mathfrak{K} > 1$ such that for all $x \in \Lambda$ the following holds:

(Exp3) If $y_1, y_2 \in B_x$ then $\|d(\widetilde{\exp_{y_1}})_{v_1} - d(\widetilde{\exp_{y_2}})_{v_2}\| \leq \mathfrak{K} d_{\rm Sas}(v_1, v_2)$, for all $\|v_1\|, \|v_2\| \leq 2\mathfrak{r}$, and $\|\tau(y_1, z_1) - \tau(y_2, z_2)\| \leq \mathfrak{K}[d(y_1, y_2) + d(z_1, z_2)]$ for all $z_1, z_2 \in B_x$.

(Exp4) If $y_1, y_2 \in B_x$ then the map $\tau(y_1, \cdot) - \tau(y_2, \cdot) : B_x \to \mathscr{L}_x$ has Lipschitz constant $\leq \mathfrak{K} d(y_1, y_2)$.

Condition (Exp3) controls the Lipschitz constants of the derivatives of $\exp_x$, and (Exp4) controls the Lipschitz constants of their second derivatives. The existence of $\mathfrak{K}$ is guaranteed whenever the curvature tensor of $\widehat{\Lambda}$ is uniformly bounded, and this happens because $\widehat{\Lambda}$ is the restriction to a compact subset of a finite union of ρ–transverse (open) discs.

2.4. Induced linear Poincaré flows

Classically, the linear Poincaré flow is the $\mathbb{R}$–cocycle induced by $d\varphi$ in the bundle orthogonal to X. In this paper we employ a different definition: we fix a 1–form θ and consider parallel projections to X onto the bundle $\mathrm{Ker}(\theta)$. We begin choosing a suitable 1–form.

Lemma 2.2. *If $\widehat{\Lambda}$ is a proper section of size $\rho/2$, there exists a 1–form θ on M such that:*

(1) $\theta(X(x)) = 1$ *and* $\angle(X(x), \mathrm{Ker}(\theta_x)^\perp) < \rho$, $\forall x \in M$.
(2) $\mathrm{Ker}(\theta_x) = T_x\widehat{\Lambda}$, $\forall x \in \widehat{\Lambda}$.

Proof. Take $\eta(v) = \frac{\langle v, X(x)\rangle}{\|X(x)\|^2}$ for $v \in T_xM$. Clearly η is a 1–form on M satisfying (1) above. Let U be a small neighborhood of $\widehat{\Lambda}$. By the tubular neighborhood theorem, there exists a 1–form ζ on U such that $\zeta(X(x)) = 1$ and $\angle(X(x), \mathrm{Ker}(\zeta_x)^\perp) < \rho$ for all $x \in U$, and $\mathrm{Ker}(\zeta_x) = T_x\widehat{\Lambda}$ for all $x \in \widehat{\Lambda}$. Let V be a neighborhood of $\widehat{\Lambda}$ with $\widehat{\Lambda} \subset V \subset U$, and take a bump function $h : M \to [0, 1]$ such that $h\restriction_V \equiv 0$ and $h\restriction_{M\setminus U} \equiv 1$. The 1–form $\theta := h\eta + (1 - h)\zeta$ satisfies the following:

- $\theta(X(x)) = 1$, $\forall x \in M$: clear, since $\eta(X(x)) = \zeta(X(x)) = 1$.
- $\angle(X(x), \mathrm{Ker}(\theta_x)^\perp) < \rho$, $\forall x \in M$: to see this, write $\eta_x(\cdot) = \langle \cdot, v_x\rangle$ and $\zeta_x(\cdot) = \langle \cdot, w_x\rangle$, where $v_x = \frac{X(x)}{\|X(x)\|}$ and $\angle(X(x), w_x) < \rho$. Since $\mathrm{Ker}(\theta_x)^\perp$ is generated by the linear combination $h(x)v_x + (1 - h(x))w_x$, we have $\angle(X(x), \mathrm{Ker}(\theta_x)^\perp) \leq \angle(X(x), w_x) < \rho$.
- $\mathrm{Ker}(\theta_x) = T_x\widehat{\Lambda}$, $\forall x \in \widehat{\Lambda}$: since $h(x) = 0$, we have $\mathrm{Ker}(\theta_x) = \mathrm{Ker}(\zeta_x) = T_x\widehat{\Lambda}$.

The proof is complete. □

From now on, we fix a 1–form θ satisfying Lemma 2.2. Introduce the two dimensional bundle

$$N := \bigsqcup_{x\in M} \mathrm{Ker}(\theta_x).$$

For each $x \in M$, let $\mathfrak{p}_x : T_xM \to N_x$ be the projection to N_x parallel to $X(x)$. By Lemma 2.2(1), for all $x \in M$ we have:

$$\|\mathfrak{p}_x\| = \tfrac{1}{\cos\angle(X(x),\mathrm{Ker}(\theta_x)^{\perp})} < \tfrac{1}{\cos\rho} < 1+\rho.$$

Induced linear Poincaré flow: The *linear Poincaré flow of φ induced by θ* is the flow $\Phi = \{\Phi^t\}_{t\in\mathbb{R}} : N \to N$ defined by $\Phi^t(v) = \mathfrak{p}_{\varphi^t(x)}[d\varphi^t_x(v)]$ for $v \in N_x$.

When the context is clear, we will omit the subscripts x and $\varphi^t(x)$. Clearly Φ is Hölder continuous, and $\|\Phi^t_x\| \le \|\mathfrak{p}_{\varphi^t(x)}\|\|d\varphi^t_x\| \le (1+\rho)e^{|t|} < e^{\rho+|t|}$, $\forall t \in \mathbb{R}$. In particular:

$$\|\Phi^t\| = e^{\pm 4\rho},\ \forall |t| \le 2\rho. \tag{2.1}$$

Lemma 2.3. *The following hold.*

(1) *Φ is a flow: $\Phi^{t+t'} = \Phi^t \circ \Phi^{t'}$, $\forall t, t' \in \mathbb{R}$.*
(2) *If $D \subset \widehat{\Lambda}$ is a transverse disc, then for all $x \in D$ it holds $d(\mathfrak{q}_D)_x = \mathfrak{p}_x$.*

Proof. (1) If $v \in N_x$ and $t, t' \in \mathbb{R}$, then there is $\gamma \in \mathbb{R}$ such that

$$\begin{aligned}
&\Phi^{t'}(\Phi^t(v)) = \Phi^{t'}(\mathfrak{p}_{\varphi^t(x)}[d\varphi^t_x(v)]) = \Phi^{t'}(d\varphi^t_x(v) + \gamma X(\varphi^t(x)))\\
&= \mathfrak{p}_{\varphi^{t'+t}(x)}[d\varphi^{t'}_{\varphi^t(x)}(d\varphi^t_x(v) + \gamma X(\varphi^t(x)))] = \mathfrak{p}_{\varphi^{t'+t}(x)}[d\varphi^{t'+t}_x(v) + \gamma X(\varphi^{t'+t}(x))]\\
&= \mathfrak{p}_{\varphi^{t'+t}(x)}[d\varphi^{t'+t}_x(v)] = \Phi^{t'+t}(v).
\end{aligned}$$

(2) Fix $x \in D \subset \widehat{\Lambda}$. It is enough to show that $d(\mathfrak{q}_D)_x[X(x)] = 0$ and $d(\mathfrak{q}_D)_x[v] = v$ for all $v \in N_x$. We have $d(\mathfrak{q}_D)_x[X(x)] = \frac{d}{dt}\big|_{t=0}[\mathfrak{q}_D(\varphi^t(x))] = 0$ because $\mathfrak{q}_D(\varphi^t(x)) = x$ for small t. Now, since $N_x = T_x\widehat{\Lambda}$ and $\mathfrak{q}_D \restriction_{\widehat{\Lambda}}$ is the identity, $d(\mathfrak{q}_D)_x[v] = v$ for all $v \in N_x$. $\square$

2.5. Holonomy maps

We have fixed $\Lambda, \widehat{\Lambda}$, two proper sections of size $\rho/2$. From now on, write $f := f_\Lambda$. The maps f, r_Λ admit discontinuities, hence we introduce a related family of local diffeomorphisms. Recall that $\mathfrak{r} > 0$ is a fixed small parameter, and that $B_x := B(x, 2\mathfrak{r})$. Write $\widehat{\Lambda} = \bigcup_{i=1}^{\mathfrak{n}} D_i$ as the disjoint union of ρ–transverse discs D_i, and let $\mathfrak{q}_{D_i}$ as before. By Lemma 2.1, $\mathrm{Lip}(\mathfrak{q}_{D_i}) < 2$.

Assume that $x, \varphi^t(x) \in \Lambda$ for some $|t| \le \rho$, with $x \in D_i$ and $\varphi^t(x) \in D_j$. In this case, the restrictions $\mathfrak{q}_{D_j} \restriction_{B_x}$ and $\mathfrak{q}_{D_i} \restriction_{B_{\varphi^t(x)}}$ are diffeomorphisms onto their images, and one is the inverse of the other when the compositions makes sense. When this happens, we call these restrictions *holonomy maps*.

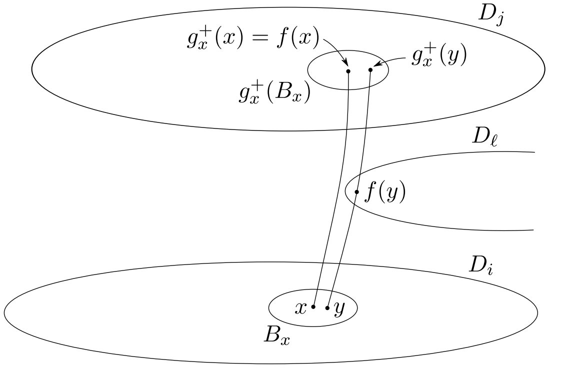

Fig. 1. The holonomy map g_x^+: it may differ from f and $f_{\widehat{\Lambda}}$.

Lemma 2.4. *Under the above conditions, the holonomy map* $\mathfrak{q}_{D_j}\restriction_{B_x}$ *is a* 2*–bi-Lipschitz* $C^{1+\beta}$ *diffeomorphism onto its image, and its derivative at* x *equals* $\Phi^t\restriction_{N_x}$.

Proof. Write $g = \mathfrak{q}_{D_j}\restriction_{B_x}$. The first statement follows from Lemma 2.1. Now, since $g = \mathfrak{q}_{D_j}\circ\varphi^t$, Lemma 2.3(2) implies $dg_x = d(\mathfrak{q}_{D_j})_{\varphi^t(x)}\circ d\varphi^t_x\restriction_{T_x\widehat{\Lambda}} = \mathfrak{p}_{\varphi^t(x)}\circ d\varphi^t_x\restriction_{N_x} = \Phi^t\restriction_{N_x}$. □

In the sequel we will investigate some particular holonomy maps, defined as follows. Let $0 < t, t' < \rho$ such that $f(x) = \varphi^t(x)\in D_j$ and $f^{-1}(x) = \varphi^{-t'}(x)\in D_k$.

Holonomy maps: The *forward holonomy map* at x is $g_x^+ := \mathfrak{q}_{D_j}\restriction_{B_x}$. Similarly, the *backward holonomy map* at x is $g_x^- := \mathfrak{q}_{D_k}\restriction_{B_x}$.

Note that g_x^+ differs from f and from the Poincaré return to $\widehat{\Lambda}$, see Fig. 1. Also, $(g_x^+)^{-1} = g^-_{f(x)}$.

3. The non-uniformly hyperbolic locus

Up to now, we have fixed $\varphi, \chi, \rho, \Lambda, \widehat{\Lambda}$ and θ. In this section, we:

(1) Define the set NUH of points that exhibit a hyperbolicity of strength at least χ. We fix $\varepsilon > 0$ small enough, and associate to each $x \in$ NUH a number $Q(x)\in(0,1)$ that approaches zero as the quality of the hyperbolicity at x deteriorates.
(2) Introduce numbers $q(x)\in[0,1)$, that measure how fast $Q(\varphi^t(x))$ decreases to zero as $|t|\to\infty$. We also associate analogous number $q^s(x)$ and $q^u(x)$ for future and past orbits.
(3) Define the set NUH$^\#$ of points $x\in$ NUH whose hyperbolicity satisfies a recurrence property: there is $c(x) > 0$ such that $q(\varphi^t(x)) > c(x)$ for some values of t arbitrarily close to $\pm\infty$. This set carries all χ–hyperbolic measures.
(4) Define Pesin charts Ψ_x for each $x\in\Lambda\cap$ NUH. We then prove that, in Pesin charts, the holonomy maps $g_x^\pm$ are close to hyperbolic linear maps.

3.1. *The non-uniformly hyperbolic locus* NUH

NON-UNIFORMLY HYPERBOLIC LOCUS $\mathrm{NUH} = \mathrm{NUH}(\varphi, \chi, \rho, \theta)$: It is the invariant set of points $x \in M$ for which there are unitary vectors n^s_x, n^u_x with the following properties:

(NUH1) s–DIRECTION: $\liminf_{t\to+\infty} \frac{1}{t}\log\|\Phi^{-t} n^s_x\| > 0$, $\limsup_{t\to+\infty} \frac{1}{t}\log\|\Phi^{t} n^s_x\| \le -\chi$, and

$$s(x) := 2e^{2\rho}\left(\int_0^{+\infty} e^{2\chi t}\|\Phi^t n^s_x\|^2 dt\right)^{1/2} < +\infty.$$

(NUH2) u–DIRECTION: $\liminf_{t\to+\infty} \frac{1}{t}\log\|\Phi^{t} n^u_x\| > 0$, $\limsup_{t\to+\infty} \frac{1}{t}\log\|\Phi^{-t} n^u_x\| \le -\chi$, and

$$u(x) := 2e^{2\rho}\left(\int_0^{+\infty} e^{2\chi t}\|\Phi^{-t} n^u_x\|^2 dt\right)^{1/2} < +\infty.$$

It is clear that n^s_x, n^u_x are unique up to a choice of signs. We choose the sign so that their angle is less than or equal to $\pi/2$, and make the following definition.

ANGLE: $\alpha(x) := \angle(n^s_x, n^u_x)$.

Let us remind that $\chi \in (0,1)$, see Section 1.3. From the estimate before (2.1), we have

$$\int_0^{+\infty} e^{2\chi t}\|\Phi^t n^s_x\|^2 dt \ge \int_0^{+\infty} e^{2\chi t} e^{-4\rho-2t} dt = \tfrac{e^{-4\rho}}{2(1-\chi)} > \tfrac{e^{-4\rho}}{2},$$

therefore for each $x \in \mathrm{NUH}$ we have $s(x), u(x) \in [\sqrt{2}, +\infty)$ and $\alpha(x) \neq 0$.

Conditions (NUH1)–(NUH2) are weaker than Lyapunov regularity, hence NUH contains all Lyapunov regular points with exponents greater than χ in absolute value. Moreover, a periodic point x is in NUH iff all of its exponents are greater than χ in absolute value. But NUH might contain points with Lyapunov exponents equal to $\pm\chi$, and even points which are not Lyapunov regular, where the contraction rates oscillate infinitely often.

Proposition 3.1. *If μ is a χ–hyperbolic measure, then $\mu[\mathrm{NUH}] = 1$.*

Proof. Fix a χ–hyperbolic measure μ. By the Oseledets theorem, there is a set $X \subset M$ with $\mu[X] = 1$ such that for all $x \in X$ there are unitary vectors $e^s_x, e^u_x \in T_xM$ satisfying:

(1) $\lim_{t\to\pm\infty} \frac{1}{t}\log\|d\varphi^t e^s_x\| < -\chi$ and $\lim_{t\to\pm\infty} \frac{1}{t}\log\|d\varphi^t e^u_x\| > \chi$.

(2) $\lim_{t\to\pm\infty} \frac{1}{t}\log|\sin\angle(e^s_{\varphi^t(x)}, e^u_{\varphi^t(x)})| = \lim_{t\to\pm\infty} \frac{1}{t}\log|\sin\angle(X_{\varphi^t(x)}, e^{s/u}_{\varphi^t(x)})| = 0.$

If $x \in X$ then $e_x^{s/u} \notin \operatorname{span}(X_x)$, hence there are scalars $\gamma^{s/u}(x)$, $\delta^{s/u}(x)$ such that

$$n_x^s = \gamma^s(x)e_x^s + \delta^s(x)X_x \ \text{ and } \ n_x^u = \gamma^u(x)e_x^u + \delta^u(x)X_x \tag{3.1}$$

are unitary vectors in N_x. If $X(x) \perp N_x$ then $\gamma^{s/u}(x) = \pm\frac{1}{\sin\angle(X_x, e_x^{s/u})}$. Since by construction we have $\angle(N_x, X(x)^\perp) < \rho$ (see Lemma 2.2), we conclude that $\gamma^{s/u}(x) = \pm\frac{e^{\pm 4\rho}}{\sin\angle(X_x, e_x^{s/u})}$ and so condition (2) translates to

$$\lim_{t\to\pm\infty} \tfrac{1}{t}\log|\gamma^s(\varphi^t(x))| = \lim_{t\to\pm\infty} \tfrac{1}{t}\log|\gamma^u(\varphi^t(x))| = 0.$$

We claim that $X \subset \text{NUH}$, and we prove this showing that

$$\lim_{t\to\pm\infty} \tfrac{1}{t}\log\|\Phi^t n_x^s\| < -\chi \text{ and } \lim_{t\to\pm\infty} \tfrac{1}{t}\log\|\Phi^{-t} n_x^u\| < -\chi.$$

We show the first estimate (the second is analogous). For that, we claim that $\|\Phi^t n_x^s\| = \frac{|\gamma^s(x)|}{|\gamma^s(\varphi^t(x))|}\|d\varphi^t e_x^s\|$ for all $x \in X$. By (3.1),

$$d\varphi^t n_x^s = d\varphi^t[\gamma^s(x)e_x^s + \delta^s(x)X_x] = \gamma^s(x)\|d\varphi^t e_x^s\| e^s_{\varphi^t(x)} + \delta^s(x)X_{\varphi^t(x)},$$

hence

$$\begin{aligned}\Phi^t n_x^s &= \mathfrak{p}_{\varphi^t(x)}[\gamma^s(x)\|d\varphi^t e_x^s\| e^s_{\varphi^t(x)} + \delta^s(x)X_{\varphi^t(x)}] = \gamma^s(x)\|d\varphi^t e_x^s\|\mathfrak{p}_{\varphi^t(x)}[e^s_{\varphi^t(x)}]\\ &= \gamma^s(x)\|d\varphi^t e_x^s\|\mathfrak{p}_{\varphi^t(x)}\left[\tfrac{1}{\gamma^s(\varphi^t(x))}n^s_{\varphi^t(x)} - \tfrac{\delta^s(\varphi^t(x))}{\gamma^s(\varphi^t(x))}X_{\varphi^t(x)}\right] = \tfrac{\gamma^s(x)}{\gamma^s(\varphi^t(x))}\|d\varphi^t e_x^s\| n^s_{\varphi^t(x)}.\end{aligned}$$

Taking norms, we get that $\|\Phi^t n_x^s\| = \frac{|\gamma^s(x)|}{|\gamma^s(\varphi^t(x))|}\|d\varphi^t e_x^s\|$. Hence

$$\begin{aligned}&\lim_{t\to\pm\infty} \tfrac{1}{t}\log\|\Phi^t n_x^s\| = -\lim_{t\to\pm\infty} \tfrac{1}{t}\log|\gamma^s(\varphi^t(x))| + \lim_{t\to\pm\infty} \tfrac{1}{t}\log\|d\varphi^t e_x^s\|\\ &= \lim_{t\to\pm\infty} \tfrac{1}{t}\log\|d\varphi^t e_x^s\| < -\chi. \quad \square\end{aligned}$$

3.2. Oseledets-Pesin reduction

Let $e_1 = (1,0), e_2 = (0,1)$ be the canonical basis of $\mathbb{R}^2$. We define a change of coordinates that diagonalizes the induced linear Poincaré flow.

Linear map $C(x)$: For $x \in \text{NUH}$, let $C(x) : \mathbb{R}^2 \to N_x$ be the linear map defined by

$$C(x) : e_1 \mapsto \frac{n_x^s}{s(x)}\ ,\ C(x) : e_2 \mapsto \frac{n_x^u}{u(x)}.$$

Lemma 3.2. *The following holds for all $x \in \text{NUH}$.*

(1) $\|C(x)\| \leq \|C(x)\|_{\text{Frob}} \leq 1$ *and* $\|C(x)^{-1}\|_{\text{Frob}} = \frac{\sqrt{s(x)^2+u(x)^2}}{|\sin\alpha(x)|}$.
(2) $C(\varphi^t(x))^{-1} \circ \Phi^t \circ C(x)$ *is a diagonal matrix with diagonal entries* $A_t(x), B_t(x)$ *satisfying:*

$$e^{-4\rho} < |A_t(x)| < e^{-\chi t} \text{ and } e^{\chi t} < |B_t(x)| < e^{4\rho},\ \forall\, 0 < t \leq 2\rho.$$

(3) *For all* $|t| \leq 2\rho$*:*

$$\frac{s(\varphi^t(x))}{s(x)} = e^{\pm 10\rho},\ \frac{u(\varphi^t(x))}{u(x)} = e^{\pm 10\rho},\ \frac{|\sin\alpha(\varphi^t(x))|}{|\sin\alpha(x)|} = e^{\pm 8\rho}.$$

In particular,

$$\frac{\|C(\varphi^t(x))^{-1}\|_{\text{Frob}}}{\|C(x)^{-1}\|_{\text{Frob}}} = e^{\pm 18\rho}.$$

The proof is in Appendix A. Part (2) is known as Oseledets-Pesin reduction, and represents the diagonalization of Φ.

3.3. Quantification of hyperbolicity: the parameters $\mathbf{Q(x), q(x), q^{s/u}(x)}$

We now introduce another small parameter $\varepsilon \in (0, \mathfrak{r})$ such that $\varepsilon \ll \rho \ll 1$ (each symbol $\ll$ is defined by means of a finite number of inequalities that need to be satisfied throughout the paper). Instead of working with $\|C(x)^{-1}\|_{\text{Frob}}$, it is more convenient to introduce:

THE PARAMETER $Q(x)$: For $x \in \text{NUH}$, let $Q(x) := \varepsilon^{3/\beta}\|C(x)^{-1}\|_{\text{Frob}}^{-12/\beta}$.

The choice of the powers $3/\beta$ and $12/\beta$ is not canonical but just an artifact of the proof, and any choice of powers larger than these values also makes the proof work. The hyperbolicity degenerates as Q goes to 0. Lemma 3.2 immediately implies that

$$\tfrac{Q(\varphi^t(x))}{Q(x)} = e^{\pm\frac{250\rho}{\beta}},\quad \forall x \in \text{NUH}, \forall\, 0 < t \leq 2\rho, \tag{3.2}$$

and the following result.

Proposition 3.3. *An invariant set* $K \subset \text{NUH}$ *is uniformly hyperbolic if and only if* $\inf_{x\in K} Q(x) > 0$.

It will be important to identify the orbits in NUH whose hyperbolicity satisfies some recurrence (to ensure e.g. the existence of stable and unstable manifolds) and ask how fast $Q(\varphi^t(x))$ can go to zero when $k \to \pm\infty$. For that reason, we introduce:

THE PARAMETERS $q(x), q^s(x), q^u(x)$: For $x \in \mathrm{NUH}$, define:

$$\begin{aligned} q(x) &:= \varepsilon \inf\{e^{\varepsilon|t|}Q(\varphi^t(x)) : t \in \mathbb{R}\} \\ q^s(x) &:= \varepsilon \inf\{e^{\varepsilon|t|}Q(\varphi^t(x)) : t \geq 0\} \\ q^u(x) &:= \varepsilon \inf\{e^{\varepsilon|t|}Q(\varphi^t(x)) : t \leq 0\}. \end{aligned}$$

Clearly $0 \leq q(x), q^s(x), q^u(x) \leq \varepsilon Q(x)$, hence these parameters are much smaller than $Q(x)$. Also, $q^s(x) \wedge q^u(x) = q(x)$. The families $\{q^s(\varphi^t(x))\}_{t\in\mathbb{R}}$ and $\{q^u(\varphi^t(x))\}_{t\in\mathbb{R}}$ represent the *local quantifications of hyperbolicity* along the orbit $\{\varphi^t(x)\}_{t\in\mathbb{R}}$. We collect the following simple lemma, for later use.

Lemma 3.4. *For all $x \in \mathrm{NUH}$ and $t \in \mathbb{R}$, it holds $q(\varphi^t(x)) = e^{\pm\varepsilon|t|}q(x)$.*

Proof. Using that $|t'| = |t' + t| \pm |t|$, we have

$$q(\varphi^t(x)) = e^{\pm\varepsilon|t|}\varepsilon \inf\{e^{\varepsilon|t'+t|}Q(\varphi^{t'+t}(x)) : t' \in \mathbb{R}\} = e^{\pm\varepsilon|t|}q(x).$$

The proof is complete. □

3.4. The recurrently non-uniformly hyperbolic locus $\mathrm{NUH}^\#$

RECURRENTLY NON-UNIFORMLY HYPERBOLIC LOCUS $\mathrm{NUH}^\# = \mathrm{NUH}^\#(\varphi, \chi, \rho, \theta, \varepsilon)$: It is the invariant set of points $x \in \mathrm{NUH}$ such that:

(NUH3) $q(x) > 0$.
(NUH4) $\limsup_{t\to+\infty} q(\varphi^t(x)) > 0$ and $\limsup_{t\to-\infty} q(\varphi^t(x)) > 0$.

Note that if (NUH3) holds then $q(\varphi^t(x)), q^s(\varphi^t(x)), q^u(\varphi^t(x))$ are positive for all $t \in \mathbb{R}$. Condition (NUH4) requires that these values do not degenerate to zero in the limit. The set $\mathrm{NUH}^\#$ carries all χ–hyperbolic measures, as we now prove.

Proposition 3.5. *If μ is a φ–invariant probability measure with $\mu[\mathrm{NUH}] = 1$, then $\mu[\mathrm{NUH}^\#] = 1$. In particular, if μ is χ–hyperbolic then $\mu[\mathrm{NUH}^\#] = 1$.*

Proof. Note that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log Q(\varphi^{n\rho}(x)) = 0 \implies \lim_{t\to\pm\infty} \frac{1}{t} \log Q(\varphi^t(x)) = 0 \implies q(x) > 0.$$

To establish the first limit, we use the following basic fact of ergodic theory.

FACT: Let (X, μ, T) be an invertible probability-preserving system, and $u : X \to (0, +\infty)$ measurable. If there is $C > 0$ such that $C^{-1} \leq \frac{u(Tx)}{u(x)} \leq C$ for μ–a.e. $x \in X$, then $\lim_{n\to\pm\infty} \frac{1}{n} \log u(T^n x) = 0$ for μ–a.e. $x \in X$.

Proof of the fact. By the Poincaré recurrence theorem, $\liminf_{n\to\pm\infty} u(T^n x) < +\infty$ a.e., hence $\liminf_{n\to\pm\infty} \frac{1}{n}\log u(T^n x) = 0$ a.e. Now, applying the Birkhoff ergodic theorem to the bounded function $U := \log u \circ T - \log u$, $\lim_{n\to\pm\infty} \frac{1}{n}\log u(T^n x)$ exists a.e. Therefore

$$\lim_{n\to\pm\infty} \frac{1}{n}\log u(T^n x) = \liminf_{n\to\pm\infty} \frac{1}{n}\log u(T^n x) = 0$$

for μ–a.e. $x \in X$. $\square$

Now we prove the proposition. Assume that $\mu[\text{NUH}] = 1$. By (3.2), $\frac{Q(\varphi^\rho(x))}{Q(x)} = e^{\pm\frac{250\rho}{\beta}}$ for all $x \in \text{NUH}$. Applying the Fact to the transformation $T = \varphi^\rho$ and the function Q, we conclude that (NUH3) holds μ–a.e. Finally, by the Poincaré recurrence theorem, (NUH4) also holds μ–a.e. $\square$

3.5. *The $\mathbb{Z}$–indexed versions of* $\mathbf{q^{s/u}(x)}$*: the parameters* $\mathbf{p^{s/u}(x)}$

We now define discrete time approximate versions of $q^s(x), q^u(x)$ that satisfy recursive explicit formulas, which we call $\mathbb{Z}$*–indexed versions* of $q^s(x), q^u(x)$. Recall that r_Λ is the Poincaré return time of the proper section Λ of size $\rho/2$. In particular, $0 < \inf(r_\Lambda) \leq \sup(r_\Lambda) \leq \rho/2$.

$\mathbb{Z}$–INDEXED VERSIONS OF q^s, q^u: Let $x \in \text{NUH}$. For each sequence $\mathcal{T} = \{t_n\}_{n\in\mathbb{Z}}$ of real numbers with $\frac{1}{2}\inf(r_\Lambda) \leq t_{n+1} - t_n \leq 2\sup(r_\Lambda)$, define:

$$p^s(x, \mathcal{T}, n) := \varepsilon \inf\{e^{\varepsilon(t_m - t_n)} Q(\varphi^{t_m}(x)) : m \geq n\}$$
$$p^u(x, \mathcal{T}, n) := \varepsilon \inf\{e^{\varepsilon(t_n - t_m)} Q(\varphi^{t_m}(x)) : m \leq n\}.$$

Clearly, $p^{s/u}(x, \mathcal{T}, n) \geq q^{s/u}(\varphi^{t_n}(x))$. As the choice of $\mathcal{T}$ will be always clear in the context, we will simply write $p^{s/u}(\varphi^{t_n}(x))$ for $p^{s/u}(x, \mathcal{T}, n)$. As a matter of fact, although the values $p^{s/u}(\varphi^{t_n}(x))$ do depend on the choice of $\mathcal{T}$, they are not very sensitive to this choice.

Proposition 3.6. *The following holds for all $x \in \text{NUH}^\#$ and $\mathcal{T} = \{t_n\}_{n\in\mathbb{Z}}$ with $\frac{1}{2}\inf(r_\Lambda) \leq t_{n+1} - t_n \leq 2\sup(r_\Lambda)$.*

(1) *ROBUSTNESS: Let $\mathfrak{H} := \varepsilon\rho + \frac{250\rho}{\beta}$. For all $n \in \mathbb{Z}$ and $t \in [t_n, t_{n+1}]$, it holds:*

$$\frac{p^{s/u}(\varphi^{t_n}(x))}{q^{s/u}(\varphi^t(x))} = e^{\pm\mathfrak{H}}.$$

(2) *GREEDY ALGORITHM: For all $n \in \mathbb{Z}$ it holds:*

$$p^s(\varphi^{t_n}(x)) = \min\left\{e^{\varepsilon(t_{n+1}-t_n)} p^s(\varphi^{t_{n+1}}(x)), \varepsilon Q(\varphi^{t_n}(x))\right\}$$

$$p^u(\varphi^{t_n}(x)) = \min\left\{e^{\varepsilon(t_n-t_{n-1})}p^u(\varphi^{t_{n-1}}(x)), \varepsilon Q(\varphi^{t_n}(x))\right\}.$$

In particular:

$$\begin{aligned}
\varepsilon Q(\varphi^{t_n}(x)) \geq\ & p^s(\varphi^{t_n}(x)) \geq e^{-\varepsilon(t_n-t_m)}p^s(\varphi^{t_m}(x)),\ \forall n \geq m,\\
\varepsilon Q(\varphi^{t_n}(x)) \geq\ & p^u(\varphi^{t_n}(x)) \geq e^{-\varepsilon(t_m-t_n)}p^s(\varphi^{t_m}(x)),\ \forall m \geq n.
\end{aligned}$$

(3) *Maximality: $p^s(\varphi^{t_n}(x)) = \varepsilon Q(\varphi^{t_n}(x))$ for infinitely many $n > 0$, and $p^u(\varphi^{t_n}(x)) = \varepsilon Q(\varphi^{t_n}(x))$ for infinitely many $n < 0$.*

Proof. We prove the statements for p^s (the proofs for p^u are analogous).

(1) Fix $x \in \mathrm{NUH}^\#$, $n \in \mathbb{Z}$, $t \in [t_n, t_{n+1}]$. By Lemma 3.4, we have $\frac{p^s(\varphi^{t_n}(x))}{q^s(\varphi^t(x))} = \frac{p^s(\varphi^{t_n}(x))}{q^s(\varphi^{t_n}(x))} \cdot \frac{q^s(\varphi^{t_n}(x))}{q^s(\varphi^t(x))} = e^{\pm\varepsilon\rho}\frac{p^s(\varphi^{t_n}(x))}{q^s(\varphi^{t_n}(x))}$, hence we need to estimate $\frac{p^s(\varphi^{t_n}(x))}{q^s(\varphi^{t_n}(x))}$. For $m \geq n$, let $\gamma_m := e^{\varepsilon(t_m-t_n)}Q(\varphi^{t_m}(x))$ and $\delta_m := \inf\{e^{\varepsilon(t-t_n)}Q(\varphi^t(x)) : t_m \leq t \leq t_{m+1}\}$. By definition, we have $p^s(\varphi^{t_n}(x)) = \varepsilon \inf\{\gamma_m : m \geq n\}$ and $q^s(\varphi^{t_n}(x)) = \varepsilon \inf\{\delta_m : m \geq n\}$. Since $\frac{Q(\varphi^t(x))}{Q(\varphi^{t_m}(x))} = e^{\pm\frac{250\rho}{\beta}}$ for $t_m \leq t \leq t_{m+1}$ (see (3.2)), we get:

$$\begin{aligned}
&\gamma_m \geq \delta_m = e^{\varepsilon(t_m-t_n)}\inf\{e^{\varepsilon(t-t_m)}Q(\varphi^t(x)) : t_m \leq t \leq t_{m+1}\}\\
&\geq e^{\varepsilon(t_m-t_n)}e^{-\frac{250\rho}{\beta}}Q(\varphi^{t_m}(x)) = e^{-\frac{250\rho}{\beta}}\gamma_m.
\end{aligned}$$

Hence $1 \leq \frac{p^s(\varphi^{t_n}(x))}{q^s(\varphi^{t_n}(x))} \leq e^{\frac{250\rho}{\beta}}$ and so $\frac{p^s(\varphi^{t_n}(x))}{q^s(\varphi^t(x))} = e^{\pm\mathfrak{H}}$.

(2) We have

$$\begin{aligned}
&p^s(\varphi^{t_n}(x)) = \varepsilon \inf\left\{e^{\varepsilon(t_m-t_n)}Q(\varphi^{t_m}(x)) : m \geq n\right\}\\
&= \min\left\{\varepsilon \inf\left\{e^{\varepsilon(t_m-t_n)}Q(\varphi^{t_m}(x)) : m \geq n+1\right\}, \varepsilon Q(\varphi^{t_n}(x))\right\}\\
&= \min\left\{e^{\varepsilon(t_{n+1}-t_n)}p^s(\varphi^{t_{n+1}}(x)), \varepsilon Q(\varphi^{t_n}(x))\right\},
\end{aligned}$$

which proves the recursive relation. Clearly $p^s \leq \varepsilon Q$. For, the other side of the inequality, note that if $n \geq m$ then:

$$\begin{aligned}
&p^s(\varphi^{t_n}(x)) = \varepsilon \inf\{e^{\varepsilon(t_\ell-t_n)}Q(\varphi^{t_\ell}(x)) : \ell \geq n\}\\
&= e^{-\varepsilon(t_n-t_m)}\varepsilon \inf\{e^{\varepsilon(t_\ell-t_m)}Q(\varphi^{t_\ell}(x)) : \ell \geq n\}\\
&\geq e^{-\varepsilon(t_n-t_m)}\varepsilon \inf\{e^{\varepsilon(t_\ell-t_m)}Q(\varphi^{t_\ell}(x)) : \ell \geq m\} = e^{-\varepsilon(t_n-t_m)}p^s(\varphi^{t_m}(x)).
\end{aligned}$$

(3) The proof is based on [36, Prop. 8.3]. Since $x \in \mathrm{NUH}^\#$, $\limsup_{t\to\infty} q^s(\varphi^t(x)) > 0$. By part (1), $\limsup_{n\to\infty} p^s(\varphi^{t_n}(x)) > 0$ hence $\exists\delta_0 > 0$ such that $p^s(\varphi^{t_n}(x)) > \delta_0$ for infinitely many $n > 0$. By contradiction, assume $\exists n_0 > 0$ such that $p^s(\varphi^{t_n}(x)) <$

$\varepsilon Q(\varphi^{t_n}(x))$ for all $n \geq n_0$. By the greedy algorithm in part (2), $p^s(\varphi^{t_n}(x)) = e^{\varepsilon(t_{n+1}-t_n)} p^s(\varphi^{t_{n+1}}(x))$ for all $n \geq n_0$. This implies that $p^s(\varphi^{t_{n_0}}(x)) = e^{\varepsilon(t_{n_0+\ell}-t_{n_0})} p^s(\varphi^{t_{n_0+\ell}}(x))$ for all $\ell \geq 0$, hence $p^s(\varphi^{t_{n_0}}(x)) > e^{\varepsilon(t_{n_0+\ell}-t_{n_0})}\delta_0$ for infinitely many $\ell \geq 0$, which is a contradiction since $e^{\varepsilon(t_{n_0+\ell}-t_{n_0})} \to \infty$ as $\ell \to \infty$. □

3.6. Pesin charts Ψ_x

Recall that $R[\mathfrak{r}] := [-\mathfrak{r}, \mathfrak{r}]^2 \subset \mathbb{R}^2$. We define Pesin charts for $x \in \Lambda \cap \mathrm{NUH}$.

Pesin chart at x: It is the map $\Psi_x : R[\mathfrak{r}] \to \widehat{\Lambda}$ defined by $\Psi_x := \exp_x \circ C(x)$.

The center x of the Pesin chart Ψ_x always belongs to the reference section Λ, while its image is contained in the security section $\widehat{\Lambda}$. In particular, when x is close to the boundary of Λ, the image of Ψ_x is *not* contained in Λ. This definition is different from [28], and it is the first step to bypass the boundary effect mentioned in Section 1.1.

For $x \in \widehat{\Lambda}$, let $\iota_x : T_x\widehat{\Lambda} \to \mathbb{R}^2$ be an isometry. If $x \in \Lambda, y \in \widehat{\Lambda}$ with $d(x,y) \leq 2\mathfrak{r}$, we consider as in section 1.2.3 an isometry $P_{y,x} : T_yM \to T_xM$. If $A : \mathbb{R}^2 \to T_y\widehat{\Lambda}$ is a linear map, we define $\widetilde{A} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\widetilde{A} := \iota_x \circ P_{y,x} \circ A$. The map $\widetilde{A}$ depends on x but $\|\widetilde{A}\|$ does not.

Lemma 3.7. *For all $x \in \Lambda \cap \mathrm{NUH}$, the Pesin chart Ψ_x is a diffeomorphism onto its image and:*

1. *Ψ_x is 2–Lipschitz and Ψ_x^{-1} is $2\|C(x)^{-1}\|$–Lipschitz.*
2. *$\|\widetilde{d(\Psi_x)_{v_1}} - \widetilde{d(\Psi_x)_{v_2}}\| \leq \mathfrak{K}\|v_1 - v_2\|$ for all $v_1, v_2 \in R[\mathfrak{r}]$.*

Proof. Since $C(x)$ is a contraction, $C(x)R[\mathfrak{r}] \subset B_x[2\mathfrak{r}]$ and so Ψ_x is well-defined with inverse $C(x)^{-1} \circ \exp_x^{-1}$. It is a diffeomorphism because $C(x)$ and $\exp_x$ are.

(1) $C(x)$ is a contraction and $\exp_x$ is 2–bi-Lipschitz in $B_x[2\mathfrak{r}]$. Therefore Ψ_x is 2–Lipschitz and Ψ_x^{-1} is $2\|C(x)^{-1}\|$–Lipschitz.

(2) Since $C(x)v_i \in B_x[2\mathfrak{r}]$, condition (Exp3) gives that

$$
\begin{aligned}
&\|\widetilde{d(\Psi_x)_{v_1}} - \widetilde{d(\Psi_x)_{v_2}}\| = \|\widetilde{d(\exp_x)_{C(x)v_1}} \circ C(x) - \widetilde{d(\exp_x)_{C(x)v_2}} \circ C(x)\| \\
&\leq \mathfrak{K}\|C(x)v_1 - C(x)v_2\| \leq \mathfrak{K}\|v_1 - v_2\|.
\end{aligned}
$$

The proof is complete. □

3.7. Holonomy maps $g_x^{\pm}$ in Pesin charts

The parameter $Q(x)$ defines the size of the domain where we can control $g_x^{\pm}$ in Pesin charts: in these coordinates, $g_x^{\pm}$ are small perturbations of hyperbolic linear maps.

Theorem 3.8. *The following holds for all $\varepsilon > 0$ small enough. For all $x \in \Lambda \cap \mathrm{NUH}$ the map $f_x^+ := \Psi_{f(x)}^{-1} \circ g_x^+ \circ \Psi_x$ is well-defined on $R[10Q(x)]$ and satisfies:*

(1) $d(f_x^+)_0 = C(f(x))^{-1} \circ \Phi^{r_\Lambda(x)} \circ C(x)$ *and* $e^{-4\rho} < m(d(f_x^+)_0) \leq \|d(f_x^+)_0\| < e^{4\rho}$.

(2) $f_x^+ = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + H$ *where:*

 (a) $e^{-4\rho} < |A| < e^{-\chi r_\Lambda(x)}$ *and* $e^{\chi r_\Lambda(x)} < |B| < e^{4\rho}$, *cf. Lemma 3.2(2).*

 (b) $H(0) = 0$ *and* $dH_0 = 0$.

 (c) $\|H\|_{C^{1+\frac{\beta}{2}}} < \varepsilon$.

A similar statement holds for $f_x^- := \Psi_x^{-1} \circ g_{f(x)}^- \circ \Psi_{f(x)}$.

The proof is in Appendix A.

3.8. The overlap condition

We now control the coordinate change from Ψ_x to Ψ_y when x, y are "sufficiently close". This can only be made when both x, y and $C(x), C(y)$ are very close. In the sequel we will make extensive use of Pesin charts with different domains.

Pesin chart Ψ_x^η: It is the restriction of Ψ_x to $R[\eta]$, where $0 < \eta \leq Q(x)$.

Recall that d is the distance on $\widehat{\Lambda}$ associated to the induced Riemannian metric.

ε–overlap: We say that two Pesin charts $\Psi_{x_1}^{\eta_1}, \Psi_{x_2}^{\eta_2}$ *ε–overlap* if $\frac{\eta_1}{\eta_2} = e^{\pm\varepsilon}$ and $d(x_1, x_2) + \|\widetilde{C(x_1)} - \widetilde{C(x_2)}\| < (\eta_1\eta_2)^4$. In particular, x_1, x_2 belong to the same local connected component of Λ. We write $\Psi_{x_1}^{\eta_1} \overset{\varepsilon}{\approx} \Psi_{x_2}^{\eta_2}$.

Lemma 3.9. *The following holds for $\varepsilon > 0$ small. If $\Psi_{x_1}^{\eta_1} \overset{\varepsilon}{\approx} \Psi_{x_2}^{\eta_2}$, then*

$$\Psi_{x_i}(R[10Q(x_i)]) \subset B_{x_1} \cap B_{x_2}.$$

In particular, it makes sense to consider $\|\widetilde{C(x_1)} - \widetilde{C(x_2)}\|$.

Proof. Let $i = 1$. By Lemma 3.7(1), $\Psi_{x_1}(R[10Q(x_1)]) \subset B(x_1, 40Q(x_1))$. This latter ball is contained in B_{x_1} since $40Q(x_1) < 40\varepsilon^{3/\beta} < 2\mathfrak{r}$ when $\varepsilon > 0$ is small. Also:

$$\Psi_{x_1}(R[10Q(x_1)]) \subset B(x_1, 40Q(x_1)) \subset B(x_2, 40Q(x_1) + d(x_1, x_2)).$$

Since $40Q(x_1) + d(x_1, x_2) < 40\varepsilon^{3/\beta} + \varepsilon^{24/\beta} < 2\mathfrak{r}$ for small $\varepsilon > 0$, $\Psi_{x_1}(R[10Q(x_1)]) \subset B_{x_2}$. $\square$

The next result guarantees that the ε–overlap of Pesin charts allows to change coordinates while maintaining a good control on the dynamics and geometry of the charts.

Proposition 3.10. *The following holds for $\varepsilon > 0$ small. If $\Psi_{x_1}^{\eta_1} \overset{\varepsilon}{\approx} \Psi_{x_2}^{\eta_2}$ then:*

(1) Control of s, u: $\frac{s(x_1)}{s(x_2)} = e^{\pm(\eta_1\eta_2)^3}$ *and* $\frac{u(x_1)}{u(x_2)} = e^{\pm(\eta_1\eta_2)^3}$.
(2) Control of α: $\frac{|\sin\alpha(x_1)|}{|\sin\alpha(x_2)|} = e^{\pm(\eta_1\eta_2)^3}$.
(3) Overlap: $\Psi_{x_i}(R[e^{-2\varepsilon}\eta_i]) \subset \Psi_{x_j}(R[\eta_j])$ *for* $i, j = 1, 2$.
(4) Change of coordinates: *For $i, j = 1, 2$, the map $\Psi_{x_i}^{-1} \circ \Psi_{x_j}$ is well-defined in $R[\mathfrak{r}]$, and $\|\Psi_{x_i}^{-1} \circ \Psi_{x_j} - \mathrm{Id}\|_{C^2} < \varepsilon(\eta_1\eta_2)^2$ where the norm is taken in $R[\mathfrak{r}]$.*

The proof is in Appendix A.

3.9. The maps $f_{x,y}^+, f_{x,y}^-$

Let $x, y \in \Lambda \cap \mathrm{NUH}$ such that $\Psi_{f(x)}^{\eta} \overset{\varepsilon}{\approx} \Psi_y^{\eta'}$. In this section, we change $\Psi_{f(x)}$ by Ψ_y in the definition of f_x^+ and obtain a result similar to Theorem 3.8.

The maps $f_{x,y}^+$ and $f_{x,y}^-$: If $\Psi_{f(x)}^{\eta} \overset{\varepsilon}{\approx} \Psi_y^{\eta'}$, we define the map $f_{x,y}^+ := \Psi_y^{-1} \circ g_x^+ \circ \Psi_x$. If $\Psi_x^{\eta} \overset{\varepsilon}{\approx} \Psi_{f^{-1}(y)}^{\eta'}$, we define $f_{x,y}^- := \Psi_x^{-1} \circ g_y^- \circ \Psi_y$.

Since any meaningful estimate of $f_{x,y}^{\pm}$ in the $C^{1+\beta/2}$ norm cannot be better than that of Theorem 3.8, and to keep estimates of size ε, we consider the $C^{1+\beta/3}$ norm of $f_{x,y}^{\pm}$.

Theorem 3.8'. *The following holds for all $\varepsilon > 0$ small enough. If $x, y \in \Lambda \cap \mathrm{NUH}$ and $\Psi_{f(x)}^{\eta} \overset{\varepsilon}{\approx} \Psi_y^{\eta'}$, then $f_{x,y}^+$ is well-defined on $R[10Q(x)]$ and can be written as $f_{x,y}^+ = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + H$ where:*

(1) $e^{-4\rho} < |A| < e^{-\chi r_\Lambda(x)}$, $e^{\chi r_\Lambda(x)} < |B| < e^{4\rho}$, *cf. Lemma 3.2(2).*
(2) *For $i = 1, 2$, it holds $\|H(0)\| < \varepsilon\eta$, $\|dH_0\| < \varepsilon\eta^{\beta/3}$, $\mathrm{H\ddot{o}l}_{\beta/3}(dH) < \varepsilon$.*

If $\Psi_x^{\eta} \overset{\varepsilon}{\approx} \Psi_{f^{-1}(y)}^{\eta'}$ then a similar statement holds for $f_{x,y}^-$.

Proof. We write $f_{x,y}^+ = (\Psi_y^{-1} \circ \Psi_{f(x)}) \circ f_x^+ =: g \circ f_x^+$ and see it as a small perturbation of f_x^+. By Theorem 3.8,

$$f_x^+(0) = 0,\ \|d(f_x^+)\|_{C^0} < 2e^{4\rho},\ \|d(f_x^+)_v - d(f_x^+)_w\| \le \varepsilon\|v - w\|^{\beta/2}, \forall v, w \in R[10Q(x)],$$

where the C^0 norm is taken in $R[10Q(x)]$, and by Proposition 3.10(4) we have

$$\|g - \mathrm{Id}\| < \varepsilon(\eta\eta')^2,\ \|d(g - \mathrm{Id})\|_{C^0} < \varepsilon(\eta\eta')^2,\ \|dg_v - dg_w\| \le \varepsilon(\eta\eta')^2\|v - w\|^{\beta/2}$$

for $v, w \in R[\mathfrak{r}]$, where the C^0 norm is taken in $R[\mathfrak{r}]$.

We first prove that $f^+_{x,y}$ is well-defined on $R[10Q(x)]$. For $\varepsilon > 0$ small enough we have $f^+_x(R[10Q(x)]) \subset B(0, 40e^{4\rho}Q(x)) \subset R[\mathfrak{r}]$ since $40e^{4\rho}Q(x) < 40e^{4\rho}\varepsilon^{3/\beta} < \mathfrak{r}$. By Proposition 3.10(4), $f^+_{x,y}$ is well-defined.

Letting A, B as in Lemma 3.2, part (1) is clear, so we focus on part (2). We have $\|H(0)\| = \|g(0)\| < \varepsilon(\eta\eta')^2 < \varepsilon\eta$ and for $\varepsilon > 0$ small enough:

$$\|dH_0\| \leq \|dg_0 \circ d(f^+_x)_0 - d(f^+_x)_0\| \leq \|d(g - \mathrm{Id})_0\|\|d(f^+_x)_0\| < \varepsilon(\eta\eta')^2 e^{4\rho} < \varepsilon\eta^{\beta/3}.$$

Finally, since $f^+_x(R[10Q(x)]) \subset R[\mathfrak{r}]$, if $\varepsilon > 0$ is small then for $v, w \in R[10Q(x)]$ it holds:

$$\begin{aligned}
&\|dH_v - dH_w\| = \|dg_{f^+_x(v)} \circ d(f^+_x)_v - dg_{f^+_x(w)} \circ d(f^+_x)_w\| \\
&\leq \|dg_{f^+_x(v)} - dg_{f^+_x(w)}\|\|d(f^+_x)_v\| + \|dg_{f^+_x(w)}\|\|d(f^+_x)_v - d(f^+_x)_w\| \\
&\leq \varepsilon(\eta\eta')^2\|f^+_x(v) - f^+_x(w)\|^{\beta/2}\|d(f^+_x)\|_{C^0} + \varepsilon\|dg\|_{C^0}\|v - w\|^{\beta/2} \\
&\leq \left[\varepsilon(\eta\eta')^2\|d(f^+_x)\|_{C^0}^{1+\beta/2} + 40\varepsilon\|dg\|_{C^0}Q(x)^{\beta/6}\right]\|v - w\|^{\beta/3} \\
&\leq \left[\eta^2\eta'^2(2e^{4\rho})^{1+\beta/2} + 80Q(x)^{\beta/6}\right]\varepsilon\|v - w\|^{\beta/3} \\
&\leq \left[\varepsilon^{12/\beta}(2e^{4\rho})^{1+\beta/2} + 80\varepsilon^{1/2}\right]\varepsilon\|v - w\|^{\beta/3} < \varepsilon\|v - w\|^{\beta/3}.
\end{aligned}$$

The proof is now complete. □

4. Invariant manifolds and shadowing

Up to now, we have fixed $\varphi, \chi, \rho, \Lambda, \widehat{\Lambda}, \theta$ and ε, where ρ, ε are small parameters. In this section, we:

(1) Define *ε–double charts* $\Psi_x^{p^s,p^u}$, which are double versions of Pesin charts whose stable and unstable sizes p^s, p^u may differ. The parameters p^s/p^u control separately the local stable/unstable hyperbolicity at x.
(2) Define *generalized pseudo-orbit*, which is a sequence $\underline{v}$ of ε–double charts satisfying *edge conditions*, which are nearest neighbor conditions relating the parameters of consecutive ε–double charts.
(3) Associate to each generalized pseudo-orbit its *local stable and unstable manifolds* $V^s[\underline{v}]$ and $V^u[\underline{v}]$. As a consequence, we obtain a *shadowing lemma*.

4.1. Pseudo-orbits

ε–DOUBLE CHART: An *ε–double chart* is a pair of Pesin charts $\Psi_x^{p^s,p^u} = (\Psi_x^{p^s}, \Psi_x^{p^u})$ where $0 < p^s, p^u \leq \varepsilon Q(x)$.

The parameters p^s/p^u are local quantifications of the hyperbolicity at x. One can think of them as a definite size for the stable and unstable manifolds at x.

Transition time: For two ε–double charts $v=\Psi_x^{p^s,p^u}$, $w=\Psi_y^{q^s,q^u}$ we define $T(v,w)$ by

$$\min\left\{\min\{T^+(z):z\in\Psi_x(R[\tfrac{1}{20}(p^s\wedge p^u)])\},\,\min\{-T^-(z):z\in\Psi_y(R[\tfrac{1}{20}(q^s\wedge q^u)])\}\right\},$$

where $T^+:B_x\to\mathbb{R}$ and $T^-:B_y\to\mathbb{R}$ are the $C^{1+\beta}$ functions satisfying $g_x^+=\varphi^{T^+}$, $g_{f^{-1}(y)}^-=\varphi^{T^-}$ with $T^+(x)=r_\Lambda(x)$ and $T^-(y)=-r_\Lambda(f^{-1}(y))$.

Edge $v\overset{\varepsilon}{\to}w$: Given two ε–double charts $v=\Psi_x^{p^s,p^u}$, $w=\Psi_y^{q^s,q^u}$, we draw an *edge* from v to w if the two following conditions are satisfied:

(GPO1) $\Psi_{f(x)}^{q^s\wedge q^u}\overset{\varepsilon}{\approx}\Psi_y^{q^s\wedge q^u}$ and $\Psi_{f^{-1}(y)}^{p^s\wedge p^u}\overset{\varepsilon}{\approx}\Psi_x^{p^s\wedge p^u}$.

(GPO2) The following estimates hold:

$$e^{-\varepsilon p^s}\min\{e^{\varepsilon T(v,w)}q^s,e^{-\varepsilon}\varepsilon Q(x)\}\le p^s\le\min\{e^{\varepsilon T(v,w)}q^s,\varepsilon Q(x)\}\tag{4.1}$$

$$e^{-\varepsilon q^u}\min\{e^{\varepsilon T(v,w)}p^u,e^{-\varepsilon}\varepsilon Q(y)\}\le q^u\le\min\{e^{\varepsilon T(v,w)}p^u,\varepsilon Q(y)\}.\tag{4.2}$$

Remark 4.1. In the above notation, if $v\overset{\varepsilon}{\to}w$ then by Theorem 3.8' we have

$$g_y^-(\Psi_y(R[\tfrac{1}{20}(q^s\wedge q^u)]))\subset\Psi_x(R[\tfrac{1}{15}(p^s\wedge p^u)])$$

and so $T(v,w)=T^+(z)$ for some $z\in\Psi_x(R[\frac{1}{15}(p^s\wedge p^u)])$. In particular, $T(v,w)\le\rho$.

ε–generalized pseudo-orbit (ε–gpo): An *ε–generalized pseudo-orbit (ε–gpo)* is a sequence $\underline{v}=\{v_n\}_{n\in\mathbb{Z}}$ of ε–double charts such that $v_n\overset{\varepsilon}{\to}v_{n+1}$ for all $n\in\mathbb{Z}$. We say that $\underline{v}$ is *regular* if there are v,w such that $v_n=v$ for infinitely many $n>0$ and $v_n=w$ for infinitely many $n<0$.

Positive and negative ε–gpo: A *positive ε–gpo* is a sequence $\underline{v}^+=\{v_n\}_{n\ge0}$ of ε–double charts such that $v_n\overset{\varepsilon}{\to}v_{n+1}$ for all $n\ge0$. A *negative ε–gpo* is a sequence $\underline{v}^-=\{v_n\}_{n\le0}$ of ε–double charts such that $v_n\overset{\varepsilon}{\to}v_{n+1}$ for all $n\le-1$.

Condition (GPO1) allows to pass from an ε–double chart at x to an ε–double chart at y and vice-versa. Condition (GPO2) is a greedy recursion that implies that the local quantifications of hyperbolicity are "as large as possible". The need of (GPO2) will be clear in the proof of Theorem 5.1 (coarse graining) and Theorem 6.1 (inverse theorem).

Lemma 4.2. *If $v=\Psi_x^{p^s,p^u},w=\Psi_y^{q^s,q^u}$ are ε–double charts satisfying* (GPO2) *then $\frac{p^s\wedge p^u}{q^s\wedge q^u}=e^{\pm2\varepsilon}$.*

Proof. We have $e^{-\varepsilon p^s}\min\{e^{\varepsilon T(v,w)}q^s,e^{-\varepsilon}\varepsilon Q(x)\}\le p^s\le\min\{e^{\varepsilon T(v,w)}q^s,\varepsilon Q(x)\}$, therefore $e^{-\varepsilon p^s}\min\{e^{\varepsilon T(v,w)}q^s,e^{-\varepsilon}p^u\}\le p^s\wedge p^u\le\min\{e^{\varepsilon T(v,w)}q^s,p^u\}$ and so

$$e^{-\varepsilon-\varepsilon p^s}\min\{e^{\varepsilon T(v,w)}q^s,p^u\}\le p^s\wedge p^u\le\min\{e^{\varepsilon T(v,w)}q^s,p^u\}.$$

By the same reason, $e^{-\varepsilon-\varepsilon q^u}\min\{e^{\varepsilon T(v,w)}p^u,q^s\}\leq q^s\wedge q^u\leq \min\{e^{\varepsilon T(v,w)}p^u,q^s\}$ hence

$$e^{-\varepsilon-\varepsilon q^u-\varepsilon T(v,w)}\min\{e^{\varepsilon T(v,w)}q^s,p^u\}\leq q^s\wedge q^u\leq e^{\varepsilon T(v,w)}\min\{e^{\varepsilon T(v,w)}q^s,p^u\}.$$

Together, these inequalities imply that

$$e^{-\varepsilon[1+p^s+T(v,w)]}\leq \tfrac{p^s\wedge p^u}{q^s\wedge q^u}\leq e^{\varepsilon[1+q^u+T(v,w)]}.$$

Since $p^s,q^u<\varepsilon<0.25$ and $T(v,w)\leq\rho<0.25$, it follows that $\frac{p^s\wedge p^u}{q^s\wedge q^u}=e^{\pm 2\varepsilon}$. $\square$

Remark 4.3. There is a big difference between (GPO2) above and all previous definitions used in [36,27,3,28,29,2]. The first is that we only require inequalities, while previous work required equalities. One reason is the following: while for diffeomorphisms the hyperbolicity acquired in an edge $v\overset{\varepsilon}{\to}w$ is at least e^{ε}, for flows it is at least $e^{\varepsilon T(v,w)}$. Since $T(v,w)$ usually does not belong to a countable set, neither does $\min\{e^{\varepsilon T(v,w)}q^s,\varepsilon Q(x)\}$. Therefore, instead of requiring p^s to be equal to this minimum we relax the assumption to an "approximate equality". This approximate equality implies that either p^s is of the order of $e^{\varepsilon T(v,w)}q^s$ and/or it is essentially maximal (of the order of $\varepsilon Q(x)$). The conditions we consider are weak enough to code all relevant orbits (Theorem 5.1(2)) but still strong enough for the coding to be "unique up to bounded error" (Theorem 6.1).

4.2. Graph transforms and invariant manifolds

Let $v=\Psi_x^{p^s,p^u}$ be an ε–double chart.

ADMISSIBLE MANIFOLDS: An *s–admissible manifold at v* is a set of the form

$$V=\Psi_x\{(t,F(t)):|t|\leq p^s\}$$

where $F:[-p^s,p^s]\to\mathbb{R}$ is a $C^{1+\beta/3}$ function such that:

(AM1) $|F(0)|\leq 10^{-3}(p^s\wedge p^u)$.
(AM2) $|F'(0)|\leq\frac{1}{2}(p^s\wedge p^u)^{\beta/3}$.
(AM3) $\|F'\|_{C^0}+\text{Höl}_{\beta/3}(F')\leq\frac{1}{2}$ where the norms are taken in $[-p^s,p^s]$.

The function F is called the *representing function* of V. Similarly, a *u–admissible manifold at v* is a set of the form $\Psi_x\{(G(t),t):|t|\leq p^u\}$ where $G:[-p^u,p^u]\to\mathbb{R}$ is a $C^{1+\beta/3}$ function satisfying (AM1)–(AM3), with norms taken in $[-p^u,p^u]$.

If V_1,V_2 are two s–admissible manifolds at v, with representing functions F_1,F_2, for $i\geq 0$ define $d_{C^i}(V_1,V_2):=\|F_1-F_2\|_{C^i}$ where the norm is taken in $[-p^s,p^s]$. The same applies to u–admissible manifolds.

In the sequel, we introduce *graph transforms*, which is the tool used to construct invariant manifolds. Since the proofs are adaptations of [36], we restrict the discussion to stable manifolds. The main result of this section, Theorem 4.5, collects the basic properties of invariant manifolds. Given a ε–double chart $v = \Psi_x^{p^s,p^u}$, we denote by $\mathscr{M}^s(v)$ the set of its s–admissible manifolds.

THE GRAPH TRANSFORM $\mathscr{F}^s_{v,w}$: To any edge $v \overset{\varepsilon}{\to} w$ between ε–double charts $v = \Psi_x^{p^s,p^u}$ and $w = \Psi_y^{q^s,q^u}$, we associate the *graph transform* $\mathscr{F}^s_{v,w} : \mathscr{M}^s(w) \to \mathscr{M}^s(v)$ as being the map that sends an s–admissible manifold at w with representing function $F : [-q^s, q^s] \to \mathbb{R}$ to the unique s–admissible manifold at v with representing function $G : [-p^s, p^s] \to \mathbb{R}$ such that $\{(t, G(t)) : |t| \le p^s\} \subset f^-_{x,y}\{(t, F(t)) : |t| \le q^s\}$.

Lemma 4.4. *If $\varepsilon > 0$ is small enough, then $\mathscr{F}^s_{v,w}$ is well-defined for any edge $v \overset{\varepsilon}{\to} w$. Furthermore, if $V_1, V_2 \in \mathscr{M}^s(w)$ then:*

(1) $d_{C^0}(\mathscr{F}^s_{v,w}(V_1), \mathscr{F}^s_{v,w}(V_2)) \le e^{-\chi \inf(r_\Lambda)/2} d_{C^0}(V_1, V_2)$.
(2) $d_{C^1}(\mathscr{F}^s_{v,w}(V_1), \mathscr{F}^s_{v,w}(V_2)) \le e^{-\chi \inf(r_\Lambda)/2} (d_{C^1}(V_1, V_2) + d_{C^0}(V_1, V_2)^{\beta/3})$.

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [36, Prop. 4.12 and 4.14]. The same proofs work by changing C_f and χ in [36] to $e^{4\rho}$ and $\chi \inf(r_\Lambda)$ in our case, and observing that by Lemma 3.2(2) and Theorem 3.8'(1) we have $e^{-4\rho} < |A| < e^{-\chi \inf(r_\Lambda)}$ and $e^{\chi \inf(r_\Lambda)} < |B| < e^{4\rho}$.

THE STABLE MANIFOLD OF POSITIVE ε–GPO: The *stable manifold* of a positive ε–gpo $\underline{v}^+ = \{v_n\}_{n \ge 0}$ is

$$V^s[\underline{v}^+] := \lim_{n \to +\infty} (\mathscr{F}^s_{v_0,v_1} \circ \cdots \circ \mathscr{F}^s_{v_{n-2},v_{n-1}} \circ \mathscr{F}^s_{v_{n-1},v_n})(V_n)$$

for some (any) choice $(V_n)_{n \ge 0}$ with $V_n \in \mathscr{M}^s(v_n)$. The convergence occurs in the C^1 topology.

The proof of the good definition and C^1 convergence is done as in [36, Prop. 4.15, part (1)]. Similarly, we introduce the *unstable manifold* $V^u[\underline{v}^-]$ of a negative ε–gpo. We then arrive at the basic properties of $V^s[\underline{v}^+]$ and $V^u[\underline{v}^-]$.

Theorem 4.5 *(Stable manifold theorem). The following holds for all $\varepsilon > 0$ small enough. Let $\underline{v}^+ = \{v_n\}_{n \ge 0} = \{\Psi_{x_n}^{p^s_n,p^u_n}\}_{n \ge 0}$ be a positive ε–gpo.*

(1) *ADMISSIBILITY. The set $V^s[\underline{v}^+]$ is an s–admissible manifold at v_0, equal to*

$$V^s[\underline{v}^+] = \{x \in \Psi_{x_0}(R[p^s_0]) : (g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0})(x) \in \Psi_{x_n}(R[10Q(x_n)]), \forall n \ge 0\}.$$

(2) *INVARIANCE.* $g^+_{x_0}(V^s[\{v_n\}_{n \ge 0}]) \subset V^s[\{v_n\}_{n \ge 1}]$.
(3) *HYPERBOLICITY. For all y, y' in $V^s[\underline{v}^+]$ and all $n \ge 0$:*

$$d(g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0}(y), g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0}(y')) \le d(\Psi^{-1}_{x_0}(y), \Psi^{-1}_{x_0}(y'))\, e^{-\frac{\chi \inf(r_\Lambda)}{2} n}.$$

For any unit vector w tangent to $V^s[\underline{v}^+]$ at a point y and all $n \geq 0$:

$$\|d(g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0})_y w\| \leq 8\|C(x_0)^{-1}\|\, e^{-\frac{\chi\ \inf(r_\Lambda)}{2}n} \quad \text{and}$$
$$\|d(g^-_{x_{-n+1}} \circ \cdots \circ g^-_{x_0})_y w\| \geq \tfrac{1}{8}(p^s_0 \wedge p^u_0)^{\frac{\beta}{12}}\, e^{\left(\frac{\chi\ \inf(r_\Lambda)}{2} - \frac{\beta\varepsilon}{6}\right)n}.$$

(4) *Bounded distortion. For all y, y' in $V^s[\underline{v}^+]$, unit vectors w, w' tangent to $V^s[\underline{v}^+]$ at y, y' respectively and all $n \geq 0$,*

$$\Big|\log\|d(g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0})_y w\| - \log\|d(g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0})_{y'} w'\|\Big| \leq Q(x_0)^{\beta/4}.$$

(5) *Hölder property. The map $\underline{v}^+ \mapsto V^s[\underline{v}^+]$ is Hölder continuous: There are $K > 0$ and $\theta \in (0,1)$ such that for all $N \geq 0$, if $\underline{v}^+, \underline{w}^+$ are positive ε–gpo's with $v_n = w_n$ for $n = 0, \ldots, N$ then $d_{C^1}(V^s[\underline{v}^+], V^s[\underline{w}^+]) \leq K\theta^N$.*

The curve $V^s[\underline{v}^+]$ is called local stable manifold *of $\underline{v}^+$. A similar statement holds for unstable manifold $V^u[\underline{v}^-]$ of a negative ε-gpo $\underline{v}^-$.*

The above theorem is a strengthening of the Pesin stable manifold theorem [32]. Its statement is similar to [36], and its proof is perfomed exactly as in [36, Prop. 4.15 and 6.3], noting that in Pesin charts the composition $g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0}$ is represented by $f^+_{x_{n-1},x_n} \circ \cdots \circ f^+_{x_0,x_1}$. Since each $f^+_{x_i,x_{i+1}}$ is hyperbolic (Theorem 3.8') and each $\mathscr{F}^s_{v_i,v_{i+1}}$ is contracting (Lemma 4.4), the proof follows. We note that the second estimate of part (3) is proved as in [36, Prop. 6.5], see also the proof of [2, Prop. 4.11].

4.3. Shadowing

We say that an ε–gpo $\{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\in\mathbb{Z}}$ *shadows* a point $x \in \widehat{\Lambda}$ if:

$$(g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0})(x) \in \Psi_{x_n}(R[p^s_n \wedge p^u_n]) \text{ for all } n \geq 0,$$
$$(g^-_{x_{n+1}} \circ \cdots \circ g^-_{x_0})(x) \in \Psi_{x_n}(R[p^s_n \wedge p^u_n]) \text{ for all } n \leq 0.$$

An important property is the following.

Proposition 4.6. *If ε is small enough, then every ε–gpo $\underline{v}$ shadows a unique point $\{x\} = V^s[\underline{v}] \cap V^u[\underline{v}]$.*

The proof uses the following property of admissible manifolds.

Lemma 4.7. *The following holds for all $\varepsilon > 0$ small enough. If $v = \Psi^{p^s,p^u}_x$ is an ε–double chart, then for every $V^{s/u} \in \mathscr{M}^{s/u}(v)$ it holds:*

(1) *V^s and V^u intersect at a single point $P \in \Psi_x(R[10^{-2}(p^s \wedge p^u)])$.*
(2) $\frac{\sin\angle(V^s,V^u)}{\sin\alpha(x)} = e^{\pm(p^s\wedge p^u)^{\beta/4}}$ *and* $|\cos\angle(V^s,V^u) - \cos\alpha(x)| < 2(p^s \wedge p^u)^{\beta/4}$*, where* $\angle(V^s,V^u)$ *is the angle of intersection of V^s and V^u at P.*

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, the above lemma is [36, Prop. 4.11]. The same proof works in our case, since inside $\Psi_x(R[10Q(x)])$ the estimates (Exp1)–(Exp4) hold.

Proof of Proposition 4.6. Let $\underline{v} = \{v_n\}_{n\in\mathbb{Z}} = \{\Psi_{x_n}^{p_n^s,p_n^u}\}_{n\in\mathbb{Z}}$ be an ε–gpo. The proof is the same as that of [36, Theorem 4.2], and follows the steps below:

- By Theorem 4.5(1), any point shadowed by $\underline{v}$ must lie in $V^s[\{v_n\}_{n\geq 0}]\cap V^u[\{v_n\}_{n\leq 0}]$. By Lemma 4.7(1), this intersection is a single point $\{x\}$. We claim that $\underline{v}$ shadows x.
- The definition of shadowing is equivalent to the following weaker definition: $\underline{v}$ shadows x if and only if

$$(g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0})(x) \in \Psi_{x_n}(R[10Q(x_n)]) \text{ for all } n \geq 0,$$
$$(g^-_{x_{n+1}} \circ \cdots \circ g^-_{x_0})(x) \in \Psi_{x_n}(R[10Q(x_n)]) \text{ for all } n \leq 0.$$

- By Theorem 4.5(2), if $n \geq 0$ then $g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0}(x) \in V^s[\{v_{n+k}\}_{k\geq 0}] \subset \Psi_{x_n}(R[10Q(x_n)])$, and if $n \geq 0$ then $(g^-_{x_{n+1}} \circ \cdots \circ g^-_{x_0})(x) \in V^u[\{v_{n+k}\}_{k\leq 0}] \subset \Psi_{x_n}(R[10Q(x_n)])$, and so the weaker definition of shadowing holds.

This concludes the proof. □

4.4. Additional properties

Now, we relate stable/unstable manifolds of ε–gpo's with stable/unstable manifolds of the flow φ.

Proposition 4.8. *The following holds for all $\varepsilon > 0$ small enough. Let $\underline{v} = \{v_n\}_{n\geq 0}$ be a positive ε–gpo with $v_0 = \Psi_x^{p^s,p^u}$, and let $F : [-p^s, p^s] \to \mathbb{R}$ be the representing function of $V^s = V^s[\underline{v}^+]$. Then there exists a function $\Delta : [-p^s, p^s] \to \mathbb{R}$ with $\Delta(0) = 0$ such that the curve $\widetilde{V}^s := \{\varphi^{\Delta(t)}[\Psi_x(t, F(t))] : |t| \leq p^s\}$ satisfies $d(\varphi^t(\widetilde{y}), \varphi^t(\widetilde{z})) \leq e^{-\frac{\chi \inf(r_\Lambda)}{2\sup(r_\Lambda)}t}$ for all $\widetilde{y}, \widetilde{z} \in \widetilde{V}^s$ and $t \geq 0$. An analogous statement holds for negative ε–gpo's.*

In other words, $\widetilde{V}^s$ is a lift of V^s to a curve that contracts in the future under the flow (we are not claiming $\widetilde{V}^s$ is the local stable manifold of φ at x).

Proof. Write $v_n = \Psi_{x_n}^{p_n^s,p_n^u}$ with $\Psi_{x_0}^{p_0^s,p_0^u} = \Psi_x^{p^s,p^u}$. The idea is simple: Δ is the cumulative shear of a point of V^s under iterations of the maps $g^+_{x_n}$. Write $g^+_{x_n} = \varphi^{T_n}$ where T_n :

$B_{x_n} \to \mathbb{R}$ is a $C^{1+\beta}$ function with $T_n(x_n) = r_\Lambda(x_n)$. Let $G_0 = \mathrm{Id}$ and $G_n := g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0}$, $n \geq 1$. For $n \geq 0$, define $\tau_n : [-p^s, p^s] \to \mathbb{R}$ by

$$\tau_n(t) := \sum_{k=0}^{n-1} T_k(G_k[\Psi_x(t, F(t))]),$$

equal to the flow displacement of the point $\Psi_x(t, F(t))$ under the maps $g^+_{x_0}, g^+_{x_1}, \ldots, g^+_{x_{n-1}}$. Define $\Delta_n : [-p^s, p^s] \to \mathbb{R}$ by $\Delta_n(t) := \tau_n(t) - \tau_n(0)$ for $n \geq 0$, and $\Delta : [-p^s, p^s] \to \mathbb{R}$ by $\Delta(t) := \lim_{n\to+\infty} \Delta_n(t)$. We have:

- $\mathrm{Lip}(T_n) < 1$, by Lemma 2.1(3).
- $\|\Delta - \Delta_n\|_{C^0} < \varepsilon e^{-\frac{\chi}{2}n}$ for all $n \geq 0$, since

$$\begin{aligned}\|\Delta - \Delta_n\|_{C^0} &\leq \sum_{k=n}^{\infty} \|T_k(G_k[\Psi_x(\cdot, F(\cdot))]) - T_k(G_k[\Psi_x(0, F(0))])\|_{C^0} \\ &\overset{!}{\leq} \sum_{k=n}^{\infty} \mathrm{Lip}(T_k) 6p^s e^{-\frac{\chi \inf(r_\Lambda)}{2}k} \leq \frac{6p^s}{1 - e^{-\frac{\chi \inf(r_\Lambda)}{2}}} e^{-\frac{\chi \inf(r_\Lambda)}{2}n} \overset{!!}{<} \varepsilon e^{-\frac{\chi \inf(r_\Lambda)}{2}n},\end{aligned}$$

  where in $\overset{!}{\leq}$ we used Theorem 4.5(3) and in $\overset{!!}{<}$ we used that $\frac{6p^s}{1-e^{-\frac{\chi \inf(r_\Lambda)}{2}}} < \frac{6\varepsilon^{3/\beta}}{1-e^{-\frac{\chi \inf(r_\Lambda)}{2}}} < \varepsilon$ when $\varepsilon > 0$ is small enough.

Let $\widetilde{V}^s := \{\varphi^{\Delta(t)}[\Psi_x(t, F(t))] : |t| \leq p^s\}$. Fix $\widetilde{y}, \widetilde{z} \in \widetilde{V}^s$, say $\widetilde{y} = \varphi^{\Delta(t_0)}[\Psi_x(t_0, F(t_0))] = \varphi^{\Delta(t_0)}(y)$ and $\widetilde{z} = \varphi^{\Delta(t_1)}[\Psi_x(t_1, F(t_1))] = \varphi^{\Delta(t_1)}(z)$ with $t_0, t_1 \in [-p^s, p^s]$. By definition, $y, z \in V^s$. Fix $t \geq 0$, and take the unique $n \geq 0$ such that $\tau_{n-1}(0) < t \leq \tau_n(0)$. For such n, write $\Delta = \Delta_n + E$, with $\|E\|_{C^0} < \varepsilon e^{-\frac{\chi \inf(r_\Lambda)}{2}n}$. Therefore

$$\varphi^t(\widetilde{y}) = \varphi^{t+\Delta(t_0)}(y) = \varphi^{t+\Delta_n(t_0)+E(t_0)}(y) = \varphi^{t-\tau_n(0)+E(t_0)}[G_n(y)],$$

and similarly $\varphi^t(\widetilde{z}) = \varphi^{t-\tau_n(0)+E(t_1)}[G_n(z)]$, hence

$$\begin{aligned} d(\varphi^t(\widetilde{y}), \varphi^t(\widetilde{z})) &\leq d(\varphi^{t-\tau_n(0)+E(t_0)}[G_n(y)], \varphi^{t-\tau_n(0)+E(t_0)}[G_n(z)]) + \\ &\quad d(\varphi^{t-\tau_n(0)+E(t_0)}[G_n(z)], \varphi^{t-\tau_n(0)+E(t_1)}[G_n(z)]) \\ &\leq \sup_{|\zeta|\leq 1} \mathrm{Lip}(\varphi^\zeta) d(G_n(y), G_n(z)) + \|X\|_{C^0} |E(t_0) - E(t_1)| \\ &\leq \left[6p^s \sup_{|\zeta|\leq 1} \mathrm{Lip}(\varphi^\zeta) + 2\varepsilon \|X\|_{C^0}\right] e^{-\frac{\chi \inf(r_\Lambda)}{2}n} \leq e^{-\frac{\chi \inf(r_\Lambda)}{2}n} \end{aligned}$$

for $\varepsilon > 0$ small. Since $t \leq \tau_n(0) \leq n \sup(r_\Lambda)$, we get that $d(\varphi^t(\widetilde{y}), \varphi^t(\widetilde{z})) \leq e^{-\frac{\chi \inf(r_\Lambda)}{2\sup(r_\Lambda)}t}$. $\square$

We note two important facts. Firstly, the choice of $\Delta(0) = 0$ is arbitrary: given $y = \Psi_x(t, F(t)) \in V^s$, we can choose Δ so that $\Delta(t) = 0$. The resulting smooth curve $\widetilde{V}^s \ni y$ also satisfies Proposition 4.8.

The second is more relevant. Given $y \in V^s = V^s[\underline{v}]$, lift V^s to $\widetilde{V}^s \ni y$, and let $\widetilde{e}^s_y$ be a unitary vector tangent to $\widetilde{V}^s$ at y (it is defined up to a sign). By construction, the projection of $\widetilde{e}^s_y$ in the flow direction is a multiple of $\widetilde{n}^s_y$. Taking the angle $\angle(N_y, X(y))$ into account, we can prove that $d\varphi^t \widetilde{e}^s_y$ contracts exponentially fast as $t \to +\infty$, i.e. $\limsup\limits_{t\to+\infty} \frac{1}{t}\log\|d\varphi^t \widetilde{e}^s_y\| < 0$ (just a lim sup, not necessarily a lim). The same holds for u–admissible manifolds $V^u[\underline{v}]$. Therefore, given an ε–gpo $\underline{v} = \{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\in\mathbb{Z}}$, if $V^s[\underline{v}] \cap V^u[\underline{v}] = \{x\}$, there are two smooth curves $\widetilde{V}^s, \widetilde{V}^u$ passing through x satisfying the following:

- If $\widetilde{e}^s_x$ is a unitary vector tangent to $\widetilde{V}^s$ at x, then $\limsup\limits_{t\to+\infty} \frac{1}{t}\log\|d\varphi^t \widetilde{e}^s_x\| < 0$.
- If $\widetilde{e}^u_x$ is a unitary vector tangent to $\widetilde{V}^u$ at x, then proceeding as in [36, Prop. 6.5] we show that $\limsup\limits_{t\to+\infty} \frac{1}{t}\log\|d\varphi^t \widetilde{e}^u_x\| > 0$.[2]

These two properties above uniquely define the directions $\widetilde{e}^s_x, \widetilde{e}^u_x$ (up to a sign). Therefore we can consider $\alpha(x), s(x), u(x)$, although we do not know that $s(x), u(x)$ are finite. Remember that $\widetilde{n}^s_x, \widetilde{n}^u_x \in N_x$, the tangent vectors to V^s, V^u at x, are the projections of $\widetilde{e}^s_x, \widetilde{e}^u_x$ in the flow direction.

We finish this section proving another property about invariant manifolds.

Proposition 4.9. *The following holds for $\varepsilon > 0$ small enough. Let $\underline{v}^+ = \{v_n\}_{n\ge 0}$ and $\underline{w} = \{w_n\}_{n\ge 0}$ be positive ε–gpo's, with $v_0 = \Psi^{p^s,p^u}_x$ and $w_0 = \Psi^{q^s,q^u}_x$. Then either $V^s[\underline{v}^+], V^s[\underline{w}^+]$ are disjoint or one contains the other.*

Proof. For $C^{1+\beta}$ surface diffeomorphism, this is [36, Prop. 6.4]. We apply a similar idea, using Proposition 4.8. Write $V^s = V^s[\underline{v}^+]$ and $U^s = V^s[\underline{w}^+]$. If $V^s \cap U^s = \emptyset$, we are done, so assume there is $z \in V^s \cap U^s$. Assuming without loss of generality that $q^s \le p^s$, we will prove that $U^s \subset V^s$. The proof will follow from three claims as in [36, Prop. 6.4]. Write $\underline{v}^+ = \{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\ge 0}$. We continue using the same terminology of the previous proposition, with $g^+_{x_n} = \varphi^{T_n}$ for $n \ge 0$, $G_0 = \mathrm{Id}$, and $G_n = g^+_{x_{n-1}} \circ \cdots \circ g^+_{x_0}$ for $n \ge 1$.

Claim 1: If n is large enough then $G_n(V^s) \subset \Psi_{x_n}(R[\frac{1}{2}Q(x_n)])$.

Proof of Claim 1. Same as [36, Prop. 6.4], using that the representation of $g^+_{x_n}$ in Pesin charts satisfies Theorem 3.8'. □

Claim 2: If n is large enough then $G_n(U^s) \subset \Psi_{x_n}(R[Q(x_n)])$.

---

[2] One important ingredient in the proof of [36, Prop. 6.5] is the estimate $p^s_{n+1} \wedge p^u_{n+1} \le e^{\varepsilon}(p^s_n \wedge p^u_n)$. We have a similar result, by Lemma 4.2.

Proof of Claim 2. Lift U^s to a curve $\widetilde{U}^s$ passing through z and satisfying Proposition 4.8. Fix $n \geq 0$, and let $t_n = \sum_{k=0}^{n-1} T_k(G_k(z))$ be the total flow time of z under G_n. Let $z_n = G_n(z) = \varphi^{t_n}(z)$. If $D \subset \widehat{\Lambda}$ is the disc containing x_n then

$$G_n(U^s) = \mathfrak{q}_D[\varphi^{t_n}(\widetilde{U}^s)].$$

Let $c := \inf(r_\Lambda)^2/2\sup(r_\Lambda)$. Since $\mathfrak{q}_D$ is 2–Lipschitz (Lemma 2.1(2)), Lemma 2.4 and Proposition 4.8 imply that

$$\operatorname{diam}(G_n(U^s)) = \operatorname{diam}(\mathfrak{q}_D[\varphi^{t_n}(\widetilde{U}^s)]) \leq 2\operatorname{diam}(\varphi^{t_n}(\widetilde{U}^s)) \leq 2e^{-\frac{\chi \inf(r_\Lambda)}{2\sup(r_\Lambda)}t_n} \leq 2e^{-\chi cn},$$

since $t_n \geq \inf(r_\Lambda)n$. Hence $\Psi_{x_n}^{-1}[G_n(U^s)]$ is contained in the ball with center $\Psi_{x_n}^{-1}(z_n)$ and radius $4\|C(x_n)^{-1}\|e^{-\chi cn}$. Since by Claim 1 we have $\Psi_{x_n}^{-1}(z_n) \in R[\frac{1}{2}Q(x_n)]$, it is enough to prove that $4\|C(x_n)^{-1}\|e^{-\chi cn} < \frac{1}{2}Q(x_n)$. Using that $Q(x_n) < \|C(x_n)^{-1}\|^{-1}$, we just need to prove that $8Q(x_n)^{-2}e^{-\chi cn} < 1$. We claim that $Q(x_n)^{-2}e^{-\chi cn}$ converges to zero exponentially fast as n increases. Indeed, by Lemma 4.2 we have $Q(x_n) \geq p_n^s \wedge p_n^u \geq e^{-2\varepsilon n}(p_0^s \wedge p_0^u)$ and so

$$Q(x_n)^{-2}e^{-\chi cn} \leq e^{4\varepsilon n}(p_0^s \wedge p_0^u)^{-2}e^{-\chi cn} = (p_0^s \wedge p_0^u)^{-2}e^{-(\chi c-4\varepsilon)n}$$

which converges to zero if $\varepsilon > 0$ is small enough. □

By Theorem 4.5(1), we conclude that $G_n(U^s) \subset V^s[\{\Psi_{x_k}^{p_k^s,p_k^u}\}_{k\geq n}]$ for every n large enough.

Claim 3: $U^s \subset V^s$.

Proof of Claim 3. Fix n large enough so that $G_n(U^s) \subset V^s[\{\Psi_{x_k}^{p_k^s,p_k^u}\}_{k\geq n}]$, and proceed as in Claim 3 of [36, Prop. 6.4]. □

The proof of the proposition is complete. □

5. First coding

Up to now, we have fixed $\varphi, \chi, \rho, \Lambda, \widehat{\Lambda}, \theta, \varepsilon$ such that $\varepsilon \ll \rho \ll 1$, and we have constructed invariant manifolds for ε–gpo's. We also defined shadowing. In this section, we:

- Construct a countable family of ε–double charts whose ε–gpo's they define shadow the whole set $\Lambda \cap \mathrm{NUH}^\#$.
- Define a first coding, that is usually infinite-to-one.

5.1. Coarse graining

This self-contained section comprises an important part of this work that cannot be obtained using the methods of [36,28,27]. Indeed, condition (GPO2) in our definition of edge between ε–double charts is a set of inequalities, so we need to show it is loose enough to shadow all points of $\Lambda \cap \mathrm{NUH}^{\#}$. The proof of this fact requires an analysis of orbits at hyperbolic times, where parameters are essentially uniquely defined.

Theorem 5.1 *(Coarse graining). For all $0 < \varepsilon \ll \rho \ll 1$, there exists a countable family $\mathscr{A}$ of ε–double charts with the following properties:*

(1) *Discreteness: For all $t > 0$, the set $\{\Psi_x^{p^s,p^u} \in \mathscr{A} : p^s, p^u > t\}$ is finite.*
(2) *Sufficiency: If $x \in \Lambda \cap \mathrm{NUH}^{\#}$ then there is a regular ε–gpo $\underline{v} \in \mathscr{A}^{\mathbb{Z}}$ that shadows x.*
(3) *Relevance: For each $v \in \mathscr{A}$, $\exists \underline{v} \in \mathscr{A}^{\mathbb{Z}}$ an ε–gpo with $v_0 = v$ that shadows a point in $\Lambda \cap \mathrm{NUH}^{\#}$.*

Recall that $\underline{v} = \{v_n\}_{n\in\mathbb{Z}}$ is regular if there are v, w such that $v_n = v$ for infinitely many $n > 0$ and $v_n = w$ for infinitely many $n < 0$. According to Proposition 3.5 and part (2) above, the ε–gpo's in $\mathscr{A}$ shadow almost every point with respect to every χ–hyperbolic measure.

Proof. When M is a closed surface and f is a diffeomorphism, the above statement is consequence of Propositions 3.5, 4.5 and Lemmas 4.6, 4.7 of [36]. When M is a compact surface with boundary and f is a local diffeomorphism with bounded derivatives, this is Proposition 4.3 of [28]. When M is a surface and f is a local diffeomorphism with unbounded derivatives, this is Theorem 5.1 of [27]. Our proof follows a similar strategy of [36,28] but the implementation is significantly harder, since the definition of edge is more complicated. In particular, we need to control the cumulative shear between an orbit and an ε–gpo.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $X := \Lambda^3 \times \mathrm{GL}(2,\mathbb{R})^3 \times (0,1]$. For $x \in \Lambda \cap \mathrm{NUH}^{\#}$, let $\Gamma(x) = (\underline{x}, \underline{C}, \underline{Q}) \in X$ with

$$\underline{x} = (f^{-1}(x), x, f(x)),\ \underline{C} = (C(f^{-1}(x)), C(x), C(f(x))),\ \underline{Q} = (Q(x), q(x)).$$

Let $Y = \{\Gamma(x) : x \in \Lambda \cap \mathrm{NUH}^{\#}\}$. We want to construct a countable dense subset of Y. Since the maps $x \mapsto C(x), Q(x), q(x)$ are usually just measurable, we apply a precompactness argument. For each $\underline{\ell} = (\ell_{-1}, \ell_0, \ell_1) \in \mathbb{N}_0^3$ and $m, j \in \mathbb{N}_0$, define

$$Y_{\underline{\ell},m,j} := \left\{ \Gamma(x) \in Y : \begin{array}{c} e^{\ell_i} \le \|C(f^i(x))^{-1}\| < e^{\ell_i+1}, \quad -1 \le i \le 1 \\ e^{-m-1} \le Q(x) < e^{-m} \\ e^{-j-1} \le q(x) < e^{-j} \end{array} \right\}.$$

Claim 1: $Y = \bigcup_{\substack{\underline{\ell}\in\mathbb{N}_0^3\\ m,j\in\mathbb{N}_0}} Y_{\underline{\ell},m,j}$, and each $Y_{\underline{\ell},m,j}$ is precompact in X.

Proof of Claim 1. The first statement is clear. We focus on precompactness. Fix $\underline{\ell}\in\mathbb{N}_0^3$, $m,j\in\mathbb{N}_0$, and take $\Gamma(x)\in Y_{\underline{\ell},m,j}$. Then $\underline{x}\in\Lambda^3$, a precompact subset of M^3. For $|i|\le 1$, $C(f^i(x))$ is an element of $\mathrm{GL}(2,\mathbb{R})$ with norm ≤ 1 and inverse norm $\le e^{\ell_i+1}$, hence it belongs to a compact subset of $\mathrm{GL}(2,\mathbb{R})$. This guarantees that $\underline{C}$ belongs to a compact subset of $\mathrm{GL}(2,\mathbb{R})^3$. Also, $\underline{Q}\in[e^{-m-1},1]\times[e^{-j-1},1]$, a compact subinterval of $(0,1]$. Since the product of precompact sets is precompact, the claim is proved. $\square$

By Claim 1, there exists a finite set $Z_{\underline{\ell},m,j}\subset Y_{\underline{\ell},m,j}$ such that for every $\Gamma(x)\in Y_{\underline{\ell},m,j}$ there exists $\Gamma(y)\in Z_{\underline{\ell},m,j}$ with:

(a) $d(f^i(x),f^i(y))+\|\widetilde{C(f^i(x))}-\widetilde{C(f^i(y))}\|<\frac{1}{2}q(x)^8$, $|i|\le 1$.
(b) $\frac{Q(x)}{Q(y)}=e^{\pm\varepsilon/3}$ and $\frac{q(x)}{q(y)}=e^{\pm\varepsilon/3}$.

A fortiori, (a) implies that $f^i(x),f^i(y)$ belong to the same disc of Λ, for $|i|\le 1$. For $\eta>0$, let $I_{\varepsilon,\eta}:=\{e^{-\varepsilon^2\eta k}:k\ge 0\}$, a countable discrete set whose "thickness" depends on η.

The alphabet $\mathscr{A}$: Let $\mathscr{A}$ be the countable family of $\Psi_x^{p^s,p^u}$ such that:

(CG1) $\Gamma(x)\in Z_{\underline{\ell},m,j}$ for some $(\underline{\ell},m,j)\in\mathbb{N}_0^3\times\mathbb{N}_0\times\mathbb{N}_0$.
(CG2) $0<p^s,p^u\le\varepsilon Q(x)$ and $p^s,p^u\in I_{\varepsilon,q(x)}$.
(CG3) $e^{-\mathfrak{H}-1}\le\frac{p^s\wedge p^u}{q(x)}\le e^{\mathfrak{H}+1}$, where $\mathfrak{H}$ is given by Proposition 3.6(1).

Proof of discreteness. Fix $t>0$, and let $\Psi_x^{p^s,p^u}\in\mathscr{A}$ with $p^s,p^u>t$. If $\Gamma(x)\in Z_{\underline{\ell},m,j}$ then:

- Finiteness of $\underline{\ell}$: we have $e^{\ell_0}\le\|C(x)^{-1}\|<Q(x)^{-1}<t^{-1}$, hence $\ell_0<|\log t|$. By Lemma 3.2(3), for $i=\pm1$ we have
$$e^{\ell_i}\le\|C(f^i(x))^{-1}\|\le\|C(f^i(x))^{-1}\|_{\mathrm{Frob}}\le e^{18\rho}\|C(x)^{-1}\|_{\mathrm{Frob}}<e^{18\rho}t^{-1},$$
hence $\ell_{-1},\ell_1<18\rho+|\log t|=:T_t$, which is bigger than $|\log t|$.
- Finiteness of m: $e^{-m}>Q(x)>t$, hence $m<|\log t|$.
- Finiteness of j: $e^{-j}>q(x)\ge e^{-\mathfrak{H}-1}(p^s\wedge p^u)>e^{-\mathfrak{H}-1}t$, hence $j\le|\log t|+\mathfrak{H}+1$.

Therefore
$$\#\left\{\Gamma(x):\Psi_x^{p^s,p^u}\in\mathscr{A}\text{ s.t. }p^s,p^u>t\right\}\le\sum_{j=0}^{\lceil|\log t|+\mathfrak{H}\rceil+1}\sum_{m=0}^{\lceil|\log t|\rceil}\sum_{\substack{-1\le i\le 1\\ \ell_i=0}}^{T_t}\#Z_{\underline{\ell},m,j}$$

is the finite sum of finite terms, hence finite. For each such $\Gamma(x)$,

$$\#\{(p^s,p^u):\Psi_x^{p^s,p^u}\in\mathscr{A}\text{ s.t. }p^s,p^u>t\}\le(\#I_{\varepsilon,q(x)}\cap(t,1))^2$$

is finite, hence

$$\#\left\{\Psi_x^{p^s,p^u}\in\mathscr{A}:p^s,p^u>t\right\}\le\sum_{j=0}^{\lceil|\log t|+\mathfrak{H}\rceil+1}\sum_{m=0}^{\lceil|\log t|\rceil}\sum_{\substack{-1\le i\le 1\\ \ell_i=0}}^{T_t}\sum_{\Gamma(x)\in Z_{\underline{\ell},m,j}}(\#I_{\varepsilon,q(x)}\cap(t,1))^2$$

is the finite sum of finite terms, hence finite. This proves the discreteness of $\mathscr{A}$.

Proof of sufficiency. Let $x\in\Lambda\cap\mathrm{NUH}^{\#}$. Take $(\ell_i)_{i\in\mathbb{Z}},(m_i)_{i\in\mathbb{Z}},(j_i)_{i\in\mathbb{Z}}$ such that:

$$\begin{aligned}&\|C(f^i(x))^{-1}\|\in[e^{\ell_i},e^{\ell_i+1}),Q(f^i(x))\in[e^{-m_i-1},e^{-m_i}),\\&q(f^i(x))\in[e^{-j_i-1},e^{-j_i}).\end{aligned}$$

For $n\in\mathbb{Z}$, let $\underline{\ell}^{(n)}=(\ell_{n-1},\ell_n,\ell_{n+1})$. Then $\Gamma(f^n(x))\in Y_{\underline{\ell}^{(n)},m_n,j_n}$. Take $\Gamma(x_n)\in Z_{\underline{\ell}^{(n)},m_n,j_n}$ such that:

(a$_n$) $d(f^i(f^n(x)),f^i(x_n))+\|\widetilde{C(f^i(f^n(x)))}-\widetilde{C(f^i(x_n))}\|<\frac{1}{2}q(f^n(x))^8$, $|i|\le 1$.
(b$_n$) $\frac{Q(f^n(x))}{Q(x_n)}=e^{\pm\varepsilon/3}$ and $\frac{q(f^n(x))}{q(x_n)}=e^{\pm\varepsilon/3}$.

From now on the proof differs from [36,28,27]. Take $\{t_n\}_{n\in\mathbb{Z}}$ such that $f^n(x)=\varphi^{t_n}(x)$, with $t_0=0$ and $g^+_{x_n}[f^n(x)]=\varphi^{t_{n+1}-t_n}[f^n(x)]$. Define

$$\begin{aligned}P^s_n&:=\varepsilon\inf\{e^{\varepsilon|t_{n+k}-t_n|}Q(x_{n+k}):k\ge 0\},\\P^u_n&:=\varepsilon\inf\{e^{\varepsilon|t_{n+k}-t_n|}Q(x_{n+k}):k\le 0\}.\end{aligned}$$

There is no reason for $\Psi_{x_n}^{P^s_n,P^u_n}$ belonging to $\mathscr{A}$ nor for $\{\Psi_{x_n}^{P^s_n,P^u_n}\}_{n\in\mathbb{Z}}$ being an ε–gpo. Indeed, with the above definitions one of the inequalities in (GPO2) holds in the reverse direction. To satisfy (GPO2), we will slightly decrease each P^s_n,P^u_n. Below we show how to make this "surgery" for P^s_n (the method for P^u_n is symmetric).

Start noting the greedy recursion $P^s_n=\min\{e^{\varepsilon(t_{n+1}-t_n)}P^s_{n+1},\varepsilon Q(x_n)\}$ and that

$$P^s_n=e^{\pm\frac{\varepsilon}{3}}\varepsilon\inf\{e^{\varepsilon|t_{n+k}-t_n|}Q(f^{n+k}(x)):k\ge 0\}=e^{\pm\frac{\varepsilon}{3}}p^s(x,\mathcal{T},n)=e^{\pm(\mathfrak{H}+\frac{\varepsilon}{3})}q^s(f^n(x)),$$

by (b$_n$) above and Proposition 3.6(1), where $\mathcal{T}=\{t_n\}_{n\in\mathbb{Z}}$. We fix $\lambda:=\exp[\varepsilon^{1.5}]$ and divide the indices $n\in\mathbb{Z}$ into two groups:

n is *growing* if $P^s_n\ge\lambda P^s_{n+1}$ and it is *maximal* otherwise.

Note that λ has an exponent with order smaller than ε. The definition of growing/maximal indices is motivated by the following: the parameter P_n^s gives a choice on the size of the stable manifold at x_n, therefore we expect P_n^s to be larger than P_{n+1}^s at least by a multiplicative factor bigger than λ, unless it reaches the maximal size $\varepsilon Q(x_n)$. In the first case the index is growing, and in the second it is maximal. Assuming that $\varepsilon > 0$ is sufficiently small, we note two properties of this notion:

- If n is maximal then $P_n^s = \varepsilon Q(x_n)$: otherwise $P_n^s = e^{\varepsilon(t_{n+1}-t_n)} P_{n+1}^s \geq e^{\varepsilon \inf(r_\Lambda)} P_{n+1}^s > \lambda P_{n+1}^s$, which contradicts the assumption that n is maximal.
- There are infinitely many maximal indices $n > 0$, and infinitely many maximal indices $n < 0$: the first claim follows exactly as in the proof of Proposition 3.6(3) (remember we are assuming that $x \in \mathrm{NUH}^\#$ and so $\limsup_{n\to+\infty} P_n^s > 0$). The second claim follows from direct computation: if there is n_0 such that every $n < n_0$ is growing then $P_n^s \geq \lambda^{n_0-n} P_{n_0}^s$ for all $n < n_0$, which cannot hold since $\lambda^{n_0-n} \to \infty$ as $n \to -\infty$.

We define $p_n^s = a_n P_n^s$ where $e^{-\varepsilon} < a_n \leq 1$ are appropriately chosen. We first define a_n for the maximal indices $n \in \mathbb{Z}$ as the largest value in $(0, 1]$ with $a_n P_n^s \in I_{\varepsilon,q(x_n)}$. In particular, $e^{-\varepsilon^2 q(x_n)} \leq a_n \leq 1$. Then we define a_n for the growing indices. Fix two consecutive maximal indices $n < m$ and define $a_{n+1}, \dots, a_{m-1}$ with a backwards induction as follows. If $n < k < m$ and a_{k+1} is well-defined then we choose a_k largest as possible satisfying:

(i) $e^{-\frac{\varepsilon}{4}P_k^s} a_{k+1} \leq e^{\frac{\varepsilon}{4}P_k^s} a_k \leq a_{k+1}$;
(ii) $a_k P_k^s \in I_{\varepsilon,q(x_k)}$.

This choice is possible because the interval $(e^{-\frac{\varepsilon}{4}P_k^s} a_{k+1}, a_{k+1}]$ intersects $I_{\varepsilon,q(x_k)}$, since $\frac{\varepsilon}{4}P_k^s \geq \frac{\varepsilon}{4} e^{-(\mathfrak{H}+\frac{\varepsilon}{3})} q^s(f^k(x)) \geq \frac{\varepsilon}{4} e^{-(\mathfrak{H}+\frac{\varepsilon}{3})} q(f^k(x)) \geq \frac{\varepsilon}{4} e^{-(\mathfrak{H}+\frac{2\varepsilon}{3})} q(x_k) > \varepsilon^2 q(x_k)$. The first condition implies that $0 < a_{n+1} \leq \dots \leq a_{m-1} \leq a_m \leq 1$. The maximality on the choice of a_k indeed implies the inequality $e^{-\varepsilon^2 q(x_k)} a_{k+1} \leq e^{\frac{\varepsilon}{4}P_k^s} a_k \leq a_{k+1}$ for every growing k (this is stronger than (i)).

Before continuing, we collect some estimates relating $q(x_k), P_k^s, p_k^s$. Fix two consecutive maximal indices $n < m$. Then the following holds for all $\varepsilon > 0$ small enough:

- $\sum_{k=n+1}^{m} P_k^s < \varepsilon^{\frac{3}{\beta}-1}$: every $k = n+1, \dots, m-1$ is growing, thus $P_k^s \leq \lambda^{n+1-k} P_{n+1}^s$ for $k = n+1, \dots, m$. This implies that

$$\sum_{k=n+1}^{m} P_k^s \leq P_{n+1}^s \sum_{i=0}^{m-n-1} \lambda^{-i} < \varepsilon^{\frac{3}{\beta}+1} \frac{1}{1-\lambda^{-1}} < 2\varepsilon^{\frac{3}{\beta}-0.5} < \varepsilon^{\frac{3}{\beta}-1},$$

since $\lim\limits_{\varepsilon\to 0}\frac{\varepsilon^{1.5}}{1-\lambda^{-1}}=1$.

- $\sum\limits_{k=n+1}^{m} q(x_k)<\varepsilon^{\frac{3}{\beta}-1}$: by the previous item,

$$\sum_{k=n+1}^{m} q(x_k)\leq e^{\mathfrak{H}+\frac{2\varepsilon}{3}}\sum_{k=n+1}^{m} P_k^s<2e^{\mathfrak{H}+\frac{2\varepsilon}{3}}\varepsilon^{\frac{3}{\beta}-0.5}<\varepsilon^{\frac{3}{\beta}-1}.$$

- $a_{n+1}>\lambda^{-1}$: using that $a_m\geq e^{-\varepsilon^2 q(x_m)}>e^{-\varepsilon P_m^s}$ and that $e^{-\varepsilon P_k^s}a_{k+1}\leq a_k$ for every growing k, we have

$$a_{n+1}\geq \exp\left[-\varepsilon\sum_{k=n+1}^{m-1}P_k^s\right]a_m\geq \exp\left[-\varepsilon\sum_{k=n+1}^{m}P_k^s\right]>\exp\left[-\varepsilon^{\frac{3}{\beta}}\right]>\lambda^{-1},$$

since $\varepsilon^{\frac{3}{\beta}}<\varepsilon^{1.5}$.

In particular, $a_k>\lambda^{-1}>e^{-\varepsilon}$ for all $k\in\mathbb{Z}$.

Claim 2: $\Psi_{x_n}^{p_n^s,p_n^u}\in\mathscr{A}$ for all $n\in\mathbb{Z}$.

Proof of Claim 2. We have to check (CG1)–(CG3).

(CG1) By definition, $\Gamma(x_n)\in Z_{\underline{\ell}^{(n)},m_n,j_n}$.

(CG2) We have $p_n^s\leq P_n^s\leq\varepsilon Q(x_n)$, and the same holds for p_n^u. By definition, $p_n^s,p_n^u\in I_{\varepsilon,q(x_n)}$.

(CG3) Since $e^{-\varepsilon}<a_n\leq 1$ and $P_n^s=e^{\pm(\mathfrak{H}+\frac{2\varepsilon}{3})}q^s(x_n)$, we have $e^{-\mathfrak{H}-2\varepsilon}\leq\frac{p_n^s}{q^s(x_n)}\leq e^{\mathfrak{H}+\varepsilon}$. By the same reason, ${}^{-\mathfrak{H}-2\varepsilon}\leq\frac{p_n^u}{q^u(x_n)}\leq e^{\mathfrak{H}+\varepsilon}$. These inequalities imply that $e^{-\mathfrak{H}-2\varepsilon}\leq\frac{p_n^s\wedge p_n^u}{q(x_n)}\leq e^{\mathfrak{H}+\varepsilon}$ and so $e^{-\mathfrak{H}-1}\leq\frac{p_n^s\wedge p_n^u}{q(x_n)}\leq e^{\mathfrak{H}+1}$. □

Claim 3: $\Psi_{x_n}^{p_n^s,p_n^u}\overset{\varepsilon}{\to}\Psi_{x_{n+1}}^{p_{n+1}^s,p_{n+1}^u}$ for all $n\in\mathbb{Z}$.

Proof of Claim 3. We have to check (GPO1)–(GPO2).

(GPO1) By (a_n) with $i=1$ and (a_{n+1}) with $i=0$, we have

$$\begin{aligned}
&d(f(x_n),x_{n+1})+\|C(\widetilde{f(x_n)})-C(\widetilde{x_{n+1}})\|\\
&\leq d(f^{n+1}(x),f(x_n))+\|C(\widetilde{f^{n+1}}(x))-C(\widetilde{f(x_n)})\|\\
&\quad+d(f^{n+1}(x),x_{n+1})+\|C(\widetilde{f^{n+1}}(x))-C(\widetilde{x_{n+1}})\|\\
&<\tfrac{1}{2}q(f^n(x))^8+\tfrac{1}{2}q(f^{n+1}(x))^8\overset{!}{\leq}\tfrac{1}{2}(1+e^{8\varepsilon})q(f^{n+1}(x))^8\\
&\overset{!!}{\leq}\tfrac{1}{2}e^{8\mathfrak{H}+\frac{56\varepsilon}{3}}(1+e^{8\varepsilon})(p_{n+1}^s\wedge p_{n+1}^u)^8\overset{!!!}{<}(p_{n+1}^s\wedge p_{n+1}^u)^8,
\end{aligned}$$

where in $\overset{!}{\leq}$ we used Lemma 3.4, in $\overset{!!}{\leq}$ we used (b_n) and the estimate used to prove (CG3) in the previous paragraph, and in $\overset{!!!}{<}$ we used that $\frac{1}{2}e^{8\mathfrak{H}+\frac{56\varepsilon}{3}}(1+e^{8\varepsilon})<1$ when $\varepsilon,\rho>0$ are sufficiently small. This proves that $\Psi_{f(x_n)}^{p^s_{n+1}\wedge p^u_{n+1}} \overset{\varepsilon}{\approx} \Psi_{x_{n+1}}^{p^s_{n+1}\wedge p^u_{n+1}}$. Similarly, we prove that $\Psi_{f^{-1}(x_{n+1})}^{p^s_n\wedge p^u_n} \overset{\varepsilon}{\approx} \Psi_{x_n}^{p^s_n\wedge p^u_n}$.

(GPO2) We show that relation (4.1) holds for all $k\in\mathbb{Z}$:

$$e^{-\varepsilon p^s_k}\min\{e^{\varepsilon T(v_k,v_{k+1})}p^s_{k+1}, e^{-\varepsilon}\varepsilon Q(x_k)\} \leq p^s_k \leq \min\{e^{\varepsilon T(v_k,v_{k+1})}p^s_{k+1}, \varepsilon Q(x_k)\}.$$

Relation (4.2) is proved similarly. For ease of notation, write $T_k = T(v_k, v_{k+1})$ and $\Delta_k = (t_{k+1}-t_k)-T_k$. Since T_k is the minimal time, we have $\Delta_k \geq 0$. Using Lemma 2.1(3), condition (a_n) and Remark 4.1, we also have the following upper bound for Δ_k:

$$\Delta_k \leq \mathrm{diam}(R[\tfrac{1}{15}(p^s_k\wedge p^u_k)]) = \tfrac{\sqrt{2}}{15}(p^s_k\wedge p^u_k) < \tfrac{p^s_k}{4}.$$

We fix two consecutive maximal indices $n<m$ and establish the above inequality for $k=n,\dots,m-1$. We divide the proof into two cases: $k=n$ and $k\neq n$. Assume first that $k=n$. For $\varepsilon>0$ small enough (remember $a_{n+1}>\lambda^{-1}$),

$$e^{\varepsilon T_n}p^s_{n+1} = e^{\varepsilon T_n}a_{n+1}P^s_{n+1} > \exp\left[\inf(r_\Lambda)\varepsilon-\varepsilon^{1.5}\right]P^s_{n+1} > \lambda P^s_{n+1} > P^s_n = \varepsilon Q(x_n).$$

Therefore

$$e^{-\varepsilon p^s_n}\min\{e^{\varepsilon T_n}p^s_{n+1}, e^{-\varepsilon}\varepsilon Q(x_n)\} = e^{-\varepsilon p^s_n}e^{-\varepsilon}\varepsilon Q(x_n) < e^{-\varepsilon}\varepsilon Q(x_n) < a_n P^s_n = p^s_n$$

and

$$\min\{e^{\varepsilon T_n}p^s_{n+1}, \varepsilon Q(x_n)\} = \varepsilon Q(x_n) = P^s_n \geq p^s_n.$$

This proves (4.1) for $k=n$.

Now let $k\neq n$, and call $\mathrm{I}=\min\{e^{\varepsilon T_k}p^s_{k+1}, e^{-\varepsilon}\varepsilon Q(x_k)\}$, $\mathrm{II}=\min\{e^{\varepsilon T_k}p^s_{k+1}, \varepsilon Q(x_k)\}$. We wish to show that $e^{-\varepsilon p^s_k}\mathrm{I}\leq p^s_k\leq \mathrm{II}$. Since $a_{k+1}\geq e^{-\varepsilon\Delta_k}a_{k+1} > \exp\left[-\varepsilon\frac{p^s_k}{4}-\varepsilon^{1.5}\right] > \exp[-\varepsilon]$, we have

$$\begin{aligned}\mathrm{I} &= \min\{e^{-\varepsilon\Delta_k}a_{k+1}e^{\varepsilon(t_{k+1}-t_k)}P^s_{k+1}, e^{-\varepsilon}\varepsilon Q(x_k)\}\\ &\leq a_{k+1}\min\{e^{\varepsilon(t_{k+1}-t_k)}P^s_{k+1}, \varepsilon Q(x_k)\} = a_{k+1}P^s_k.\end{aligned}$$

Therefore $e^{-\varepsilon p^s_k}\mathrm{I}\leq e^{-\frac{\varepsilon}{2}P^s_k}a_{k+1}P^s_k \leq a_k P^s_k = p^s_k$, where in the second inequality we used property (i) in the definition of a_k.

For the other inequality, start observing that

$$p^s_k = a_k P^s_k = a_k\min\{e^{\varepsilon(t_{k+1}-t_k)}P^s_{k+1}, \varepsilon Q(x_k)\} = \min\{e^{\varepsilon(t_{k+1}-t_k)}a_k P^s_{k+1}, a_k\varepsilon Q(x_k)\}.$$

Clearly $a_k \varepsilon Q(x_k) \leq \varepsilon Q(x_k)$. Using that $\Delta_k \leq \frac{P^s_k}{4}$, we have $e^{\varepsilon \Delta_k} a_k \leq e^{\frac{\varepsilon}{4} P^s_k} a_k \leq a_{k+1}$, where in the last passage we used property (i) in the definition of a_k. Hence

$$e^{\varepsilon(t_{k+1}-t_k)} a_k P^s_{k+1} = e^{\varepsilon T_k} e^{\varepsilon \Delta_k} a_k P^s_{k+1} \leq e^{\varepsilon T_k} a_{k+1} P^s_{k+1} = e^{\varepsilon T_k} p^s_{k+1}.$$

The conclusion is that $p^s_k \leq$ II. The proof of Claim 3 is now complete. □

Claim 4: $\{\Psi^{p^s_n, p^u_n}_{x_n}\}_{n\in\mathbb{Z}}$ is regular.

Proof of Claim 4. Since $x \in \mathrm{NUH}^\#$ and $\frac{p^s_n \wedge p^u_n}{q(f^n(x))} = e^{\pm(\mathfrak{H}+1)}$, we have $\limsup_{n\to+\infty} p^s_n \wedge p^u_n > 0$ and $\limsup_{n\to-\infty} p^s_n \wedge p^u_n > 0$. By the discreteness of $\mathscr{A}$, it follows that $\Psi^{p^s_n, p^u_n}_{x_n}$ repeats infinitely often in the future and infinitely often in the past. □

Claim 5: $\{\Psi^{p^s_n, p^u_n}_{x_n}\}_{n\in\mathbb{Z}}$ shadows x.

Proof of Claim 5. By (a_n) with $i = 0$, we have $\Psi^{p^s_n \wedge p^u_n}_{f^n(x)} \overset{\varepsilon}{\approx} \Psi^{p^s_n \wedge p^u_n}_{x_n}$, hence by Proposition 3.10(3) we have $f^n(x) = \Psi_{f^n(x)}(0) \in \Psi_{x_n}(R[p^s_n \wedge p^u_n])$, thus $\{\Psi^{p^s_n, p^u_n}_{x_n}\}_{n\in\mathbb{Z}}$ shadows x. This concludes the proof of sufficiency. □

Proof of relevance. The alphabet $\mathscr{A}$ might not a priori satisfy the relevance condition, but we can easily reduce it to a sub-alphabet $\mathscr{A}'$ satisfying (1)–(3). Call $v \in \mathscr{A}$ relevant if there is $\underline{v} \in \mathscr{A}^\mathbb{Z}$ with $v_0 = v$ such that $\underline{v}$ shadows a point in $\Lambda \cap \mathrm{NUH}^\#$. Since $\mathrm{NUH}^\#$ is φ–invariant, every v_i is relevant. Hence $\mathscr{A}' = \{v \in \mathscr{A} : v \text{ is relevant}\}$ is discrete because $\mathscr{A}' \subset \mathscr{A}$, it is sufficient and relevant by definition. □

5.2. First coding

Let Σ be the TMS associated to the graph with vertex set $\mathscr{A}$ given by Theorem 5.1 and edges $v \overset{\varepsilon}{\to} w$. An element $\underline{v} \in \Sigma$ is an ε–gpo, so let $\pi : \Sigma \to \widehat{\Lambda}$ by

$$\{\pi(\underline{v})\} := V^s[\underline{v}] \cap V^u[\underline{v}].$$

Here are the main properties of the triple (Σ, σ, π).

Proposition 5.2. *The following holds for all* $0 < \varepsilon \ll \rho \ll 1$.

(1) *Each* $v \in \mathscr{A}$ *has finite ingoing and outgoing degree, hence* Σ *is locally compact.*
(2) $\pi : \Sigma \to \widehat{\Lambda}$ *is Hölder continuous.*
(3) $\pi[\Sigma^\#] \supset \Lambda \cap \mathrm{NUH}^\#$.

Part (1) follows from Lemma 4.2 and Theorem 5.1(1), part (2) follows from Theorem 4.5(5), and part (3) follows from Theorem 5.1(2). It is important noting that

(Σ, σ, π) is *not* the TMS that satisfies the Main Theorem, since π might be (and usually is) infinite-to-one. We use π to induce a locally finite cover of $\Lambda \cap \mathrm{NUH}^{\#}$, which will then be refined to generate a new TMS whose TMF is the one satisfying the Main Theorem.

We finish this section introducing the TMF generated by (Σ, σ, π). Remember that $r_\Lambda : \Lambda \to (0, \rho/2]$ is the first return time to Λ.

The roof function $r : \Sigma \to (0, \rho)$: Given $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}} \in \Sigma$, let $x = \pi(\underline{v})$ and assume that x_1 belongs to the disc $D \subset \widehat{\Lambda}$. Define $r(\underline{v}) := r_\Lambda(x_0) - \mathfrak{t}_D[\varphi^{r_\Lambda(x_0)}(x)]$.

Since $g_{x_0}^+ = \mathfrak{q}_D \circ \varphi^{r_\Lambda(x_0)}$, $r(\underline{v})$ is the time increment for φ between the points $\pi(\underline{v})$ and $g_{x_0}^+[\pi(\underline{v})]$. In particular, $\varphi^{r(\underline{v})}[\pi(\underline{v})] = \pi[\sigma(\underline{v})]$ belongs to $\widehat{\Lambda}$ but not necessarily to Λ. (Note: even if $\pi(\underline{v}), \varphi^{r(\underline{v})}[\pi(\underline{v})] \in \Lambda$, the values of $r(\underline{v})$ and $r_\Lambda[\pi(\underline{v})]$ may be different.)

The triple $(\Sigma_r, \sigma_r, \pi_r)$: We take (Σ_r, σ_r) to be the TMF associated to the TMS (Σ, σ) and roof function r, and $\pi_r : \Sigma_r \to M$ to be the map defined by $\pi_r[(\underline{v}, t)] = \varphi^t[\pi(\underline{v})]$.

The next proposition collects the main properties of $(\Sigma_r, \sigma_r, \pi_r)$.

Proposition 5.3. *The following holds for all* $0 < \varepsilon \ll \rho \ll 1$.

(1) $\pi_r \circ \sigma_r^t = \varphi^t \circ \pi_r$, *for all* $t \in \mathbb{R}$.
(2) π_r *is Hölder continuous with respect to the Bowen-Walters distance.*
(3) $\pi_r[\Sigma_r^{\#}] \supset \mathrm{NUH}^{\#}$.

Proof. Part (1) is direct from the definition of π_r. The proof of Part (2) uses Proposition 5.2(2), and follows by the same methods used in the proof of [28, Lemma 5.9]. To prove part (3), let $S := \Sigma^{\#} \times \{0\} \subset \Sigma_r^{\#}$. By Proposition 5.2(3), $\pi_r[S] \supset \Lambda \cap \mathrm{NUH}^{\#}$. Since $\pi_r[\Sigma_r^{\#}] = \bigcup_{t\in\mathbb{R}} \varphi^t[\pi_r(S)]$ and $\mathrm{NUH}^{\#} = \bigcup_{t\in\mathbb{R}} \varphi^t[\Lambda \cap \mathrm{NUH}^{\#}]$, we get that $\pi_r[\Sigma_r^{\#}] \supset \mathrm{NUH}^{\#}$. □

6. Inverse theorem

In the previous section, we have constructed a first coding $\pi : \Sigma \to \widehat{\Lambda}$. As mentioned, it is usually infinite-to-one. In this section, we investigate how π loses injectivity: if $\underline{v} \in \Sigma$ and $x = \pi(\underline{v})$, what is the relation between the parameters defining $\underline{v}$ and those associated to the orbit of x? Our goal is to analyze this as an *inverse problem*: fixed $x \in \widehat{\Lambda}$, the parameters of $\underline{v}$ are defined "up to bounded error". The answer to this inverse problem is what we call an *inverse theorem*. From now on, we require that $\underline{v} \in \Sigma^{\#}$, where $\Sigma^{\#}$ is the *regular set* of Σ:

$$\Sigma^{\#} := \left\{ \underline{v} \in \Sigma : \exists v, w \in V \text{ s.t. } \begin{array}{l} v_n = v \text{ for infinitely many } n > 0 \\ v_n = w \text{ for infinitely many } n < 0 \end{array} \right\}.$$

Recall $r : \Sigma \to (0, \rho)$, the roof function defined before Proposition 5.3. Let r_n denote its n–th Birkhoff sum with respect to the shift map $\sigma : \Sigma \to \Sigma$. Let $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}} \in \Sigma$, and let $x = \pi(\underline{v})$. Then:

- $\varphi^{r_n(\underline{v})}(x) = \pi[\sigma^n(\underline{v})]$, a point in $\widehat{\Lambda}$ that is close to x_n.
- $g_{x_n}^+[\varphi^{r_n(\underline{v})}(x)] = \varphi^{r_{n+1}(\underline{v})}(x)$.

Let $p^{s/u}(\varphi^{r_n(\underline{v})}(x))$ be the $\mathbb{Z}$–indexed version of the parameter $q^{s/u}$ with respect to the sequence of times $\{r_n(\underline{v})\}_{n\in\mathbb{Z}}$ (see Section 3.5 for the definition).

Theorem 6.1 *(Inverse theorem). The following holds for all $0 < \varepsilon \ll \rho \ll 1$. If $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}} \in \Sigma^\#$ and $x = \pi(\underline{v})$, then $x \in \mathrm{NUH}^\#$ and the following are true.*

(1) $d(\varphi^{r_n(\underline{v})}(x), x_n) < 50^{-1}(p_n^s \wedge p_n^u)$.
(2) $\frac{\sin\alpha(x_n)}{\sin\alpha(\varphi^{r_n(\underline{v})}(x))} = e^{\pm(p_n^s\wedge p_n^u)^{\beta/4}}$, $|\cos\alpha(x_n) - \cos\alpha(\varphi^{r_n(\underline{v})}(x))| < 2(p_n^s \wedge p_n^u)^{\beta/4}$.
(3) $\frac{s(x_n)}{s(\varphi^{r_n(\underline{v})}(x))} = e^{\pm\sqrt{\varepsilon}}$ *and* $\frac{u(x_n)}{u(\varphi^{r_n(\underline{v})}(x))} = e^{\pm\sqrt{\varepsilon}}$.
(4) $\frac{Q(x_n)}{Q(\varphi^{r_n(\underline{v})}(x))} = e^{\pm\sqrt[3]{\varepsilon}}$.
(5) $\frac{p_n^s}{p^s(\varphi^{r_n(\underline{v})}(x))} = e^{\pm\sqrt[3]{\varepsilon}}$ *and* $\frac{p_n^u}{p^u(\varphi^{r_n(\underline{v})}(x))} = e^{\pm\sqrt[3]{\varepsilon}}$.
(6) $\Psi_{x_n}^{-1} \circ \Psi_{\varphi^{r_n(\underline{v})}(x)}$ *and* $\Psi_{\varphi^{r_n(\underline{v})}(x)}^{-1} \circ \Psi_{x_n}$ *can be written in the form* $(-1)^\sigma v + \delta + \Delta(v)$ *for* $v \in R[10Q(\varphi^{r_n(\underline{v})}(x))]$, *where* $\sigma \in \{0, 1\}$, δ *is a vector with* $\|\delta\| < 50^{-1}(p_0^s \wedge p_0^u)$ *and* Δ *is a vector field such that* $\Delta(0) = 0$ *and* $\|d\Delta\|_{C^0} < \sqrt[3]{\varepsilon}$ *on* $R[10Q(\varphi^{r_n(\underline{v})}(x))]$.

Part (1) is a direct consequence of Lemma 4.7. Indeed, since $\varphi^{r_n(\underline{v})}(x) = \pi[\sigma^n(\underline{v})]$, this point is the intersection of a s–admissible and a u–admissible manifold at $\Psi_{x_n}^{p_n^s, p_n^u}$. By Lemma 4.7(1) and since Pesin charts are 2–Lipschitz, we get that $d(\varphi^{r_n(\underline{v})}(x), x_n) < 50^{-1}(p_n^s \wedge p_n^u)$.

6.1. An improvement lemma

This section comprises the core of the proof that $x \in \mathrm{NUH}$ and of part (3) above. It states that the graph transforms $\mathscr{F}^s/\mathscr{F}^u$ improve the ratios of s/u–parameters, therefore we call it an *improvement lemma.*

Lemma 6.2 *(Improvement lemma). The following holds for all $0 < \varepsilon \ll \rho \ll 1$. Let $v \overset{\varepsilon}{\to} w$ with $v = \Psi_x^{p^s, p^u}$, $w = \Psi_y^{q^s, q^u}$, let $W^s \in \mathscr{M}^s[w]$ be the stable manifold of a positive ε–gpo, and let $V^s = \mathscr{F}_{v,w}^s(W^s)$, then:*

(1) *If $s(z) < \infty$ for some (every) $z \in W^s$, then $s(z') < \infty$ for every $z' \in V^s$.*
(2) *Let $z \in W^s$ with $g_y^-(z) \in V^s$. For $\xi \geq \sqrt{\varepsilon}$, if $\frac{s(z)}{s(y)} = e^{\pm\xi}$ then $\frac{s(g_y^-(z))}{s(x)} = e^{\pm(\xi - Q(y)^{\beta/4})}$.*

We note that the ratio improves.

Proof. When M is a closed surface and f is a $C^{1+\beta}$ diffeomorphism, this is [36, Lemma 7.2]. When M is a surface (possibly with boundary) and f is a local diffeomorphism with unbounded derivatives, this is [27, Lemma 6.3]. The main difference from these results to what we will do below is that our parameters s, u involve integrals instead of sums. So we need to be careful on how to split the integrals in a way that we can control each part reasonably. In the sequel, we will use the parallel transports $P_{z,y}$ and the maps $\widetilde{A}$ defined in the beginning of Section 1.2.3. We will also use estimate (2.1), which states that $\|\Phi^t\| = e^{\pm 4\rho}$ for $|t| \leq 2\rho$.

Claim 1: $\exists \mathfrak{C} = \mathfrak{C}(M, \varphi, \theta) > 0$ such that if $z \in B_y$ and $v \in T_y\Lambda, w \in T_z\Lambda$ with $\|v\| = \|w\| = 1$ then for all $|t| \leq 2\rho$:

$$|\|\Phi^t(v)\| - \|\Phi^t(w)\|| \leq \mathfrak{C}[d(y,z)^\beta + \|v - P_{z,y}w\|] \text{ and}$$
$$\left|\frac{\|\Phi^t(v)\|}{\|\Phi^t(w)\|} - 1\right| \leq \mathfrak{C}[d(y,z)^\beta + \|v - P_{z,y}w\|].$$

In particular $|\log \|\Phi^t(v)\| - \log \|\Phi^t(w)\|| \leq \mathfrak{C}[d(y,z)^\beta + \|v - P_{z,y}w\|]$.

Proof of Claim 1. The inequalities are direct consequences of the Hölder continuity of Φ, as follows: if $\mathfrak{C}_0 = \mathfrak{C}_0(M, \varphi, \theta) > 0$ is a constant such that

$$|\|\Phi^t(v)\| - \|\Phi^t(w)\|| \leq \mathfrak{C}_0[d(y,z)^\beta + \|v - P_{z,y}w\|]$$

for all y, z, v, w as above, then the claim holds with $\mathfrak{C} := e^{4\rho}\mathfrak{C}_0$. $\square$

Now we start the proof of the lemma. We have $g_y^-(y) = f^{-1}(y)$, therefore $\frac{s(g_y^-(z))}{s(x)} = \frac{s(g_y^-(z))}{s(g_y^-(y))} \cdot \frac{s(f^{-1}(y))}{s(x)}$. Since $(p^s \wedge p^u)^3(q^s \wedge q^u)^3 \ll Q(y)^{\beta/4}$, Proposition 3.10(1) implies $\frac{s(f^{-1}(y))}{s(x)} = e^{\pm Q(y)^{\beta/4}}$. Thus it is enough to show that $\frac{s(g_y^-(z))}{s(g_y^-(y))} = e^{\pm(\xi - 2Q(y)^{\beta/4})}$. We show one side of the inequality (the other is similar). Note that this is the term that gives the improvement.

Write $g_y^- = \varphi^{T^-}$ where T^- is a $C^{1+\beta}$ function with $T^-(y) = -r_\Lambda(f^{-1}(y))$. Then $g_y^-(y) = \varphi^{T^-(y)}(y)$ and $g_y^-(z) = \varphi^{T^-(z)}(z)$. For simplicity of notation, let $t_0 = -T^-(y)$ and $t_1 = -T^-(z)$, then $g_y^-(y) = \varphi^{-t_0}(y)$ and $g_y^-(z) = \varphi^{-t_1}(z)$. In the proof of Lemma 3.2 (see Appendix A), we saw that

$$s(x)^2 = 4e^{4\rho}\int_0^t e^{2\chi t'}\|\Phi^{t'}n_x^s\|^2 dt' + e^{2\chi t}\|\Phi^t n_x^s\|^2 s(\varphi^t(x))^2$$

for $x \in \text{NUH}$ and $t \in \mathbb{R}$. Therefore we can decompose $s(g_y^-(y))^2$ and $s(g_y^-(z))^2$ as follows:

$$s(g_y^-(y))^2 = \underbrace{4e^{4\rho}\int_0^{t_0} e^{2\chi t}\|\Phi^t n^s_{g_y^-(y)}\|^2 dt}_{=:I_1} + \underbrace{e^{2\chi t_0}\|\Phi^{t_0} n^s_{g_y^-(y)}\|^2}_{=:I_2} s(y)^2 =: I_1 + I_2 s(y)^2$$

$$s(g_y^-(z))^2 = \underbrace{4e^{4\rho}\int_0^{t_1} e^{2\chi t}\|\Phi^t n^s_{g_y^-(z)}\|^2 dt}_{=:I_3} + \underbrace{e^{2\chi t_1}\|\Phi^{t_1} n^s_{g_y^-(z)}\|^2}_{=:I_4} s(z)^2 =: I_3 + I_4 s(z)^2.$$

Using that $\|\Phi^{t_0} n^s_{g_y^-(y)}\| = \|\Phi^{-t_0} n^s_y\|^{-1}$ and an analogous equation for z, we have

$$I_1 = 4e^{4\rho}\int_0^{t_0} e^{2\chi t}\|\Phi^{-t} n^s_y\|^{-2} dt\,, \quad I_2 = e^{2\chi t_0}\|\Phi^{-t_0} n^s_y\|^{-2},$$

$$I_3 = 4e^{4\rho}\int_0^{t_1} e^{2\chi t}\|\Phi^{-t} n^s_z\|^{-2} dt\,, \quad I_4 = e^{2\chi t_1}\|\Phi^{-t_1} n^s_z\|^{-2}.$$

Before continuing, we need to make some estimates.

Claim 2: $d(y,z) < Q(y)$ and $\|n^s_y - P_{z,y} n^s_z\| < 4\varepsilon^{1/4} Q(y)^{\beta/4}$.

Proof of Claim 2. We proceed as in [27, Lemma 6.3]. Let F, G be the representing functions of V^s, W^s respectively, and let $z = \Psi_y(t, G(t))$. Since $\mathrm{Lip}(G) < \varepsilon$, we have $\|\left[\begin{smallmatrix} t \\ G(t)\end{smallmatrix}\right]\| \leq |t| + |G(t)| \leq |t|(1 + \mathrm{Lip}(G)) + |G(0)| \leq (1+\varepsilon)q^s + 10^{-3}(q^s \wedge q^u) < 2q^s$, therefore $d(y,z) < 4q^s \leq 4\varepsilon Q(y) < Q(y)$ for small $\varepsilon > 0$.

To bound the second term, we first estimate $\sin\angle(n^s_y, P_{z,y} n^s_z)$. Since n^s_y is the unitary vector in the direction of $d(\Psi_y)_0 \left[\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right] = d(\exp_y)_0 \circ C(y) \left[\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right]$ and n^s_z is the unitary vector in the direction of $d(\Psi_y)_{(t,G(t))} \left[\begin{smallmatrix} 1 \\ G'(t)\end{smallmatrix}\right] = d(\exp_y)_{C(y)\left[\begin{smallmatrix} t \\ G(t)\end{smallmatrix}\right]} \circ C(y) \left[\begin{smallmatrix} 1 \\ G'(t)\end{smallmatrix}\right]$, the angles they define are the same. In other words, if

$$A = \widetilde{d(\exp_y)_0} \circ C(y), B = \widetilde{d(\exp_y)_{C(y)\left[\begin{smallmatrix} t \\ G(t)\end{smallmatrix}\right]}} \circ C(y), v_1 = \begin{bmatrix} 1 \\ 0\end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ G'(t)\end{bmatrix}$$

then $\sin\angle(n^s_y, P_{z,y} n^s_z) = \sin\angle(Av_1, Bv_2)$. Using (A.3) with $L = A$, $v = v_1$, $w = A^{-1}Bv_2$, we get

$$\begin{aligned} |\sin\angle(Av_1, Bv_2)| &\leq \|A\|\|A^{-1}\||\sin\angle(v_1, A^{-1}Bv_2)| \\ &\leq \|C(y)^{-1}\| \left[|\sin\angle(v_1, v_2)| + |\sin\angle(v_2, A^{-1}Bv_2)|\right]. \end{aligned}$$

We have $|\sin\angle(v_1, v_2)| \leq |G'(t)| \leq (q^s \wedge q^u)^{\beta/3} \leq Q(y)^{\beta/3}$. Also, by (Exp3):

$$\|A^{-1}B - \mathrm{Id}\| \le \|A^{-1}\|\|A - B\| \le \|C(y)^{-1}\| \left\| \widetilde{d(\exp_y)_0} - \widetilde{d(\exp_y)}_{C(y)\left[\begin{smallmatrix} t \\ G(t)\end{smallmatrix}\right]} \right\|$$

$$\le 2\mathfrak{K} q^s \|C(y)^{-1}\| \le 2\mathfrak{K}\varepsilon^{1/4} Q(y)^{1-\beta/12} < \tfrac{1}{4} Q(y)^{\beta/3} \ll 1.$$

This implies that $v_2, A^{-1}Bv_2$ are almost unitary vectors, therefore

$$|\sin\angle(v_2, A^{-1}Bv_2)| \le 2\|v_2 - A^{-1}Bv_2\| \le 4\|A^{-1}B - \mathrm{Id}\| < Q(y)^{\beta/3},$$

and so $|\sin\angle(n_y^s, P_{z,y}n_z^s)| < 2\|C(y)^{-1}\|Q(y)^{\beta/3}$. Since $\|n_y^s\| = \|P_{z,y}n_z^s\| = 1$ and the angle between them is small, we conclude that for small $\varepsilon > 0$:

$$\|n_y^s - P_{z,y}n_z^s\| \le 2|\sin\angle(n_y^s, P_{z,y}n_z^s)| < 4\|C(y)^{-1}\|Q(y)^{\beta/3} \le 4\varepsilon^{1/4}Q(y)^{\beta/4}. \quad \square$$

Claim 3: $\frac{I_1}{I_3} = \exp[\pm Q(y)^{\beta/4}]$ and $\frac{I_2}{I_4} = \exp[\pm Q(y)^{\beta/4}]$.

Proof of Claim 3. We first bound $\frac{I_1}{I_3}$. Since $t_0, t_1 \ge \frac{\inf(r_\Lambda)}{2}$, we have $I_1, I_3 \ge 4e^{4\rho}\frac{\inf(r_\Lambda)}{2} \times e^{-8\rho} = 2e^{-4\rho}\inf(r_\Lambda)$ are uniformly bounded away from zero. We have

$$I_1 - I_3 = 4e^{4\rho}\int_0^{t_0} e^{2\chi t}(\|\Phi^{-t}n_y^s\|^{-2} - \|\Phi^{-t}n_z^s\|^{-2})dt - 4e^{4\rho}\int_{t_0}^{t_1} e^{2\chi t}\|\Phi^{-t}n_z^s\|^{-2}dt$$

We estimate each integral separately.

- By Claims 1 and 2:

$$4e^{4\rho}\int_0^{t_0} e^{2\chi t}\left(\|\Phi^{-t}n_y^s\|^{-2} - \|\Phi^{-t}n_z^s\|^{-2}\right)dt \le 4e^{4\rho}\int_0^{t_0} e^{2\rho}2e^{12\rho}\left|\|\Phi^{-t}n_y^s\| - \|\Phi^{-t}n_z^s\|\right|dt$$

$$\le 8\rho e^{18\rho}\mathfrak{C}[d(y,z)^\beta + \|n_y^s - P_{z,y}n_z^s\|] \le 16\rho e^{18\rho}\mathfrak{C}Q(y)^{\beta/3} < \varepsilon^{1/8}Q(y)^{\beta/4}.$$

- By Lemma 2.1(3) and the proof of Claim 2:

$$4e^{4\rho}\int_{t_0}^{t_1} e^{2\chi t}\|\Phi^{-t}n_z^s\|^{-2}dt \le 4e^{14\rho}|t_1 - t_0| \le 4e^{14\rho}d(y,z) < 16e^{14\rho}\varepsilon Q(y) < Q(y).$$

Therefore $|I_1 - I_3| < \varepsilon^{1/8}Q(y)^{\beta/4} + Q(y) < 2\varepsilon^{1/8}Q(y)^{\beta/4}$, and so

$$\left|\frac{I_1}{I_3} - 1\right| < [2e^{-4\rho}\inf(r_\Lambda)]^{-1}2\varepsilon^{1/8}Q(y)^{\beta/4} < \tfrac{1}{2}Q(y)^{\beta/4}.$$

Since $e^{-2t} < 1 - t < 1 + t < e^{2t}$ for small $t > 0$, the above inequality implies that $\frac{I_1}{I_3} = \exp[\pm Q(y)^{\beta/4}]$.

The estimate of $\frac{I_2}{I_4}$ is easier. We have $\frac{I_4}{I_2} = e^{2\chi(t_1-t_0)}\frac{\|\Phi^{-t_0}n_y^s\|^2}{\|\Phi^{-t_1}n_z^s\|^2}$, and:

- $2\chi(t_1 - t_0) = \pm 4\chi d(y,z) = \pm 4\chi Q(y) = \pm\frac{1}{4}Q(y)^{\beta/4}$, hence $e^{2\chi(t_1-t_0)} = \exp[\pm\frac{1}{4}Q(y)^{\beta/4}]$.
- By Claim 1,

$$\left|\frac{\|\Phi^{-t_0}n_y^s\|}{\|\Phi^{-t_0}n_z^s\|} - 1\right| \le \mathfrak{C}[d(y,z)^\beta + \|n_y^s - P_{z,y}n_z^s\|] < 2\mathfrak{C}Q(y)^{\beta/3} < \tfrac{1}{8}Q(y)^{\beta/4},$$

therefore $\frac{\|\Phi^{-t_0}n_y^s\|}{\|\Phi^{-t_0}n_z^s\|} = \exp[\pm\frac{1}{4}Q(y)^{\beta/4}]$ and so $\frac{\|\Phi^{-t_0}n_y^s\|^2}{\|\Phi^{-t_0}n_z^s\|^2} = \exp[\pm\frac{1}{2}Q(y)^{\beta/4}]$.

These two items together imply that $\frac{I_2}{I_4} = \exp[\pm Q(y)^{\beta/4}]$. □

Now we complete the proof of the lemma. By Claim 3, we can write $\frac{s(g_y^-(z))^2}{s(g_y^-(y))^2} = \frac{I_3+I_4 s(z)^2}{I_1+I_2 s(y)^2} = \exp[\pm Q(y)^{\beta/4}]\frac{I_1+I_2 s(z)^2}{I_1+I_2 s(y)^2}$. Since we want to show that $\frac{s(g_y^-(z))^2}{s(g_y^-(y))^2} = \exp[\pm(2\xi - 4Q(y)^{\beta/4})]$, it remains to prove that $\frac{I_1+I_2 s(z)^2}{I_1+I_2 s(y)^2} = \exp[\pm(2\xi - 5Q(y)^{\beta/4})]$. We show one side of the inequality and leave the other to the reader. By assumption, $s(z) \le e^\xi s(y)$, hence

$$\frac{I_1+I_2 s(z)^2}{I_1+I_2 s(y)^2} \le \frac{I_1+e^{2\xi}I_2 s(y)^2}{I_1+I_2 s(y)^2} = e^{2\xi} - \frac{I_1(e^{2\xi}-1)}{I_1+I_2 s(y)^2} = e^{2\xi}\left[1 - \frac{I_1(1-e^{-2\xi})}{I_1+I_2 s(y)^2}\right].$$

It is enough to show that $\frac{I_1(1-e^{-2\xi})}{I_1+I_2 s(y)^2} > 5Q(y)^{\beta/4}$, since this implies

$$e^{2\xi}\left[1 - \frac{I_1(1-e^{-2\xi})}{I_1+I_2 s(y)^2}\right] < e^{2\xi}(1 - 5Q(y)^{\beta/4}) < e^{2\xi - 5Q(y)^{\beta/4}}.$$

Note that:

- $I_1 \ge 2e^{-4\rho}\inf(r_\Lambda)$, as established in the proof of Claim 3.
- $1 - e^{-2\xi} \ge 1 - e^{-2\varepsilon^{1/2}} > \varepsilon^{1/2}$ when $\varepsilon > 0$ is small enough.
- Since $\sup(r_\Lambda) < 1$, we have $I_1 < 4e^{14\rho}$ and $I_2 \le e^{10\rho}$. Since $s(y) \ge \sqrt{2}$, it follows that $I_1 + I_2 s(y)^2 < 5e^{14\rho}s(y)^2$.

Altogether, we get that

$$\begin{aligned}
\frac{I_1(1-e^{-2\xi})}{I_1+I_2 s(y)^2} &> \tfrac{2}{5}e^{-18\rho}\inf(r_\Lambda)\varepsilon^{1/2}s(y)^{-2} \ge \tfrac{2}{5}e^{-18\rho}\inf(r_\Lambda)\varepsilon^{1/2}\|C(y)^{-1}\|^{-2}\\
&\ge \tfrac{2}{5}e^{-18\rho}\inf(r_\Lambda)Q(y)^{\beta/6} = \tfrac{2}{5}e^{-18\rho}\inf(r_\Lambda)Q(y)^{-\beta/12}Q(y)^{\beta/4}\\
&\ge \tfrac{2}{5}e^{-18\rho}\inf(r_\Lambda)\varepsilon^{-1/4}Q(y)^{\beta/4} > 5Q(y)^{\beta/4},
\end{aligned}$$

since $\frac{2}{5}e^{-18\rho}\inf(r_\Lambda)\varepsilon^{-1/4} > 5$ for $\varepsilon > 0$ small enough. □

Now we can prove that $x \in \mathrm{NUH}$.

Proposition 6.3. *If* $0 < \varepsilon \ll \rho \ll 1$*, then* $\pi[\Sigma^\#] \subset \mathrm{NUH}$.

Proof. Let $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}} \in \Sigma^\#$, and let $x = \pi(\underline{v})$. We need to prove properties (NUH1) and (NUH2) for x. We prove the first property (the second is symmetric). The proof that $s(x) < +\infty$ for surface diffeomorphisms is contained in Claims 1 and 2 in [36, Prop. 7.3], and uses four facts, which we also have here:

- The derivative of the diffeomorphism is continuous: in our context, the induced linear Poincaré flow Φ is continuous.
- Every vertex of the alphabet $\mathscr{A}$ is relevant: in our context, this is Theorem 5.1(3).
- Bounded distortion along invariant manifolds: in our context, this is Theorem 4.5(4).
- Improvement lemma: in our context, this is Lemma 6.2.

Let us give the details. Let $n_k \to +\infty$ such that $(v_{n_k})_{k\geq 0}$ is constant. Since $\pi[\Sigma^\#]$ and NUH are invariant, we can assume that $n_0 = 0$. Since v_0 is relevant, there is $\underline{w} = \{w_n\}_{n\in\mathbb{Z}}$ with $w_0 = v_0$ such that $y = \pi(\underline{w}) \in \mathrm{NUH}^\#$. In particular, $s(y) < +\infty$. Let $V := V^s[\underline{w}]$. We claim that $\sup_{y'\in V} s(y') < +\infty$. To prove this, fix $y' \in V$. Using the same notation of Proposition 4.9, let

$$t_n = \sum_{k=0}^{n-1} T_k[(g_{x_{k-1}}^+ \circ \cdots \circ g_{x_0}^+)(y)] \ \text{ and } \ t'_n = \sum_{k=0}^{n-1} T_k[(g_{x_{k-1}}^+ \circ \cdots \circ g_{x_0}^+)(y')].$$

In particular, $(g_{x_{n-1}}^+ \circ \cdots \circ g_{x_0}^+)(y) = \varphi^{t_n}(y)$ and $(g_{x_{n-1}}^+ \circ \cdots \circ g_{x_0}^+)(y') = \varphi^{t'_n}(y')$. By Lemma 2.1(3) and Theorem 4.5(3), we have

$$\begin{aligned} |t_n - t'_n| &\leq \sum_{k=0}^{n-1} \mathrm{Lip}(T_k) d((g_{x_{k-1}}^+ \circ \cdots \circ g_{x_0}^+)(y), (g_{x_{k-1}}^+ \circ \cdots \circ g_{x_0}^+)(y')) \\ &\leq d(\Psi_{x_0}^{-1}(y), \Psi_{x_0}^{-1}(y')) \sum_{k=0}^{n-1} e^{-\frac{\chi \inf(r_\Lambda)}{2}k} < \frac{4p_0^s}{1 - e^{-\frac{\chi \inf(r_\Lambda)}{2}}} \ll \varepsilon \ll \rho. \end{aligned}$$

Since $\|d(g_{x_{n-1}}^+ \circ \cdots \circ g_{x_0}^+)_y w\| = \|\Phi^{t_n} n_y^s\|$ and $\|d(g_{x_{n-1}}^+ \circ \cdots \circ g_{x_0}^+)_{y'} w'\| = \|\Phi^{t'_n} n_{y'}^s\|$, it follows from Theorem 4.5(4) and estimate (2.1) that $\frac{\|\Phi^{t_n} n_y^s\|}{\|\Phi^{t_n} n_{y'}^s\|} = \frac{\|\Phi^{t_n} n_y^s\|}{\|\Phi^{t'_n} n_{y'}^s\|} \cdot \frac{\|\Phi^{t'_n} n_{y'}^s\|}{\|\Phi^{t_n} n_{y'}^s\|} = e^{\pm(Q(x_0)^{\beta/4} + 4\rho)} = e^{\pm 6\rho}$. Now we interpolate this estimate. Given $t \geq 0$, let n such that $t_n \leq t \leq t_{n+1}$. Since $|t_{n+1} - t_n| \leq \rho$, using estimate (2.1) again gives that

$$\frac{\|\Phi^t n_y^s\|}{\|\Phi^t n_{y'}^s\|} = \frac{\|\Phi^t n_y^s\|}{\|\Phi^{t_n} n_y^s\|} \cdot \frac{\|\Phi^{t_n} n_y^s\|}{\|\Phi^{t_n} n_{y'}^s\|} \cdot \frac{\|\Phi^{t_n} n_{y'}^s\|}{\|\Phi^t n_{y'}^s\|} = e^{\pm 14\rho}.$$

This implies that $\frac{s(y)}{s(y')} = e^{\pm 14\rho}$. Since $y' \in V$ is arbitrary, $L_0 := \sup_{y'\in V} s(y') < +\infty$.

The next step is to prove that $s(x) < +\infty$. Recalling that $V \in \mathscr{M}^s(v_0) = \mathscr{M}^s(v_{n_k})$, define $V_k := (\mathscr{F}^s_{v_0,v_1} \circ \mathscr{F}^s_{v_1,v_2} \circ \cdots \circ \mathscr{F}^s_{v_{n_k-1},v_{n_k}})[V]$. By Section 4.2, $(V_k)_{k\geq 0}$ converges in the C^1 topology to $V^s[\underline{v}]$. In other words, if G is the representing function of $V^s[\underline{v}]$ and G_k is the representing function of V_k, then $(G_k)_{k\geq 0}$ converges to G in the C^1 topology. Writing $x = \Psi_{x_0}(t, G(t))$, let $z_k = \Psi_{x_0}(t, G_k(t)) \in V_k$ and $y_k = (g^+_{x_{n_k-1}} \circ \cdots \circ g^+_{x_0})(z_k)$. By Theorem 4.5(2), we have $y_k \in V$ and so $s(y_k) \leq L_0$. Consider the ratio $\frac{s(y_k)}{s(x_0)}$, which is bounded by $\frac{L_0}{s(x_0)}$. Since $x_0 = x_{n_k}$ by our choice of n_k, we can apply Lemma 6.2 along the sequence of edges $v_0 \to v_1 \to \cdots \to v_{n_k}$. We obtain that $\frac{s(z_k)}{s(x_0)} \leq \max\left\{e^{\sqrt{\varepsilon}}, \frac{L_0}{s(x_0)}\right\} =: L_1$. Since Φ is continuous and $n^s_{z_k} \to n^s_x$ as $k \to +\infty$, for every $T \geq 0$ we have

$$4e^{4\rho}\int_0^T e^{2\chi t}\|\Phi^t n^s_x\|^2 dt \leq \limsup_{k\to+\infty} 4e^{4\rho}\int_0^T e^{2\chi t}\|\Phi^t n^s_{z_k}\|^2 \leq s(z_k)^2 \leq L_1^2 s(x_0)^2.$$

Taking $T \to +\infty$, we conclude that $s(x) \leq L_1 s(x_0)$.

Now we prove that $\liminf_{t\to+\infty} \frac{1}{t}\log\|\Phi^{-t} n^s_x\| > 0$. Let $t_n = r_n(\underline{v})$ (see before the statement of Theorem 6.1 for the definition of $r_n(\underline{v})$). For $n \geq 0$, we have $0 \leq -t_{-n} \leq n\sup(r_\Lambda)$, hence it is enough to prove that $\liminf_{n\to+\infty} \frac{1}{n}\log\|\Phi^{t_{-n}} n^s_x\| > 0$. This can also be done as in the case of diffeomorphisms, as follows. The second estimate of Theorem 4.5(3) gives that $\|\Phi^{r_n(\underline{v})} n^s_x\| \leq 8\|C(x_0)^{-1}\| e^{-\frac{\chi\inf(r_\Lambda)}{2}n}$ for every $n \geq 0$. Applying this to $\sigma^{-n}(\underline{v})$ and $G_{-n}(x) = \pi[\sigma^{-n}(\underline{v})]$, we get that

$$\|\Phi^{t_{-n}} n^s_x\| = \|\Phi^{-t_{-n}} n^s_{G_{-n}(x)}\|^{-1} \geq \tfrac{1}{8}\|C(x_{-n})^{-1}\|^{-1} e^{\frac{\chi\inf(r_\Lambda)}{2}n}.$$

Since $\|C(x_{-n})^{-1}\|^{-1} \geq Q(x_{-n})^{\frac{\beta}{12}} \geq (p^s_{-n} \wedge p^u_{-n})^{\frac{\beta}{12}} \geq (e^{-2\varepsilon n} p^s_0 \wedge p^u_0)^{\frac{\beta}{12}}$, we have that

$$\liminf_{n\to+\infty} \tfrac{1}{n}\log\|\Phi^{t_{-n}} n^s_x\| \geq \tfrac{\chi\inf(r_\Lambda)}{2} - \tfrac{\beta\varepsilon}{6},$$

which is positive if $\varepsilon > 0$ is small enough. □

6.2. *Control of $\alpha(x_n), s(x_n), u(x_n), Q(x_n)$*

We now prove parts (2)–(4) of Theorem 6.1. Part (2) follows directly from 4.7(2), as follows: since $\varphi^{r_n(\underline{v})}(x) = \pi[\sigma^n(\underline{v})]$ is the intersection point of a s–admissible and a u–admissible manifold at $\Psi^{p^s_n,p^u_n}_{x_n}$, we have

$$\frac{\sin\alpha(x_n)}{\sin\alpha(\varphi^{r_n(\underline{v})}(x))} = e^{\pm(p^s_n\wedge p^u_n)^{\beta/4}} \text{ and } |\cos\alpha(x_n) - \cos\alpha(\varphi^{r_n(\underline{v})}(x))| < 2(p^s_n \wedge p^u_n)^{\beta/4}.$$

Now we proceed to control $s(x_n)$ and $u(x_n)$.

Proposition 6.4. *The following holds for all $0 < \varepsilon \ll \rho \ll 1$. If $\underline{v} = \{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\in\mathbb{Z}} \in \Sigma^\#$ and $x = \pi(\underline{v})$ then for all $n \in \mathbb{Z}$:*

$$\tfrac{s(x_n)}{s(\varphi^{r_n(\underline{v})}(x))}=e^{\pm\sqrt{\varepsilon}} \text{ and } \tfrac{u(x_n)}{u(\varphi^{r_n(\underline{v})}(x))}=e^{\pm\sqrt{\varepsilon}}.$$

Proof. When M is a compact surface and f is a $C^{1+\beta}$ diffeomorphism, this is [36, Prop. 7.3], and our proof follows the same methods. To ease notation, write $z_n=\varphi^{r_n(\underline{v})}(x)$, $n\in\mathbb{Z}$. We sketch the proof for the first estimate:

- By Proposition 6.3, $\pi[\Sigma^\#]\subset$ NUH hence $s(x)<\infty$.
- As in Claim 1 of [36, Prop. 7.3], there is $\xi\geq\sqrt{\varepsilon}$ and a sequence $n_k\to+\infty$ such that $\frac{s(x_{n_k})}{s(z_{n_k})}=e^{\pm\xi}$.
- Since $g^-_{x_n}(z_n)=z_{n-1}$, we can apply Lemma 6.2 along $\underline{v}$ and the points z_n: if $v_n=v$ for infinitely many $n>0$, then the ratio improves at each of these indices.

The conclusion is that $\frac{s(x_n)}{s(z_n)}=e^{\pm\sqrt{\varepsilon}}$ for all $n\in\mathbb{Z}$. $\square$

Part (4) is consequence of parts (2) and (3). Remind that

$$Q(x):=\varepsilon^{3/\beta}\|C(x)^{-1}\|_{\rm Frob}^{-12/\beta}=\varepsilon^{3/\beta}\left(\tfrac{\sqrt{s(x)^2+u(x)^2}}{|\sin\alpha(x)|}\right)^{-12/\beta}.$$

By part (2), $\frac{\sin\alpha(x_n)}{\sin\alpha(z_n)}=e^{\pm\sqrt{\varepsilon}}$. By part (3), $\frac{\sqrt{s(x_n)^2+u(x_n)^2}}{\sqrt{s(z_n)^2+u(z_n)^2}}=e^{\pm\sqrt{\varepsilon}}$. Therefore $\frac{\|C(x_n)^{-1}\|_{\rm Frob}}{\|C(z_n)^{-1}\|_{\rm Frob}}=e^{\pm2\sqrt{\varepsilon}}$, and so $\frac{Q(x_n)}{Q(z_n)}=\frac{\|C(x_n)^{-1}\|_{\rm Frob}^{-12/\beta}}{\|C(z_n)^{-1}\|_{\rm Frob}^{-12/\beta}}=\exp[\pm\frac{24}{\beta}\sqrt{\varepsilon}]=\exp[\pm\sqrt[3]{\varepsilon}]$ when $\varepsilon>0$ is small enough.

6.3. Control of p^s_n,p^u_n

Up to now, we have proved that $x\in$ NUH and Parts (1)–(4) of Theorem 6.1. Now we prove Part (5). In particular, it follows that $x\in$ NUH$^\#$. We continue to write $z_n=\varphi^{r_n(\underline{v})}(x)$, as in the previous section. The control of $p^{s/u}_n$ consists on proving that it is comparable to $p^{s/u}(z_n)$. To have the control from below, we will use that $\{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\in\mathbb{Z}}\in\Sigma^\#$ implies that the parameters $p^{s/u}_n$ are almost maximal infinitely often. Proposition 3.6(3) is the statement of maximality for $p^{s/u}(z_n)$. The statement for $p^{s/u}_n$ is in the next lemma. For simplicity of notation, write $T_k=T(v_k,v_{k+1})$.

Lemma 6.5. *If $\{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\in\mathbb{Z}}\in\Sigma^\#$ then $\min\{e^{\varepsilon T_n}p^s_{n+1},e^{-\varepsilon}\varepsilon Q(x_n)\}=e^{-\varepsilon}\varepsilon Q(x_n)$ for infinitely many $n>0$, and $\min\{e^{\varepsilon T_n}p^u_n,e^{-\varepsilon}\varepsilon Q(x_{n+1})\}=e^{-\varepsilon}\varepsilon Q(x_{n+1})$ for infinitely many $n<0$.*

Proof. The strategy is the same used in the proof of Proposition 3.6(3). We prove the first statement (the second is identical). By contradiction, assume that there exists $n\in\mathbb{Z}$ such that $\min\{e^{\varepsilon T_N}p^s_{N+1},e^{-\varepsilon}\varepsilon Q(x_N)\}=e^{\varepsilon T_N}p^s_{N+1}$ for all $N\geq n$. By (GPO2), it follows that $p^s_N\geq e^{\varepsilon(T_N-p^s_N)}p^s_{N+1}$ for all $N\geq n$. Let $\lambda=\exp[\varepsilon^{1.5}]$, then $\varepsilon(T_N-p^s_N)\geq\varepsilon(\inf(r_\Lambda)-\varepsilon)>$

λ when $\varepsilon > 0$ is sufficiently small. Hence $p^s_N > \lambda p^s_{N+1}$ for all $N \geq n$, and so $p^s_n \geq \lambda^{N-n} p^s_N$ for all $N \geq n$. This is a contradiction, since $p^s_n < \varepsilon$ and $\limsup_{N\to+\infty} p^s_N > 0$. $\square$

Now we prove Theorem 6.1(5). We will prove the statement for p^s_n and $p^s(z_n)$ (the proof for p^u_n and $p^u(z_n)$ is identical).

Step 1. $p^s_n \geq e^{-\sqrt[3]{\varepsilon}} p^s(z_n)$ for all $n \in \mathbb{Z}$.

We divide the proof into two cases, according to whether n satisfies Lemma 6.5 or not. Assume first that it does, i.e. $\min\{e^{\varepsilon T_n} p^s_{n+1}, e^{-\varepsilon}\varepsilon Q(x_n)\} = e^{-\varepsilon}\varepsilon Q(x_n)$. By (GPO2), we have $p^s_n \geq e^{-\varepsilon p^s_n} e^{-\varepsilon}\varepsilon Q(x_n) \geq e^{-2\varepsilon}\varepsilon Q(x_n)$. By Theorem 6.1(4), we get that

$$p^s_n \geq e^{-2\varepsilon}\varepsilon Q(x_n) \geq e^{-2\varepsilon - O(\sqrt{\varepsilon})}\varepsilon Q(z_n) \geq e^{-2\varepsilon - O(\sqrt{\varepsilon})} p^s(z_n) \geq e^{-\sqrt[3]{\varepsilon}} p^s(z_n).$$

Now assume that n does not satisfy Lemma 6.5. Take the smallest $m > n$ that satisfies Lemma 6.5. Hence $\min\{e^{\varepsilon T_k} p^s_{k+1}, e^{-\varepsilon}\varepsilon Q(x_k)\} = e^{\varepsilon T_k} p^s_{k+1}$ for $k = n, \ldots, m-1$. By (GPO2), we get that $p^s_k \geq e^{\varepsilon(T_k - p^s_k)} p^s_{k+1} > \lambda p^s_{k+1}$ for $k = n, \ldots, m-1$. Therefore $p^s_k \leq \lambda^{n-k} p^s_n$ for $k = n, \ldots, m-1$. Writing $\Delta_k = (t_{k+1} - t_k) - T_k \geq 0$, this latter estimate gives two consequences:

- $\sum_{k=n}^{m-1} p^s_k < \varepsilon$: indeed,

$$\sum_{k=n}^{m-1} p^s_k \leq p^s_n \sum_{k=n}^{m-1} \lambda^{n-k} \leq \varepsilon^{\frac{3}{\beta}} \frac{1}{1-\lambda^{-1}} < 2\varepsilon^{\frac{3}{\beta}-1.5} < \varepsilon,$$

  since $\lim_{\varepsilon\to 0} \frac{\varepsilon^{1.5}}{1-\lambda^{-1}} = 1$.
- $\sum_{k=n}^{m-1} \Delta_k < \varepsilon$: since the transition time from x_k to x_{k+1} is 2–Lipschitz (Lemma 2.1(3)), we have

$$\sum_{k=n}^{m-1} \Delta_k \leq 4 \sum_{k=n}^{m-1} p^s_k < 8\varepsilon^{\frac{3}{\beta}-1.5} < \varepsilon.$$

Using that $p^s_k \geq e^{\varepsilon(T_k - p^s_k)} p^s_{k+1} = e^{-\varepsilon(p^s_k + \Delta_k)} e^{\varepsilon(t_{k+1}-t_k)} p^s_{k+1}$ for $k = n, \ldots, m-1$, we conclude that

$$\begin{aligned} p^s_n &\geq \exp\left[-\varepsilon \sum_{k=n}^{m-1} p^s_k - \varepsilon \sum_{k=n}^{m-1} \Delta_k\right] e^{\varepsilon(t_m - t_n)} p^s_m \\ &\geq \exp\left[-2\varepsilon^2 - 2\varepsilon - O(\sqrt{\varepsilon})\right] e^{\varepsilon(t_m - t_n)} p^s(z_m) \geq e^{-\sqrt[3]{\varepsilon}} p^s(z_n), \end{aligned}$$

where in the last inequality we used Proposition 3.6(2).

Step 2. $p^s(z_n) \geq e^{-\sqrt[3]{\varepsilon}} p^s_n$ for all $n \in \mathbb{Z}$.

The motivation for this inequality is that $p^s(z_n)$ grows at least as much as p^s_n, since $p^s(z_n)$ satisfies the recursive equality $p^s(z_n) = \min\{e^{\varepsilon(t_{n+1}-t_n)} p^s(z_{n+1}), \varepsilon Q(z_n)\}$ while by (GPO2) we have the recursive inequality $p^s_n \leq \min\{e^{\varepsilon T_n} p^s_{n+1}, \varepsilon Q(x_n)\}$ and $t_{n+1}-t_n \geq T_n$. For ease of notation, let $n = 0$ (the general case is identical). By the above recurve equality and inequality, we have

$$p^s(z_0) = \varepsilon \inf\{e^{\varepsilon t_n} Q(z_n) : n \geq 0\} \ \text{ and } \ p^s_0 \leq \varepsilon \inf\{e^{\varepsilon(T_0+\cdots+T_{n-1})} Q(x_n) : n \geq 0\}.$$

Using Part (4) and that $t_n = \sum_{k=0}^{n-1}(t_{k+1}-t_k) \geq \sum_{k=0}^{n-1} T_k$, we conclude that

$$p^s(z_0) = \varepsilon \inf\{e^{\varepsilon t_n} Q(z_n) : n \geq 0\} \geq e^{-\sqrt[3]{\varepsilon}} \varepsilon \inf\{e^{\varepsilon(T_0+\cdots+T_{n-1})} Q(x_n) : n \geq 0\} = e^{-\sqrt[3]{\varepsilon}} p^s_0.$$

Steps 1 and 2 conclude the proof of Part (5). In particular, since $\{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\in\mathbb{Z}} \in \Sigma^\#$, it follows that $x \in \mathrm{NUH}^\#$.

6.4. Control of $\Psi^{-1}_{x_0} \circ \Psi_x$

In the case of diffeomorphisms, this is [36, Thm. 5.2], whose idea of proof is the following: if $\underline{v} = \{\Psi^{p^s_n,p^u_n}_{x_n}\}_{n\in\mathbb{Z}}, \underline{w} = \{\Psi^{q^s_n,q^u_n}_{y_n}\}_{n\in\mathbb{Z}} \in \Sigma^\#$ with $\pi(\underline{v}) = \pi(\underline{w}) = x$, then the parameters of $\Psi^{p^s_n,p^u_n}_{x_n}$ and $\Psi^{q^s_n,q^u_n}_{y_n}$ are comparable, hence $\Psi^{-1}_{y_n} \circ \Psi_{x_n}$ is close to $\pm\mathrm{Id}$. In our case, we know that $x \in \mathrm{NUH}^\#$, hence the Pesin charts along the orbit of x are well-defined. By parts (1)–(5), the parameters of $\Psi^{p^s_0,p^u_0}_{x_0}$ and $\Psi^{q^s(x),q^u(x)}_x$ are comparable, therefore we can apply the same proof of [36, Thm 5.2] to conclude that both $\Psi^{-1}_{x_0} \circ \Psi_x$ and $\Psi^{-1}_x \circ \Psi_{x_0}$ can be written in the form $(-1)^\sigma v + \delta + \Delta(v)$ for $v \in R[10Q(x)]$, where $\sigma \in \{0,1\}$ and Δ is a vector field such that $\Delta(0) = 0$ and $\|d\Delta\|_{C^0} < \sqrt[3]{\varepsilon}$ on $R[10Q(x)]$. The proof will be complete once we estimate $\|\delta\|$.

Assume that $(\Psi^{-1}_{x_0} \circ \Psi_x)(v) = (-1)^\sigma v + \delta + \Delta(v)$ as above, and write $p = p^s_0 \wedge p^u_0$. By Lemma 4.7(1), $x = \Psi_{x_0}(\eta)$ for some $\eta \in R[10^{-2}p]$. In particular $\|\eta\| \leq 10^{-2}\sqrt{2}p < 50^{-1}p$. Since $\Psi_x(0) = x$, taking $v = 0$ we conclude that $\eta = \delta$, hence $\|\delta\| < 50^{-1}p$. Similarly, if $\Psi^{-1}_x \circ \Psi_{x_0} = (-1)^\sigma v + \delta + \Delta(v)$ then $v = \eta$ gives $0 = (-1)^\sigma \eta + \delta + \Delta(\eta)$ and so

$$\|\delta\| \leq \|\eta\| + \|\Delta(\eta)\| \leq (1 + \|d\Delta\|_{C^0})\|\eta\| \leq (1 + \sqrt[3]{\varepsilon})10^{-2}\sqrt{2}p < 50^{-1}p.$$

7. A countable locally finite section

Up to now, we have:

- Constructed a countable family $\mathscr{A}$ of ε–double charts, see Theorem 5.1.
- Letting Σ be the TMS defined by $\mathscr{A}$ with the edge condition defined in Section 4.1, we constructed a Hölder continuous map $\pi : \Sigma \to \widehat{\Lambda}$ that "captures" all orbits in $\mathrm{NUH}^\#$, see Propositions 5.2 and 5.3. The map π is defined as $\{\pi(\underline{v})\} := V^s[\underline{v}] \cap V^u[\underline{v}]$.

- Although π is not finite-to-one, we solved the inverse problem by analyzing when π loses injectivity, see Theorem 6.1.

We now use these information to construct a countable family $\mathscr{Z}$ of subsets of $\widehat{\Lambda}$ with the following properties:

- The union of elements of $\mathscr{Z}$, from now on also denoted by $\mathscr{Z}$, is a section that contains $\Lambda \cap \mathrm{NUH}^{\#}$.
- $\mathscr{Z}$ is *locally finite*: each point $x \in \mathscr{Z}$ belongs to at most finitely many rectangles $Z \in \mathscr{Z}$.
- Every element $Z \in \mathscr{Z}$ is a *rectangle*: each point $x \in Z$ has *invariant fibers* $W^s(x, Z)$, $W^u(x, Z)$ in Z, and these fibers induce a local product structure on Z.
- $\mathscr{Z}$ satisfies a *symbolic Markov property.*

In this section, all statements assume that $0 < \varepsilon \ll \rho \ll 1$, so we will omit this information.

7.1. The Markov cover $\mathscr{Z}$

Let $\mathscr{Z} := \{Z(v) : v \in \mathscr{A}\}$, where

$$Z(v) := \{\pi(\underline{v}) : \underline{v} \in \Sigma^{\#} \text{ and } v_0 = v\}.$$

In other words, $\mathscr{Z}$ is the family of sets induced by π under the natural partition of $\Sigma^{\#}$ into cylinders at the zeroth position. Using admissible manifolds, we define *invariant fibers* inside each $Z \in \mathscr{Z}$. Let $Z = Z(v)$.

s/u–FIBRES IN $\mathscr{Z}$: Given $x \in Z$, let $W^s(x, Z) := V^s[\{v_n\}_{n\geq 0}] \cap Z$ be the *s–fiber* of x in Z for some (any) $\underline{v} = \{v_n\}_{n\in\mathbb{Z}} \in \Sigma^{\#}$ such that $\pi(\underline{v}) = x$ and $v_0 = v$. Similarly, let $W^u(x, Z) := V^u[\{v_n\}_{n\leq 0}] \cap Z$ be the *u–fiber* of x in Z.

By Proposition 4.9, the above definitions do not depend on the choice of $\underline{v}$, and any two s–fibers (u–fibers) either coincide or are disjoint. We also define $V^s(x, Z) := V^s[\{v_n\}_{n\geq 0}]$ and $V^u(x, Z) := V^u[\{v_n\}_{n\leq 0}]$. We can make two distinctions between $V^{s/u}(x, Z)$ and $W^{s/u}(x, Z)$:

- $V^{s/u}(x, Z)$ are smooth curves, while $W^{s/u}(x, Z)$ are usually fractal sets.
- $V^{s/u}(x, Z)$ are *not* subsets of Z, while $W^{s/u}(x, Z)$ are.

7.2. Fundamental properties of $\mathscr{Z}$

Although $\mathscr{Z}$ is usually a fractal set (and hence not a proper section), we can still define its Poincaré return map. Indeed, if $x = \pi(\underline{v}) \in \mathscr{Z}$ with $\underline{v} \in \Sigma^{\#}$ then $\varphi^{r_n(\underline{v})}(x) =$

$\pi[\sigma^n(\underline{v})] \in \mathscr{Z}$ for all $n \in \mathbb{N}$. Define $r_{\mathscr{Z}} : \mathscr{Z} \to (0, \rho)$ by $r_{\mathscr{Z}}(x) := \min\{t > 0 : \varphi^t(x) \in \mathscr{Z}\}$.

THE RETURN MAP H: It is the map $H : \mathscr{Z} \to \mathscr{Z}$ defined by $H(x) := \varphi^{r_{\mathscr{Z}}}(x)$.

Below we collect the main properties of $\mathscr{Z}$.

Proposition 7.1. *The following are true.*

(1) *COVERING PROPERTY: $\mathscr{Z}$ is a cover of $\Lambda \cap \mathrm{NUH}^{\#}$.*
(2) *LOCAL FINITENESS: For every $Z \in \mathscr{Z}$,*

$$\#\left\{Z' \in \mathscr{Z} : \left(\bigcup_{|n|\leq 1} H^n[Z]\right) \cap Z' \neq \emptyset\right\} < \infty.$$

(3) *LOCAL PRODUCT STRUCTURE: For every $Z \in \mathscr{Z}$ and every $x, y \in Z$, the intersection $W^s(x, Z) \cap W^u(y, Z)$ consists of a single point, and this point belongs to Z.*
(4) *SYMBOLIC MARKOV PROPERTY: If $x = \pi(\underline{v}) \in \mathscr{Z}$ with $\underline{v} = \{v_n\}_{n\in\mathbb{Z}} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}} \in \Sigma^{\#}$, then*

$$g_{x_0}^+(W^s(x, Z(v_0))) \subset W^s(g_{x_0}^+(x), Z(v_1)) \text{ and}$$
$$g_{x_1}^-(W^u(g_{x_0}^+(x), Z(v_1))) \subset W^u(x, Z(v_0)).$$

Before proceeding to the proof, we use part (3) to give the following definition: for $x, y \in Z$, let $[x, y]_Z :=$ intersection point of $W^s(x, Z)$ and $W^u(y, Z)$, and call it the *Smale bracket* of x, y in Z.

Proof. We have $\mathscr{Z} = \pi[\Sigma^{\#}]$. Since $\pi[\Sigma^{\#}] \supset \Lambda \cap \mathrm{NUH}^{\#}$ by Proposition 5.2(3), it follows that $\mathscr{Z}$ contains $\Lambda \cap \mathrm{NUH}^{\#}$. This proves (1).

(2) Write $Z = Z[\Psi_x^{p^s, p^u}]$, and take $Z' = Z[\Psi_y^{q^s, q^u}]$ such that

$$\left(\bigcup_{|n|\leq 1} H^n[Z]\right) \cap Z' \neq \emptyset.$$

We will estimate the ratio $\frac{p^s \wedge p^u}{q^s \wedge q^u}$. By assumption, there is $x \in Z$ such that $x' = H^n(x) \in Z'$ for some $|n| \leq 1$. Let $\underline{v} \in \Sigma^{\#}$ with $v_0 = \Psi_x^{p^s, p^u}$ such that $x = \pi(\underline{v})$. Recalling that $p^{s/u}(x) = p^{s/u}(x, \mathcal{T}, 0)$ for $\mathcal{T} = \{R_n(\underline{v})\}_{n\in\mathbb{Z}}$, the following holds:

- $x \in Z$, hence by Theorem 6.1(5) we have $\frac{p^s}{p^s(x)} = e^{\pm\sqrt[3]{\varepsilon}}$ and $\frac{p^u}{p^u(x)} = e^{\pm\sqrt[3]{\varepsilon}}$, and so $\frac{p^s \wedge p^u}{p^s(x) \wedge p^u(x)} = e^{\pm\sqrt[3]{\varepsilon}}$. By Proposition 3.6(1), we have $\frac{p^s(x) \wedge p^u(x)}{q(x)} = e^{\pm\mathfrak{H}}$. The conclusion is that $\frac{p^s \wedge p^u}{q(x)} = e^{\pm(\sqrt[3]{\varepsilon} + \mathfrak{H})}$.

- $x' \in Z'$, hence by the same reason $\frac{q^s \wedge q^u}{q(x')} = e^{\pm(\sqrt[3]{\varepsilon}+\mathfrak{H})}$.
- $x' = \varphi^t(x)$ with $|t| \leq 2\rho$, hence by Lemma 3.4 we have $\frac{q(x)}{q(x')} = e^{\pm 2\varepsilon}$.

Altogether, we conclude that $\frac{p^s \wedge p^u}{q^s \wedge q^u} = e^{\pm 2(\sqrt[3]{\varepsilon}+\varepsilon+\mathfrak{H})}$, and so

$$\left\{ Z' \in \mathscr{Z} : \left(\bigcup_{|n|\leq 1} H^n[Z] \right) \cap Z' \neq \emptyset \right\} \subset \{\Psi_y^{q^s,q^u} \in \mathscr{A} : (q^s \wedge q^u) \geq e^{-2(\sqrt[3]{\varepsilon}+\varepsilon+\mathfrak{H})}(p^s \wedge p^u)\}.$$

By Theorem 5.1(1), this latter set is finite.

(3) We proceed as in [36, Prop. 10.5]. Let $Z = Z(v)$, and take $x, y \in Z$, say $x = \pi(\underline{v}), y = \pi(\underline{w})$ with $\underline{v}, \underline{w} \in \Sigma^\#$, where $\underline{v} = \{v_n\}_{n\in\mathbb{Z}} = \{\Psi_{x_n}^{p_n^s,p_n^u}\}_{n\in\mathbb{Z}}$ and $\underline{w} = \{w_n\}_{n\in\mathbb{Z}} = \{\Psi_{y_n}^{q_n^s,q_n^u}\}_{n\in\mathbb{Z}}$ with $v_0 = w_0 = v$. We let $z = \pi(\underline{u})$ where $\underline{u} = \{u_n\}_{n\in\mathbb{Z}}$ is defined by

$$u_n = \begin{cases} v_n & , n \geq 0 \\ w_n & , n \leq 0. \end{cases}$$

We claim that $\{z\} = W^s(x, Z) \cap W^u(y, Z)$. To prove this, first remember that $V^s[\{u_n\}_{n\geq 0}] \cap V^u[\{u_n\}_{n\leq 0}]$ intersects at a single point (Lemma 4.7(1)), and that z belongs to such intersection. Therefore, it is enough to show that $z \in \pi[\Sigma^\#]$, which is clear since $\underline{u} \in \Sigma^\#$.

(4) Proceed exactly as in [36, Prop. 10.9]. □

Let $Z = Z(v), Z' = Z(w)$ where $v = \Psi_x^{p^s,p^u}, w = \Psi_y^{q^s,q^u} \in \mathscr{A}$, and assume that $Z \cap \varphi^{[-2\rho,2\rho]} Z' \neq \emptyset$. Let D, D' be the connected components of $\widehat{\Lambda}$ such that $Z \subset D$ and $Z' \subset D'$. We wish to compare s–fibers of Z with u–fibers of Z' and vice-versa. To do that, we apply the holonomy maps $\mathfrak{q}_D$ and $\mathfrak{q}_{D'}$. Given $z \in Z, z' \in Z'$, define

$$\{[z, z']_Z\} := V^s(z, Z) \cap \mathfrak{q}_D[V^u(z', Z')]$$
$$\{[z, z']_{Z'}\} := \mathfrak{q}_{D'}[V^s(z, Z)] \cap V^u(z', Z').$$

The next proposition proves that $[z, z']_Z$ and $[z, z']_{Z'}$ consist of single points, and some compatibility properties that will be used in the next section.

Proposition 7.2. *Let $Z = Z(v), Z' = Z(w)$ where $v = \Psi_x^{p^s,p^u}, w = \Psi_y^{q^s,q^u} \in \mathscr{A}$, and assume that $Z \cap \varphi^{[-2\rho,2\rho]} Z' \neq \emptyset$. Let D, D' be the connected components of $\widehat{\Lambda}$ such that $Z \subset D$ and $Z' \subset D'$. The following are true.*

(1) $\mathfrak{q}_{D'} \circ \Psi_x(R[\frac{1}{2}(p^s \wedge p^u)]) \subset \Psi_y(R[q^s \wedge q^u])$.
(2) *If $z \in Z$ with $z' = \mathfrak{q}_{D'}(z) \in Z'$, then $\mathfrak{q}_{D'}[W^{s/u}(z, Z)] \subset V^{s/u}(z', Z')$.*
(3) *If $z \in Z, z' \in Z'$ then $[z, z']_Z$, $[z, z']_{Z'}$ are points with $[z, z']_Z = \mathfrak{q}_D([z, z']_{Z'})$.*

When M is a compact surface and f is a diffeomorphism, this is [36, Lemmas 10.8 and 10.10]. A very similar method of proof works in our case: Theorem 3.8 also works when we change g_x^+ to $\mathfrak{q}_{D'}$, so we can control the composition $\Psi_y^{-1} \circ \mathfrak{q}_{D'} \circ \Psi_x$. The details are in Appendix A. We will also need more information regarding the Smale product of nearby charts.

Proposition 7.3. *Let Z, Z', Z'' such that $Z \cap \varphi^{[-2\rho,2\rho]} Z' \neq \emptyset$, $Z \cap \varphi^{[-2\rho,2\rho]} Z'' \neq \emptyset$, and let D be the connected components of $\widehat{\Lambda}$ such that $Z \subset D$. Assume that $z' \in Z'$ such that $\varphi^t(z') \in Z''$ for some $|t| \leq 2\rho$. For every $z \in Z$, it holds*

$$[z, z']_Z = [z, \varphi^t(z')]_Z.$$

Note that $[z, z']_Z$ is defined by Z, Z' while $[z, \varphi^t(z')]_Z$ is defined by Z, Z''. The equality shows a compatibility of the Smale product along small flow displacements. It holds because such displacements barely change the sizes of invariant fibers, hence the unique intersection is preserved.

8. A refinement procedure

Up to now, we have constructed a countable family $\mathscr{Z}$ of subsets of $\widehat{\Lambda}$ with the following properties:

- The union of elements of $\mathscr{Z}$, from now on also denoted by $\mathscr{Z}$, is a section that contains $\Lambda \cap \mathrm{NUH}^{\#}$.
- $\mathscr{Z}$ is locally finite: each point $x \in \mathscr{Z}$ belongs to at most finitely many rectangles $Z \in \mathscr{Z}$.
- Every element $Z \in \mathscr{Z}$ is a rectangle: each point $x \in Z$ has invariant fibers $W^s(x, Z)$, $W^u(x, Z)$ in Z, and these fibers induce a local product structure on Z.
- $\mathscr{Z}$ satisfies a *symbolic* Markov property.

In this section, we will refine $\mathscr{Z}$ to generate a countable family of disjoint sets $\mathscr{R}$ that satisfy a *geometrical* Markov property. We stress the difference from a symbolic to a geometrical Markov property: by Proposition 7.1(4), $g_{x_0}^{\pm}$ satisfy a symbolic Markov property; our goal is to obtain a Markov property for the first return map H. In general the orbit of x can intersect $\mathscr{Z}$ between x and $g_{x_0}^+(x)$, in which case we will have that $g_{x_0}^+(x) \neq H(x)$. Therefore the symbolic Markov property of Proposition 7.1(4) does not directly translate into a geometrical Markov property for H. To accomplish this latter property, we will use a refinement procedure developed by Bowen [7], motivated by the work of Sinaĭ [37,38]. The difference from our setup to Bowen's is that, while in Bowen's case all families are finite, in ours it is usually countable. Fortunately, as implemented in [36], the refinement procedure works well for countable covers with the local finiteness property, which we have by Proposition 7.1(2).

8.1. The partition $\mathscr{R}$

We first see that the map $g^+_{x_0}$ can be deduced from H by a bounded time change.

Lemma 8.1. *There exists $N \geq 1$ such that for any $x = \pi(\underline{v}) \in \mathscr{Z}$ there exists $0 < n < N$ such that $g^+_{x_0}(x) = H^n(x)$.*

Proof. We have $g^+_{x_0}(x) \in \mathscr{Z}$, so $g^+_{x_0}(x) = H^n(x)$ for some $n > 0$. Remember that $\widehat{\Lambda} = \bigcup_{i=1}^{n} D_i$ is a proper section of size $\rho/2$ (see Section 2 for the definitions). In particular $\inf(r_{\widehat{\Lambda}}) > 0$. Since $\mathscr{Z} \subset \widehat{\Lambda}$, every hit of x to $\mathscr{Z}$ is also a hit to $\widehat{\Lambda}$. Writing $g^+_{x_0}(x) = \varphi^t(x)$ for some $t \leq \rho$, we conclude that $n \inf(r_{\widehat{\Lambda}}) \leq t \leq \rho$, therefore $n \leq \left\lceil \frac{\rho}{\inf(r_{\widehat{\Lambda}})} \right\rceil$. We thus define $N := \left\lceil \frac{\rho}{\inf(r_{\widehat{\Lambda}})} \right\rceil + 1$. □

Therefore Proposition 7.1(4) implies that for every $x \in \mathscr{Z}$ there are $0 < k, \ell < N$ such that $H^k(x)$ satisfies a Markov property in the stable direction and $H^{-\ell}(x)$ satisfies a Markov property in the unstable direction.

At this point, it is worth mentioning the method that Bowen used to construct Markov partitions for Axiom A flows [7]:

(1) Fix a global section for the flow; inside this section, construct a finite family of rectangles (sets that are closed under the Smale bracket operation). Let H be the Poincaré return map of this family.
(2) Apply the method of Sinaĭ of successive approximations to get a new family of rectangles $\mathscr{Z}$ with the following property: if H is the Poincaré return map of $\mathscr{Z}$, then for every $x \in \mathscr{Z}$ there are $k, \ell > 0$ such that $H^k(x)$ satisfies a Markov property in the stable direction and $H^{-\ell}(x)$ satisfies a Markov property in the unstable direction. In addition, there is a global constant $N > 0$ such that $k, \ell < N$.
(3) Apply a refinement procedure to $\mathscr{Z}$ such that the resulting partition $\mathscr{R}$ is a disjoint family of rectangles satisfying the Markov property for H.

The attentive reader might have note that, so far, we did implement steps (1) and (2) above, with the difference that while Bowen used the method of successive approximations, we used the method of ε–gpo's. It remains to establish step (3), and we will do this closely following Bowen [7].

For each $Z \in \mathscr{Z}$, let

$$\mathscr{I}_Z := \left\{ Z' \in \mathscr{Z} : \varphi^{[-\rho,\rho]} Z \cap Z' \neq \emptyset \right\}.$$

By Theorem 6.1, $\mathscr{I}_Z$ is finite. Let D be the connected component of $\widehat{\Lambda}$ such that $Z \subset D$. By continuity, having chosen the discs D_i small enough the following property holds:

$$\text{If } Z' \in \mathscr{I}_Z \text{ then } Z' \subset \varphi^{[-2\rho,2\rho]} D. \tag{8.1}$$

Therefore $\mathfrak{q}_D(Z')$ is a well-defined subset of D. For each $Z' \in \mathscr{I}_Z$ we consider the partition of Z into four subsets as follows:

$$
\begin{aligned}
E^{su}_{Z,Z'} &= \{x \in Z : W^s(x,Z) \cap \mathfrak{q}_D(Z') \neq \emptyset, W^u(x,Z) \cap \mathfrak{q}_D(Z') \neq \emptyset\}\\
E^{s\emptyset}_{Z,Z'} &= \{x \in Z : W^s(x,Z) \cap \mathfrak{q}_D(Z') \neq \emptyset, W^u(x,Z) \cap \mathfrak{q}_D(Z') = \emptyset\}\\
E^{\emptyset u}_{Z,Z'} &= \{x \in Z : W^s(x,Z) \cap \mathfrak{q}_D(Z') = \emptyset, W^u(x,Z) \cap \mathfrak{q}_D(Z') \neq \emptyset\}\\
E^{\emptyset\emptyset}_{Z,Z'} &= \{x \in Z : W^s(x,Z) \cap \mathfrak{q}_D(Z') = \emptyset, W^u(x,Z) \cap \mathfrak{q}_D(Z') = \emptyset\}.
\end{aligned}
$$

Call this partition $\mathscr{P}_{Z,Z'} := \{E^{su}_{Z,Z'}, E^{s\emptyset}_{Z,Z'}, E^{\emptyset u}_{Z,Z'}, E^{\emptyset\emptyset}_{Z,Z'}\}$. Clearly, $E^{su}_{Z,Z'} = Z \cap \mathfrak{q}_D(Z')$.

THE PARTITION $\mathscr{E}_Z$: It is the coarser partition of Z that refines all of $\mathscr{P}_{Z,Z'}$, $Z' \in \mathscr{I}_Z$.

To define a partition of $\mathscr{Z}$, we define an equivalence relation on $\mathscr{Z}$.

EQUIVALENCE RELATION $\overset{N}{\sim}$ ON $\mathscr{Z}$: For $x, y \in \mathscr{Z}$, we write $x \overset{N}{\sim} y$ if for any $|k| \leq N$:

(i) For all $Z \in \mathscr{Z}$: $H^k(x) \in Z \Leftrightarrow H^k(y) \in Z$.
(ii) For all $Z \in \mathscr{Z}$ such that $H^k(x), H^k(y) \in Z$, the points $H^k(x), H^k(y)$ belong to the same element of $\mathscr{E}_Z$.

Clearly $\overset{N}{\sim}$ is an equivalence relation in $\mathscr{Z}$, hence it defines a partition of $\mathscr{Z}$. Before proceeding, let us state a fact that will be used in the sequel: if $x \overset{N}{\sim} y$ with $x \in Z = Z(\Psi^{p^s_0,p^u_0}_{x_0}) \in \mathscr{Z}$, then there exists $|k| \leq N$ such that $g^+_{x_0}(x) = H^k(x)$ and $g^+_{x_0}(y) = H^k(y)$. To see this, write $x = \pi(\underline{v})$ with $v_0 = \Psi^{p^s_0,p^u_0}_{x_0}$, and let D' be the connected component of $\widehat{\Lambda}$ with $Z(v_1) \subset D'$. On one hand, $g^+_{x_0}(y) = \mathfrak{q}_{D'}(y)$. On the other hand, since $H^k(x) \in Z(v_1) \subset D'$ for some $|k| \leq N$, the definition of $\overset{N}{\sim}$ implies that $H^k(y) \in Z(v_1) \subset D'$, hence $H^k(y) = \mathfrak{q}_{D'}(y)$. A similar result holds for $g^-_{x_0}$.

THE MARKOV PARTITION $\mathscr{R}$: It is the partition of $\mathscr{Z}$ whose elements are the equivalence classes of $\overset{N}{\sim}$.

By definition, $\mathscr{R}$ is a refinement of $\mathscr{Z}$.

Lemma 8.2. *The partition $\mathscr{R}$ satisfies the following properties.*

(1) *For every $Z \in \mathscr{Z}$, $\#\{R \in \mathscr{R} : R \subset \varphi^{[-\rho,\rho]}Z\} < \infty$.*
(2) *For every $R \in \mathscr{R}$, $\#\{Z \in \mathscr{Z} : R \subset \varphi^{[-\rho,\rho]}Z\} < \infty$.*

Proof. (1) Start noting that, for every $Z \in \mathscr{Z}$, $\#\{R \in \mathscr{R} : R \subset Z\} \leq 4^{\#\mathscr{I}_Z}$. Hence

$$\#\{R \in \mathscr{R} : R \subset \varphi^{[-\rho,\rho]}Z\} \leq \sum_{Z' \in \mathscr{I}_Z} \#\{R \in \mathscr{R} : R \subset Z'\} \leq \sum_{Z' \in \mathscr{I}_Z} 4^{\#\mathscr{I}_{Z'}} < +\infty$$

since the last summand is the finite sum of finite numbers.

(2) For any $Z' \in \mathscr{Z}$ such that $Z' \supset R$, we have $\{Z \in \mathscr{Z} : R \subset \varphi^{[-\rho,\rho]}Z\} \subset \mathscr{I}_{Z'}$. Since each $\mathscr{I}_{Z'}$ is finite, the result follows. □

8.2. The Markov property

The final step in the refinement procedure is to show that $\mathscr{R}$ is a Markov partition for the map H, in the sense of Sinaĭ [38].

s/u–FIBRES IN $\mathscr{R}$: Given x in $R \in \mathscr{R}$, we define the *s–fiber* and *u–fiber* of x by:

$$W^s(x,R) := \bigcap_{Z \in \mathscr{Z}: Z \supset R} V^s(x,Z) \cap R, \quad W^u(x,R) := \bigcap_{Z \in \mathscr{Z}: Z \supset R} V^u(x,Z) \cap R.$$

By Proposition 4.9, any two s–fibers (u–fibers) either coincide or are disjoint.

Proposition 8.3. *The following are true.*

1. *PRODUCT STRUCTURE: For every $R \in \mathscr{R}$ and every $x, y \in R$, the intersection $W^s(x,R) \cap W^u(y,R)$ is a single point, and this point is in R. Denote it by $[x,y]$.*
2. *HYPERBOLICITY: If $z, w \in W^s(x,R)$ then $d(H^n(z), H^n(w)) \xrightarrow[n\to\infty]{} 0$, and if $z, w \in W^u(x,R)$ then $d(H^n(z), H^n(w)) \xrightarrow[n\to-\infty]{} 0$. The rates are exponential.*
3. *GEOMETRICAL MARKOV PROPERTY: Let $R_0, R_1 \in \mathscr{R}$. If $x \in R_0 \cap H^{-1}(R_1)$ then*

$$H(W^s(x,R_0)) \subset W^s(H(x),R_1) \text{ and } H^{-1}(W^u(H(x),R_1)) \subset W^u(x,R_0).$$

Proof. The sets $R \in \mathscr{R}$ are defined from the sets $Z \in \mathscr{Z}$ and the partitions $\mathscr{E}_Z$. By Proposition 7.1 and by the definition of the partitions $\mathscr{P}_{Z,Z'}$, each Z and each element of $\mathscr{E}_Z$ is a rectangle. Note that rectangles are preserved under the holonomy maps $\mathfrak{q}_{D_i}$ and that rectangles contained in a same disc D_i are preserved under intersections. Consequently the sets $R \in \mathscr{R}$ are also rectangles and so part (1) follows. Part (2) is a direct consequence of the properties of the stable and unstable manifolds obtained in Theorem 4.5(3). It remains to prove part (3).

Fix $R_0, R_1 \in \mathscr{R}$ and $x \in R_0 \cap H^{-1}(R_1)$. We check that $H(W^s(x,R_0)) \subset W^s(H(x),R_1)$ (the other inclusion is proved similarly). Let $y \in W^s(x,R_0)$. By Proposition 7.2(2) and the definition of $W^s(H(x),R_1)$, it is enough to check that $H(x) \overset{N}{\sim} H(y)$. Since $x \overset{N}{\sim} y$, we already know that $H^k(x), H^k(y)$ satisfy the properties (i) and (ii) defining the relation $\overset{N}{\sim}$ when $-N \le k \le N$, hence it is enough to prove that this is also true for $k = N+1$. The property (ii) for $k = N$ says that $H^N(x), H^N(y)$ belong to the same elements of the partitions $\mathscr{E}_Z$. We claim that this implies that $H^{N+1}(x), H^{N+1}(y)$ belong to the same sets $Z \in \mathscr{Z}$, which gives (i) for $k = N+1$. To see this, let $Z' \in \mathscr{Z}$ such that $H^{N+1}(x) \in Z'$, and let D' be the connected component of $\widehat{\Lambda}$ that contains Z'. Let $Z \in \mathscr{Z}$ containing $H^N(x), H^N(y)$. Noting that $H^N(x) \in E^{su}_{Z,Z'}$, it follows from property (ii) for $k = N$ that $H^N(y) \in E^{su}_{Z,Z'}$, hence $\mathfrak{q}_{D'}(H^N(y)) \in Z'$. If $\mathfrak{q}_{D'}(H^N(y)) = H^{N+1}(y)$, the claim is proved.

If not, there is $Z'' \in \mathscr{Z}$ such that $H^{N+1}(y) \in Z''$, and so repeating the same argument with the roles of x, y interchanged gives that $\mathfrak{q}_{D''}(H^N(x)) \in Z''$, a contradiction since the time transition from Z to Z'' is smaller than time transitions from Z to Z'. Hence property (i) for $k = N+1$ is proved, and it remains to prove property (ii) for $k = N+1$.

Let $Z \in \mathscr{Z}$ be a rectangle which contains $H^{N+1}(x), H^{N+1}(y)$ and let D be the connected component of $\widehat{\Lambda}$ that contains Z. We need to show that $H^{N+1}(x), H^{N+1}(y)$ belong to the same element of $\mathscr{E}_Z$. We first note that $W^s(H^{N+1}(x), Z) = W^s(H^{N+1}(y), Z)$: since x, y belong to the same s–fiber of a rectangle in $\mathscr{Z}$, this can be checked by applying Proposition 7.2(2) inductively. In particular, we have the following property:

$$\forall Z' \in \mathscr{I}_Z, \quad W^s(H^{N+1}(x), Z) \cap \mathfrak{q}_D(Z') \neq \emptyset \iff W^s(H^{N+1}(y), Z) \cap \mathfrak{q}_D(Z') \neq \emptyset. \tag{8.2}$$

We then prove the analogous property for the sets $W^u(H^{N+1}(x), Z)$, $W^u(H^{N+1}(y), Z)$. In Fig. 2 we draw the points we will define below.

Let us consider $Z' \in \mathscr{I}_Z$ and assume for instance that $W^u(H^{N+1}(x), Z) \cap \mathfrak{q}_D(Z')$ contains a point z (the case when $W^u(H^{N+1}(y), Z) \cap \mathfrak{q}_D(Z') \neq \emptyset$ is treated analogously). Write $H^{N+1}(x) = \pi(\underline{v})$ with $\underline{v} = \{v_n\}_{n\in\mathbb{Z}} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}} \in \Sigma^\#$ and $Z = Z(v_0)$. By Lemma 8.1, there exists $0 \leq k \leq N$ such that the point $\widetilde{x} := H^k(x)$ coincides with $\pi[\sigma^{-1}(\underline{v})]$. The rectangle $\widetilde{Z} := Z(v_{-1})$ contains $\widetilde{x}$. The symbolic Markov property in Proposition 7.1(4) implies that the image of $W^u(\widetilde{x}, \widetilde{Z})$ under $g_{x_{-1}}^+$ contains $W^u(H^{N+1}(x), Z)$, hence the point z. In particular, the backward orbit of z under the flow intersects $W^u(\widetilde{x}, \widetilde{Z})$ at some point $\widetilde{z}$.

By the definition of z and Property 8.1, we have $\varphi^s(z) \in Z'$ for some $|s| \leq 2\rho$, thus we can write $\varphi^s(z) = \pi(\underline{w})$ with $\underline{w} = \{w_n\}_{n\in\mathbb{Z}} \in \Sigma^\#$ and $Z' = Z(w_0)$. Since all transition times of holonomy maps are bounded by ρ, necessarily the piece of orbit $\varphi^{[0,\rho]}(\widetilde{z})$ contains some $\pi[\sigma^{-b}(\underline{w})]$ with $b \geq 1$. Let $b \geq 1$ and $0 \leq \widetilde{s} \leq \rho$ with $\pi[\sigma^{-b}(\underline{w})] = \varphi^{\widetilde{s}}(\widetilde{z})$. Consequently the rectangle $\widetilde{Z}' := Z(w_{-b})$ belongs to $\mathscr{I}_{\widetilde{Z}}$. Moreover, $\widetilde{z}$ belongs to the intersection between $W^u(\widetilde{x}, \widetilde{Z})$ and $\mathfrak{q}_{\widetilde{D}}(\widetilde{Z}')$, where $\widetilde{D}$ is the connected component of $\widehat{\Lambda}$ containing $\widetilde{Z}$.

By the induction assumption, the point $\widetilde{y} := H^k(y)$ also belongs to $\widetilde{Z}$ and to the same element of the partition $\mathscr{P}_{\widetilde{Z},\widetilde{Z}'}$ as $\widetilde{x}$. Since $W^u(\widetilde{x}, \widetilde{Z})$ intersects $\mathfrak{q}_{\widetilde{D}}(\widetilde{Z}')$, the u–fiber $W^u(\widetilde{y}, \widetilde{Z})$ intersects it as well at some point $\widetilde{t}$. Note that $[\widetilde{z}, \widetilde{t}]_{\widetilde{Z}} = [\widetilde{z}, \widetilde{y}]_{\widetilde{Z}}$ also belongs to $W^u(\widetilde{y}, \widetilde{Z})$ and to $\mathfrak{q}_{\widetilde{D}}(\widetilde{Z}')$ (this latter property follows from Proposition 7.2(3), noting that $\widetilde{z}, \widetilde{t} \in \widetilde{Z} \cap \mathfrak{q}_{\widetilde{D}}(\widetilde{Z}')$), hence we can replace $\widetilde{t}$ by any point in $W^u(\widetilde{y}, \widetilde{Z}) \cap \mathfrak{q}_{\widetilde{D}}(\widetilde{Z}')$. Take $\widetilde{t} := [\widetilde{z}, \widetilde{y}]_{\widetilde{Z}}$.

Let $0 < r \leq 2\rho$ such that $\varphi^r(\widetilde{t}) \in W^s(\varphi^{\widetilde{s}}(\widetilde{z}), \widetilde{Z}')$. The symbolic Markov property in Proposition 7.1(4) then implies that its forward orbit under the flow will meet the rectangles $Z(w_{-b}), \ldots, Z(w_0)$.

Note that $\widetilde{z} \in \widetilde{Z} = Z(v_{-1})$ and $z = g_{x_{-1}}^+(\widetilde{z}) \in Z = Z(v_0)$. The same property holds for $\widetilde{y}$ and $H^{N+1}(y) = g_{x_{-1}}^+(\widetilde{y})$ since the points $H^i(x)$ and $H^i(y)$ belong to the same rectangles in $\mathscr{Z}$ for each $i = k, \ldots, N+1$. Using Proposition 7.2(3), it follows that

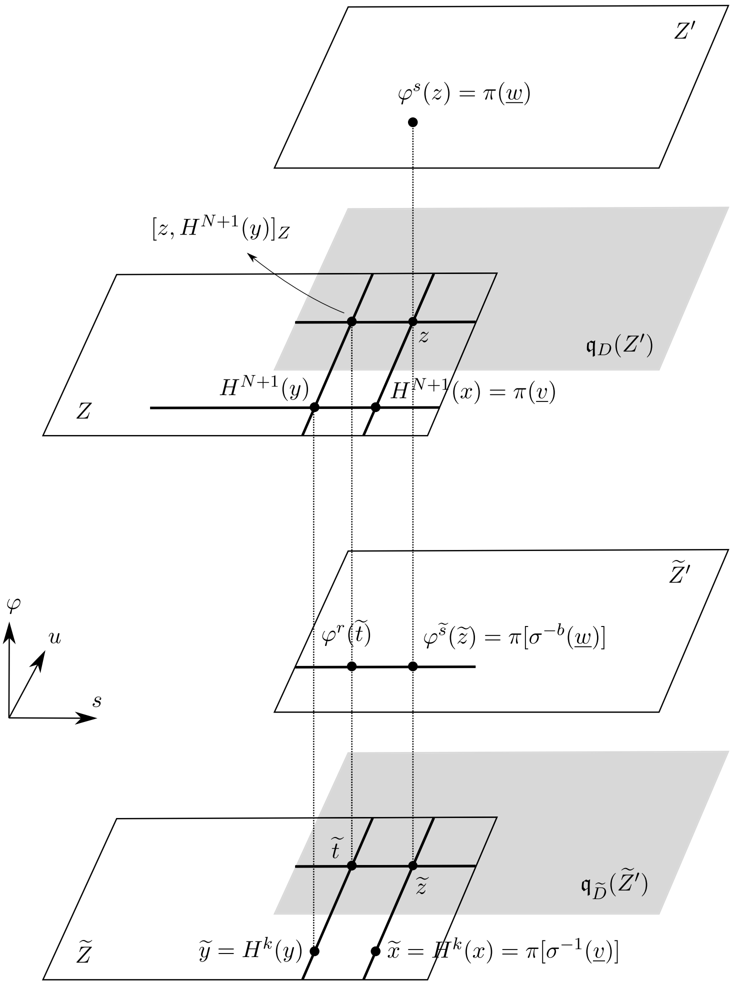

Fig. 2. Proof of the Markov property.

the image of $\widetilde{t} = [\widetilde{z}, \widetilde{y}]_{\widetilde{Z}}$ by $g^+_{x_{-1}}$ belongs to Z and coincides with the Smale product $[z, H^{N+1}(y)]_Z$.

The properties found in the two previous paragraphs imply that $W^u(H^{N+1}(y), Z)$ intersects $\mathfrak{q}_D(Z')$ at a point of the orbit of $\widetilde{t}$, contained in $W^s(z, Z)$. In particular, the intersection $W^u(H^{N+1}(y), Z) \cap \mathfrak{q}_D(Z')$ is non-empty. We have thus shown:

$$\forall Z' \in \mathscr{I}_Z, \quad W^u(H^{N+1}(x), Z) \cap \mathfrak{q}_D(Z') \neq \emptyset \iff W^u(H^{N+1}(y), Z) \cap \mathfrak{q}_D(Z') \neq \emptyset. \quad (8.3)$$

Properties (8.2) and (8.3) mean that $H^{N+1}(x)$ and $H^{N+1}(y)$ belong to the same element of $\mathscr{E}_Z$ for any rectangle $Z \in \mathscr{Z}$ containing $H^{N+1}(x), H^{N+1}(y)$. This concludes the proof that $H(x) \overset{N}{\sim} H(y)$, and of part (3) of the proposition. $\square$

9. A finite-to-one extension

In this section, we construct a finite-to-one extension and deduce the Main Theorem. We rely on the family of disjoint sets $\mathscr{R}$ satisfying a geometrical Markov property. This family was obtained in the previous section as a refinement of the family $\mathscr{Z}$ constructed in Section 7, which was itself induced by the coding π introduced in Section 5.2. One important property of $\mathscr{Z}$ is that, due to the inverse theorem (Theorem 6.1), it satisfies a local finiteness property, see Proposition 7.1(2). Having these facts in mind, we construct a symbolic coding of the return map H.

9.1. A detailed statement

The theorem below implies the Main Theorem and includes additional properties that will be useful for some applications, including the one we will obtain in Section 10. We begin defining a Bowen relation for flows. This notion was formalized for diffeomorphisms in [10], and the following is an adaptation for flows. We refer to [15] for a discussion on the notion, and in particular on the non-uniqueness of such a relation.

Let $T_r : S_r \to S_r$ be a suspension flow over a symbolic system S that is an extension of some flow $U : X \to X$ by a semiconjugacy map $\pi : S_r \to X$, i.e. $U^t \circ \pi = \pi \circ T_r^t$ for all $t \in \mathbb{R}$.

Bowen relation: A *Bowen relation* $\sim$ for (T_r, π, U) is a symmetric binary relation on the alphabet of S satisfying the following two properties:

(i) $\forall \omega, \omega' \in S_r,\ \ \pi(\omega) = \pi(\omega') \implies \mathrm{v}(\omega) \sim \mathrm{v}(\omega')$, where $\mathrm{v}(x,t) := x_0$ for $x \in S$;
(ii) $\exists \gamma > 0$ with the following property:

$$\forall \omega, \omega' \in S_r,\ \ \left[\forall t \in \mathbb{R},\ \mathrm{v}(T_r^t\omega) \sim \mathrm{v}(T_r^t\omega')\right] \implies \left[\exists |t| < \gamma, \pi(\omega) = U^t(\pi(\omega'))\right].$$

Theorem 9.1. *Let X be a non-singular $C^{1+\beta}$ vector field ($\beta > 0$) on a closed 3-manifold M. Given $\chi > 0$, there exist a locally compact topological Markov flow $(\widehat{\Sigma}_{\widehat{r}}, \widehat{\sigma}_{\widehat{r}})$ with graph $\widehat{\mathscr{G}} = (\widehat{V}, \widehat{E})$ and roof function $\widehat{r}$ and a map $\widehat{\pi}_{\widehat{r}} : \widehat{\Sigma}_{\widehat{r}} \to M$ such that $\widehat{\pi}_{\widehat{r}} \circ \widehat{\sigma}_{\widehat{r}}^t = \varphi^t \circ \widehat{\pi}_{\widehat{r}}$, for all $t \in \mathbb{R}$, and satisfying:*

(1) *$\widehat{r}$ and $\widehat{\pi}_{\widehat{r}}$ are Hölder continuous.*
(2) *$\widehat{\pi}_{\widehat{r}}[\widehat{\Sigma}_{\widehat{r}}^{\#}] = \mathrm{NUH}^{\#}$ has full measure for every χ–hyperbolic measure; for every ergodic χ–hyperbolic measure μ, there is an ergodic $\widehat{\sigma}_{\widehat{r}}$–invariant measure $\overline{\mu}$ on $\widehat{\Sigma}_{\widehat{r}}$ such that $\overline{\mu} \circ \widehat{\pi}_{\widehat{r}}^{-1} = \mu$ and $h_{\overline{\mu}}(\widehat{\sigma}_{\widehat{r}}) = h_\mu(\varphi)$.*
(3) *If $(\underline{R}, t) \in \widehat{\Sigma}_{\widehat{r}}^{\#}$ satisfies $R_n = R$ and $R_m = S$ for infinitely many $n < 0$ and $m > 0$, then $\mathrm{Card}\{z \in \widehat{\Sigma}_{\widehat{r}}^{\#} : \widehat{\pi}_{\widehat{r}}(z) = \widehat{\pi}_{\widehat{r}}(\underline{R}, t)\}$ is bounded by a number $C(R,S)$, depending only on R, S.*
(4) *There is $\lambda > 0$ and for $x \in \widehat{\pi}_{\widehat{r}}(\widehat{\Sigma}_{\widehat{r}})$ there is a unique splitting $N_x = N_x^s \oplus N_x^u$ such that:*

$$\limsup_{t\to+\infty}\tfrac{1}{t}\log\|\Phi^t|_{N^s_x}\|\leq-\lambda \quad \text{and} \quad \liminf_{t\to+\infty}\tfrac{1}{t}\log\|\Phi^{-t}|_{N^s_x}\|\geq\lambda$$

$$\limsup_{t\to+\infty}\tfrac{1}{t}\log\|\Phi^{-t}|_{N^u_x}\|\leq-\lambda \quad \text{and} \quad \liminf_{t\to+\infty}\tfrac{1}{t}\log\|\Phi^{t}|_{N^u_x}\|\geq\lambda.$$

The splitting is Φ–equivariant, and the maps $z\mapsto N^{s/u}_{\widehat{\pi}_{\widehat{r}}(z)}$ are Hölder continuous on $\widehat{\Sigma}_{\widehat{r}}$.

(5) *For every $z\in\widehat{\Sigma}_{\widehat{r}}$, there are C^1 submanifolds $V^{cs}(z),V^{cu}(z)$ passing through $x:=\widehat{\pi}_{\widehat{r}}(z)$ such that:*
  (a) *$T_xV^{cs}(z)=N^s_x+\mathbb{R}\cdot X(x)$ and $T_xV^{cu}(z)=N^u_x+\mathbb{R}\cdot X(x)$.*
  (b) *For all $y\in V^{cs}(z)$, there is $\tau\in\mathbb{R}$ such that $d(\varphi^t(x),\varphi^{t+\tau}(y))\leq e^{-\lambda t}$, $\forall t\geq 0$.*
  (c) *For all $y\in V^{cu}(z)$, there is $\tau\in\mathbb{R}$ such that $d(\varphi^{-t}(x),\varphi^{-t+\tau}(y))\leq e^{-\lambda t}$, $\forall t\geq 0$.*

(6) *There is a symmetric binary relation $\sim$ on the alphabet $\widehat{V}$ satisfying:*
  (a) *For any $R\in\widehat{V}$, the set $\{S\in\widehat{V}:R\sim S\}$ is finite.*
  (b) *The relation $\sim$ is a Bowen relation for $(\widehat{\sigma}_{\widehat{r}},\widehat{\pi}_{\widehat{r}}|_{\widehat{\Sigma}^\#_{\widehat{r}}},\varphi)$.*

(7) *There exists a measurable set $\mathscr{R}$ with a measurable partition indexed by $\widehat{V}$, which we denote by $\{R:R\in\widehat{V}\}$, such that:*
  (a) *The orbit of any point $x\in\mathrm{NUH}^\#$ intersects $\mathscr{R}$.*
  (b) *The first return map $H\colon\mathscr{R}\to\mathscr{R}$ induced by φ is a well-defined bijection.*
  (c) *For any $x\in\mathscr{R}$, if $\underline{R}=\{R_n\}_{n\in\mathbb{Z}}$ satisfies $H^n(x)\in R_n$ for all $n\in\mathbb{Z}$, then $(\underline{R},0)\in\widehat{\Sigma}^\#_{\widehat{r}}$ and $\widehat{\pi}_{\widehat{r}}(\underline{R},0)=x$.*

(8) *For any compact transitive invariant hyperbolic set $K\subset M$ whose ergodic φ–invariant measures are all χ–hyperbolic, there is a* transitive *invariant compact set $X\subset\widehat{\Sigma}_{\widehat{r}}$ such that $\widehat{\pi}_{\widehat{r}}(X)=K$.*

Part (6) provides a combinatorial characterization of the noninjectivity of the coding. It is an adaptation for flows of the *Bowen property*, which was introduced in [10] for diffeomorphisms and motivated by the work of Bowen [9]. Note that, in contrast to [9], we *do not* claim that the flow restricted to $\widehat{\pi}_{\widehat{r}}[\widehat{\Sigma}^\#_{\widehat{r}}]$ is topologically equivalent to the corresponding quotient dynamics.

The relation $\sim$ will be the *affiliation*, which will be introduced in Section 9.3, following a similar notion introduced in [36]. Note that the assumption $\big[\mathrm{v}(\widehat{\sigma}^t_{\widehat{r}}(z))\sim\mathrm{v}(\widehat{\sigma}^t_{\widehat{r}}(z'))$ for all $t\in\mathbb{R}\big]$ consists of countably many affiliation conditions: if $z=(\underline{R},s)$ and $z'=(\underline{S},s')$, then varying t in the interval $[\widehat{r}_n(\underline{R}),\widehat{r}_{n+1}(\underline{R}))$ provides $i\leq\frac{\sup(\widehat{r})}{\inf(\widehat{r})}$ affiliations of the form $R_n\sim S_{m+1},\ldots,R_n\sim S_{m+i}$.

Part (7) provides for any $x\in\mathrm{NUH}^\#$ a particular pair $(\underline{R},t)\in\widehat{\Sigma}^\#_{\widehat{r}}$ such that $\widehat{\pi}_{\widehat{r}}(\underline{R},t)=x$ (here t is the smallest non-negative number such that $\varphi^{-t}(x)\in\mathscr{R}$). We call the pair $(\underline{R},t)$ the *canonical lift* of x. This is a measurable embedding of $\mathrm{NUH}^\#$ into $\widehat{\Sigma}_{\widehat{r}}$.

Part (8) is a version of [17, Proposition 3.9] in our context, and the proof is very similar, see Section 9.4.

9.2. Second coding

Let $\widehat{\mathscr{G}} = (\widehat{V}, \widehat{E})$ be the oriented graph with vertex set $\widehat{V} = \mathscr{R}$ and edge set $\widehat{E} = \{R \to S : R, S \in \mathscr{R} \text{ s.t. } H(R) \cap S \neq \emptyset\}$, and let $(\widehat{\Sigma}, \widehat{\sigma})$ be the TMS induced by $\widehat{\mathscr{G}}$. We note that the ingoing and outgoing degree of every vertex in $\widehat{\Sigma}$ is finite. We show this for the outgoing edges, since the proof for the ingoing edges is symmetric. Fix $R \in \mathscr{R}$, and fix $Z \in \mathscr{Z}$ such that $Z \supset R$. If $(R, S) \in \widehat{E}$ then $\varphi^{[0,\rho]}(R) \cap S \neq \emptyset$, hence for any $Z' \in \mathscr{Z}$ with $S \subset Z'$, we have $Z' \in \mathscr{I}_Z$. In particular,

$$\#\{(R, S) \in \widehat{E}\} \leq \sum_{Z' \in \mathscr{I}_Z} \#\{S \in \mathscr{R} : S \subset Z'\} < +\infty,$$

since both $\mathscr{I}_Z$ and each $\{S \in \mathscr{R} : S \subset Z'\}$ are finite sets (see Lemma 8.2(1)).

For $\ell \in \mathbb{Z}$ and a path $R_m \to \cdots \to R_n$ on $\widehat{\mathscr{G}}$ define

$${}_\ell[R_m, \ldots, R_n] := H^{-\ell}(R_m) \cap \cdots \cap H^{-\ell-(n-m)}(R_n),$$

the set of points whose itinerary under H from ℓ to $\ell + (n - m)$ visits the rectangles $R_m, \ldots, R_n$ respectively. The crucial property that gives the new coding is that ${}_\ell[R_m, \ldots, R_n] \neq \emptyset$. This follows by induction, using the Markov property of $\mathscr{R}$ (Proposition 8.3(3)).

The map π defines similar sets: for $\ell \in \mathbb{Z}$ and a path $v_m \overset{\varepsilon}{\to} \cdots \overset{\varepsilon}{\to} v_n$ on Σ, let

$$Z_\ell[v_m, \ldots, v_n] := \{\pi(\underline{w}) : \underline{w} \in \Sigma^\# \text{ and } w_\ell = v_m, \ldots, w_{\ell+(n-m)} = v_n\}.$$

There is a relation between these sets we just defined. Before stating such a relation, we will define the coding of H, and then collect some of its properties.

The map $\widehat{\pi} : \widehat{\Sigma} \to M$: Given $\underline{R} = \{R_n\}_{n\in\mathbb{Z}} \in \widehat{\Sigma}$, $\widehat{\pi}(\underline{R})$ is defined by the identity

$$\{\widehat{\pi}(\underline{R})\} := \bigcap_{n\geq 0} \overline{{}_{-n}[R_{-n}, \ldots, R_n]}.$$

Note that $\widehat{\pi}$ is well-defined, because the right hand side is an intersection of nested compact sets with diameters going to zero. The proposition below states relations between Σ and $\widehat{\Sigma}$, and between π and $\widehat{\pi}$. For $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}} \in \Sigma$, define

$$G_{\underline{v}}^n = \begin{cases} g_{x_{n-1}}^+ \circ \cdots \circ g_{x_0}^+ & , n \geq 0 \\ g_{x_{n+1}}^- \circ \cdots \circ g_{x_0}^- & , n < 0. \end{cases}$$

Recall the integer N introduced in Lemma 8.1.

Proposition 9.2. *For each $\underline{R} = (R_n)_{n\in\mathbb{Z}} \in \widehat{\Sigma}$ and $Z \in \mathscr{Z}$ with $Z \supset R_0$, there are an ε–gpo $\underline{v} = \{v_k\}_{k\in\mathbb{Z}} \in \Sigma$ with $Z(v_0) = Z$ and a sequence $(n_k)_{k\in\mathbb{Z}}$ of integers with $n_0 = 0$ and $1 \leq n_k - n_{k-1} \leq N$ for all $k \in \mathbb{Z}$ such that:*

(1) *For each* $k \geq 1$,

$$ {}_{n_{-k}}[R_{n_{-k}}, \ldots, R_{n_k}] \subset Z_{-k}[v_{-k}, \ldots, v_k]. $$

In particular, $\widehat{\pi}(\underline{R}) = \pi(\underline{v})$. *Moreover,* $R_{n_k} \subset Z(v_k)$ *for all* $k \in \mathbb{Z}$.

(2) *The map* $\widehat{\pi}$ *is Hölder continuous over* $\widehat{\Sigma}$. *In fact,* $\{v_i\}_{|i|\leq k}$ *depends only on* $\{R_j\}_{|j|\leq kN}$ *for each* $k \geq 1$.

(3) *If* $\underline{R} \in \widehat{\Sigma}^{\#}$, *then* $\underline{v} \in \Sigma^{\#}$.

(4) *The two codings have the same regular image:* $\pi[\Sigma^{\#}] = \widehat{\pi}[\widehat{\Sigma}^{\#}]$.

For diffeomorphisms, the above lemma is [36, Lemma 12.2]. The difference from the case of diffeomorphisms relies on our definitions of $\mathscr{G}$ and $\widehat{\mathscr{G}}$. While the edges of $\widehat{\mathscr{G}}$ correspond to possible time evolutions of H, the edges of $\mathscr{G}$ correspond to ε–overlaps. In particular, not every edge of $\widehat{\mathscr{G}}$ corresponds to an edge of $\mathscr{G}$, and this is the reason we have to introduce the sequence $(n_k)_{k\in\mathbb{Z}}$. In fact, each edge $v_k \to v_{k+1}$ of $\mathscr{G}$ corresponds to a sequence of edges $R_{n_k} \to \cdots \to R_{n_{k+1}}$ of $\widehat{\mathscr{G}}$.

Proof. We begin proving part (1). Fix $\{R_n\}_{n\in\mathbb{Z}} \in \widehat{\Sigma}$. The proof consists of successive uses of the following fact.

CLAIM: *For all* $i \in \mathbb{Z}$ *and* $v \in \mathscr{A}$ *such that* $R_i \subset Z(v)$, *there are* $1 \leq k \leq N$ *and* $w \in \mathscr{A}$ *such that* ${}_0[R_i, \ldots, R_{i+k}] \subset Z_0[v, w]$ *and* $R_{i+k} \subset Z(w)$. *Similarly, there are* $1 \leq \ell \leq N$ *and* $u \in \mathscr{A}$ *such that* ${}_0[R_{i-\ell}, \ldots, R_i] \subset Z_0[u, v]$ *and* $R_{i-\ell} \subset Z(u)$.

Proof of the claim. We prove the first statement (the second is proved similarly). Let $v = \Psi_x^{p^s,p^u} \in \mathscr{A}$ such that $R_i \subset Z(v)$. Since $\underline{R} \in \widehat{\Sigma}$, there is $y^* \in {}_0[R_i, \ldots, R_{i+N}]$. Moreover, there is $\underline{v}^* \in \Sigma^{\#}$ such that $\pi(\underline{v}^*) = y^*$ and $v_0^* = v$. We set $w := v_1^*$ so that $v \to w$. By construction, $g_x^+(\pi(\underline{v}^*)) = \pi(\sigma(\underline{v}^*))$ so $Z(w)$ contains $g_x^+(y^*)$. Also, there is $1 \leq k \leq N$ such that $g_x^+(y^*) = H^k(y^*)$.

We claim that ${}_0[R_i, \ldots, R_{i+k}] \subset Z_0[v, w]$. To see that, let $y \in {}_0[R_i, \ldots, R_{i+k}]$. We have $y \overset{N}{\sim} y^*$, thus the following occur:

- $y \in {}_0[R_i, \ldots, R_{i+k}] \subset R_i \subset Z(v)$, hence $y = \pi(\underline{v})$ for some $\underline{v} \in \Sigma^{\#}$ with $v_0 = v$.
- $g_x^+(y^*) = H^k(y^*) \in Z(w) \Rightarrow g_x^+(y) = H^k(y) \in Z(w)$, hence $\pi(\sigma(\underline{v})) = g_x^+(y) = \pi(\underline{w})$ for some $\underline{w} \in \Sigma^{\#}$ with $w_0 = w$.

Define $\underline{u} = \{u_n\}_{n\in\mathbb{Z}}$ by

$$ u_n = \begin{cases} v_n & , n \leq 0 \\ w_{n-1} & , n \geq 1. \end{cases} $$

Note that $\underline{u}$ belongs to $\widehat{\Sigma}^{\#}$ since $\underline{v}, \underline{w} \in \widehat{\Sigma}^{\#}$ and $v \to w$ on $\mathscr{G}$. To prove that $y = \pi(\underline{u})$, note that:

- If $n \le 0$, then $G^n_{\underline{u}}(y) = G^n_{\underline{v}}(y) \in Z(v_n)$.
- If $n \ge 1$, then $G^n_{\underline{u}}(y) = G^{n-1}_{\underline{w}}[g^+_x(y)] \in Z(w_{n-1})$.

By Proposition 4.6, it follows that $y = \pi(\underline{u}) \in Z_0[v, w]$, proving the inclusion.

The rest of the claim follows by symmetry, replacing g^+_x, H, σ by $g^-_x, H^{-1}, \sigma^{-1}$ and noting that $\overset{N}{\sim}$ considers H^k for all $|k| \le N$. $\square$

Now we prove part (1). Fix $n_0 = 0$ and $v_0 \in \mathscr{A}$ such that $R_0 \subset Z(v_0)$. Applying the claim for $i = 0$ and v_0, we get $0 < n_1 \le N$ and $v_1 \in \mathscr{A}$ such that ${}_0[R_0, \ldots, R_{n_1}] \subset Z_0[v_0, v_1]$ and $R_{n_1} \subset Z(v_1)$. By induction, we obtain an increasing sequence $n_0 = 0 < n_1 < n_2 < \cdots$ such that $n_k < n_{k+1} \le n_k + N$, ${}_0[R_{n_k}, \ldots, R_{n_{k+1}}] \subset Z_0[v_k, v_{k+1}]$, and $R_{n_k} \subset Z(v_k)$ for all $k \ge 0$. Doing the same for negative iterates, we get a decreasing sequence $n_0 = 0 > n_{-1} > n_{-2} > \cdots$ such that $n_k - N \le n_{k-1} < n_k$, ${}_0[R_{n_k}, \ldots, R_{n_{k+1}}] \subset Z_0[v_k, v_{k+1}]$, and $R_{n_k} \subset Z(v_k)$ for all $k < 0$. We claim that the sequence $\underline{v} = \{v_k\}_{k\in\mathbb{Z}}$ satisfies the proposition.

Fix $k \ge 0$. We wish to show that ${}_{n_{-k}}[R_{n_{-k}}, \ldots, R_{n_k}] \subset Z_{-k}[v_{-k}, \ldots, v_k]$, i.e. given $y \in {}_{n_{-k}}[R_{n_{-k}}, \ldots, R_{n_k}]$ we want to find $\underline{u} \in \Sigma^\#$ such that $(u_{-k}, \ldots, u_k) = (v_{-k}, \ldots, v_k)$ and $\pi(\underline{u}) = y$. Since $H^{n_{-k}}(y) \in R_{n_{-k}} \subset Z(v_{-k})$, there is $\underline{w}^- \in \Sigma^\#$ with $w^-_0 = v_{-k}$ and $H^{n_{-k}}(y) = \pi(\underline{w}^-)$. Similarly, since $H^{n_k}(y) \in R_{n_k} \subset Z(v_k)$, there is $\underline{w}^+ \in \Sigma^\#$ with $w^+_0 = v_k$ and $H^{n_k}(y) = \pi(\underline{w}^+)$. Define $\underline{u} = \{u_i\}_{i\in\mathbb{Z}}$ by:

$$u_i = \begin{cases} w^-_{i+k} & , i \le -k \\ v_i & , i = -k, \ldots, k \\ w^+_{i-k} & , i \ge k. \end{cases}$$

Clearly $\underline{u} \in \Sigma^\#$. We claim that $\pi(\underline{u}) = y$. Indeed:

- $-k \le i \le k$: we have $G^i_{\underline{u}}(y) = H^{n_i}(y) \in R_{n_i} \subset Z(v_i)$.
- $i \le -k$: since $G^{-k}_{\underline{u}}(y) = H^{n_{-k}}(y)$ and $G^{i+k}_{\sigma^{-k}(\underline{u})} = G^{i+k}_{\underline{w}^-}$ (the sequences $\sigma^{-k}(\underline{u})$ and $\underline{w}^-$ coincide in the past), we have $G^i_{\underline{u}}(y) = G^{i+k}_{\sigma^{-k}(\underline{u})}[G^{-k}_{\underline{u}}(y)] = G^{i+k}_{\underline{w}^-}[H^{n_{-k}}(y)] \in Z(w^-_{i+k}) = Z(u_i)$.
- $i \ge k$: as in the previous case, $G^i_{\underline{u}}(y) \in Z(u_i)$.

Therefore $G^i_{\underline{u}}(y) \in Z(u_i)$ for all $i \in \mathbb{Z}$, hence by Proposition 4.6 it follows that $\pi(\underline{u}) = y$.

Now we show that $\widehat{\pi}(\underline{R}) = \pi(\underline{v})$. Indeed, since $n_k \to \pm\infty$ as $k \to \pm\infty$, we have

$$\{\widehat{\pi}(\underline{R})\} = \bigcap_{k\ge0} \overline{{}_{n_{-k}}[R_{n_{-k}}, \ldots, R_{n_k}]} \subset \bigcap_{k\ge0} \overline{Z_{-k}[v_{-k}, \ldots, v_k]}.$$

On one hand, this latter set is, by Theorem 4.5(3), the intersection of a descending chain of closed sets with diameter going to zero, hence it is a singleton. On the other hand, it contains $\bigcap_{k\geq 0} Z_{-k}[v_{-k},\dots,v_k]=\{\pi(\underline{v})\}$. Thus $\widehat{\pi}(\underline{R})=\pi(\underline{v})$, which concludes the proof of part (1).

To check part (2), note that its second statement is immediate from the above argument. It implies the rest, since π is Hölder-continuous.

We turn to part (3). Assume that $\underline{R}\in\widehat{\Sigma}^\#$. Let $R\in\mathscr{R}$ and $m_j\to+\infty$ such that $R_{m_j}=R$ for all j. Since $\widehat{\Sigma}$ is locally compact (the degrees of $\widehat{\mathscr{G}}$ are all finite), the set

$$\mathscr{P}=\{S\in\mathscr{R}:\exists\text{ path } S_0=R\to S_1\to\cdots\to S_i=S\text{ with } i\leq N\}$$

is finite. Given j, let $k=k(j)$ be the unique integer such that $n_{k-1}<m_j\leq n_k$. Since $n_k-n_{k-1}\leq N$, it follows that $R_{n_k}\in\mathscr{P}$. By Lemma 8.2(2), it follows that v_k belongs to the finite set $\{Z\in\mathscr{Z}:\exists S\in\mathscr{P}\text{ such that } S\subset Z\}$, and so there is a sequence $k_i\to+\infty$ such that $\{v_{k_i}\}_{i\geq 0}$ is a constant sequence. Proceeding similarly for the negative indices, we conclude that $\underline{v}\in\Sigma^\#$. This proves part (3).

Now we prove part (4). By part (3), we have $\widehat{\pi}[\widehat{\Sigma}^\#]\subset\pi[\Sigma^\#]$. To prove the converse inclusion, let $\underline{v}=\{v_n\}_{n\in\mathbb{Z}}\in\Sigma^\#$ and write $x=\pi(\underline{v})$. Let $R_n\in\mathscr{R}$ such that $H^n(x)\in R_n$. Clearly, $\underline{R}=\{R_n\}\in\widehat{\Sigma}$ and $x=\widehat{\pi}(\underline{R})$. It remains to prove that $\underline{R}\in\widehat{\Sigma}^\#$. Let $v\in\mathscr{A}$ and $k_i\to+\infty$ such that $v_{k_i}=v$ for all $i\geq 0$. Letting $m_i:=n_{k_i}\to+\infty$ so that $H^{m_i}(x)=\pi[\sigma^{k_i}(\underline{v})]$, we have $H^{m_i}(x)\in R_{m_i}\cap Z(v)$ and so $R_{m_i}\subset Z(v)$. By Lemma 8.2(1), there is a subsequence m_{ℓ_j} such that $(R_{m_{\ell_j}})$ is constant. Proceeding similarly for negative indices, it follows that $\underline{R}\in\widehat{\Sigma}^\#$ and so $\pi[\Sigma^\#]\subset\widehat{\pi}[\widehat{\Sigma}^\#]$. This concludes the proof of part (4), and of the proposition. □

We now define the topological Markov flow (TMF) and coding that satisfy the Main Theorem. For that, recall the definition of TMF in Section 1.2.

The triple $(\widehat{\Sigma}_{\widehat{r}},\widehat{\sigma}_{\widehat{r}},\widehat{\pi}_{\widehat{r}})$: The topological Markov flow $(\widehat{\Sigma}_{\widehat{r}},\widehat{\sigma}_{\widehat{r}})$ is the suspension of $(\widehat{\Sigma},\widehat{\sigma})$ by the roof function $\widehat{r}:\widehat{\Sigma}\to(0,\rho)$ defined by

$$\widehat{r}(\underline{R}):=\min\{t>0:\varphi^t(\widehat{\pi}(\underline{R}))=\widehat{\pi}(\widehat{\sigma}(\underline{R}))\},$$

and the factor map $\widehat{\pi}_{\widehat{r}}:\widehat{\Sigma}_{\widehat{r}}\to M$ is given by $\widehat{\pi}_{\widehat{r}}(\underline{R},s):=\varphi^s(\widehat{\pi}(\underline{R}))$.

As claimed above, we have $\sup\widehat{r}<\rho$. Indeed, by Proposition 9.2 there is $\underline{v}=\{v_n\}_{n\in\mathbb{Z}}\in\Sigma$ such that $\widehat{\pi}(\underline{R})=\pi(\underline{v})$, and there are integers $n_{-1}<0<n_1$ such that ${}_{n_{-1}}[R_{n_{-1}},\dots,R_{n_1}]\subset Z_{-1}[v_{-1},v_0,v_1]$, hence $\widehat{r}(\underline{R})\leq\widehat{r}_{n_1}(\underline{R})=r(\underline{v})<\rho$. The rest of this section is devoted to proving that $(\widehat{\Sigma}_{\widehat{r}},\widehat{\sigma}_{\widehat{r}},\widehat{\pi}_{\widehat{r}})$ satisfies Theorem 9.1. We start with some fundamental properties.

Proposition 9.3. *The following holds for all $\varepsilon>0$ small enough.*

(1) $\widehat{r}:\widehat{\Sigma}\to(0,\infty)$ *is well-defined and Hölder continuous.*
(2) $\widehat{\pi}_{\widehat{r}}\circ\widehat{\sigma}^t_{\widehat{r}}=\varphi^t\circ\widehat{\pi}_{\widehat{r}}$, *for all $t\in\mathbb{R}$.*

(3) $\widehat{\pi}_{\widehat{r}}$ *is Hölder continuous with respect to the Bowen-Walters distance.*
(4) $\widehat{\pi}_{\widehat{r}}[\widehat{\Sigma}^\#_{\widehat{r}}] = \mathrm{NUH}^\#$.

Proof. To prove part (1), note that, by construction of $\widehat{\sigma}$ and of the sections $\Lambda \subset \widehat{\Lambda}$, $\widehat{r}$ is well-defined over $\widehat{\Sigma}$. Now, let $\underline{R} \in \widehat{\Sigma}$ and notice that $U := \{\underline{S} \in \widehat{\Sigma} : (S_0, S_1) = (R_0, R_1)\}$ is a neighborhood of $\underline{R}$. Moreover, there are $v, w \in \mathscr{A}$ and discs D_i, D_j from $\widehat{\Lambda}$ such that $\widehat{\pi}(U) \subset Z(v) \subset D_i$ and $\widehat{\pi}(\widehat{\sigma}(U)) \subset Z(w) \subset D_j$. Setting $\tau(x) = \inf\{t > 0 : \varphi^t(x) \in D_j\}$ for x on a neighborhood of $Z(v)$, we have $\widehat{r} = \tau \circ \widehat{\pi}$ on U. Since τ is a continuous passage time between the two smooth disks, transverse to the flow, it is well-defined and smooth, see Lemma 2.1(3). To finish the proof of part (1), recall that $\widehat{\pi}$ is Hölder continuous by Proposition 9.2(2).

Part (2) follows from the definition of $\widehat{r}$ by a routine argument, which we quickly recall. For $n \in \mathbb{Z}$, let $\widehat{r}_n$ be the n–th Birkhoff sum of $\widehat{r}$ (see Section 1.2). Let $(\underline{R}, s) \in \widehat{\Sigma}_{\widehat{r}}$. Given $t \in \mathbb{R}$, let $n \in \mathbb{Z}$ be defined by $\widehat{r}_n(\underline{R}) \le t + s < \widehat{r}_{n+1}(\underline{R})$ so that $\widehat{\sigma}^t_{\widehat{r}}(\underline{R}, s) = (\widehat{\sigma}^n(\underline{R}), t + s - \widehat{r}_n(\underline{R}))$. We have

$$
\begin{aligned}
(\widehat{\pi}_{\widehat{r}} \circ \widehat{\sigma}^t_{\widehat{r}})(\underline{R}, s) &= \widehat{\pi}_{\widehat{r}}\left(\widehat{\sigma}^n(\underline{R}), t + s - \widehat{r}_n(\underline{R})\right) = \varphi^{t+s-\widehat{r}_n(\underline{R})}(\widehat{\pi}(\widehat{\sigma}^n(\underline{R})))\\
&= \varphi^{t+s-\widehat{r}_n(\underline{R})}(\varphi^{\widehat{r}_n(\underline{R})}(\widehat{\pi}(\underline{R}))) = \varphi^{t+s}(\widehat{\pi}(\underline{R})) = (\varphi^t \circ \widehat{\pi}_{\widehat{r}})(\underline{R}, s),
\end{aligned}
$$

and so part (2) is established.

Now we prove part (3). By Proposition 9.2(2), $\widehat{\pi}$ is Hölder continuous. Applying the same arguments of [28, Lemma 5.9], we conclude that $\widehat{\pi}_{\widehat{r}}$ is Hölder continuous with respect to the Bowen-Walters distance.

We finally arrive at part (4). Recall from Proposition 9.2(4) that $\widehat{\pi}[\widehat{\Sigma}^\#] = \pi[\Sigma^\#]$, hence Proposition 5.2(3) rewrites as $\widehat{\pi}[\widehat{\Sigma}^\#] \supset \Lambda \cap \mathrm{NUH}^\#$. The flow saturation of $\widehat{\pi}[\widehat{\Sigma}^\#]$ is $\widehat{\pi}_{\widehat{r}}[\widehat{\Sigma}^\#_{\widehat{r}}]$ by definition, and the flow saturation of $\Lambda \cap \mathrm{NUH}^\#$ is $\mathrm{NUH}^\#$ since Λ is a global section and $\mathrm{NUH}^\#$ is φ–invariant. Therefore $\widehat{\pi}_{\widehat{r}}[\widehat{\Sigma}^\#_{\widehat{r}}] \supset \mathrm{NUH}^\#$. Reversely, $\widehat{\pi}[\widehat{\Sigma}^\#] = \pi[\Sigma^\#]$ is contained in $\mathrm{NUH}^\#$ by Theorem 6.1. Saturating this inclusion under the flow, we obtain that $\widehat{\pi}_{\widehat{r}}[\widehat{\Sigma}^\#_{\widehat{r}}] \subset \mathrm{NUH}^\#$. This concludes the proof of part (4). □

By Proposition 3.5, the above proposition establishes Parts (1) and (2) of the Main Theorem. In the next sections, we focus on proving part (3) and the other properties stated in Theorem 9.1.

9.3. The map $\widehat{\pi}_r$ is finite-to-one

Given $Z \in \mathscr{Z}$, remember that $\mathscr{I}_Z = \{Z' \in \mathscr{Z} : \varphi^{[-\rho,\rho]}Z \cap Z' \neq \emptyset\}$. The loss of injectivity of $\widehat{\pi}_{\widehat{r}}$ is related to the following notion.

Affiliation: We say that two rectangles $R, S \in \mathscr{R}$ are *affiliated*, and write $R \sim S$, if there are $Z, Z' \in \mathscr{Z}$ such that $R \subset Z$, $S \subset Z'$ and $Z' \in \mathscr{I}_Z$. This is a symmetric relation.

Lemma 9.4. *If* $\widehat{\pi}(\underline{R}) = \varphi^t[\widehat{\pi}(\underline{S})]$ *with* $\underline{R}, \underline{S} \in \widehat{\Sigma}^\#$ *and* $|t| \leq \rho$, *then* $R_0 \sim S_0$. *More precisely, if* $\underline{v}, \underline{w} \in \Sigma^\#$ *are such that* $\pi(\underline{v}) = \widehat{\pi}(\underline{R})$ *and* $\pi(\underline{w}) = \widehat{\pi}(\underline{S})$, *then* $R_0 \subset Z(v_0)$ *and* $S_0 \subset Z(w_0)$ *with* $Z(w_0) \in \mathscr{I}_{Z(v_0)}$.

Proof. Let $y = \widehat{\pi}(\underline{R})$ and $z = \widehat{\pi}(\underline{S})$, so that $y = \varphi^t(z)$. Applying Proposition 9.2 to $\underline{R}$ and $\underline{S}$, we find two ε–gpo's $\underline{v}, \underline{w} \in \Sigma^\#$ such that:

- $\pi(\underline{v}) = y$ and $R_0 \subset Z(v_0)$,
- $\pi(\underline{w}) = z$ and $S_0 \subset Z(w_0)$.

The lemma thus follows with $Z = Z(v_0)$ and $Z' = Z(w_0)$, since $\varphi^t(z) \in Z(v_0)$. □

Remark 9.5. We observe that the condition $\widehat{\pi}(\underline{R}) = \varphi^t[\widehat{\pi}(\underline{S})]$ in the above lemma actually implies more than just $R_0 \sim S_0$. It implies a *strong* affiliation: for *any* $Z, Z' \in \mathscr{Z}$ such that $Z \supset R_0$ and $Z' \supset S_0$, we have $Z' \in \mathscr{I}_Z$. Indeed, if $\underline{R}, \underline{S} \in \widehat{\Sigma}^\#$ and $|t| \leq \rho$ satisfy $\widehat{\pi}(\underline{R}) = \varphi^t[\widehat{\pi}(\underline{S})]$ and $Z, Z' \in \mathscr{Z}$ satisfy $Z \supset R_0$ and $Z' \supset S_0$, Proposition 9.2 gives the existence of $\underline{v}, \underline{w} \in \Sigma^\#$ such that $\pi(\underline{v}) = \widehat{\pi}(\underline{R})$ and $\pi(\underline{w}) = \widehat{\pi}(\underline{S})$ with $Z(v_0) = Z$ and $Z(w_0) = Z'$, and so $Z' \in \mathscr{I}_Z$.

For each $R \in \mathscr{R}$, define

$$A(R) := \{(S, Z') \in \mathscr{R} \times \mathscr{Z} : R \sim S \text{ and } S \subset Z'\} \text{ and } N(R) := \#A(R).$$

We can use Lemma 8.2 and proceed as in the proof of [36, Lemma 12.7] to show that $N(R) < \infty$, $\forall R \in \mathscr{R}$. Having this in mind, we now prove the finiteness-to-one property of $\widehat{\pi}_{\widehat{r}}$, i.e. part (3) of the Main Theorem and of Theorem 9.1.

Theorem 9.6. *Every* $x \in \widehat{\pi}_{\widehat{r}}[\widehat{\Sigma}^\#_{\widehat{r}}]$ *has finitely many* $\widehat{\pi}_{\widehat{r}}$*–preimages inside* $\widehat{\Sigma}^\#_{\widehat{r}}$. *More precisely, if* $x = \widehat{\pi}_{\widehat{r}}(\underline{R}, t)$ *with* $R_n = R$ *for infinitely many* $n > 0$ *and* $R_n = S$ *for infinitely many* $n < 0$, *then* $\#\{(\underline{S}, t') \in \widehat{\Sigma}^\#_{\widehat{r}} : \widehat{\pi}_{\widehat{r}}(\underline{S}, t') = x\} \leq N(R)N(S)$.

Proof. The proof is by contradiction. Assuming that $\#\{(\underline{S}, t') \in \widehat{\Sigma}^\#_{\widehat{r}} : \widehat{\pi}_{\widehat{r}}(\underline{S}, t') = x\}$ contains $N(R)N(S) + 1$ distinct elements $(\underline{R}^{(i)}, t_i)$, we are going to show that, up to permutation of these preimages, there are arbitrarily large integers $k < 0 < \ell$ such that

$$(R^{(1)}_k, \ldots, R^{(1)}_\ell) = (R^{(2)}_k, \ldots, R^{(2)}_\ell), \tag{9.1}$$

i.e. $\underline{R}^{(1)}$ and $\underline{R}^{(2)}$ agree between positions $k \to -\infty$ and $\ell \to +\infty$. This implies that $\underline{R}^{(1)} = \underline{R}^{(2)}$ and so $t_1 \neq t_2$. But then x is periodic with period $|t_2 - t_1| < \widehat{r}(\underline{R}^{(1)}) < \rho$, a contradiction to the choice of ρ (see Section 2.1).

The proof of equality (9.1) uses, as in [36, Theorem 12.8], an idea of Bowen [9, pp. 13–14]: it exploits the (non-uniform) expansiveness of φ, expressed in terms of the uniqueness of shadowing (Proposition 4.6). For simplicity of notation, we assume without loss

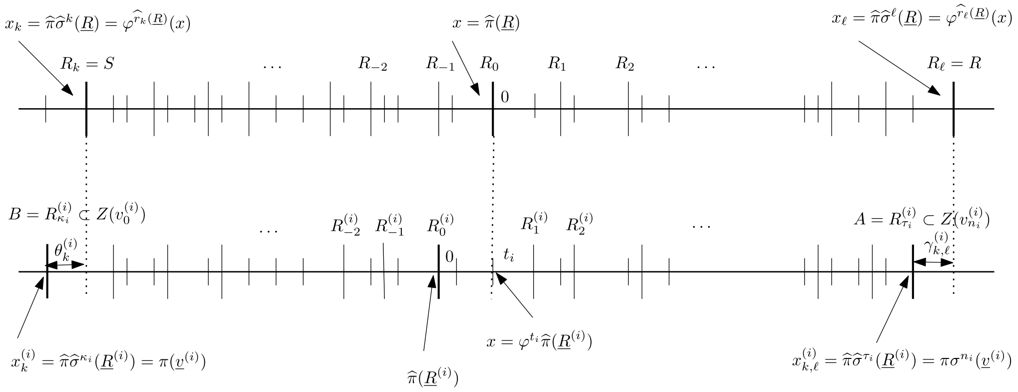

Fig. 3. The objects in the proof of Theorem 9.6. The line above depicts the points associated to $x = \widehat{\pi}(\underline{R})$ and the line below to $x = \varphi^{t_i}\widehat{\pi}(\underline{R}^{(i)})$. Vertical segments represent visits to the section (long segments correspond to the symbols from $\underline{R}$ or $\underline{R}^{(i)}$). The origins $\widehat{\pi}(\underline{R})$ and $\widehat{\pi}(\underline{R}^{(i)})$ are marked by a zero.

of generality that $t = 0$. Recall that r_n and $\widehat{r}_n$ denote Birkhoff sums for $n \in \mathbb{Z}$, see Section 1.2.

Let $x_n := \varphi^{\widehat{r}_n(\underline{R})}(x) = \widehat{\pi}[\widehat{\sigma}^n(\underline{R})]$, a point in the trajectory of x. Fix two integers $k < 0 < \ell$ such that $R_k = S$ and $R_\ell = R$. (See Fig. 3.)

For each $i = 1, \dots, N(R)N(S) + 1$, consider the following objects:

- Let $\kappa_i \in \mathbb{Z}$ be the unique integer such that $\widehat{r}_{\kappa_i}(\underline{R}^{(i)}) \le \widehat{r}_k(\underline{R}) + t_i < \widehat{r}_{\kappa_i+1}(\underline{R}^{(i)})$, so that $(\varphi^{t_i} \circ \widehat{\pi} \circ \widehat{\sigma}^k)(\underline{R})$ belongs to the orbit segment between $(\widehat{\pi} \circ \widehat{\sigma}^{\kappa_i})(\underline{R}^{(i)})$ and $(\widehat{\pi} \circ \widehat{\sigma}^{\kappa_i+1})(\underline{R}^{(i)})$.
- Let $\theta^{(i)}_k := \widehat{r}_k(\underline{R}) + t_i - \widehat{r}_{\kappa_i}(\underline{R}^{(i)})$, then $0 \le \theta^{(i)}_k < \rho$.
- Let $x^{(i)}_k := \varphi^{\widehat{r}_{\kappa_i}(\underline{R}^{(i)})-t_i}(x) = \widehat{\pi}[\widehat{\sigma}^{\kappa_i}(\underline{R}^{(i)})]$, a point in the trajectory of x. Note that $\varphi^{\theta^{(i)}_k}(x^{(i)}_k) = \varphi^{\widehat{r}_k(\underline{R})}(x) = x_k$ and $\varphi^{\theta^{(i)}_k+\widehat{r}_\ell(\underline{R})-\widehat{r}_k(\underline{R})}(x^{(i)}_k) = x_\ell$.
- By Proposition 9.2, there is an ε–gpo $\underline{v}^{(i)} \in \Sigma^\#$ such that $\pi[\underline{v}^{(i)}] = x^{(i)}_k$ and $R^{(i)}_{\kappa_i} \subset Z(v^{(i)}_0)$.
- Let n_i be the unique integer such that $r_{n_i}(\underline{v}^{(i)}) \le \theta^{(i)}_k + \widehat{r}_\ell(\underline{R}) - \widehat{r}_k(\underline{R}) < r_{n_i+1}(\underline{v}^{(i)})$. Hence x_ℓ belongs to the orbit segment between $(\pi \circ \sigma^{n_i})(\underline{v}^{(i)})$ and $(\pi \circ \sigma^{n_i+1})(\underline{v}^{(i)})$.
- Let $\tau_i > \kappa_i$ be the unique integer such that $\widehat{r}_{\tau_i-\kappa_i}[\widehat{\sigma}^{\kappa_i}(\underline{R}^{(i)})] = r_{n_i}(\underline{v}^{(i)})$. The existence of such an integer is ensured by Proposition 9.2 which also gives $R^{(i)}_{\tau_i} \subset Z(v^{(i)}_{n_i})$.
- Let $x^{(i)}_{k,\ell} := \varphi^{r_{n_i}(\underline{v}^{(i)})}(x^{(i)}_k) = \widehat{\pi}[\widehat{\sigma}^{\tau_i}(\underline{R}^{(i)})]$, a point in the trajectory of x.
- Let $\gamma^{(i)}_{k,\ell} := \theta^{(i)}_k + \widehat{r}_\ell(\underline{R}) - \widehat{r}_k(\underline{R}) - r_{n_i}(\underline{v}^{(i)})$, then $|\gamma^{(i)}_{k,\ell}| < \rho$. This is the time displacement between $x^{(i)}_{k,\ell}$ and x_ℓ, i.e. $x_\ell = \varphi^{\gamma^{(i)}_{k,\ell}}(x^{(i)}_{k,\ell})$.

Therefore, for each i, we have:

- $(R^{(i)}_{\kappa_i}, Z(v^{(i)}_0)) \in A(S)$: this follows from Lemma 9.4, since $x^{(i)}_k = \widehat{\pi}[\widehat{\sigma}^{\kappa_i}(\underline{R}^{(i)})]$, $x_k = \widehat{\pi}[\widehat{\sigma}^k(\underline{R})]$ and $x_k = \varphi^{\theta^{(i)}_k}(x^{(i)}_k)$.

◦ $(R^{(i)}_{\tau_i}, Z(v^{(i)}_{n_i})) \in A(R)$: this also follows from Lemma 9.4, since $x^{(i)}_{k,\ell} = \widehat{\pi}[\widehat{\sigma}^{\tau_i}(\underline{R}^{(i)})]$, $x_\ell = \widehat{\pi}[\widehat{\sigma}^\ell(\underline{R})]$ and $x_\ell = \varphi^{\gamma^{(i)}_{k,\ell}}(x^{(i)}_{k,\ell})$.

The previous paragraph implies that every quadruple $(R^{(i)}_{\kappa_i}, Z(v^{(i)}_0); R^{(i)}_{\tau_i}, Z(v^{(i)}_{n_i}))$ we constructed belongs to the cartesian product $A(R)\times A(S)$. This latter set has cardinality $N(R)N(S)$, hence by the pigeonhole principle there are distinct i, j such that

$$(R^{(i)}_{\kappa_i}, Z(v^{(i)}_0); R^{(i)}_{\tau_i}, Z(v^{(i)}_{n_i})) = (R^{(j)}_{\kappa_j}, Z(v^{(j)}_0); R^{(j)}_{\tau_j}, Z(v^{(j)}_{n_j})).$$

For simplicity of notation, we assume $i = 1$ and $j = 2$ and write $R^{(1)}_{\kappa_1} = R^{(2)}_{\kappa_2} =: B$ and $R^{(1)}_{\tau_1} = R^{(2)}_{\tau_2} =: A$.

Set $\alpha_i := \widehat{r}_{\kappa_i}(\underline{R}) - t_i$ and $\beta_i := \widehat{r}_{\tau_i}(\underline{R}) - t_i$ for $i = 1, 2$. By definition, we have $\alpha_i \in [\widehat{r}_k(\underline{R}) - \rho, \widehat{r}_k(\underline{R})]$, and so $|\alpha_1 - \alpha_2| \le \rho$. Since $\varphi^{\alpha_1}(x) = \widehat{\pi}[\widehat{\sigma}^{\kappa_1}(\underline{R}^{(1)})]$ and $\varphi^{\alpha_2}(x) = \widehat{\pi}[\widehat{\sigma}^{\kappa_2}(\underline{R}^{(2)})]$ both belong to $\overline{B}$, we must have $\alpha_1 = \alpha_2$. An analogous argument shows that $\beta_1 = \beta_2$. We denote these common values by α, β.

Since $R^{(1)}_{\kappa_1} \to \cdots \to R^{(1)}_{\tau_1}$ and $R^{(2)}_{\kappa_2} \to \cdots \to R^{(2)}_{\tau_2}$ are admissible paths on $\widehat{\Sigma}$, we can find non-periodic points

$$y \in {}_0[R^{(1)}_{\kappa_1}, \dots, R^{(1)}_{\tau_1}] \text{ and } z \in {}_0[R^{(2)}_{\kappa_2}, \dots, R^{(2)}_{\tau_2}].$$

Let $y' = H^{\tau_1-\kappa_1}(y)$ and $z' = H^{\tau_2-\kappa_2}(z)$. We have $y, z \in B$ and $y', z' \in A$. By Proposition 8.3(1), we can define two points w, w' by the equalities

$$\{w\} := \{[y, z]\} = W^s(y, B) \cap W^u(z, B)$$
$$\{w'\} := \{[y', z']\} = W^s(y', A) \cap W^u(z', A).$$

Note that neither w nor w' can be periodic.

Claim: w, w' belong to the same trajectory of φ. More precisely, $w' = H^{\tau_1-\kappa_1}(w)$.

Proof of the claim. This is a consequence of Proposition 7.3: we can obtain w' from w by applying small flow displacements of Smale products of points at nearby rectangles.

To implement this idea, we first divide the interval $[\alpha, \beta]$ by visits to the rectangles $\left\{R^{(1)}_k\right\}_{\kappa_1\le k\le\tau_1}$ and $\left\{R^{(2)}_k\right\}_{\kappa_2\le k\le\tau_2}$. Since these visits are ρ–dense in this interval, we can select times:

$$\delta_0 = \alpha < \varepsilon_0 < \delta_1 < \varepsilon_1 < \cdots < \delta_T \le \varepsilon_T = \beta \text{ such that } 0 < \varepsilon_s - \delta_s, \delta_{s+1} - \varepsilon_s \le \rho \quad (9.2)$$

where each $\delta_t = \widehat{r}_m(\underline{R}^{(1)})$ for some $m = m(t) \in [\kappa_1, \tau_1]$ and each $\varepsilon_t = \widehat{r}_n(\underline{R}^{(2)})$ for some $n = n(t) \in [\kappa_2, \tau_2]$.

By Lemma 9.4, this implies that the successive rectangles implied by eq. (9.2) are affiliated: $R^{(1)}_{m(t)} \sim R^{(2)}_{n(t)}$ and $R^{(2)}_{n(t)} \sim R^{(1)}_{m(t+1)}$. Applying Proposition 9.2, find rectangles

Z, Z', Z'' of $\mathscr{Z}$ that contain $R^{(1)}_{m(t)}, R^{(2)}_{n(t)}, R^{(1)}_{m(t+1)}$ and satisfy the conditions of Proposition 7.3. The same applies to the three rectangles $R^{(2)}_{n(t)}, R^{(1)}_{m(t+1)}, R^{(2)}_{n(t+1)}$.

Now let $y_k = H^{k-\kappa_1}(y)$ for $\kappa_1 \le k \le \tau_1$ and $z_\ell = H^{\ell-\kappa_2}(z)$ for $\kappa_2 \le \ell \le \tau_2$. For each $t = 0, 1, \dots, T$, note that $y_{m(t)} \in R^{(1)}_{m(t)}$ and $z_{n(t)} \in R^{(2)}_{n(t)}$. We let $D^{(1)}_t$ and $D^{(2)}_t$ be the connected components of $\widehat{\Lambda}$ containing $R^{(1)}_{m(t)}$ and $R^{(2)}_{n(t)}$ respectively.

On the one hand, since $y_{m(t+1)} = \varphi^u(y_{m(t)})$ with $0 \le u \le \rho$, Proposition 7.3 implies that

$$[y_{m(t)}, z_{n(t)}]_{D^{(2)}_t} = [y_{m(t+1)}, z_{n(t)}]_{D^{(2)}_t}.$$

On the other hand, Proposition 7.2(3) yields:

$$\begin{aligned}
[y_{m(t+1)}, z_{n(t)}]_{D^{(1)}_{t+1}} &= \mathfrak{q}_{D^{(1)}_{t+1}}([y_{m(t+1)}, z_{n(t)}]_{D^{(2)}_t}) \\
&= \mathfrak{q}_{D^{(1)}_{t+1}}([y_{m(t)}, z_{n(t)}]_{D^{(2)}_t}) = (\mathfrak{q}_{D^{(1)}_{t+1}} \circ \mathfrak{q}_{D^{(2)}_t})([y_{m(t)}, z_{n(t)}]_{D^{(1)}_t}).
\end{aligned}$$

Finally, applying Proposition 7.3 again, we conclude that

$$[y_{m(t+1)}, z_{n(t+1)}]_{D^{(1)}_{t+1}} = [y_{m(t+1)}, z_{n(t)}]_{D^{(1)}_{t+1}} = (\mathfrak{q}_{D^{(1)}_{t+1}} \circ \mathfrak{q}_{D^{(2)}_t})([y_{m(t)}, z_{n(t)}]_{D^{(1)}_t}).$$

Proceeding inductively,

$$\begin{aligned}
w' = [y', z'] &= \mathfrak{q}_{D^{(2)}_T}([y_{m(T)}, z_{n(T)}]_{D^{(1)}_T}) \\
&= (\mathfrak{q}_{D^{(2)}_T} \circ \mathfrak{q}_{D^{(1)}_T} \circ \mathfrak{q}_{D^{(2)}_{T-1}})([y_{m(T-1)}, z_{n(T-1)}]_{D^{(1)}_{T-1}}) \\
&= \cdots \\
&= (\mathfrak{q}_{D^{(2)}_T} \circ \mathfrak{q}_{D^{(1)}_T} \circ \cdots \circ \mathfrak{q}_{D^{(1)}_1} \circ \mathfrak{q}_{D^{(2)}_0})([y_{m(0)}, z_{n(0)}]_{D^{(1)}_0}) \\
&= (\mathfrak{q}_{D^{(2)}_T} \circ \mathfrak{q}_{D^{(1)}_T} \circ \cdots \circ \mathfrak{q}_{D^{(1)}_1} \circ \mathfrak{q}_{D^{(2)}_0})([y, z]_{D^{(1)}_0}) \\
&= (\mathfrak{q}_{D^{(2)}_T} \circ \mathfrak{q}_{D^{(1)}_T} \circ \cdots \circ \mathfrak{q}_{D^{(1)}_1} \circ \mathfrak{q}_{D^{(2)}_0})(w),
\end{aligned}$$

which proves that w and w' belong to the same trajectory. Repeating the argument using the holonomy maps corresponding to the sequence $(R^{(1)}_{\kappa_1}, \dots, R^{(1)}_{\tau_1})$, we get that their composition sends w to w'. By the Markov property in the stable direction, these holonomy maps correspond to first returns. This proves that $w' = H^{\tau_1-\kappa_1}(w)$. $\square$

Now it is easy to conclude the proof of the theorem. A symmetric version of the claim implies that $w = H^{-(\tau_2-\kappa_2)}(w')$. Since w is not periodic, we obtain $\tau_1 - \kappa_1 = \tau_2 - \kappa_2$. It follows that $(R^{(1)}_{\kappa_1}, \dots, R^{(1)}_{\tau_1}) = (R^{(2)}_{\kappa_2}, \dots, R^{(2)}_{\tau_2})$, since both correspond to the rectangles in $\mathscr{R}$ that contain $H^{\kappa_1}(w), \dots, H^{\tau_1}(w)$. This concludes the proof. $\square$

9.4. Conclusion of the proof of Theorem 9.1

We already proved parts (1) and the first half of part (2). Also, Theorem 9.6 establishes part (3). For the second half of part (2), we note that every point of $\mathrm{NUH}^{\#}$ has a finite and nonzero number of lifts to $\widehat{\Sigma}^{\#}_{\widehat{r}}$, hence every ergodic χ–hyperbolic measure on M, which is supported in $\mathrm{NUH}^{\#}$, can be lifted to an ergodic $\widehat{\sigma}_{\widehat{r}}$–invariant measure $\overline{\mu}$, exactly as in the argument performed in [36, Section 13]. This concludes the proof of part (2) of Theorem 9.1.

We now prove the remaining parts (4)–(8) stated in Theorem 9.1.

Part (4) Using Theorem 4.5, we define $N^{s/u}_z$ as follows:

- For $z = (\underline{R}, 0) \in \widehat{\Sigma}_{\widehat{r}}$, define first $V^{s/u}(z) = W^{s/u}(\widehat{\pi}(\underline{R}), R_0)$ and $N^{s/u}_z = T_{\widehat{\pi}(\underline{R})}V^{s/u}(z)$. By definition, $V^s(z)$ and $V^u(z)$ are transverse.
- For $z = (\underline{R}, t) \in \widehat{\Sigma}_{\widehat{r}}$, define $N^{s/u}_z = \Phi^t\left(N^{s/u}_{(\underline{R},0)}\right)$. Since Φ is an isomorphism, $N_{\widehat{\pi}_{\widehat{r}}(\underline{R},t)} = N^s_z \oplus N^u_z$.

The geometrical Markov property of Proposition 8.3(3) implies that the families $\{N^{s/u}_z\}$ are invariant under Φ. The convergence rates along $N^{s/u}_z$ follow from Theorem 4.5(3), taking $\lambda := \frac{1}{\sup(r_\Lambda)}\left(\frac{\chi \inf(r_\Lambda)}{2} - \frac{\beta\varepsilon}{6}\right)$. These estimates show, in particular, that these spaces only depend on $x := \widehat{\pi}_{\widehat{r}}(z)$, hence one can set $N^{s/u}_x := N^{s/u}_z$. Finally, the Hölder continuity follows from Theorem 4.5(5). This concludes the proof of part (4).

Part (5) For any $z = (\underline{R}, 0) \in \widehat{\Sigma}$, Theorem 4.5 associates curves $V^{s/u}(z)$ tangent to $\widehat{\Lambda}$, hence transverse to the flow direction. For general $z = (\underline{R}, t) \in \widehat{\Sigma}_{\widehat{r}}$, one then defines the manifolds $V^{cs/cu}(z) := \varphi^{[t-1,t+1]}(V^{s/u}(\underline{R}, 0))$. By construction, $V^{cs/cu}(z)$ is tangent to $N^{s/u}_z + \mathbb{R}\cdot X(\widehat{\pi}_{\widehat{r}}(z))$. Moreover, by Proposition 4.8, for any $y \in V^{cs}(z)$ there exists $\tau \in \mathbb{R}$ such that $d(\varphi^t(\widehat{\pi}_{\widehat{r}}(z)), \varphi^{t+\tau}(y)) \le \exp(-\lambda t)$ for all $t \ge 0$. The same holds for $V^{cu}(z)$, thus concluding the proof of Part (5).

Part (7) The proof of this part is almost automatic. The measurable set $\mathscr{Z} = \mathscr{R}$ contains $\Lambda \cap \mathrm{NUH}^{\#}$, hence the orbit of any point $x \in \mathrm{NUH}^{\#}$ intersects $\mathscr{R}$, which proves item (a). Item (b) was proved in the beginning of Section 7.2. Finally, any $x \in \mathscr{R}$ defines $\{R_n\}_{n\in\mathbb{Z}}$ such that $H^n(x) \in R_n$ for all $n \in \mathbb{Z}$. In particular, $H(R_n) \cap R_{n+1} \neq \emptyset$ for all $n \in \mathbb{Z}$ and so $\underline{R} = \{R_n\} \in \widehat{\Sigma}$. Since $\mathscr{R} = \pi[\Sigma^{\#}]$, we also have $x = \pi(\underline{v})$ for some $\underline{v} = \{v_n\}_{n\in\mathbb{Z}} \in \Sigma^{\#}$. For each $k \in \mathbb{Z}$, the point $\pi[\sigma^k(\underline{v})]$ is a return of x to $\mathscr{R}$, hence there is an increasing sequence such that $\pi[\sigma^k(\underline{v})] = H^{n_k}(x)$. Therefore $R_{n_k} \subset Z(v_k)$. Using that $\underline{v} \in \Sigma^{\#}$ and Lemma 8.2(1), it follows that $\underline{R} \in \widehat{\Sigma}^{\#}$.

Part (8) Assume $K \subset M$ is a compact, transitive, invariant, hyperbolic set such that all φ–invariant measures supported by it are χ–hyperbolic. Let $TK = E^s \oplus X \oplus E^u$

be the continuous hyperbolic splitting. Proceeding as in [17, Proposition 2.8], there are constants $C > 0$ and $\kappa > \chi$ such that

$$\|d\varphi^t v^s\| \leq Ce^{-\kappa t}\|v^s\| \text{ and } \|d\varphi^{-t} v^u\| \leq Ce^{-\kappa t}\|v^u\|, \quad \text{for all } v^s \in E^s, v^u \in E^u \text{ and } t \geq 0.$$

Now we proceed as in the proof of Proposition 3.1. Using the notation of equation (3.1), the functions $x \in K \mapsto \gamma^{s/u}(x)$ are continuous. Therefore there is a constant $C_1 = C_1(K)$ such that $s(x), u(x) < C_1$ and $\alpha(x) = \angle(n_x^s, n_x^u) > C_1^{-1}$ for all $x \in K$. This implies that $\inf_{x\in K} Q(x) > 0$, which in turn implies that $\inf_{x\in K} q(x) > 0$. In particular, $K \subset \mathrm{NUH}^\#$. This is enough to reproduce the method of proof of [17, Prop. 3.9], as follows. We recall that $X \subset \widehat{\Sigma}_{\widehat{r}}$ is $\widehat{\sigma}_{\widehat{r}}$–invariant if $\widehat{\sigma}_{\widehat{r}}^t(X) = X$ for all $t \in \mathbb{R}$.

Step 1: There is a $\widehat{\sigma}_{\widehat{r}}$–invariant compact set $X_0 \subset \widehat{\Sigma}_{\widehat{r}}$ such that $\widehat{\pi}_{\widehat{r}}(X_0) \supset K$.

Proof of Step 1. For each $x \in K \cap \mathscr{R}$, consider its canonical coding $\underline{R}(x) = \{R_n(x)\}_{n\in\mathbb{Z}}$. Since $\inf_{x\in K} q(x) > 0$, K intersects finitely many rectangles of $\mathscr{R}$. Hence there is a finite set $V_0 \subset \mathscr{R}$ such that $R_0(x) \in V_0$ for all $x \in K \cap \mathscr{R}$. By invariance, the same happens for all $n \in \mathbb{Z}$, i.e. $R_n(x) \in V_0$ for all $x \in K \cap \mathscr{R}$. Therefore the subshift Σ_0 induced by V_0, which is compact since V_0 is finite, satisfies $\widehat{\pi}(\Sigma_0) \supset K \cap \mathscr{R}$. Let X_0 be the TMF defined by (Σ_0, σ) with roof function $\widehat{r}\restriction_{\Sigma_0}$. Saturating the latter inclusion under φ and using part (7)(a), we conclude that $\widehat{\pi}_{\widehat{r}}(X_0) \supset K$. □

Step 2: There is a transitive $\widehat{\sigma}_{\widehat{r}}$–invariant compact subset $X \subset X_0$ such that $\widehat{\pi}_{\widehat{r}}(X) = K$.

Proof of Step 2. Among all compact $\widehat{\sigma}_{\widehat{r}}$–invariant sets $X \subset X_0$ with $\widehat{\pi}_{\widehat{r}}(X) \supset K$, consider one which is minimal for the inclusion (it exists by Zorn's lemma). We claim that such an X satisfies Step 2. To see that, let $z \in K$ whose forward orbit is dense in K, let $x \in X$ be a lift of z, and let Y be the ω–limit set of the forward orbit of x,

$$Y = \{y \in \widehat{\Sigma}_{\widehat{r}} : \exists t_n \to +\infty \text{ s.t. } \widehat{\sigma}_{\widehat{r}}^{t_n}(x) \to y\}.$$

For any $n \geq 1$, the set $Y_n := \{\sigma_{\widehat{r}}^t(x), t \geq n\} \cup Y \subset X$ is compact and forward invariant. Hence the projection $\widehat{\pi}_{\widehat{r}}(Y_n)$ is compact and contains $\{\varphi^t(z), t \geq n\}$. Since the forward orbit of z is dense in K, we have $\widehat{\pi}_{\widehat{r}}(Y_n) \supset K$. Taking the intersection over n, one deduces that the projection of the $\sigma_{\widehat{r}}^t$–invariant compact set Y contains K. By the minimality of X, it follows that $X = Y$. □

This concludes the proof of Part (8).

Part (6), items (a) and (b)-(i) Item (a) of Part (6), the local finiteness of the affiliation, was proved at the beginning of Section 9.3. Item (b) claims that the affiliation $\sim$ is a Bowen relation. This splits into two properties (i) and (ii).

To prove item (i) of the Bowen relation, let $(\underline{R}, t), (\underline{S}, s) \in \widehat{\Sigma}_{\widehat{r}}^\#$ with $\widehat{\pi}_{\widehat{r}}(\underline{R}, t) = \widehat{\pi}_{\widehat{r}}(\underline{S}, s)$, i.e. $\widehat{\pi}(\underline{R}) = \varphi^{s-t}\widehat{\pi}(\underline{S})$. Since $|s - t| \leq \sup(\widehat{r}) \leq \rho$, Lemma 9.4 implies that $R_0 \sim S_0$.

Part (6)*, item (b)-(ii)* We turn to property (ii) of a Bowen relation. Fig. 4 contains the involved objects in the proof. We take $\gamma = 3\rho$. Let $z, z' \in \widehat{\Sigma}^{\#}_{\widehat{r}}$ such that $\mathrm{v}(\widehat{\sigma}^t_{\widehat{r}} z) \sim \mathrm{v}(\widehat{\sigma}^t_{\widehat{r}} z')$ for all $t \in \mathbb{R}$. By flowing the two orbits, we can assume that $z = (\underline{R}, 0)$ and $z' = (\underline{S}, s)$. Let $x = \widehat{\pi}(\underline{R})$ and $y = \widehat{\pi}(\underline{S})$. We wish to show that $x = \varphi^{t+s}(y)$ for some $|t| < \gamma$. We will deduce from the affiliation condition that the orbit of y must be shadowed by an ε–gpo that shadows x. By Proposition 4.6, the two orbits are equal and the time shift between x and $\varphi^s(y)$ will be easily bounded.

To do this, we first apply Proposition 9.2(1) and get ε–gpo's $\underline{v}, \underline{w} \in \Sigma^{\#}$ such that $x = \widehat{\pi}(\underline{R}) = \pi(\underline{v})$ and $y = \widehat{\pi}(\underline{S}) = \pi(\underline{w})$ with $R_0 \subset Z(v_0)$ and $S_0 \subset Z(w_0)$. Moreover, there are increasing integer sequences $(n_i)_{i\in\mathbb{Z}}$, $(\widetilde{m}_i)_{i\in\mathbb{Z}}$ such that $R_{n_i} \subset Z(v_i)$ and $S_{\widetilde{m}_i} \subset Z(w_i)$. For each $i \in \mathbb{Z}$, we locate affiliated symbols in the codings of x and y as follows.

We start with $\varphi^t(x) \in Z(v_i)$ for $t = r_i(\underline{v}) = \widehat{r}_{n_i}(\underline{R})$. We have $\widehat{\sigma}^t_{\widehat{r}}(\underline{R}, 0) = (\widehat{\sigma}^{n_i}(\underline{R}), 0)$, hence $\mathrm{v}(\widehat{\sigma}^t_{\widehat{r}}(z)) = R_{n_i}$. We also have $\widehat{\sigma}^t_{\widehat{r}}(\underline{S}, s) = (\widehat{\sigma}^{\ell_i}(\underline{S}), t + s - \widehat{r}_{\ell_i}(\underline{S}))$, where ℓ_i is the unique integer such that $\widehat{r}_{\ell_i}(\underline{S}) \le t + s < \widehat{r}_{\ell_i+1}(\underline{S})$. Thus $\mathrm{v}(\widehat{\sigma}^t_{\widehat{r}}(z')) = S_{\ell_i}$ and, by assumption, $R_{n_i} \sim S_{\ell_i}$.

Let $a_i \in \mathbb{Z}$ be the largest integer such that $m_i := \widetilde{m}_{a_i} \le \ell_i$. Hence, $S_{m_i} \subset Z(w_{a_i})$. We have $R_{n_i} \subset Z(v_i) \subset D_i$ and likewise $S_{m_i} \subset Z(w_{a_i}) \subset E_i$ for some unique connected components D_i, E_i of the section $\widehat{\Lambda}$.

We write $\Psi^{P^s_i, P^u_i}_{X_i}$ for v_i and $\Psi^{Q^s_i, Q^u_i}_{Y_i}$ for w_{a_i} for all $i \in \mathbb{Z}$. Finally, we set $\widetilde{y}_i := \pi(\sigma^{a_i}\underline{w}) \in Z(w_{a_i})$ and $y_i := \mathfrak{q}_{D_i}(\widetilde{y}_i)$. We are going to show that, for all $i \in \mathbb{Z}$:

(1) y_i is well-defined, and for $i = 0$ we have $y_0 = \varphi^u(\widetilde{y}_0)$ with $|u| \le 2\rho$;
(2) $y_{i+1} = g^+_{X_i}(y_i)$.

Proposition 4.6 will then imply that $x = y_0 = \varphi^u(\widetilde{y}_0) = \varphi^u(y) = \varphi^{u-s}(\widehat{\pi}_{\widehat{r}}(\underline{S}, s))$, where $|u - s| \le 2\rho + \sup \widehat{r} < 3\rho$. Property (ii) and therefore the Bowen relation claimed by Part (6)(b) will be established.

It remains to prove the above identities. They require checking that some holonomies along the flow are compatible. We will prove this using that affiliation implies that charts have comparable parameters and their images fall inside $\widehat{\Lambda}$ far from its boundary. The claims below are not sharp but enough for our purposes. We begin by proving some variants of Proposition 7.2(1).

Claim 1: Let $Z_1, Z_2 \in \mathscr{Z}$ such that $Z_1 \cap \varphi^{[-\rho,\rho]} Z_2 \neq \emptyset$. Write $Z_i = Z(\Psi^{p^s_i, p^u_i}_{x_i})$ and let D_i be the connected component of $\widehat{\Lambda}$ containing Z_i. Then $\frac{p^s_1 \wedge p^u_1}{p^s_2 \wedge p^u_2} = e^{\pm(O(\sqrt[3]{\varepsilon}) + O(\rho))}$ and

$$\mathfrak{q}_{D_1}(\Psi_{x_2}(R[c(p^s_2 \wedge p^u_2)])) \subset \Psi_{x_1}(R[2c(p^s_1 \wedge p^u_1)])$$

for all $1 \le c \le 64$.

Proof of Claim 1. Same of Proposition 7.2(1). □

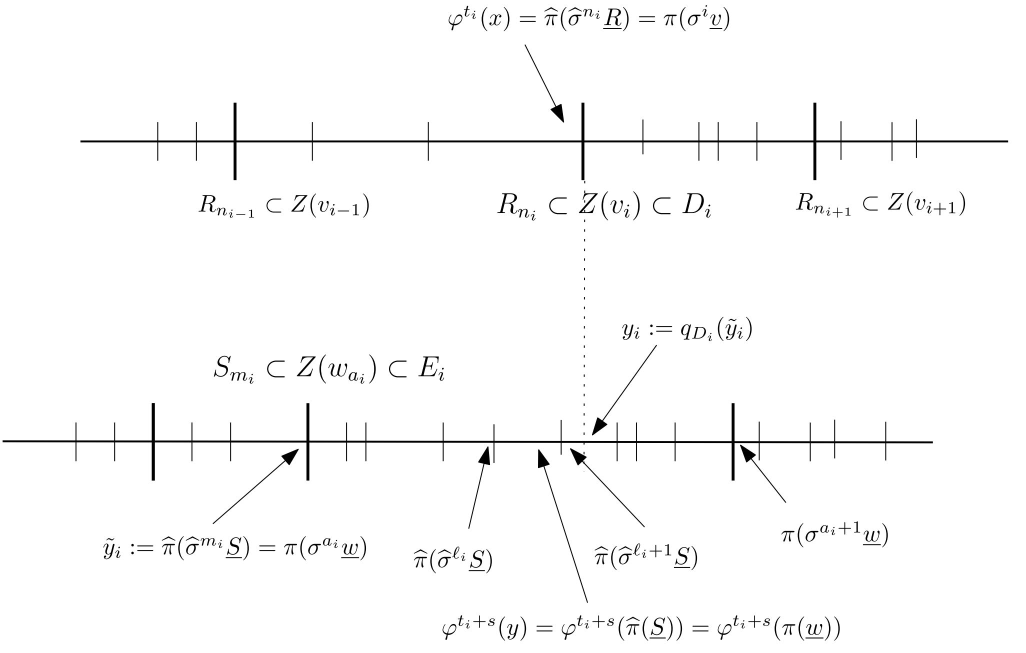

Fig. 4. The objects in the proof of Theorem 9.1, part (6)(b). The time t_i is $r_i(\underline{v}) = \widehat{r}_{n_i}(\underline{R})$ for an arbitrary $i \in \mathbb{Z}$. The line above depicts the coding of $x = \widehat{\pi}(\underline{R}) = \pi(\underline{v})$: large vertical lines correspond to $R_{n_j} \subset Z(v_j)$, shorter ones to other R_n's. The line below is related to the coding of $y = \widehat{\pi}(\underline{S}) = \pi(\underline{w})$ with $S_{m_i} \subset Z(w_{a_i})$, the symbol that our proof relates to $R_{n_i} \subset Z(v_i)$. By construction $R_{n_i} \sim S_{\ell_i}$ and $S_{\ell_i} \sim S_{m_i}$. The point y_i is the trace of the orbit of y on D_i, the connected component of the section containing $Z(v_i)$, figured by a dotted line.

CLAIM 2: Let $R_1, R_2 \in \mathscr{R}$ such that $R_1 \sim R_2$. For $i = 1, 2$, let D_i be the connected component of $\widehat{\Lambda}$ containing R_i, and let $Z_i = Z(\Psi_{x_i}^{p_i^s, p_i^u}) \in \mathscr{Z}$ such that $Z_i \supset R_i$. Then $\frac{p_1^s \wedge p_1^u}{p_2^s \wedge p_2^u} = e^{\pm(O(\sqrt[3]{\varepsilon})+O(\rho))}$ and

$$\mathfrak{q}_{D_1}(\Psi_{x_2}(R[c(p_2^s \wedge p_2^u)])) \subset \Psi_{x_1}(R[8c(p_1^s \wedge p_1^u)]),$$

for all $1 \le c \le 16$.

Proof of Claim 2. Since $R_1 \sim R_2$, there are $W_1, W_2 \in \mathscr{Z}$ such that $W_i \supset R_i$ and $W_1 \cap \varphi^{[-\rho,\rho]}W_2 \neq \emptyset$. Write $W_i = Z(\Psi_{y_i}^{q_i^s, q_i^u})$. We apply Claim 1 three times:

- Since $W_2, Z_2 \supset R_2$, we have $W_2 \cap Z_2 \neq \emptyset$, hence $\frac{p_2^s \wedge p_2^u}{q_2^s \wedge q_2^u} = e^{\pm(O(\sqrt[3]{\varepsilon})+O(\rho))}$ and

$$\Psi_{x_2}(R[c(p_2^s \wedge p_2^u)]) \subset \Psi_{y_2}(R[2c(q_2^s \wedge q_2^u)]).$$

- Since $W_1 \cap \varphi^{[-\rho,\rho]}W_2 \neq \emptyset$, we have $\frac{q_2^s \wedge q_2^u}{q_1^s \wedge q_1^u} = e^{\pm(O(\sqrt[3]{\varepsilon})+O(\rho))}$ and

$$\mathfrak{q}_{D_1}(\Psi_{y_2}(R[2c(q_2^s \wedge q_2^u)])) \subset \Psi_{y_1}(R[4c(q_1^s \wedge q_1^u)]).$$

- Since $W_1, Z_1 \supset R_1$, we have $W_1 \cap Z_1 \neq \emptyset$, hence $\frac{q_1^s \wedge q_1^u}{p_1^s \wedge p_1^u} = e^{\pm(O(\sqrt[3]{\varepsilon})+O(\rho))}$ and

$$\Psi_{y_1}(R[4c(q_1^s \wedge q_1^u)]) \subset \Psi_{x_1}(R[8c(p_1^s \wedge p_1^u)]).$$

Plugging these inclusions together, Claim 2 is proved. □

CLAIM 3: Let $R_1, R_2, R_3 \in \mathscr{R}$ such that $R_1 \sim R_2$ and $R_2 \sim R_3$. For $i = 1, 2, 3$, let D_i be the connected component of $\widehat{\Lambda}$ containing R_i, and let $Z_i = Z(\Psi_{x_i}^{p_i^s, p_i^u}) \in \mathscr{Z}$ such that $Z_i \supset R_i$. Then $\frac{p_3^s \wedge p_3^u}{p_1^s \wedge p_1^u} = e^{\pm(O(\sqrt[3]{\varepsilon})+O(\rho))}$ and

$$(\mathfrak{q}_{D_1} \circ \mathfrak{q}_{D_2})(\Psi_{x_3}(R[c(p_3^s \wedge p_3^u)])) = \mathfrak{q}_{D_1}(\Psi_{x_3}(R[c(p_3^s \wedge p_3^u)])) \subset \Psi_{x_1}(R[64c(p_1^s \wedge p_1^u)])$$

for all $1 \leq c \leq 2$.

Proof of Claim 3. The estimate $\frac{p_3^s \wedge p_3^u}{p_1^s \wedge p_1^u} = e^{\pm(O(\sqrt[3]{\varepsilon})+O(\rho))}$ follows directly from Claim 2. Also by Claim 2, we have the inclusions $\mathfrak{q}_{D_2}(\Psi_{x_3}(R[c(p_3^s \wedge p_3^u)])) \subset \Psi_{x_2}(R[8c(p_2^s \wedge p_2^u)])$ and $\mathfrak{q}_{D_1}(\Psi_{x_2}(R[8c(p_2^s \wedge p_2^u)])) \subset \Psi_{x_1}(R[64c(p_1^s \wedge p_1^u)])$. This implies that $(\mathfrak{q}_{D_1} \circ \mathfrak{q}_{D_2})(\Psi_{x_3}(R[c(p_3^s \wedge p_3^u)])) \subset \Psi_{x_1}(R[64c(p_1^s \wedge p_1^u)])$. In particular, it proves that we can project $\Psi_{x_3}(R[c(p_3^s \wedge p_3^u)])$ to D_1, and so the equality follows. □

Now we apply the above claims to our particular situation. Write $v_i = \Psi_{x_i}^{p_i^s, p_i^u}$ and $w_i = \Psi_{z_i}^{q_i^s, q_i^u}$, so that $Q_i^{s/u} = q_{a_i}^{s/u}$.

CLAIM 4: Let $i \in \mathbb{Z}$. We have

$$\mathfrak{q}_{E_{i+1}}(\Psi_{Y_i}(R[Q_i^s \wedge Q_i^u])) \subset \Psi_{Y_{i+1}}(R[2(Q_{i+1}^s \wedge Q_{i+1}^u)]).$$

Proof of Claim 4. By Lemma 4.2 and Claim 3, we have

$$\frac{q_{a_{i+1}}^s \wedge q_{a_{i+1}}^u}{q_{a_i}^s \wedge q_{a_i}^u} = \frac{q_{a_{i+1}}^s \wedge q_{a_{i+1}}^u}{p_{i+1}^s \wedge p_{i+1}^u} \cdot \frac{p_{i+1}^s \wedge p_{i+1}^u}{p_i^s \wedge p_i^u} \cdot \frac{p_i^s \wedge p_i^u}{q_{a_i}^s \wedge q_{a_i}^u} = e^{\pm(O(\sqrt[3]{\varepsilon})+O(\rho))}.$$

This estimate allows to apply the same proof of Proposition 7.2(1), and so we can obtain the claimed inclusion in the same manner. □

CLAIM 5: Let $i \in \mathbb{Z}$. Restricted to the set $\Psi_{Y_i}(R[Q_i^s \wedge Q_i^u])$, we have the equality $\mathfrak{q}_{D_{i+1}} \circ \mathfrak{q}_{E_{i+1}} = \mathfrak{q}_{D_{i+1}} = g_{X_i}^+ \circ \mathfrak{q}_{D_i}$. A similar statement holds for $i \leq 0$.

Proof of Claim 5. It is enough to prove the equality for $i = 0$, i.e. that $\mathfrak{q}_{D_1} \circ \mathfrak{q}_{E_1} = \mathfrak{q}_{D_1} = g_{X_0}^+ \circ \mathfrak{q}_{D_0}$ when restricted to $\Psi_{Y_0}(R[Q_0^s \wedge Q_0^u])$. By Claim 4, $\mathfrak{q}_{E_1}[\Psi_{Y_0}(R[Q_0^s \wedge Q_0^u])] \subset \Psi_{Y_1}(R[2(Q_1^s \wedge Q_1^u)])$. Applying Claim 3 with $c = 2$ to the triple $(R_{n_1}, S_{\ell_1}, S_{m_1})$, we get that $\mathfrak{q}_{D_1}[\Psi_{Y_1}(R[2(Q_1^s \wedge Q_1^u)])]$ is well-defined, hence $\mathfrak{q}_{D_1} \circ \mathfrak{q}_{E_1} = \mathfrak{q}_{D_1}$ when restricted to $\Psi_{Y_0}(R[Q_0^s \wedge Q_0^u])$. On the other hand, applying Claim 3 with $c = 1$ to the triple

$(R_{n_0}, S_{\ell_0}, S_{m_0})$, we have that $\mathfrak{q}_{D_0}[\Psi_{Y_0}(R[Q_0^s \wedge Q_0^u])] \subset \Psi_{X_0}(R[64(P_0^s \wedge P_0^u)])$. By definition, $g^+_{X_0} = \mathfrak{q}_{D_1}$ when restricted to $R[64(P_0^s \wedge P_0^u)]$. Therefore, $g^+_{X_0} \circ \mathfrak{q}_{D_0} = \mathfrak{q}_{D_1}$ when restricted to $\Psi_{Y_0}(R[Q_0^s \wedge Q_0^u])$. This proves Claim 5. $\square$

We now complete the proof of identities (1) and (2) of page 73, which in turn will complete the proof of part (6) of Theorem 9.1. For that, we use the claims we just proved.

Firstly we check that $y_i := q_{D_i}(\widetilde{y}_i)$ is well-defined. By assumption $R_{n_i} \sim S_{\ell_i}$, and by construction the orbit of y between S_{m_i} and S_{ℓ_i} flows for a time at most $\sup(r) < \rho$, hence $S_{\ell_i} \sim S_{m_i}$. This allows us to apply Claim 3 for $c = 1$ and get that $y_i := \mathfrak{q}_{D_i}(\widetilde{y}_i)$ is well-defined. To calculate the time displacement for $i = 0$, recall that $m_0 = \ell_0 = 0$. Since $R_0 \sim S_0$, inclusion (8.1) implies that $y_0 = \varphi^u(\widehat{y}_0)$ with $|u| \leq 2\rho$.

Finally, Claim 5 implies that

$$g^+_{X_i}(y_i) = g^+_{X_i} \circ \mathfrak{q}_{D_i}(\widetilde{y}_i) = \mathfrak{q}_{D_{i+1}} \circ \mathfrak{q}_{E_{i+1}}(\widetilde{y}_i) = \mathfrak{q}_{D_{i+1}}(\widetilde{y}_{i+1}) = y_{i+1},$$

finishing the proof of Theorem 9.1.

10. Homoclinic classes of measures

In this final section, we prove Theorem 1.1 stated in the introduction, as well as Corollary 1.2.

10.1. The homoclinic relation

For any hyperbolic measure μ and μ–a.e. x, the stable set $W^s(x)$ of the orbit of x is the set of points y such that there exists an increasing homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ satisfying $d(\varphi^t(x), \varphi^{h(t)}(y)) \to 0$ as $t \to +\infty$. This is an injectively immersed submanifold which is tangent to $E^s_x \oplus X(x)$ and invariant under the flow. We define similarly the unstable manifold $W^u(x)$ by considering past orbits.

Homoclinic relation of measures: We say that two ergodic hyperbolic measures μ, ν are *homoclinically related* if for μ–a.e. x and ν–a.e. y there exist transverse intersections $W^s(x) \pitchfork W^u(y) \neq \emptyset$ and $W^u(x) \pitchfork W^s(y) \neq \emptyset$, i.e., points $z_1 \in W^s(x) \cap W^u(y)$ and $z_2 \in W^u(x) \cap W^s(y)$ satisfying $T_{z_1}M = T_{z_1}W^s(x) + T_{z_1}W^u(y)$ and $T_{z_2}M = T_{z_2}W^u(x) + T_{z_2}W^s(y)$.

Note that the invariance of the stable and unstable manifolds makes this notion slightly *simpler* than it is for diffeomorphisms. Since any hyperbolic periodic orbit supports a (unique) ergodic measure, the above homoclinic relation is also defined between hyperbolic periodic orbits, in which case it coincides with the classical notion, see, e.g., [30].

Proposition 10.1. *The homoclinic relation is an equivalence relation among ergodic hyperbolic measures.*

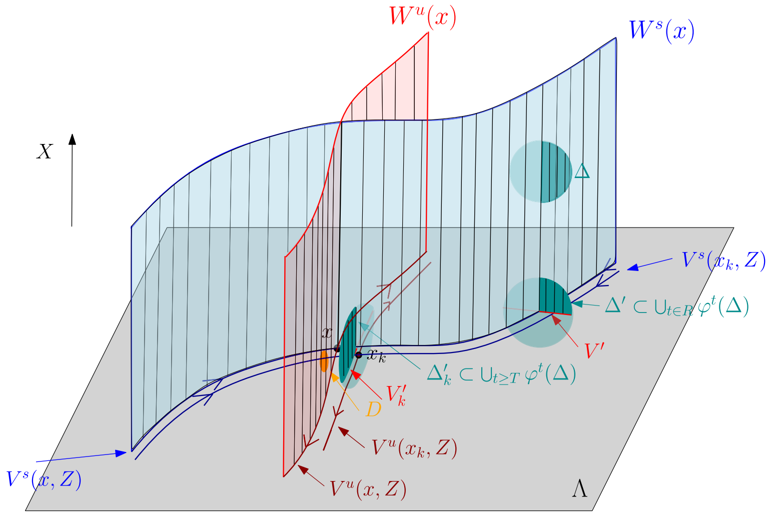

Fig. 5. The objects in the proof of the inclination lemma.

Proof. The only property that is not obvious is the transitivity of the relation. Its proof uses the following standard lemma.

Inclination lemma. *For any hyperbolic measure μ, there is a set $Y \subset M$ of full μ–measure satisfying the following: if $x \in Y$, $D \subset W^u(x)$ is a two-dimensional disc and Δ is a two-dimensional disc tangent to X having a transverse intersection point with $W^s(x)$, then there are discs $\Delta_k \subset \varphi_{(k,+\infty)}(\Delta)$ which converge to D in the C^1 topology.*

Sketch of the proof. Taking $\chi > 0$ small, the measure μ is χ–hyperbolic and the constructions done in the other sections apply. Consequently one may replace x by an iterate in the section Λ and assume that it is the projection under π of a regular sequence $\underline{v} \in \Sigma^\#$. We denote $Z = Z(v_0)$ and let $n_k \to +\infty$ such that $\sigma^{n_k}(\underline{v}) \to \underline{v}$.

We let $x_k := \pi(\sigma^{n_k}(\underline{v}))$. We consider the curves $V^s(x_k, Z) \to V^s(x, Z)$, $V^u(x_k, Z) \to V^u(x, Z)$ as in Section 4.2 and especially Theorem 4.5(5). The intersections $W^s(x) \cap \Lambda$ and $W^u(x) \cap \Lambda$ contain the stable and unstable curves $V^s(x, Z)$ and $V^u(x, Z)$. See Fig. 5 for the various objects.

Now, the orbit of Δ contains a disk Δ' transversally intersecting $V^s(x, Z) \subset \Lambda$. Thus Δ' transversally intersects Λ along some curve V'. This curve V' intersects the stable curve $V^s(x, Z)$ transversally inside the section Λ.

Hence, if k is large enough, then the curve $V^s(x_k, Z)$ also intersects Δ' transversally inside Λ along the curve V'. Now, the graph transform argument in Section 4.2 shows that the images of V' (by suitable holonomies of the flow mapping x_k to $x_{k'}$ for $k' \gg k$) contain curves V'_k that C^1–approximate $V^u(x_{k'}, Z)$.

It follows that the orbit of Δ contains a curve which is arbitrarily C^1–close to $V^u(x, Z)$. By invariance, this orbit contains discs which are arbitrarily C^1–close to the arbitrary subset $D \subset W^u(x)$. $\square$

In order to prove the proposition, let us consider three measures μ_1, μ_2, μ_3 such that μ_1, μ_2 are homoclinically related and μ_2, μ_3 are homoclinically related. For each measure μ_i, let x_i be a point in the full measure set implied by the homoclinic relation. In particular, there exist a disc $\Delta \subset W^u(x_1)$ which intersects transversally $W^s(x_2)$ and a disc $D \subset W^u(y_2)$ which intersects transversally $W^s(x_3)$. By the inclination lemma, the orbit of Δ contains discs that converge to D for the C^1–topology. This proves that $W^u(x_1)$ has a transverse intersection point with $W^s(x_3)$. The same argument shows that $W^u(x_3)$ has a transverse intersection with $W^s(x_1)$. Hence μ_1 and μ_3 are homoclinically related. □

Homoclinic classes of measures: The equivalence classes for the homoclinic relation on the set of hyperbolic measures are called *homoclinic classes of measures.*

10.2. Proof of Theorem 1.1

The proof follows closely the argument in [17, Section 3]. We consider the setting of the Main Theorem and especially a topological Markov flow $(\widehat{\Sigma}_{\widehat{r}}, \widehat{\sigma}_{\widehat{r}})$ satisfying the properties stated in Theorem 9.1.

We begin by some preliminary lemmas. The first two correspond to properties (C6), (C7) in [17].

Lemma 10.2. *For any two ergodic measures supported on a common irreducible component of $\widehat{\Sigma}_{\widehat{r}}$, their projections under $\widehat{\pi}_{\widehat{r}}$ are hyperbolic ergodic measures that are homoclinically related.*

Proof. Let us consider two ergodic measures $\overline{\mu}$ and $\overline{\nu}$ on a same irreducible component of $\widehat{\Sigma}_{\widehat{r}}$ and their projections $\mu = \overline{\mu} \circ \widehat{\pi}_{\widehat{r}}^{-1}$ and $\nu = \overline{\nu} \circ \widehat{\pi}_{\widehat{r}}^{-1}$. These two measures are obviously ergodic. They are hyperbolic by Theorem 9.1(4).

Let x, y be points in full measure sets for μ and ν respectively: they are the projections of points $\overline{x}, \overline{y}$ which are in the irreducible component supporting the measures $\overline{\mu}, \overline{\nu}$. Note that one can replace $x, y, \overline{x}, \overline{y}$ by iterates and assume that $\overline{x} = (\underline{R}, 0)$, $\overline{y} = (\underline{S}, 0)$. Since $\overline{x}, \overline{y}$ belong to the same irreducible component, there exists a finite word $w = w_0 w_1 \cdots w_\ell$ such that $w_0 = R_0$ and $w_\ell = S_0$. One can thus consider the point $\overline{z} = (\underline{T}, 0)$ such that $T_{-n} = R_{-n}$ and $T_{\ell+n} = S_n$ for any $n \geq 0$ and $T_n = w_n$ for $1 \leq n \leq \ell$. One deduces from the Hölder-continuity of $\widehat{\pi}_{\widehat{r}}$ that the projection $z = \widehat{\pi}_{\widehat{r}}(\overline{z}) = \widehat{\pi}(\underline{T})$ belongs to the intersection between $W^s(x)$ and $W^u(y)$. In particular $W^s(z) = W^s(x)$, hence using Theorem 9.1(4)(5) we have

$$\limsup_{t \to +\infty} \tfrac{1}{t} \log \|\Phi^t|_{T_z(W^s(x) \cap \Lambda)}\| \leq -\lambda < 0.$$

Therefore $N_z^s = T_z(W^s(x) \cap \Lambda)$, and similarly $N_z^u = T_z(W^u(x) \cap \Lambda)$. Since $N_z^s \oplus N_z^u = N_z$, one deduces that the intersection between $W^s(x)$ and $W^u(y)$ at z is transverse. By the same argument, one finds a transverse intersection between $W^u(x)$ and $W^s(y)$. Since the points x and y can be taken in full measure sets for μ and ν respectively, this proves that μ and ν are homoclinically related. $\square$

Lemma 10.3. *For any $\chi' > 0$, the set of ergodic measures on $\widehat{\Sigma}_{\widehat{r}}$ whose projection is χ'–hyperbolic is open for the weak–* topology.*

Proof. The two Lyapunov exponents of the projection of any measure $\overline{\mu}$ on $\widehat{\Sigma}_{\widehat{r}}$ are obtained by integration of the bounded continuous functions $\overline{x} \mapsto \log \|\Phi^t|_{N_x^s}\|$ and $\overline{x} \mapsto \log \|\Phi^t|_{N_x^u}\|$ $(x = \widehat{\pi}_{\widehat{r}}(\overline{x}))$. Hence they vary continuously with the measure $\overline{\mu}$ in the weak–* topology. $\square$

The next lemma finds an irreducible component that lifts periodic orbits.

Lemma 10.4. *There exists an irreducible component $\widehat{\Sigma}'_{\widehat{r}} \subset \widehat{\Sigma}_{\widehat{r}}$ to which one can lift all χ–hyperbolic periodic orbits that are homoclinically related to μ.*

Proof. Periodic orbits that are homoclinically related to μ are homoclinically related together. Hence, given any finite set of such periodic orbits, there exists a transitive χ–hyperbolic set K which contains all of them. By Theorem 9.1(8), there is a transitive invariant compact set $X \subset \widehat{\Sigma}_{\widehat{r}}$ such that $\widehat{\pi}_{\widehat{r}}(X) = K$. In particular, X is contained in an irreducible component of $\widehat{\Sigma}_{\widehat{r}}$.

Note that $X \subset \widehat{\Sigma}^{\#}_{\widehat{r}}$, the regular set of $\widehat{\Sigma}_{\widehat{r}}$, since X sees only finitely many vertices. In particular, $\widehat{\pi}_{\widehat{r}} : X \to K$ is not only onto but finite-to-one and all periodic orbits of K lift to periodic orbits of X (though with perhaps larger periods).

Let us enumerate all the χ–hyperbolic periodic orbits $\mathcal{O}_i \sim \mu$, $i = 1, 2, \dots$. For each n, the set of irreducible components which contains periodic lifts of all the periodic orbits $\mathcal{O}_i$ with $1 \le i \le n$ is non-empty (by the previous paragraph), finite (by the finiteness-to-one property of the coding) and is non-increasing with n. Hence their intersection is nonempty, and any irreducible component which belongs to it satisfies the claim. $\square$

Let ν be a χ–hyperbolic ergodic measure that is homoclinically related to μ. By Theorem 9.1(2), there exists an ergodic lift $\overline{\nu}$ of ν to $\widehat{\Sigma}_{\widehat{r}}$. Consider a point $q \in \widehat{\Sigma}_{\widehat{r}}$ that is recurrent (such that there exists a sequence of forward iterates $\widehat{\sigma}^{k_i}(q)$ which converges to q) and generic for $\overline{\nu}$, and let $x = \widehat{\pi}_{\widehat{r}}(q)$.

The recurrence of q gives rise to a sequence of periodic points q^i in $\widehat{\Sigma}_{\widehat{r}}$ which converge to q (hence are in a same irreducible component) and whose orbits weak–* converge to $\overline{\nu}$. By Lemma 10.3 the projections of these periodic orbits are χ–hyperbolic and by Lemma 10.2 they are homoclinically related to μ. Therefore there are periodic orbits p^i in the irreducible component $\widehat{\Sigma}'_{\widehat{r}}$ which have the same projections as the periodic orbits q^i.

Let us write $q^i = (\underline{R}^i, t^i)$ and $p^i = (\underline{S}^i, s^i)$. Since (q^i) is converging and $\widehat{\Sigma}$ is locally compact, the sequence $(\underline{R}^i)$ is relatively compact. The Bowen property of Theorem 9.1(6) implies that $\mathrm{v}(\widehat{\sigma}_{\widehat{r}}^t(q^i)) \sim \mathrm{v}(\widehat{\sigma}_{\widehat{r}}^t(p^i))$ for all $t \in \mathbb{R}$ so, by the local finiteness of the affiliation, the sequence $(\underline{S}^i)$ is relatively compact. This implies that (p^i) is relatively compact and (up to taking a subsequence) converges to some $p \in \widehat{\Sigma}'_{\widehat{r}}$. By continuity of the projection, $\widehat{\pi}_{\widehat{r}}(p) = \widehat{\pi}_{\widehat{r}}(q) = x$.

We claim that $p \in \widehat{\Sigma'}^{\#}_{\widehat{r}}$. This follows from the fact that q is recurrent and that the Bowen relation is locally finite. More precisely, there are some vertex $A \in \widehat{V}$ and integers $m_k, n_k \to \infty$ such that $q_{m_k} = q_{-n_k} = A$. In particular, for each $k \geq 1$ we have $q^i_{m_k} = q^i_{-n_k} = A$ for all large i. Hence $p^i_{m_k}, p^i_{-n_k}$ are related to A, and so they belong to the set $\{B \in \widehat{V} : B \sim A\}$. Since this latter set is finite, some symbol must repeat as required and this passes to the limit p, proving the claim.

We have proved that ν–almost every point has a lift in $\widehat{\Sigma'}^{\#}_{\widehat{r}}$. The finiteness-to-one property of Theorem 9.1(3) and the same averaging argument used in the proof of Theorem 9.1(2) imply that ν has a lift in $\widehat{\Sigma}'_{\widehat{r}}$. Considering the ergodic decomposition, we can choose an ergodic lift, as claimed. Theorem 1.1 is now proved. □

10.3. Proof of Corollary 1.2

Let $\mathcal{H}$ be some homoclinic class of hyperbolic ergodic measures. Let us deduce from Theorem 1.1 that there is at most one $\nu \in \mathcal{H}$ such that $h(\varphi, \nu) = \sup\{h(\varphi, \mu) : \mu \in \mathcal{H}\}$. Let $\nu, \nu' \in \mathcal{H}$ be two measures with this property. They are both hyperbolic, hence χ–hyperbolic for some $\chi > 0$. For one such fixed parameter χ, let $\pi_r : \Sigma_r \to M$ be the coding given by the Main Theorem.

By Theorem 1.1, there is an irreducible component Σ'_r of Σ_r to which both ν and ν' lift. Since the factor map π_r preserves the entropy and since the projection of any ergodic measure on Σ'_r is homoclinically related to ν and ν' by Lemma 10.2, the two lifts are measures of maximal entropy for Σ'_r. But the measure of maximal entropy of an irreducible component of a topological Markov flow with a Hölder continuous roof function r is unique (see e.g. [28, Proof of Theorem 6.2]). Hence $\nu = \nu'$, which proves Corollary 1.2.

Appendix A. Standard proofs

Remind we are assuming that $\|\nabla X\| \leq 1$, and that this implies two facts:

- Every Lyapunov exponent of φ has absolute value ≤ 1, hence we consider $\chi \in (0, 1)$.
- $\|\Phi^t\| \leq e^{2\rho + |t|}$, $\forall t \in \mathbb{R}$, see Section 2.4.

Proof of Lemma 3.2. We begin with some preliminary calculations. Fix $t \in \mathbb{R}$. We prove that e_1 is an eigenvector of $C(\varphi^t(x))^{-1} \circ \Phi^t \circ C(x)$, and calculate its eigenvalue. By the proof of Proposition 3.1, $\Phi^t n^s_x = \pm\|\Phi^t n^s_x\| n^s_{\varphi^t(x)}$, therefore $[\Phi^t \circ C(x)](e_1) = \pm\frac{\|\Phi^t n^s_x\|}{s(x)} n^s_{\varphi^t(x)}$. This implies that $[C(\varphi^t(x))^{-1} \circ \Phi^t \circ C(x)](e_1) = \pm\|\Phi^t n^s_x\| \frac{s(\varphi^t(x))}{s(x)} e_1$, hence e_1 is an eigenvector with eigenvalue $A_t(x) = \pm\|\Phi^t n_x^s\|\frac{s(\varphi^t(x))}{s(x)}$. Similarly, e_2 is an eigenvector with eigenvalue $B_t(x) = \pm\|\Phi^t n_x^u\|\frac{u(\varphi^t(x))}{u(x)}$. Note that

$$s(x)^2 = 4e^{4\rho}\int_0^t e^{2\chi t'}\|\Phi^{t'}n_x^s\|^2dt' + 4e^{4\rho}\int_t^\infty e^{2\chi t'}\|\Phi^{t'}n_x^s\|^2dt'$$

$$= 4e^{4\rho}\int_0^t e^{2\chi t'}\|\Phi^{t'}n_x^s\|^2dt' + e^{2\chi t}\|\Phi^t n_x^s\|^2 s(\varphi^t(x))^2$$

and so

$$e^{2\chi t}\|\Phi^t n_x^s\|^2\frac{s(\varphi^t(x))^2}{s(x)^2} = 1 - \frac{4e^{4\rho}}{s(x)^2}\int_0^t e^{2\chi t'}\|\Phi^{t'}n_x^s\|^2dt'.$$

When $0 < t \le 2\rho$, we have $\frac{4e^{4\rho}}{s(x)^2}\int_0^t e^{2\chi t'}\|\Phi^{t'}n_x^s\|^2dt' \le 4\rho e^{16\rho} < 5\rho$ for $\rho > 0$ small enough, therefore

$$e^{-4\rho} < e^{\chi t}\|\Phi^t n_x^s\|\frac{s(\varphi^t(x))}{s(x)} < 1. \tag{A.1}$$

Similarly,

$$u(\varphi^t(x))^2 = 4e^{4\rho}\int_0^t e^{2\chi t'}\|\Phi^{-t'}n^u_{\varphi^t(x)}\|^2dt' + 4e^{4\rho}\int_t^\infty e^{2\chi t'}\|\Phi^{-t'}n^u_{\varphi^t(x)}\|^2dt'$$

$$= 4e^{4\rho}\int_0^t e^{2\chi t'}\|\Phi^{-t'}n^u_{\varphi^t(x)}\|^2dt' + e^{2\chi t}\|\Phi^t n_x^u\|^{-2}u(x)^2$$

since $1 = \|\Phi^{-t}\Phi^t n_x^u\| = \|\Phi^t n_x^u\|\cdot\|\Phi^{-t}n^u_{\varphi^t(x)}\|$, and so

$$e^{-4\rho} < e^{\chi t}\|\Phi^t n_x^u\|^{-1}\frac{u(x)}{u(\varphi^t(x))} < 1. \tag{A.2}$$

We will use (A.1) and (A.2) to prove (2)–(3).

(1) In the basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ and the basis $\{n_x^s, (n_x^s)^\perp\}$ of N_x, $C(x)$ takes the form $\begin{bmatrix}\frac{1}{s(x)} & \frac{\cos\alpha(x)}{u(x)}\\ 0 & \frac{\sin\alpha(x)}{u(x)}\end{bmatrix}$, hence $\|C(x)\|^2_{\rm Frob} = \frac{1}{s(x)^2} + \frac{1}{u(x)^2} \le 1$. Now observe that the inverse of $C(x)$ is $\begin{bmatrix}s(x) & -\frac{s(x)\cos\alpha(x)}{\sin\alpha(x)}\\ 0 & \frac{u(x)}{\sin\alpha(x)}\end{bmatrix}$, hence $\|C(x)^{-1}\|_{\rm Frob} = \frac{\sqrt{s(x)^2+u(x)^2}}{|\sin\alpha(x)|}$.

(2) The first part was already proved, so we concentrate on the second part. Fix $0 < t \le 2\rho$. By (A.1), $e^{-4\rho} < e^{\chi t}|A_t(x)| < 1$ and so $e^{-8\rho} < |A_t(x)| < e^{-\chi t}$. Similarly, (A.2) implies that $e^{-4\rho} < e^{\chi t}|B_t(x)|^{-1} < 1$, and so $e^{\chi t} < |B_t(x)| < e^{8\rho}$.

(3) For $|t| \leq 2\rho$, we have $e^{\chi t}\|\Phi^t n_s^x\| = e^{\pm 6\rho}$, therefore by (A.1) it follows that $e^{-10\rho} < \frac{s(\varphi^t(x))}{s(x)} < e^{6\rho}$, so that $\frac{s(\varphi^t(x))}{s(x)} = e^{\pm 10\rho}$. Similarly, $\frac{u(\varphi^t(x))}{u(x)} = e^{\pm 10\rho}$. To estimate $\frac{|\sin\alpha(\varphi^t(x))|}{|\sin\alpha(x)|}$, we use the general inequality for an invertible linear transformation L:

$$\frac{1}{\|L\|\|L^{-1}\|} \leq \frac{|\sin\angle(Lv, Lw)|}{|\sin\angle(v,w)|} \leq \|L\|\|L^{-1}\|. \tag{A.3}$$

Apply this to $L = \Phi^t$, $v = n_x^s$, $w = n_x^u$ to get that $\frac{|\sin\alpha(\varphi^t(x))|}{|\sin\alpha(x)|} = e^{\pm 8\rho}$. Finally, the above estimates and part (1) imply that $\frac{\|C(\varphi^t(x))\|_{\text{Frob}}}{\|C(x)\|_{\text{Frob}}} = e^{\pm 18\rho}$. $\square$

For the proof of the next theorem we will need some estimates on $Q(x)$. By Lemma 3.2(3) proved above, $\frac{Q(\varphi^t(x))}{Q(x)} = e^{\pm\frac{200\rho}{\beta}}$ for all $x \in \text{NUH}$ and $|t| \leq 2\rho$. Therefore, if $x \in \Lambda \cap \text{NUH}$ then $\frac{Q(f(x))}{Q(x)} = e^{\pm\frac{200\rho}{\beta}}$. Hence the following bounds hold for $Q(x)$:

$$Q(x) \leq \varepsilon^{3/\beta} \text{ and } \|C(x)^{-1}\|Q(x)^{\beta/12} \leq \varepsilon^{1/4} \text{ for all } x \in \text{NUH},$$
$$Q(x)^{\beta/2} \leq e^{100\rho}Q(f(x))^{\beta/2} \text{ for all } x \in \Lambda \cap \text{NUH}.$$

Proof of Theorem 3.8. Recall that $B_x = B(x, 2\mathfrak{r})$. If $\varepsilon > 0$ is small enough then Lemma 3.7(1) implies

$$\Psi_x(R[10Q(x)]) \subset B(x, 40Q(x)) \subset B_x,$$

and in this ball (Exp1)–(Exp4) are valid. We first show that $f_x^+ : R[10Q(x)] \to \mathbb{R}^2$ is well-defined. Since $C(x)$ is a contraction, we have $C(x)R[10Q(x)] \subset B_x[20Q(x)]$. Since $C(f(x))^{-1}$ is globally defined, it is enough to show that

$$(g_x^+ \circ \exp_x)(B_x[20Q(x)]) \subset \exp_{f(x)}(B_{f(x)}[2\mathfrak{r}]).$$

For small $\varepsilon > 0$ we have:

- ◦ $20Q(x) < 2\mathfrak{r}$, hence $\exp_x$ is well-defined on $B_x[20Q(x)]$. By (Exp2), $\exp_x$ maps $B_x[20Q(x)]$ diffeomorphically into $B(x, 40Q(x))$.
- ◦ $40Q(x) < 2\mathfrak{r} \Rightarrow B(x, 40Q(x)) \subset B_x$, hence Lemma 2.4 implies that g_x^+ maps the ball $B(x, 40Q(x))$ diffeomorphically into $B(f(x), 80Q(x))$.
- ◦ $80Q(x) < 2\mathfrak{r} \Rightarrow B(f(x), 80Q(x)) \subset B_{f(x)}$. By condition (Exp2), $\exp_{f(x)}^{-1}$ maps $B(f(x), 80Q(x))$ diffeomorphically onto its image.

The conclusion is that $f_x^+ : R[10Q(x)] \to \mathbb{R}^2$ is a diffeomorphism onto its image.

Now we check (1)–(2). Using the equalities $d(\Psi_x)_0 = C(x)$, $d(\Psi_{f(x)})_0 = C(f(x))$ and Lemma 2.4, we get that $d(f_x^+)_0 = C(f(x))^{-1} \circ \Phi^{r_\Lambda(x)} \circ C(x)$. By Lemma 3.2(2),

$d(f_x^+)_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ with $e^{-4\rho} < |A| < e^{-\chi r_\Lambda(x)}$ and $e^{\chi r_\Lambda(x)} < |B| < e^{4\rho}$. This proves part (1). Items (a)–(b) of part (2) are automatic, hence we focus on (c).

CLAIM: $\|d(f_x^+)_{v_1} - d(f_x^+)_{v_2}\| \leq \frac{\varepsilon}{3}\|v_1 - v_2\|^{\beta/2}$ for all $v_1, v_2 \in R[10Q(x)]$.

Before proving the claim, we show how to conclude (c). If $\varepsilon > 0$ is small enough then $R[10Q(x)] \subset B_x[1]$. Applying the claim with $v_2 = 0$, we get $\|dH_v\| \leq \frac{\varepsilon}{3}\|v\|^{\beta/2} < \frac{\varepsilon}{3}$. By the mean value inequality, $\|H(v)\| \leq \frac{\varepsilon}{3}\|v\| < \frac{\varepsilon}{3}$, hence $\|H\|_{C^{1+\frac{\beta}{2}}} < \varepsilon$.

Proof of the claim. Let us choose $L > \text{Höl}_\beta(dg_x^+)$. For $i = 1, 2$, write $w_i = C(x)v_i$ and let

$$A_i = d(\widetilde{\exp_{f(x)}^{-1})}_{(g_x^+ \circ \exp_x)(w_i)}, \ B_i = d(\widetilde{g_x^+)}_{\exp_x(w_i)}, \ C_i = d(\widetilde{\exp_x)}_{w_i}.$$

We first estimate $\|A_1B_1C_1 - A_2B_2C_2\|$.

- By (Exp2), $\|A_i\| \leq 2$. By (Exp2), (Exp3) and Lemma 2.4:

$$\|A_1 - A_2\| \leq \mathfrak{K}d((g_x^+ \circ \exp_x)(w_1), (g_x^+ \circ \exp_x)(w_2)) \leq 4\mathfrak{K}\|w_1 - w_2\|.$$

- By Lemma 2.4, $\|B_i\| \leq 2$. By (Exp2) and Lemma 2.4:

$$\|B_1 - B_2\| \leq Ld(\exp_x(w_1), \exp_x(w_2))^\beta \leq 2L\|w_1 - w_2\|^\beta.$$

- By (Exp2), $\|C_i\| \leq 2$. By (Exp3), $\|C_1 - C_2\| \leq \mathfrak{K}\|w_1 - w_2\|$.

Applying some triangle inequalities, we get that

$$\|A_1B_1C_1 - A_2B_2C_2\| \leq 24\mathfrak{K}L\|w_1 - w_2\|^\beta \leq 24\mathfrak{K}L\|v_1 - v_2\|^\beta.$$

Now we estimate $\|d(f_x^+)_{v_1} - d(f_x^+)_{v_2}\|$:

$$\begin{aligned}\|d(f_x^+)_{v_1} - d(f_x^+)_{v_2}\| &\leq \|C(f(x))^{-1}\|\|A_1B_1C_1 - A_2B_2C_2\|\|C(x)\| \\ &\leq 24\mathfrak{K}L\|C(f(x))^{-1}\|\|v_1 - v_2\|^\beta.\end{aligned}$$

Using estimate (3.2) and that $\|v_1 - v_2\| < 40Q(x)$, we conclude that for $\varepsilon > 0$ small:

$$\begin{aligned}&24\mathfrak{K}L\|C(f(x))^{-1}\|\|v_1 - v_2\|^{\beta/2} \leq 200\mathfrak{K}L\|C(f(x))^{-1}\|Q(x)^{\beta/2} \\ &\leq 200\mathfrak{K}Le^{125\rho}\|C(f(x))^{-1}\|Q(f(x))^{\beta/2} \leq 200\mathfrak{K}Le^{125\rho}\varepsilon^{3/2}\|C(f(x))^{-1}\|^{-5} \\ &\leq 200\mathfrak{K}Le^{125\rho}\varepsilon^{3/2} < \varepsilon.\end{aligned}$$

Hence the claim is proved. $\square$

This completes the proof of the theorem. □

Remark A.1. The sole property of g_x^+ used in the above proof is Lemma 2.4. Since any holonomy map $\mathfrak{q}_{D_j}$ also satisfies this lemma, we conclude that $\mathfrak{q}_{D_j}$ satisfies a statement analogous to Theorem 3.8. We will use this fact in the proof of Proposition 7.2.

Proof of Proposition 3.10. Write $C_i = \widetilde{C(x_i)} : \mathbb{R}^2 \to T_{x_1}\Lambda$. By assumption, $d(x_1, x_2) + \|C_1 - C_2\| < (\eta_1\eta_2)^4$. Note that $\Psi_{x_i} = \exp_{x_i} \circ P_{x_1,x_i} \circ C_i$.

(1) We prove the estimate for s (the calculation for u is similar). Since $\varepsilon > 0$ is small, it is enough to prove that $\left|\frac{s(x_1)}{s(x_2)} - 1\right| < \varepsilon^{3/\beta}(\eta_1\eta_2)^3$. We have $s(x_i)^{-1} = \|C(x_i)e_1\| = \|C_i e_1\|$, hence $|s(x_1)^{-1} - s(x_2)^{-1}| = |\|C_1e_1\| - \|C_2e_1\|| \le \|C_1 - C_2\| < (\eta_1\eta_2)^4$. Also $s(x_1) = \|C(x_1)e_1\|^{-1} \le \|C(x_1)^{-1}\| \le \frac{\varepsilon^{3/\beta}}{Q(x_1)} < \frac{\varepsilon^{3/\beta}}{\eta_1\eta_2}$, therefore

$$\left|\frac{s(x_1)}{s(x_2)} - 1\right| = s(x_1)|s(x_1)^{-1} - s(x_2)^{-1}| < \varepsilon^{3/\beta}(\eta_1\eta_2)^3.$$

(2) Apply (A.3) to $L = C_1C_2^{-1}$, $v = C_2e_1$, $w = C_2e_2$ to get that

$$\frac{1}{\|C_1C_2^{-1}\|\|C_2C_1^{-1}\|} \le \frac{\sin\alpha(x_1)}{\sin\alpha(x_2)} \le \|C_1C_2^{-1}\|\|C_2C_1^{-1}\|.$$

We have $\|C_1C_2^{-1} - \mathrm{Id}\| \le \|C_1 - C_2\|\|C_2^{-1}\| < \varepsilon^{3/\beta}(\eta_1\eta_2)^3$, and by symmetry $\|C_2C_1^{-1} - \mathrm{Id}\| < \varepsilon^{3/\beta}(\eta_1\eta_2)^3$, therefore $\|C_1C_2^{-1}\|\|C_2C_1^{-1}\| < [1 + \varepsilon^{3/\beta}(\eta_1\eta_2)^3]^2 < e^{2\varepsilon^{3/\beta}(\eta_1\eta_2)^3} < e^{(\eta_1\eta_2)^3}$. The left hand side estimate is proved similarly.

(3) We prove that $\Psi_{x_1}(R[e^{-2\varepsilon}\eta_1]) \subset \Psi_{x_2}(R[\eta_2])$. If $v \in R[e^{-2\varepsilon}\eta_1]$ then $\|C(x_1)v\| \le \sqrt{2}e^{-2\varepsilon}\eta_1 < 2\mathfrak{r}$, hence by (Exp1):

$$d_{\mathrm{Sas}}(C(x_1)v, C(x_2)v) \le 2(d(x_1, x_2) + \|C_1v - C_2v\|) \le 2(\eta_1\eta_2)^4.$$

By (Exp2), $d(\Psi_{x_1}(v), \Psi_{x_2}(v)) \le 4(\eta_1\eta_2)^4 \Rightarrow \Psi_{x_1}(v) \in B(\Psi_{x_2}(v), 4(\eta_1\eta_2)^4)$. By Lemma 3.7(1), $B(\Psi_{x_2}(v), 4(\eta_1\eta_2)^4) \subset \Psi_{x_2}(B)$ where $B \subset \mathbb{R}^2$ is the ball with center v and radius $8\|C_2^{-1}\|(\eta_1\eta_2)^4$, hence it is enough to show that $B \subset R[\eta_2]$. If $w \in B$ then $\|w\|_\infty \le \|v\|_\infty + 8\|C_2^{-1}\|(\eta_1\eta_2)^4 \le (e^{-\varepsilon} + 8\varepsilon^{3/\beta})\eta_2 < \eta_2$ for $\varepsilon > 0$ small enough.

(4) The proof that $\Psi_{x_2}^{-1} \circ \Psi_{x_1}$ is well-defined in $R[\mathfrak{r}]$ is similar to the proof of (3). The only difference is in the last calculation: if $\varepsilon > 0$ is small enough then for $w \in B$ it holds

$$\|w\| \le \|v\| + 8\|C_2^{-1}\|(\eta_1\eta_2)^4 \le \sqrt{2}\mathfrak{r} + 8(\eta_1\eta_2)^3 \le [\sqrt{2} + 8\varepsilon^{3/\beta}]\mathfrak{r} < 2\mathfrak{r},$$

therefore B is contained in the ball of $\mathbb{R}^2$ with center 0 and radius $2\mathfrak{r}$, and in this latter ball Ψ_{x_2} is a diffeomorphism onto its image. Now:

$$\begin{aligned}\Psi_{x_2}^{-1}\circ\Psi_{x_1}-\mathrm{Id} &= C(x_2)^{-1}\circ\exp_{x_2}^{-1}\circ\exp_{x_1}\circ C(x_1)-\mathrm{Id}\\ &=[C_2^{-1}\circ P_{x_2,x_1}]\circ[\exp_{x_2}^{-1}\circ\exp_{x_1}-P_{x_1,x_2}]\circ[P_{x_1,x_1}\circ C_1]+C_2^{-1}(C_1-C_2)\\ &=[C_2^{-1}\circ P_{x_2,x_1}]\circ[\exp_{x_2}^{-1}-P_{x_1,x_2}\circ\exp_{x_1}^{-1}]\circ\Psi_{x_1}+C_2^{-1}(C_1-C_2).\end{aligned}$$

We calculate the C^2 norm of $[\exp_{x_2}^{-1}-P_{x_1,x_2}\circ\exp_{x_1}^{-1}]\circ\Psi_{x_1}$ in the domain $R[\mathfrak{r}]$. By Lemma 3.7, $\|d\Psi_{x_1}\|_{C^0}\le 2$ and $\mathrm{Lip}(d\Psi_{x_1})\le\mathfrak{K}$. Call $\Theta:=\exp_{x_2}^{-1}-P_{x_1,x_2}\circ\exp_{x_1}^{-1}$. For $\varepsilon>0$ small enough, inside B_{x_1} we have:

- By (Exp2), $\|\Theta(y)\|\le d_{\mathrm{Sas}}(\exp_{x_2}^{-1}(y),\exp_{x_1}^{-1}(y))\le 2d(x_1,x_2)\le 2\varepsilon^{6/\beta}(\eta_1\eta_2)^3$ thus $\|\Theta\circ\Psi_{x_1}\|_{C^0}<\varepsilon^{2/\beta}(\eta_1\eta_2)^3$.
- By (Exp3), $\|d\Theta_y\|=\|\tau(x_2,y)-\tau(x_1,y)\|\le\mathfrak{K}d(x_1,x_2)<\varepsilon^{3/\beta}(\eta_1\eta_2)^3$. Hence $\|d\Theta\|_{C^0}<\varepsilon^{3/\beta}(\eta_1\eta_2)^3$ and $\|d(\Theta\circ\Psi_{x_1})\|_{C^0}\le 2\varepsilon^{3/\beta}(\eta_1\eta_2)^3<\varepsilon^{2/\beta}(\eta_1\eta_2)^3$.
- By (Exp4),

$$\begin{aligned}\|\widetilde{d\Theta_y}-\widetilde{d\Theta_z}\|&=\|[\tau(x_2,y)-\tau(x_1,y)]-[\tau(x_2,z)-\tau(x_1,z)]\|\\ &\le\mathfrak{K}d(x_1,x_2)d(y,z)\end{aligned}$$

  hence $\mathrm{Lip}(d\Theta)\le\mathfrak{K}d(x_1,x_2)$.
- Using that $\mathrm{Lip}(d(\Theta_1\circ\Theta_2))\le\|d\Theta_1\|_{C^0}\mathrm{Lip}(d\Theta_2)+\mathrm{Lip}(d\Theta_1)\|d\Theta_2\|_{C^0}^2$, we get that

$$\begin{aligned}&\mathrm{Lip}[d(\Theta\circ\Psi_{x_1})]\le\|d\Theta\|_{C^0}\mathrm{Lip}(d\Psi_{x_1})+\mathrm{Lip}(d\Theta)\|d\Psi_{x_1}\|_{C^0}^2\\ &<\mathfrak{K}\varepsilon^{3/\beta}(\eta_1\eta_2)^3+4\mathfrak{K}(\eta_1\eta_2)^4<5\mathfrak{K}\varepsilon^{3/\beta}(\eta_1\eta_2)^3<\varepsilon^{2/\beta}(\eta_1\eta_2)^3.\end{aligned}$$

This implies that $\|\Theta\circ\Psi_{x_1}\|_{C^2}<3\varepsilon^{2/\beta}(\eta_1\eta_2)^3$, hence

$$\|C_2^{-1}\circ P_{x_2,x}\circ\Theta\circ\Psi_{x_1}\|_{C^2}\le\|C_2^{-1}\|3\varepsilon^{2/\beta}(\eta_1\eta_2)^3\le 3\varepsilon^{2/\beta}(\eta_1\eta_2)^2.$$

Thus $\|\Psi_{x_2}^{-1}\circ\Psi_{x_1}-\mathrm{Id}\|_2\le 3\varepsilon^{2/\beta}(\eta_1\eta_2)^2+\|C_2^{-1}\|(\eta_1\eta_2)^4<3\varepsilon^{2/\beta}(\eta_1\eta_2)^2+\varepsilon^{3/\beta}(\eta_1\eta_2)^3$ $<4\varepsilon^{2/\beta}(\eta_1\eta_2)^2<\varepsilon(\eta_1\eta_2)^2$. □

Proof of Proposition 7.2. Let $z\in Z$, $z'=\varphi^t(z)\in Z'$ with $|t|\le 2\rho$, and assume that $Z'\subset D'$. Define $\Upsilon:=\Psi_y^{-1}\circ\mathfrak{q}_{D'}\circ\Psi_x$. We will write Υ as a small perturbation of $\pm\mathrm{Id}$. For ease of notation, write $p:=p^s\wedge p^u$ and $q:=q^s\wedge q^u$. Start noting that, by Lemma 3.4, Proposition 3.6(1), and Theorem 6.1(5),

$$\tfrac{p}{q}=\tfrac{p}{p^s(z)\wedge p^u(z)}\cdot\tfrac{p^s(z)\wedge p^u(z)}{q(z)}\cdot\tfrac{q(z)}{q(z')}\cdot\tfrac{q(z')}{p^s(z')\wedge p^u(z')}\cdot\tfrac{p^s(z')\wedge p^u(z')}{q}=e^{\pm[O(\sqrt[3]{\varepsilon})+O(\rho)]}.$$

We have $\Upsilon=(\Psi_y^{-1}\circ\Psi_{z'})\circ(\Psi_{z'}^{-1}\circ\mathfrak{q}_{D'}\circ\Psi_z)\circ(\Psi_z^{-1}\circ\Psi_x)$. By Theorem 6.1(6), we have:

- $(\Psi_y^{-1}\circ\Psi_{z'})=(-1)^{\sigma_1}\mathrm{Id}+\Delta_1(v)$ where $\sigma_1\in\{0,1\}$, $\|\Delta_1(0)\|<50^{-1}q$, and $\|d\Delta_1\|_{C^0}<\sqrt[3]{\varepsilon}$ on $R[10Q(z')]$.

- $(\Psi_z^{-1} \circ \Psi_x) = (-1)^{\sigma_2}\mathrm{Id} + \Delta_2(v)$ where $\sigma_2 \in \{0,1\}$, $\|\Delta_2(0)\| < 50^{-1}p$, and $\|d\Delta_2\|_{C^0} < \sqrt[3]{\varepsilon}$ on $R[10Q(z)]$.

Assume, for simplicity, that $\sigma_1 = \sigma_2 = 0$. Applying Remark A.1, we conclude that $\Psi_{z'}^{-1} \circ \mathfrak{q}_{D'} \circ \Psi_z$ can be written in the form $(v_1, v_2) \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + H$, where A, B, H satisfy Theorem 3.8(2) with ρ changed to 2ρ. Assuming for simplicity that φ preserves orientation,[3] we have $AB > 0$, hence we can rewrite $\Psi_{z'}^{-1} \circ \mathfrak{q}_{D'} \circ \Psi_z = \pm[\mathrm{Id} + \Delta_3(v)]$ on $R[10Q(z)]$, where $\pm$ is the sign of A, B. Clearly $\Delta_3(0) = 0$. If $A, B > 0$ then $d(\Delta_3)_0 = \begin{bmatrix} A-1 & 0 \\ 0 & B-1 \end{bmatrix}$ and so we have $\|d(\Delta_3)_0\| = \max\{|A-1|, |B-1|\} < e^{8\rho} - 1$. The same estimate holds if $A, B < 0$. Using Theorem 3.8(2)(c), we get that $\|d\Delta_3\|_{C^0} < e^{8\rho} - 1 + O(\varepsilon)$. Therefore $\Upsilon = \pm(\mathrm{Id} + \Delta_1)(\mathrm{Id} + \Delta_3)(\mathrm{Id} + \Delta_2)$ where:

- $\|\Delta_1(0)\| < 50^{-1}q$ and $\|d\Delta_1\|_{C^0} = O(\varepsilon^{1/3})$.
- $\Delta_3(0) = 0$ and $\|d\Delta_3\|_{C^0} < e^{8\rho} - 1 + O(\varepsilon) = O(\rho) + O(\varepsilon)$.
- $\|\Delta_2(0)\| < 50^{-1}p$ and $\|d\Delta_2\|_{C^0} = O(\varepsilon^{1/3})$.

So $\Upsilon = \pm(\mathrm{Id} + \Delta)$, where $\Delta = \Delta_2 + \Delta_3(\mathrm{Id} + \Delta_2) + \Delta_1(\mathrm{Id} + \Delta_3)(\mathrm{Id} + \Delta_2)$. We have:

- $\|d\Delta\|_{C^0} \le \|d\Delta_2\|_{C^0} + 2\|d\Delta_3\|_{C^0} + 4\|d\Delta_1\|_{C^0} = O(\rho) + O(\varepsilon^{1/3})$, which implies that $\|d\Upsilon\|_{C^0} \le 1 + O(\rho) + O(\varepsilon^{1/3})$.
- $\Delta(0) = \Delta_2(0) + \Delta_3(\Delta_2(0)) + \Delta_1(\Delta_2(0) + \Delta_3(\Delta_2(0))) = \delta + \Delta_1(\delta)$, where $\delta = \Delta_2(0) + \Delta_3(\Delta_2(0))$. Letting $a_i := \|\Delta_i(0)\|$, $b_i := \mathrm{Lip}(\Delta_i)$, by direct calculation

  $$\|\delta\| \le \|\Delta_2(0)\| + \|\Delta_3(\Delta_2(0))\| \le \|\Delta_2(0)\| + \|\Delta_3(0)\| + \mathrm{Lip}(\Delta_3)\|\Delta_2(0)\| = a_2(1 + b_3)$$

  and so $\|\Delta(0)\| \le \|\delta\| + \|\Delta_1(0)\| + \mathrm{Lip}(\Delta_1)\|\delta\| \le a_1 + a_2(1 + b_1)(1 + b_3)$. Since $p \le e^{\pm[O(\sqrt[3]{\varepsilon}) + O(\rho)]}q = [1 + O(\rho) + O(\varepsilon^{1/3})]q$, it follows that

  $$\begin{aligned} \|\Delta(0)\| &\le 50^{-1}q + [1 + O(\varepsilon^{1/3})][1 + O(\rho) + O(\varepsilon)]50^{-1}p \\ &= [1 + O(\rho) + O(\varepsilon^{1/3})]25^{-1}q. \end{aligned}$$

  Hence $\|\Upsilon(0)\| \le [1 + O(\rho) + O(\varepsilon^{1/3})]25^{-1}q$.

We now proceed to prove the proposition.

(1) We have $\Upsilon(R[\frac{1}{2}p]) \subset \Upsilon(B_0[\frac{1}{\sqrt{2}}p]) \subset B_{\Upsilon(0)}[\frac{1}{\sqrt{2}}\mathrm{Lip}(\Upsilon)p] \subset B$, where $B \subset \mathbb{R}^2$ is the ball with center 0 and radius $\|\Upsilon(0)\| + \frac{1}{\sqrt{2}}\mathrm{Lip}(\Upsilon)p$. By the estimates obtained above,

$$\|\Upsilon(0)\| + \tfrac{1}{\sqrt{2}}\mathrm{Lip}(\Upsilon)p \le [1 + O(\rho) + O(\varepsilon^{1/3})]\tfrac{1}{25}q + \tfrac{1}{\sqrt{2}}[1 + O(\rho) + O(\varepsilon^{1/3})]p$$

[3] If not, we can apply an argument similar to [3].

$$\le [1 + O(\rho) + O(\varepsilon^{1/3})]\tfrac{1}{25}q + \tfrac{1}{\sqrt{2}}[1 + O(\rho) + O(\varepsilon^{1/3})]q$$
$$= \left[\tfrac{1}{25} + \tfrac{1}{\sqrt{2}}\right][1 + O(\rho) + O(\varepsilon^{1/3})]q.$$

Since $\frac{1}{25} + \frac{1}{\sqrt{2}} < 1$, for $0 < \varepsilon \ll \rho \ll 1$ we get that $B \subset B_0[q] \subset R[q]$.

(2) Fix $z \in Z$ such that $z' = \mathfrak{q}_{D'}(z) \in Z'$. We will show that $\mathfrak{q}_{D'}[W^s(z, Z)] \subset V^s(z', Z')$ (the other case is identical). Write $W = \mathfrak{q}_{D'}[W^s(z, Z)]$ and $V = V^s(z', Z')$. Our goal is to show that $W \subset V$. Let $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n\in\mathbb{Z}}, \underline{w} = \{\Psi_{y_n}^{q_n^s, q_n^u}\}_{n\in\mathbb{Z}}$ such that $z = \pi(\underline{v})$ and $z' = \pi(\underline{w})$. For $n \ge 0$, let $G_{\underline{v}}^n = g_{x_{n-1}}^+ \circ \cdots \circ g_{x_0}^+$ and $G_{\underline{w}}^n = g_{y_{n-1}}^+ \circ \cdots \circ g_{y_0}^+$. By Theorem 4.5(1), we need to show that $G_{\underline{w}}^n[W] \subset \Psi_{y_n}(R[10Q(y_n)])$ for all $n \ge 0$.

Fix $n \ge 0$. If $z' = \varphi^t(z)$, $|t| \le 2\rho$, then there is a unique $m \ge 0$ such that $r_m(\underline{v}) < r_n(\underline{w}) + t \le r_{m+1}(\underline{v})$. Let D_k be the disc containing $\varphi^{r_n(\underline{w})}(z')$. We claim that $G_{\underline{w}}^n \circ \mathfrak{q}_{D'} = \mathfrak{q}_{D_k} \circ G_{\underline{v}}^m$ wherever these maps are well-defined. To see this, firstly note that these maps are both of the form φ^τ for some continuous function τ. Secondly, we claim that they coincide at z. Indeed, $(G_{\underline{w}}^n \circ \mathfrak{q}_{D'})(z) = G_{\underline{w}}^n(z') = \varphi^{r_n(\underline{w})}(z')$ and $(\mathfrak{q}_{D_k} \circ G_{\underline{v}}^m)(z) = \mathfrak{q}_{D_k}[\varphi^{r_m(\underline{v})}(z)]$. Writing $\varphi^{r_n(\underline{w})}(z') = z_n'$ and $\varphi^{r_m(\underline{v})}(z) = z_m$, we have $z_n' = \varphi^{t'}(z_m)$ for $t' = r_n(\underline{w}) + t - r_m(\underline{v}) \in (0, \rho]$, therefore $\mathfrak{q}_{D_k}(z_m) = z_n'$. Hence $G_{\underline{w}}^n[W] = (G_{\underline{w}}^n \circ \mathfrak{q}_{D'})[W^s(z, Z)] = (\mathfrak{q}_{D_k} \circ G_{\underline{v}}^m)[W^s(z, Z)] \subset \mathfrak{q}_{D_k}[W^s(\varphi^{r_m(\underline{v})}(z), Z(v_m))]$, where we used Proposition 7.1(4) in the last inclusion. Since $W^s(\varphi^{r_m(\underline{v})}(z), Z(v_m)) \subset \Psi_{x_m}(R[10^{-2}(p_m^s \wedge p_m^u)])$, part (1) gives that $\mathfrak{q}_{D_k}[W^s(\varphi^{r_m(\underline{v})}(z), Z(v_m))] \subset \Psi_{y_n}(R[q_n^s \wedge q_n^u])$, and this last set is contained in $\Psi_{y_n}(R[10Q(y_n)])$.

(3) When M is compact and f is a $C^{1+\beta}$ surface diffeomorphism, the proof that $[z, z']_{Z'}$ is well-defined is [36, Lemma 10.8], and the proof uses that the change of coordinates from one Pesin chart to the other is so close to the identity that the representing function of an s–admissible manifold satisfies properties similar to (AM1)–(AM3), with the constants $10^{-3}, \frac{1}{2}$ slightly increased. We can apply the same method, since we showed above that our change of coordinates Υ is a small perturbation of the identity. The details can be easily carried out with the estimates we already obtained above. Similarly, $[z, z']_Z$ is well-defined. It remains to prove that $[z, z']_Z = \mathfrak{q}_D([z, z']_{Z'})$. To see this, observe that the composition $\mathfrak{q}_D \circ \mathfrak{q}_{D'}$ is the identity where it is defined, hence

$$\mathfrak{q}_D([z, z']_{Z'}) = \mathfrak{q}_D(\mathfrak{q}_{D'}[V^s(z, Z)] \cap V^u(z', Z')) = V^s(z, Z) \cap \mathfrak{q}_D[V^u(z', Z')] = [z, z']_Z.$$

This completes the proof of the proposition. $\square$

Proof of Proposition 7.3. Let Z, Z', Z'' such that $Z \cap \varphi^{[-2\rho, 2\rho]}Z' \neq \emptyset$, $Z \cap \varphi^{[-2\rho, 2\rho]}Z'' \neq \emptyset$, and assume that $z' \in Z'$ such that $\varphi^t(z') \in Z''$ for some $|t| \le 2\rho$. We are asked to show that for every $z \in Z$ it holds

$$[z, z']_Z = [z, \varphi^t(z')]_Z.$$

The idea is the following:

- $V^u(z', Z')$ and $V^u(\varphi^t(z'), Z'')$ coincide in a small window.
- If $Z = Z(\Psi_x^{p^s,p^u})$ and G is the representing function of $V^s(z, Z)$, then $[z, z']_Z = \Psi_x(s, G(s))$ for some $|s| \le \frac{1}{3}(p^s \wedge p^u)$.

The precise statements are in the next claims. Write $Z' = Z(\Psi_y^{q^s,q^u})$, $p = p^s \wedge p^u$ and $q = q^s \wedge q^u$, and let D be the connected components of $\widehat{\Lambda}$ with $Z \subset D$.

Claim 1: $\mathfrak{q}_D[V^u(z', Z') \cap \Psi_y(R[\frac{1}{2}q])]$ contains $\Psi_x\{(H(t), t) : |t| \le \frac{1}{3}p\}$ for some function $H : [-\frac{1}{3}p, \frac{1}{3}p] \to \mathbb{R}$ such that $H(0) < \frac{4}{25}p$ and $\|H'\|_{C^0} < \frac{1}{2}$. Furthermore, $[z, z']_Z = \Psi_x(s, G(s))$ for some $|s| \le \frac{1}{3}p$.

Claim 2: If D'' is the connected components of $\widehat{\Lambda}$ such that $Z'' \subset D''$, then

$$\mathfrak{q}_{D''}[V^{s/u}(z', Z') \cap \Psi_y(R[\tfrac{1}{2}q])] \subset V^{s/u}(z'', Z'').$$

Once we prove these claims, the proposition follows: Claim 2 implies that $\mathfrak{q}_D[V^u(z', Z') \cap \Psi_y(R[\frac{1}{2}q])] \subset \mathfrak{q}_D[V^u(z'', Z'')]$ and so by Claim 1

$$\begin{aligned}\{[z, z']_Z\} &= V^s(z, Z) \cap \mathfrak{q}_D[V^u(z', Z') \cap \Psi_y(R[\tfrac{1}{2}q])]\\ &\subset V^s(z, Z) \cap \mathfrak{q}_D[V^u(z'', Z'')] = \{[z, z'']_Z\}.\end{aligned}$$

Proof of Claim 1. With the estimates obtained in the beginning of the proof of Proposition 7.2, we just need to proceed as in the proof of [36, Lemma 10.8]. We will include the calculations for completeness. By the proof of Proposition 7.2, $\Upsilon := \Psi_x^{-1} \circ \mathfrak{q}_D \circ \Psi_y = \mathrm{Id} + \Delta$ where:

- $\|d\Delta\|_{C^0} \le e^{8\rho} - 1 + O(\varepsilon^{1/3}) = O(\varepsilon^{1/3}) + O(\rho)$.
- $\|\Delta(0)\| \le \frac{3}{25}\left[1 + O(\varepsilon^{1/3}) + O(\rho)\right] p$.

In particular, $\|\Delta\|_{C^0} \le \frac{3}{25}\left[1 + O(\varepsilon^{1/3}) + O(\rho)\right] p$. Write $\Delta = (\Delta_1, \Delta_2)$, and let F be the representing function of $V^u(z', Z')$, i.e. $V^u(z', Z') = \Psi_y\{(F(t), t) : |t| \le q^u\}$. Hence $V^u(z', Z') \cap \Psi_y(R[\frac{1}{2}q]) = \Psi_y\{(F(t), t) : |t| \le \frac{1}{2}q\}$, and since $\mathfrak{q}_D \circ \Psi_y = \Psi_x \circ \Upsilon$ we have

$$\begin{aligned}\mathfrak{q}_D[V^u(z', Z') \cap \Psi_y(R[\tfrac{1}{2}q])] &= (\Psi_x \circ \Upsilon)\{(F(t), t) : |t| \le \tfrac{1}{2}q\}\\ &= \Psi_x\{(F(t) + \Delta_1(F(t), t), t + \Delta_2(F(t), t)) : |t| \le \tfrac{1}{2}q\}.\end{aligned}$$

We represent the pair inside Ψ_x above as a graph on the second coordinate. Call $\tau(t) := t + \Delta_2(F(t), t))$. We have:

- $|\tau(0)| = |\Delta_2(F(0), 0)| \le \|\Delta(F(0), 0)\| \le \|\Delta(0)\| + \|d\Delta\|_{C^0}|F(0)| \le \frac{3}{25}[1 + O(\varepsilon^{1/3}) + O(\rho)]p + [O(\varepsilon^{1/3}) + O(\rho)]10^{-3}q \le \frac{3}{25}[1 + O(\varepsilon^{1/3}) + O(\rho)]p$.
- $|\tau'(t)| = 1 \pm \|d\Delta\|_{C^0}(1 + \|F'\|_{C^0}) = 1 + [O(\varepsilon^{1/3}) + O(\rho)](1 + \varepsilon) = 1 + O(\varepsilon^{1/3}) + O(\rho)$ for every $|t| \le \frac{1}{2}q$.

In particular,

$$\tau(\tfrac{1}{2}q) \geq \tfrac{1}{2}q - |\Delta_2(F(0),0)| \geq \tfrac{1}{2}q - \tfrac{3}{25}[1 + O(\varepsilon^{1/3}) + O(\rho)]p$$
$$\geq \left(\tfrac{1}{2}e^{-[O(\varepsilon^{1/3})+O(\rho)]} - \tfrac{3}{25}[1 + O(\varepsilon^{1/3}) + O(\rho)]\right) p > \tfrac{1}{3}p,$$

for $\rho, \varepsilon > 0$ small, since $\frac{1}{2} - \frac{3}{25} > \frac{1}{3}$. Therefore, the image of $\tau : [-\frac{1}{2}q, \frac{1}{2}q] \to \mathbb{R}$ contains $[-\frac{1}{3}p, \frac{1}{3}p]$.

Now, we write the first coordinate $F(t) + \Delta_1(F(t), t)$ as a function of τ. Start noting that, since the derivative of τ is positive, it has an inverse $\theta : \tau[-\frac{1}{2}q, \frac{1}{2}q] \to [-\frac{1}{2}q, \frac{1}{2}q]$ such that $|\theta'(\tau(t))| = |\tau'(t)|^{-1} = 1 + O(\varepsilon^{1/3}) + O(\rho)$ for every $\tau(t) \in \tau[-\frac{1}{2}q, \frac{1}{2}q]$. In particular,

$$|\theta(0)| = |\theta(0) - \theta(\tau(0))| \leq \|\theta'\|_{C^0}|\tau(0)| \leq \tfrac{3}{25}[1 + O(\varepsilon^{1/3}) + O(\rho)]p < \tfrac{1}{5}p.$$

Defining $H : [-\frac{1}{3}p, \frac{1}{3}p] \to \mathbb{R}$ by

$$H(\tau) = F(t) + \Delta_1(F(t), t) = F(\theta(\tau)) + \Delta_1(F(\theta(\tau)), \theta(\tau)),$$

we have:

- $|H(0)| \leq |F(\theta(0))| + |\Delta_1(F(\theta(0), \theta(0))| \leq |F(0)| + \|F'\|_{C^0}|\theta(0)| + \|\Delta\|_{C^0} \leq 10^{-3}q + \varepsilon\frac{1}{5}p + \frac{3}{25}\left[1 + O(\varepsilon^{1/3}) + O(\rho)\right] p < \frac{4}{25}p$.
- $\|H'\|_{C^0} \leq \|F'\|_{C^0}\|\theta'\|_{C^0} + \|d\Delta\|_{C^0}(1 + \|F'\|_{C^0})\|\theta'\|_{C^0} \leq 2\varepsilon + 2[O(\varepsilon^{1/3}) + O(\rho)][1+\varepsilon] = O(\varepsilon^{1/3}) + O(\rho)$ which is smaller than $\frac{1}{2}$ for $\rho, \varepsilon > 0$ small.

This proves the first part of Claim 1. For the second part, note that $|H(\tau)| \leq |H(0)| + \|H'\|_{C^0}|\tau| \leq \frac{4}{25}p + \frac{1}{2} \cdot \frac{1}{3}p < \frac{1}{3}p$, thus $H : [-\frac{1}{3}p, \frac{1}{3}p] \to [-\frac{1}{3}p, \frac{1}{3}p]$ is a contraction. We have $[z, z']_Z = \Psi_x(t, G(t))$, where t is the unique $t \in [-p^s, p^s]$ such that $(t, G(t)) = (H(\tau), \tau)$. Necessarily $H(G(t)) = t$, i.e. t is a fixed point of $H \circ G$. Using the admissibility of G and the above estimates, the restriction of $H \circ G$ to $[-\frac{1}{3}p, \frac{1}{3}p]$ is a contraction into $[-\frac{1}{3}p, \frac{1}{3}p]$, and so it has a unique fixed point in this interval, proving that $|t| \leq \frac{1}{3}p$. □

Proof of Claim 2. The proof is very similar to the proof of Proposition 4.9. Let us prove the inclusion for V^s. Let $V^s = V^s(z'', Z'') = V^s[\underline{v}^+]$ with $\underline{v}^+ = \{\Psi_{y_n}^{q_n^s, q_n^u}\}$, and let $G_n = g_{y_{n-1}}^+ \circ \cdots \circ g_{y_0}^+$. Let $U^s = \mathfrak{q}_{D''}[V^s(z', Z') \cap \Psi_y(R[\frac{1}{2}q])]$. By Proposition 7.2(1), $U^s \subset \Psi_{y_0}(R[q_0^s \wedge q_0^u])$. Now we proceed as in the proof of Proposition 4.9 to get that:

- If n is large enough then $G_n(U^s) \subset \Psi_{y_n}(R[Q(y_n)])$: this is exactly Claim 2 in the proof of Proposition 4.9.
- $U^s \subset V^s$: this is exactly Claim 3 in the proof of Proposition 4.9.

Hence Claim 2 is proved. □

The proof of the proposition is complete. □

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