



Relationship between two different approaches relating electromagnetic and scalar beam shape coefficients

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ARTICLE INFO

Keywords:

Generalized Lorenz–Mie theory
Extended Boundary Condition Method
T-matrix
Beam shape coefficients

ABSTRACT

Two different approaches, with many different conventions, have been recently published to express electromagnetic beam shape coefficients, encoding the structure of laser beams, in terms of scalar (more specifically acoustical) beam shape coefficients. One approach, call it VP1, expresses the electromagnetic fields using one single vector potential, while the second approach, call it VP2, expresses the electromagnetic fields in terms of two vector potentials. In this paper, we discuss the relationships between the VP1 and VP2 approaches, showing how the vector potential of the VP1 approach is related to the vector potentials of the VP2 approach, and express the relationships relating the beam shape coefficients of one approach to the beam shape coefficients of the other.

1. Introduction

The description of a laser beam may be encoded in two sets of coefficients named beam shape coefficients (BSCs), denoted $g_{n,TM}^m$ and $g_{n,TE}^m$ (n from 1 to ∞ , m from $(-n)$ to $(+n)$, with TM standing for “Transverse Magnetic” and TE standing for “Transverse electric”). These coefficients may be in particular used in some light scattering theories in spherical coordinates such as generalized Lorenz–Mie theories (GLMTs) for homogeneous spheres, for multilayered spheres, for aggregates of spheres, or for spheres with spherical inclusions, among others [1,2], or such as the Extended Boundary Condition Method available as well for nonspherical particles [3–5], both being T-matrix methods [6,7]. For recent reviews, the reader may refer to [8,9], and references therein.

Several methods are available to the evaluation of BSCs, which are for instance listed in the introduction of [10]. Beside these methods, another approach has been recently developed in which electromagnetic BSCs are expressed in terms of scalar BSCs. Because there are two kinds of electromagnetic BSCs and one kind only of scalar BSCs, it is expected that the evaluation of electromagnetic BSCs in terms of scalar BSCs will roughly be twice faster, than by using more conventional methods [11–13].

There exist two such approaches which have been developed independently. One of these approaches expresses the electromagnetic fields in terms of two vector potentials (let us call it the VP2 approach) while

the other approach expresses the electromagnetic fields in terms of only one vector potential (let us call it the VP1) approach. Conceptually, it is easy to understand that the VP1 approach is a simplified case of the VP2 approach. However, because they have been developed independently, they may have different conventions related to different aspects. More specifically, the VP1 approach uses a time-dependence of the form $\exp(+i\omega t)$ as is usual in the GLMT framework, while the VP2 approach uses a time-dependence of the form $\exp(-i\omega t)$, with ω being the angular frequency, and t being the time. Also, the two approaches are different by using different normalizations, different definitions of the electromagnetic BSCs, different definitions of the associated Legendre functions $P_n^m(\cos \theta)$, i.e. Ferrers’ convention in the VP2 approach versus Hobson’s convention in the VP1 approach, and the use of $P_n^m(\cos \theta)$ in the VP2 approach versus the use of $P_n^{|m|}(\cos \theta)$ in the VP1 approach. Furthermore, the expressions of the scalar fields are different, as follows.

In the VP2 approach, the scalar field reads as [11]:

$$\psi_{(2)}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} i^n (2n+1) g_{n,(2)}^m \psi_{nm}(r, \theta, \varphi) \quad (1)$$

in which:

$$\psi_{nm}(r, \theta, \varphi) = j_n(kr) P_n^m(\cos \theta) \exp(im\varphi) \quad (2)$$

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so that, as a whole:

$$\psi_{(2)}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} i^n (2n+1) g_{n,(2)}^m j_n(kr) P_n^m(\cos \theta) \exp(im\varphi) \quad (3)$$

In the Eqs. (1)–(3), (r, θ, φ) are usual spherical coordinates, k is the wavenumber, $j_n(\cdot)$ denotes spherical Bessel functions of the first kind, $P_n^m(\cos \theta)$, as already mentioned above, are associated Legendre functions, and $g_{n,(2)}^m$ are the scalar BSCs, and the subscript (2) is used to denote quantities pertaining to the VP2 approach. In [11], the scalar field is not explicitly defined as being an acoustical field.

On the contrary, the VP1 approach has been explicitly developed to relate electromagnetic and acoustical BSCs. Previous works in this framework concern the transfer of some methods used to evaluate the electromagnetic BSCs to methods used to evaluate acoustical BSCs. This concerns the localized approximations developed for electromagnetic BSCs, e.g. [14,15] and references therein dating back to [16,17], including a variant known as the integral localized approximation, e.g. [18] dating back to [19], which have been adapted to the case of acoustical BSCs in [20–23], and also the electromagnetic finite series method, e.g. [24,25] and references therein dating back to [26,27], which has been adapted to the case of acoustical BSCs in [28]. In this framework, the scalar field (more specifically the acoustical field) is written as:

$$\psi_{(1)}(\mathbf{r}) = \psi_{A0} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} (-i)^n (2n+1) g_{n,(1)}^m j_n(kr) P_n^m(\cos \theta) \exp(im\varphi) \quad (4)$$

with the subscript (1) being used to specify that we are dealing with the VP1 approach, while the expressions of the electromagnetic BSCs in terms of the scalar (acoustical) BSCs in the VP1 approach are available from [12,13].

The paper is then organized as follows. Section 2 expounds the relationship between the VP1 and the VP2 approaches, in terms of electromagnetic field expressions. Section 3 recalls how electromagnetic BSCs are expressed in terms of scalar BSCs, both in VP1 and VP2 approaches, and simplifies the expressions of the VP2 approach so that they become valid for a kind of VP1 approach, that we shall call the VP1-derived approach. Section 4 shows how the expressions of the VP1-derived and the VP1 approaches are compatible, and how one set of expressions is related to the other set, even if the conventions used in the VP1 and the VP2 approaches are different. Section 5 is a conclusion.

2. VP1 and VP2 approaches in terms of vector potentials

2.1. General approach

We begin with a general approach before considering special cases. The starting point is then made of Maxwell's equations for lossless, isotropic, linear media and in sourceless regions, which may be written as, e.g. Eqs.(1.56)–(1.59) of [2] and in [29,30]:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (5)$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (6)$$

$$\nabla \cdot \mathbf{E} = 0 \quad (7)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (8)$$

in which $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields respectively, while μ and ϵ are respectively the magnetic permeability and the electric permittivity of the medium in which the waves propagate. Now, the VP2 approach uses a time-dependence of the form $\exp(-i\omega t)$. Maxwell's equations then become:

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \quad (9)$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} \quad (10)$$

$$\nabla \cdot \mathbf{E} = 0 \quad (11)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (12)$$

We shall then consider two contributions to the electric field. For the first contribution, we first take into account the fact that the divergences of both $\mathbf{E}$ and $\mathbf{H}$ are 0 according to Eqs. (11) and (12). Because $\mathbf{H}$ is classically expressed as the curl of a potential vector, e.g. Eq.(1.110) in [2], the similarity between $\mathbf{E}$ and $\mathbf{H}$ expressed by Eqs. (11) and (12) allows one to introduce an electric field which is expressed as well using a vector potential. The first contribution to $\mathbf{E}$ is then written as:

$$\mathbf{E}_1 = \nabla \times \mathbf{A}_1 \quad (13)$$

in which $\mathbf{A}_1$ is a potential vector. A second, more classical contribution, i.e. Eq.(1.120) in [2], is derived from Eq. (10) and is written as:

$$\mathbf{E}_2 = \frac{i}{\omega\epsilon} \nabla \times \nabla \times \mathbf{A}_2 \quad (14)$$

in which $\mathbf{A}_2$ is the second vector potential of the VP2 approach. We then define the electric field by adding the two contributions, reading as:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \nabla \times \mathbf{A}_1 + \frac{i}{\omega\epsilon} \nabla \times (\nabla \times \mathbf{A}_2) \quad (15)$$

The potential vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ are now written as:

$$\mathbf{A}_1 = \alpha \psi_{(2)}(\mathbf{r}) \mathbf{p} \quad (16)$$

$$\mathbf{A}_2 = \beta \psi_{(2)}(\mathbf{r}) \mathbf{q} \quad (17)$$

in which $\mathbf{p}$ and $\mathbf{q}$ are dimensionless polarization vectors, α and β are coordinate-independent parameters to be determined, and $\psi_{(2)}(\mathbf{r})$ is the scalar function of the VP2 approach. Let us express this function using an angular spectrum decomposition, e.g. [6,31–33], and many references therein:

$$\psi(\mathbf{r}) = E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \quad (18)$$

in which $\tilde{\psi}(k_x, k_y)$ is the angular spectrum. We next insert Eq. (18) into Eqs. (13) and (16), leading to:

$$\mathbf{E}_1 = \nabla \times \mathbf{A}_1 = \nabla \times \alpha \psi(\mathbf{r}) \mathbf{p} \quad (19)$$

$$= \nabla \times [\alpha E_0 \mathbf{p} \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y]$$

$$= E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) \nabla \times [\alpha \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r})] dk_x dk_y$$

We afterward show by a direct check that:

$$\nabla \times [\alpha \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r})] = \alpha \nabla \exp(i\mathbf{k} \cdot \mathbf{r}) \times \mathbf{p} = [i\alpha \mathbf{k} \times \mathbf{p}] \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (20)$$

so that:

$$\mathbf{E}_1 = E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) [i\alpha \mathbf{k} \times \mathbf{p}] \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \quad (21)$$

For $\mathbf{E}_2$, we start from Eq. (14), use Eq. (17) and, working then similarly as for $\mathbf{E}_1$, we obtain:

$$\mathbf{E}_2 = \frac{i}{\omega\epsilon} \nabla \times E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) [i\beta \mathbf{k} \times \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \quad (22)$$

$$= \frac{iE_0}{\omega\epsilon} \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) \nabla \times [i\beta \mathbf{k} \times \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y$$

We then show by a direct check that:

$$\nabla \times [i\beta \mathbf{k} \times \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) = i\mathbf{k} \times [i\mathbf{k} \times \beta \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (23)$$

which, once inserted into Eq. (22), leads to:

$$\mathbf{E}_2 = \frac{iE_0}{\omega\epsilon} \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) [i\mathbf{k} \times (i\mathbf{k} \times \beta \mathbf{q})] \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \quad (24)$$

Let us set:

$$\mathbf{k} = k\mathbf{e}_k \quad (25)$$

We then may add $\mathbf{E}_1$ from Eq. (21) and $\mathbf{E}_2$ from Eq. (24), and take advantage of the definition of Eq. (25), to obtain:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \quad (26)$$

$$\begin{aligned} &= E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) [i\alpha k \mathbf{e}_k \times \mathbf{p} + \frac{i}{\omega\epsilon} ik \mathbf{e}_k \times (ik \mathbf{e}_k \times \beta \mathbf{q})] \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \\ &= E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) [i\alpha k \mathbf{e}_k \times \mathbf{p} + \frac{i\beta}{\omega\epsilon} (ik)^2 \mathbf{e}_k \times (\mathbf{e}_k \times \mathbf{q})] \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \end{aligned}$$

We now state that the bracketed term is dimensionless. Let us simply consider the term $i\alpha k \mathbf{e}_k \times \mathbf{p}$, in which $\mathbf{p}$ has been defined as dimensionless and $\mathbf{e}_k$ is dimensionless as well (see Eq. (25)). Then, because α has the same dimension as $1/(ik)$, we have that $i\alpha k$ is dimensionless. Therefore, $i\alpha k \mathbf{e}_k \times \mathbf{p}$ is dimensionless. This implies that $i\beta(ik)^2 \mathbf{e}_k \times (\mathbf{e}_k \times \mathbf{q})/(\omega\epsilon)$ is dimensionless as well. Then, the whole bracketed term is dimensionless as well.

Then, we may state:

$$i\alpha k = \gamma_1 \quad (27)$$

$$\frac{i\beta}{\omega\epsilon} (ik)^2 = \gamma_2 \quad (28)$$

leading to:

$$\alpha = (ik)^{-1} \gamma_1 \quad (29)$$

$$\beta = \frac{\omega\epsilon}{i} (ik)^{-2} \gamma_2 \quad (30)$$

Next, we insert Eqs. (29) and (30) into Eq. (16), (17), leading to:

$$\mathbf{A}_1 = \gamma_1 (ik)^{-1} \psi_{(2)}(\mathbf{r}) \mathbf{p} \quad (31)$$

$$\mathbf{A}_2 = \gamma_2 \frac{\omega\epsilon}{i} (ik)^{-2} \psi_{(2)}(\mathbf{r}) \mathbf{q} \quad (32)$$

Let us define:

$$\mathbf{A} = \psi_{(2)}(\mathbf{r}) \mathbf{p} \quad (33)$$

$$\mathbf{A}' = \psi_{(2)}(\mathbf{r}) \mathbf{q} \quad (34)$$

so that:

$$\mathbf{A}_1 = \gamma_1 (ik)^{-1} \mathbf{A} \quad (35)$$

$$\mathbf{A}_2 = \gamma_2 \frac{\omega\epsilon}{i} (ik)^{-2} \mathbf{A}' \quad (36)$$

Inserting Eqs. (35), (36) into Eq. (15), we obtain:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \quad (37)$$

$$= \gamma_1 (ik)^{-1} \nabla \times \mathbf{A} + \gamma_2 (ik)^{-2} \nabla \times (\nabla \times \mathbf{A}')$$

Eq. (26) may then be rewritten as:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \quad (38)$$

$$= E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) [\gamma_1 \mathbf{e}_k \times \mathbf{p} + \gamma_2 \mathbf{e}_k \times (\mathbf{e}_k \times \mathbf{q})] \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y$$

We now recall Eq. (9) leading to:

$$\mathbf{H} = \frac{1}{i\omega\mu} \nabla \times \mathbf{E} \quad (39)$$

Inserting Eq. (38) into Eq. (39), and making a direct check, similar to the ones done for Eqs. (20) and (23), leads to:

$$\mathbf{H} = \frac{ik}{i\omega\mu} E_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) \{ \mathbf{e}_k \times [\gamma_1 \mathbf{e}_k \times \mathbf{p} + \gamma_2 \mathbf{e}_k \times (\mathbf{e}_k \times \mathbf{q})] \} \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \quad (40)$$

We then use, e.g. [2,29,30]:

$$k^2 = \mu\epsilon\omega^2 \quad (41)$$

$$H_0 = E_0 \sqrt{\frac{\epsilon}{\mu}} \quad (42)$$

to obtain:

$$\frac{ik}{i\omega\mu} E_0 = \frac{k}{\omega\mu} E_0 = \frac{\omega\sqrt{\mu\epsilon}}{\omega\mu} \sqrt{\frac{\mu}{\epsilon}} H_0 = H_0 \quad (43)$$

so that Eq. (40) becomes:

$$\mathbf{H} = H_0 \int_{k_x} \int_{k_y} \tilde{\psi}(k_x, k_y) \{ \mathbf{e}_k \times [\gamma_1 \mathbf{e}_k \times \mathbf{p} + \gamma_2 \mathbf{e}_k \times (\mathbf{e}_k \times \mathbf{q})] \} \exp(i\mathbf{k} \cdot \mathbf{r}) dk_x dk_y \quad (44)$$

2.2. Special cases

If we use $\gamma_1 = \gamma_2 = 1$, Eq. (37) becomes:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= (ik)^{-1} \nabla \times \mathbf{A} + (ik)^{-2} \nabla \times (\nabla \times \mathbf{A}') \end{aligned} \quad (45)$$

which is the expression of Eq.(1) in the VP2 approach of [11], although it was introduced without explanation. Concerning the VP1 approach of [12], it deals with only one vector potential reading as, e.g. Eq.(12) in [12]:

$$\mathbf{E} = \frac{1}{i\omega\mu\epsilon} \nabla \times (\nabla \times \mathbf{A}_{(1)}) \quad (46)$$

which is a variant of Eq. (37) with $\gamma_1 = 0$, in which the subscript (1) refers to the VP1 approach. Then, using Eq. (37), we have:

$$i\omega\mathbf{A}_{(1)} = \gamma_2 \mathbf{A}' \quad (47)$$

Eq. (47) implies that $\mathbf{A}_{(1)}$ and $\mathbf{A}'$ do not have the same dimension. This may be checked as follows. Indeed, Eqs. (32) and (34) imply:

$$\mathbf{A}_2 = \gamma_2 \frac{\omega\epsilon}{i} (ik)^{-2} \mathbf{A}' \quad (48)$$

Eqs. (14) and (48) then imply:

$$\begin{aligned} \mathbf{E}_2 &= \frac{i}{\omega\epsilon} \gamma_2 \frac{\omega\epsilon}{i} (ik)^{-2} \nabla \times (\nabla \times \mathbf{A}') \\ &= \gamma_2 (ik)^{-2} \nabla \times (\nabla \times \mathbf{A}') \end{aligned} \quad (49)$$

Comparing Eqs. (46) and (49), we have:

$$\frac{1}{i\omega\mu\epsilon} \mathbf{A}_{(1)} = \gamma_2 (ik)^{-2} \mathbf{A}' \quad (50)$$

which, using Eq. (41), leads to $i\omega\mathbf{A}_{(1)} = \gamma_2 \mathbf{A}'$ which confirms Eq. (47).

2.3. A variant

In this variant, we use a special case for the scalar function, namely:

$$\psi_{(2)}(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (51)$$

We then recall Eq. (15) and, using Eqs. (16) and (17), we have:

$$\mathbf{E} = \nabla \times \alpha \psi_{(2)}(\mathbf{r}) \mathbf{p} + \frac{i}{\omega\epsilon} \nabla \times \nabla \times \beta \psi_{(2)}(\mathbf{r}) \mathbf{q} \quad (52)$$

Let us introduce:

$$\mathbf{A} = \psi_{(2)}(\mathbf{r}) \mathbf{p} \quad (53)$$

$$\mathbf{A}' = \psi_{(2)}(\mathbf{r}) \mathbf{q} \quad (54)$$

Eq. (52) becomes:

$$\mathbf{E} = \alpha \nabla \times \mathbf{A} + \frac{i\beta}{\omega\epsilon} \nabla \times (\nabla \times \mathbf{A}') \quad (55)$$

that is to say:

$$\mathbf{E} = \alpha_1 \nabla \times \mathbf{A} + \alpha_2 \nabla \times (\nabla \times \mathbf{A}') \quad (56)$$

in which:

$$\alpha_1 = \alpha \quad (57)$$

$$\alpha_2 = \frac{i\beta}{\omega\epsilon} \quad (58)$$

Using Eq. (53), (54), together with Eq. (51), we obtain:

$$\mathbf{E} = E_0 \{ \alpha_1 \nabla \times [\mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r})] + \alpha_2 \nabla \times \nabla [\mathbf{q} \exp(i\mathbf{k} \cdot \mathbf{r})] \} \quad (59)$$

But, from Eq. (20), we have:

$$\nabla \times [\mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r})] = [i\mathbf{k} \times \mathbf{p}] \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (60)$$

and, similarly:

$$\nabla \times [\mathbf{q} \exp(i\mathbf{k} \cdot \mathbf{r})] = [i\mathbf{k} \times \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (61)$$

Hence:

$$\nabla \times \nabla \times [\mathbf{q} \exp(i\mathbf{k} \cdot \mathbf{r})] = \nabla \times [i\mathbf{k} \times \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (62)$$

Then, using Eq. (23), Eq. (62) becomes:

$$\nabla \times [i\mathbf{k} \times \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) = i\mathbf{k} \times [i\mathbf{k} \times \mathbf{q}] \exp(i\mathbf{k} \cdot \mathbf{r}) \quad (63)$$

Inserting Eqs. (60) and (63) into Eq. (59) leads to:

$$\mathbf{E} = E_0 [\alpha_1 i\mathbf{k} \times \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) + \alpha_2 i\mathbf{k} \times i\mathbf{k} \times \mathbf{q} \exp(i\mathbf{k} \cdot \mathbf{r})] \quad (64)$$

Using Eq. (25), Eq. (64) becomes:

$$\mathbf{E} = E_0 [\alpha_1 i\mathbf{k} \mathbf{e}_k \times \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) + \alpha_2 (ik)^2 \mathbf{e}_k \times \mathbf{e}_k \times \mathbf{q} \exp(i\mathbf{k} \cdot \mathbf{r})] \quad (65)$$

Again, similarly as for the discussion following Eq. (26), the bracketed term should be dimensionless, implying:

$$\alpha_1 = (ik)^{-1} \gamma_1 \quad (66)$$

$$\alpha_2 = (ik)^{-2} \gamma_2 \quad (67)$$

Eq. (56) then becomes:

$$\mathbf{E} = \gamma_1 (ik)^{-1} \nabla \times \mathbf{A} + \gamma_2 (ik)^{-2} \nabla \times (\nabla \times \mathbf{A}') \quad (68)$$

With $\gamma_1 = \gamma_2 = 1$, we recover Eq.(1) of [11].

3. BSCs from VP1 and VP2 approaches

3.1. Definitions of BSCs

In [11], the spherical wave expansions of the electric and of the magnetic fields are written as:

$$\mathbf{E}(\mathbf{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_{n,(2)}^{pw} (iG_{n,TM}^m \mathbf{N}_{nm}^{(1)} - G_{n,TE}^m \mathbf{M}_{nm}^{(1)}) \quad (69)$$

$$\mathbf{H}(\mathbf{r}) = \frac{k}{\mu\omega} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_{n,(2)}^{pw} (iG_{n,TE}^m \mathbf{N}_{nm}^{(1)} + G_{n,TM}^m \mathbf{M}_{nm}^{(1)}) \quad (70)$$

with a typo in Eq.(6) of [11], namely $c_{n,(2)}^{pw}$ of Eqs. (69) and (70) above being omitted, and in which:

$$c_{n,(2)}^{pw} = \frac{i^n(n+1/2)}{n(n+1)} \quad (71)$$

Also, $G_{n,TM}^m$ and $G_{n,TE}^m$ are the TM- and TE-BSCs in the VP2 approach, and the vector spherical wave functions of Eqs. (69) and (70) are written following Stratton's notation [29] according to:

$$\mathbf{M}_{nm}^{(1)} = A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi \quad (72)$$

$$\mathbf{N}_{nm}^{(1)} = \frac{n(n+1)}{R^2} \psi_n(R) P_n^m(\cos \theta) \exp(im\varphi) \mathbf{e}_r + B_\theta \mathbf{e}_\theta + B_\phi \mathbf{e}_\phi \quad (73)$$

in which we do not need to specify the expressions for A_θ , A_ϕ , B_θ , B_ϕ and in which $R = kr$, $\psi_n(z) = z j_n(z)$.

According to Eqs. (69) and (70), and using Eq. (71)–(73), the radial components of the fields are found to read as:

$$E_r = \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} i^{n+1} (n+1/2) G_{n,TM}^m \frac{\psi_n(R)}{R^2} P_n^m(\cos \theta) \exp(im\varphi) \quad (74)$$

$$H_r = \frac{k}{\mu\omega} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} i^{n+1} (n+1/2) G_{n,TE}^m \frac{\psi_n(R)}{R^2} P_n^m(\cos \theta) \exp(im\varphi) \quad (75)$$

which, using $\psi_n(z) = z j_n(z)$ becomes:

$$E_r = \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} i^{n+1} (n+1/2) G_{n,TM}^m \frac{j_n(R)}{R} P_n^m(\cos \theta) \exp(im\varphi) \quad (76)$$

$$H_r = \frac{k}{\mu\omega} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} i^{n+1} (n+1/2) G_{n,TE}^m \frac{j_n(R)}{R} P_n^m(\cos \theta) \exp(im\varphi) \quad (77)$$

This is to be compared with the expressions of the radial components of the VP1 approach given by Eqs.(8), (9) and (6) of [12], see as well Eqs.(3.39), (3.42), (3.45), (3.48) together with Eqs.(3.3) and (3.188) of [2], reading as:

$$E_r = E_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} (-i)^{n+1} (2n+1) g_{n,TM}^m \frac{j_n(R)}{R} P_n^{|m|}(\cos \theta) \exp(im\varphi) \quad (78)$$

$$H_r = H_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} (-i)^{n+1} (2n+1) g_{n,TE}^m \frac{j_n(R)}{R} P_n^{|m|}(\cos \theta) \exp(im\varphi) \quad (79)$$

which may be taken as the definitions of the BSCs $g_{n,TM}^m$ and $g_{n,TE}^m$ in the VP1 approach.

Eqs. (78), (79) are very similar to Eqs. (76)–(77), but we recall that they differ due to the use of different conventions, namely different ways of normalizations, different definitions of the electromagnetic BSCs, different time dependence conventions, different definitions of the associated Legendre functions (Ferrers' notation versus Hobson's notation), and the use of P_n^m versus the one of $P_n^{|m|}$.

3.2. BSCs in the VP2 approach

Starting from Eqs. (76), (77), and using the definition of Eq. (3) to express the scalar function of the VP2 approach, Shen et al. [11] expressed the electromagnetic BSCs in terms of the scalar BSCs. In doing so, the dimensionless vectors of Eqs. (16) and (17) are written as:

$$\mathbf{p} = p_x \mathbf{e}_x + p_y \mathbf{e}_y + p_z \mathbf{e}_z \quad (80)$$

$$\mathbf{q} = q_x \mathbf{e}_x + q_y \mathbf{e}_y + q_z \mathbf{e}_z \quad (81)$$

in which $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors in the x , y , z directions respectively. Eqs. (80), (81) may be rewritten as:

$$\mathbf{p} = p_+ \mathbf{e}_- + p_- \mathbf{e}_+ + p_z \mathbf{e}_z \quad (82)$$

$$\mathbf{q} = q_+ \mathbf{e}_- + q_- \mathbf{e}_+ + q_z \mathbf{e}_z \quad (83)$$

in which:

$$\mathbf{e}_\pm = \mathbf{e}_x \pm i\mathbf{e}_y \quad (84)$$

$$p_\pm = (p_x \pm ip_y)/2 \quad (85)$$

$$q_\pm = (q_x \pm iq_y)/2 \quad (86)$$

With these notations, the BSCs of the VP2 approach read as:

$$G_{n,TM}^m = 2[i(p_+ u_{nm}^+ + p_- u_{nm}^- + p_z u_{nm}^z) - \frac{q_+ v_{nm}^+ + q_- v_{nm}^- + q_z v_{nm}^z}{2n+1}] \quad (87)$$

$$G_{n,TE}^m = 2[i(q_+ u_{nm}^+ + q_- u_{nm}^- + q_z u_{nm}^z) + \frac{p_+ v_{nm}^+ + p_- v_{nm}^- + p_z v_{nm}^z}{2n+1}] \quad (88)$$

in which:

$$u_{nm}^+ = (n+m+1)(n-m)g_{n,(2)}^{m+1} \quad (89)$$

$$u_{nm}^- = g_{n,(2)}^{m-1} \quad (90)$$

$$u_{nm}^z = -mg_{n,(2)}^m \quad (91)$$

and:

$$v_{nm}^+ = n(n+m+1)(n+m+2)g_{n+1,(2)}^{m+1} + (n+1)(n-m)(n-m-1)g_{n-1,(2)}^{m+1} \quad (92)$$

$$v_{nm}^- = -ng_{n+1,(2)}^{m-1} - (n+1)g_{n-1,(2)}^{m-1} \quad (93)$$

$$v_{nm}^z = n(n+m+1)g_{n+1,(2)}^m - (n+1)(n-m)g_{n-1,(2)}^m \quad (94)$$

3.3. BSCs in the VP1-derived approach

Let us derive a VP1 approach by simplifying the VP2 approach. Let us then start from Eq. (45) which, recall it, is the basic first equation of the VP2 approach in [11]. Using Ampere's law, i.e. Eq. (9), and Eq. (45), we obtain:

$$\mathbf{H}(\mathbf{r}) = (i\mu\omega)^{-1}[(ik)^{-1}\nabla \times \nabla \times \mathbf{A} + (ik)^{-2}\nabla \times \nabla \times (\nabla \times \mathbf{A}')] \quad (95)$$

We now focus on the term $\nabla \times (\nabla \times \mathbf{A}')$ in Eq. (95). But we have a classical equation:

$$\nabla \times (\nabla \times \mathbf{A}') = \nabla(\nabla \cdot \mathbf{A}') - \Delta \mathbf{A}' \quad (96)$$

which, by using the Helmholtz equation $\Delta \mathbf{A}' + k^2 \mathbf{A}' = 0$, becomes:

$$\nabla \times (\nabla \times \mathbf{A}') = \nabla(\nabla \cdot \mathbf{A}') + k^2 \mathbf{A}' \quad (97)$$

Then, the term $\nabla \times \nabla \times (\nabla \times \mathbf{A}')$ of Eq. (95) becomes:

$$\nabla \times [\nabla \times (\nabla \times \mathbf{A}')] = \nabla \times [\nabla(\nabla \cdot \mathbf{A}') + k^2 \mathbf{A}'] \quad (98)$$

But the curl of a gradient is zero, i.e. $\nabla \times [\nabla(\nabla \cdot \mathbf{A}')] = \nabla \times \nabla \varphi = 0$, so that Eq. (98) reduces to:

$$\nabla \times \nabla \times (\nabla \times \mathbf{A}') = k^2 \nabla \times \mathbf{A}' \quad (99)$$

Inserting Eq. (99) into Eq. (95) leads to:

$$\mathbf{H}(\mathbf{r}) = (i\mu\omega)^{-1}[(ik)^{-1}\nabla \times (\nabla \times \mathbf{A}) - \nabla \times \mathbf{A}'] \quad (100)$$

We then recall Eqs. (53) and (54) in which the polarization vectors are given by Eqs. (80) and (81). These vectors are specified as:

$$p_x = p_y = p_z = 0 \quad (101)$$

$$q_x = -i\omega, q_y = q_z = 0 \quad (102)$$

Note that, in this approach, $\mathbf{q}$ is not dimensionless, in contrast with what has been done for the VP2 approach, see comment after Eqs. (16), (17), in Section 2.1. Then, using Eqs. (53), (54), (101), (102), it is found that Eqs. (45) and (100) may be rewritten as:

$$\mathbf{E}(\mathbf{r}) = i\omega k^{-2} \nabla \times [\nabla \times \mathbf{e}_x \psi_{(2)}(\mathbf{r})] = \frac{-1}{i\omega \epsilon \mu} \nabla \times [\nabla \times \mathbf{e}_x \psi_{(2)}(\mathbf{r})] \quad (103)$$

$$\mathbf{H}(\mathbf{r}) = \mu^{-1} \nabla \times \mathbf{e}_x \psi_{(2)}(\mathbf{r}) \quad (104)$$

in which we used as well Eq. (41) to obtain Eq. (103). Also, the changes from $\nabla \times \mathbf{A}'$ to $\nabla \times \mathbf{e}_x \psi_{(2)}(\mathbf{r})$ may be established by an easy direct check. Eqs. (103), (104) are structurally identical to Eqs.(12) and (34) of [12] respectively, but for a difference of sign in Eq.(12) due to the use of a different time convention. We then conclude that Eqs. (101), (102) provide a link between the VP2 approach and the VP1 approach, called the VP1-derived approach. These Eqs. (101), (102) imply:

$$p_+ = p_- = p_z = 0 \quad (105)$$

$$q_+ = q_- = -i\omega/2, q_z = 0 \quad (106)$$

Using Eqs. (105)–(106), the BSCs of the VP2 approach given in Eqs. (87)–(94) simplify to:

$$\begin{aligned} G_{n,TE}^m &= i\omega \frac{v_{nm}^+ + v_{nm}^-}{2n+1} \\ &= \frac{i\omega}{2n+1} \{n[(n+m+1)(n+m+2)g_{n+1,(2)}^{m+1} - g_{n+1,(2)}^{m-1}] \\ &\quad + (n+1)[(n-m)(n-m-1)g_{n-1,(2)}^{m+1} - g_{n-1,(2)}^{m-1}]\} \end{aligned} \quad (107)$$

$$\begin{aligned} G_{n,TE}^m &= -\omega(u_{nm}^+ + u_{nm}^-) \\ &= -\omega[(n+m+1)(n-m)g_{n,(2)}^{m+1} + g_{n,(2)}^{m-1}] \end{aligned} \quad (108)$$

3.4. BSCs in the VP1 approach

In [13], after unification of the results published in [12], the corresponding solutions of the VP1 approach are found to read as:

$$g_{n,TE}^m = \frac{-k\psi_{A0}}{2\mu H_0} [(n+m+1)(n-m)g_{n,(1)}^{m+1} + g_{n,(1)}^{m-1}], m > 0 \quad (109)$$

$$g_{n,TE}^0 = \frac{-k\psi_{A0}}{2\mu H_0} n(n+1)[g_{n,(1)}^1 - g_{n,(1)}^{-1}], m = 0 \quad (110)$$

$$g_{n,TE}^m = \frac{k\psi_{A0}}{2\mu H_0} [g_{n,(1)}^{m+1} + g_{n,(1)}^{m-1}(n-|m|)(n+|m|+1)], m < 0 \quad (111)$$

$$\begin{aligned} g_{n,TE}^m &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0} \\ &\quad \times \{n[(n+m+1)(n+m+2)g_{n+1,(1)}^{m+1} - g_{n+1,(1)}^{m-1}] \\ &\quad + (n+1)[(n-m-1)(n-m)g_{n-1,(1)}^{m+1} - g_{n-1,(1)}^{m-1}]\}, m > 0 \end{aligned} \quad (112)$$

$$\begin{aligned} g_{n,TE}^0 &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0} n(n+1) \\ &\quad \times [(n-1)(g_{n-1,(1)}^1 + g_{n-1,(1)}^{-1}) + (n+2)(g_{n+1,(1)}^1 + g_{n+1,(1)}^{-1})], m = 0 \end{aligned} \quad (113)$$

$$\begin{aligned} g_{n,TE}^m &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0} \\ &\quad \times \{n[(n+|m|+1)(n+|m|+2)g_{n+1,(1)}^{m-1} - g_{n+1,(1)}^{m+1}] \\ &\quad + (n+1)[(n-|m|-1)(n-|m|)g_{n-1,(1)}^{m-1} - g_{n-1,(1)}^{m+1}]\}, m < 0 \end{aligned} \quad (114)$$

It might be remarked that Eqs. (109)–(111) on one hand, and (112)–(114) on the other hand, may be merged as shown in [12], but we preferred to keep them separated for the sake of clarity and pedagogic skills.

4. Relationship between the BSCs of the VP1-derived and the VP1 approaches

Eqs. (107)–(108) on one hand, and Eqs. (109)–(114) on the other hand cannot be identical due to the many differences of conventions already mentioned, but they must be closely and coherently related. The present section is devoted to the investigation of these relationships.

4.1. A Rossetta stone

The electromagnetic and the acoustical BSCs of the VP1 approach must be related to the electromagnetic and the acoustical BSCs of the VP1-derived approach by equations still to be established. In order to uncover the relation between these two approaches, we therefore define:

$$g_{n,(1)}^m = A_n^m g_{n,(2)}^m \quad (115)$$

$$g_{n,TE}^m = B_n^m G_{n,TE}^m \quad (116)$$

in which A_n^m is a ratio of scalar BSCs and B_n^m is a ratio of electromagnetic BSCs. In order to determine these ratios, let us begin with the case $m < 0$, and recall Eq. (111) which, using $m = -|m|$, may be rewritten as:

$$g_{n,TE}^m = \frac{k\psi_{A0}}{2\mu H_0} [g_{n,(1)}^{m+1} + (n+m)(n-m+1)g_{n,(1)}^{m-1}] \quad (117)$$

and also, using Eq. (115), as:

$$g_{n,TE}^m = \frac{k\psi_{A0}}{2\mu H_0} [A_n^{m+1} g_{n,(2)}^{m+1} + A_n^{m-1} (n+m)(n-m+1)g_{n,(2)}^{m-1}] \quad (118)$$

Using Eq. (116), we then have:

$$G_{n,TE}^m = \frac{k\psi_{A0}}{2\mu H_0} \left[\frac{A_n^{m+1}}{B_n^m} g_{n,(2)}^{m+1} + (n+m)(n-m+1) \frac{A_n^{m-1}}{B_n^m} g_{n,(2)}^{m-1} \right] \quad (119)$$

which is to be compared with Eq. (108):

$$G_{n,TE}^m = -\omega[(n+m+1)(n-m)g_{n,(2)}^{m+1} + g_{n,(2)}^{m-1}] \quad (120)$$

Such a comparison implies:

$$\frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^{m+1}}{B_n^m} = -\omega(n+m+1)(n-m) \quad (121)$$

$$\frac{k\psi_{A0}}{2\mu H_0} (n+m)(n-m+1) \frac{A_n^{m-1}}{B_n^m} = -\omega \quad (122)$$

that is to say, modifying the indices:

$$\frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^m}{B_n^{m-1}} = -\omega(n+m)(n-m+1) \quad (123)$$

$$\frac{k\psi_{A0}}{2\mu H_0} (n+m+1)(n-m) \frac{A_n^m}{B_n^{m+1}} = -\omega \quad (124)$$

from which, we may extract A_n^m and find:

$$A_n^m = -\omega(n+m)(n-m+1) \frac{2\mu H_0}{k\psi_{A0}} B_n^{m-1} = -\omega \frac{2\mu H_0}{k\psi_{A0}(n+m+1)(n-m)} B_n^{m+1} \quad (125)$$

leading to:

$$\frac{B_n^{m+1}}{B_n^{m-1}} = (n+m)(n-m+1)(n+m+1)(n-m) \quad (126)$$

Returning to Eqs. (121), (122), we may then extract the ratio A_n^{m+1}/A_n^{m-1} which is found to be given again by Eq. (126), so that, as a whole, we have:

$$\frac{A_n^{m+1}}{A_n^{m-1}} = \frac{B_n^{m+1}}{B_n^{m-1}} = (n-m)(n-m+1)(n+m)(n+m+1) \quad (127)$$

We now consider $m > 0$ and then use Eq. (109) which, using Eq. (115), becomes:

$$g_{n,TE}^m = \frac{-k\psi_{A0}}{2\mu H_0} [(n+m+1)(n-m)A_n^{m+1} g_{n,(2)}^{m+1} + A_n^{m-1} g_{n,(2)}^{m-1}] \quad (128)$$

Using Eq. (116), we then have:

$$G_{n,TE}^m = \frac{-k\psi_{A0}}{2\mu H_0} \left[(n+m+1)(n-m) \frac{A_n^{m+1}}{B_n^m} g_{n,(2)}^{m+1} + \frac{A_n^{m-1}}{B_n^m} g_{n,(2)}^{m-1} \right] \quad (129)$$

which must identify with Eq. (108), leading to:

$$\frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^{m+1}}{B_n^m} = \frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^{m-1}}{B_n^m} = \omega \quad (130)$$

Therefore:

$$A_n^{m+1} = A_n^{m-1} = K \quad (131)$$

in which K is a constant. Next, modifying the indices in Eq. (130), we also have:

$$\frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^m}{B_n^{m-1}} = \frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^m}{B_n^{m+1}} = \omega \quad (132)$$

Hence:

$$B_n^{m-1} = B_n^{m+1} = K' \quad (133)$$

in which K' is a constant. Eqs. (130) or (132) imply:

$$\frac{B_n^m}{A_n^m} = \frac{K'}{K} = \frac{k\psi_{A0}}{2\mu H_0 \omega} \quad (134)$$

Inserting Eq. (134) into Eq. (129), we obtain:

$$G_{n,TE}^m = -\omega[(n+m+1)(n-m)g_{n,(2)}^{m+1} + g_{n,(2)}^{m-1}] \quad (135)$$

which indeed identifies with Eq. (108).

For $m = 0$, we start from Eqs. (108) and (110), specified for $m = 0$, according to:

$$g_{n,TE}^0 = \frac{-k\psi_{A0}}{2\mu H_0} n(n+1)[g_{n,(1)}^1 - g_{n,(1)}^{-1}] \quad (136)$$

$$G_{n,TE}^0 = -\omega[n(n+1)g_{n,(2)}^1 + g_{n,(2)}^{-1}] \quad (137)$$

Using Eqs. (115), (116), and also Eq. (136), we obtain:

$$G_{n,TE}^0 = \frac{-k\psi_{A0}}{2\mu H_0} \left[n(n+1) \frac{A_n^1}{B_n^0} g_{n,(2)}^1 - n(n+1) \frac{A_n^{-1}}{B_n^0} g_{n,(2)}^{-1} \right] \quad (138)$$

Comparing with Eq. (137), we then have:

$$\frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^1}{B_n^0} = \omega \quad (139)$$

$$\frac{k\psi_{A0}}{2\mu H_0} n(n+1) \frac{A_n^{-1}}{B_n^0} = -\omega \quad (140)$$

Eq. (139) is a special case of Eq. (130) which was obtained for $m > 0$, and is valid as well for $m = 0$ after extraction of:

$$\frac{k\psi_{A0}}{2\mu H_0} \frac{A_n^{m+1}}{B_n^m} = \omega \quad (141)$$

Also, Eq. (140) is a special case of Eq. (122) which was established for $m < 0$ and is valid as well for $m = 0$. Then, inserting Eqs. (139) and (140) into Eq. (138), we obtain:

$$G_{n,TE}^0 = -\omega[n(n+1)g_{n,(2)}^1 + g_{n,(2)}^{-1}] \quad (142)$$

which identifies with Eq. (137) as it should.

As a summary, the results obtained up to now in this subsection are Eq. (127) for $m < 0$, Eqs. (131), (133), (134) for $m > 0$, and Eqs. (139), (140) for $m = 0$. We may then verify that all these results can be further summarized by writing:

$$g_{n,(1)}^m P_n^{|m|}(\cos \theta) = g_{n,(2)}^m P_n^m(\cos \theta) \quad (143)$$

$$g_{n,TE}^m P_n^{|m|}(\cos \theta) = G_{n,TE}^m P_n^m(\cos \theta) \quad (144)$$

leading to:

$$A_n^m = B_n^m = \frac{P_n^m(\cos \theta)}{P_n^{|m|}(\cos \theta)} \quad (145)$$

To use Eq. (145), we may recall Eq.(2.77) in [2], leading to:

$$\frac{P_n^m(\cos \theta)}{P_n^{|m|}(\cos \theta)} = (-1)^{|m|} \frac{(n-|m|)!}{(n+|m|)!} \text{ for } m < 0 \text{ and } m = 0 \quad (146)$$

Then, for $m < 0$, using Eqs. (145) and (146), we readily recover Eq. (127). For $m > 0$, Eqs. (143) and (144) readily imply $A_n^m = B_n^m = 1$, that is to say $K = K' = 1$ in agreement with Eqs. (131), (133), while Eq. (134) implies:

$$\frac{k\psi_{A0}}{2\mu H_0 \omega} = 1 \quad (147)$$

which is a renormalization factor.

For $m = 0$, Eqs. (139), (140) together with the renormalization of Eq. (147) lead to:

$$\frac{A_n^1}{B_n^0} = 1 \quad (148)$$

$$n(n+1) \frac{A_n^{-1}}{B_n^0} = -1 \quad (149)$$

so that, using Eq. (145) to evaluate $B_n^0 = 1$, we actually have:

$$A_n^1 = 1 \quad (150)$$

$$A_n^{-1} = \frac{-1}{n(n+1)} \quad (151)$$

Both of these results in Eqs. (150), (151) are readily found to agree with Eq. (145).

4.2. TM-BSCs

The examination of TE-BSCs allowed one to establish the relationship between their expressions in the VP1-derived and in the VP1 approaches which are summarized in Eqs. (143)–(144), including the fact that, in Eq. (144), TE must be changed to TM. It remains to check that these equations allow one as well to establish the relationship between the expressions of the TM-coefficients. For this, we consider three cases as follows.

(i) $m > 0$

Eqs. (143) and (144) (with TE changed to TM) imply:

$$g_{n,(1)}^m = g_{n,(2)}^m \quad (152)$$

$$g_{n,TM}^m = G_{n,TM}^m \quad (153)$$

which are inserted into Eq. (112), leading to:

$$G_{n,TM}^m = \frac{i\omega\psi_{A0}}{2(2n+1)E_0} \times \{n[(n+m+1)(n+m+2)g_{n+1,(2)}^{m+1} - g_{n+1,(2)}^{m-1}] + (n+1)[(n-m-1)(n-m)g_{n-1,(2)}^{m+1} - g_{n-1,(2)}^{m-1}]\} \quad (154)$$

which identifies with Eq. (107) when we introduce a renormalization relation reading as:

$$\psi_{A0} = 2E_0 \quad (155)$$

(ii) $m = 0$

We have Eq. (153) with $m = 0$, reading as:

$$g_{n,TM}^0 = G_{n,TM}^0 \quad (156)$$

We also have Eq. (152) with $m = 1$, and $n = \pm 1$, reading as:

$$g_{n\pm 1,(1)}^1 = g_{n\pm 1,(2)}^1 \quad (157)$$

Also, with $m = -1$, we have to use Eqs. (143) and (146), with $n = \pm 1$, leading to:

$$g_{n-1,(1)}^{-1} = \frac{-g_{n-1,(2)}^{-1}}{n(n-1)} \quad (158)$$

$$g_{n+1,(1)}^{-1} = \frac{-g_{n+1,(2)}^{-1}}{(n+1)(n+2)} \quad (159)$$

Inserting Eqs. (156)–(159) into Eq. (113), and using the renormalization of Eq. (155) leads to:

$$G_{n,TM}^0 = \frac{i\omega}{2n+1} \{n[(n+1)(n+2)g_{n+1,(2)}^1 - g_{n+1,(2)}^{-1}] + (n+1)[n(n-1)g_{n-1,(2)}^1 - g_{n-1,(2)}^{-1}]\} \quad (160)$$

which identifies with Eq. (107) after having implemented in it the fact that we are dealing with the case $m = 0$.

(iii) $m < 0$

We insert Eqs. (143), (144) (with TE changed to TM), and (146) into Eq. (114), use the renormalization condition of Eq. (155), and rearrange to obtain:

$$G_{n,TM}^m = \frac{i\omega}{2n+1} \{n[(n-|m|+1)(n-|m|+2)g_{n+1,(2)}^{m+1} - g_{n+1,(2)}^{m-1}] + (n+1)[(n-|m|-1)(n-|m|-2)g_{n-1,(2)}^{m+1} - g_{n-1,(2)}^{m-1}]\} \quad (161)$$

$$+(n+1)[(n+|m|-1)(n+|m|)g_{n-1,(2)}^{m+1} - g_{n-1,(2)}^{m-1}]\}$$

which, after changing $|m|$ to $(-m)$, identifies with Eq. (107).

5. Conclusion

This paper is somehow the conclusion of an effort to express the electromagnetic BSCs in terms of scalar (more specifically acoustical) BSCs. Due to the fact that we are facing two sets of electromagnetic BSCs and only one set of scalar BSCs, the interest of such expressions is to speed up the computations of electromagnetic BSCs by typically a factor equal to 2. There however exist two approaches in this framework, one called the VP1 approach which expresses the electromagnetic fields in terms of only one vector potential, and another one, called the VP2 approach, which expresses the electromagnetic fields in terms of two vector potentials. In this paper, we have shown how that the VP1 approach is a simpler special case of the VP2 approach and, although both approaches use different definitions and conventions, it has been possible to relate the expressions of BSCs obtained in the VP1 approach with those obtained in the VP2 approach. The detailed relationships between the VP1 and the VP2 approaches constitute the most important result of the present paper, allowing anyone entering this field of research, more specifically concerning the evaluation of beam shape coefficients, to more conveniently know immediately how they are related, therefore avoiding the burden of algebraic calculations which require a certain amount of skill and time.

CRedit authorship contribution statement

Gérard Gouesbet: Writing – original draft, Formal analysis, Conceptualization. **Jianqi Shen**: Writing – review & editing, Formal analysis, Conceptualization. **Leonardo A. Ambrosio**: Writing – review & editing, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The research was partially supported by the National Council for Scientific and Technological Development (CNPq) (406949/2021-2, 309201/2021-7).

Data availability

No data was used for the research described in the article.

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