ON INFINITELY COHOMOLOGOUS TO ZERO OBSERVABLES

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Abstract. We show that for a large class of piecewise expanding maps T, the bounded p-variation observables u_0 that admits an infinite sequence of bounded p-variation observables u_i satisfying

$$u_i = u_{i+1} \circ T - u_{i+1}$$

are constant. The method of the proof consists in to find a suitable Hilbert basis for $L^2(hm)$, where hm is the unique absolutely continuous invariant probability of T. In terms of this basis, the action of the Perron-Frobenious and the Koopman operator on $L^2(hm)$ can be easily understood. This result generalizes earlier results by Bamón, Kiwi, Rivera-Letelier and Urzúa in the case $T(x) = \ell x \ mod \ 1$, $\ell \in \mathbb{N} \setminus \{0,1\}$ and Lipschitizian observables u_0.

1. Introduction

Let $T\colon I \to I$ be a dynamical system. Consider the *cohomological operator* defined by

$$\mathcal{L}(\psi) = \psi \circ T - \psi,$$

Given an observable, that is, a function $u_0\colon I \to \mathbb{R}$, one can ask if there exists a solution u_1 to the *Lvisic cohomologous equation*

$$\mathcal{L}(u_1) = u_0.$$

Such equation was intensively studied after its introduction by the seminal work of Livsic. These studies mainly concerns to the existence and regularity of the solution u_1.

Let μ be an invariant probability measure of T. We say that a function $u : I \to \mathbb{R}$ in $L^1(\mu)$ is cohomologous to zero if there is a function $w : I \to \mathbb{R}$ in $L^1(\mu)$ such that

$$u = \mathcal{L}(w).$$

An observable u_0 is *infinitely cohomologous to zero* if there exists a sequence of functions $u_n \in L^1(\mu)$, $n \in \mathbb{N}$, such that $\mathcal{L}^n u_n = u_0$, for all $n \in \mathbb{N}$.

Bamón, Kiwi, Rivera-Letelier and Urzúa [4] consider the expanding maps defined by

$$T_\ell(x) = \ell x \ mod \ 1,$$

where $\ell \geq 2$ is an integer. The Lebesgue measure on $[0,1]$ is invariant by T_ℓ. They show that every non-constant lipschitzian function $u : I \to \mathbb{R}$ is not infinitely

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cohomologous to zero. In this work we generalize this result to a much larger class of observables and piecewise expanding maps.

In [4] the study of this problem is motivated by the following observation. Let $\lambda \in (-1,1)$, $u_0\colon I \to \mathbb{R}$ be a Lipschitz function and define

$$A\colon I\times\mathbb{R}\to I\times\mathbb{R}$$

by

$$A_{\lambda,u_0}(x,y) = (T_\ell(x), \lambda y + u_0(x)).$$

In [4] they notice that

i. If $\mathcal{L}(u_1) = u_0$ then $A_{\lambda,u_1}\circ H = H\circ A_{\lambda,u_0}$, where H is the homeomorphism

$$H(x,y) = (x, \frac{y+u_1(x)}{1-\lambda}).$$

ii. It turns out that the analysis of topological structure of the attractor of $A_{\lambda,u}$ is easier if u is *not* cohomologous to zero.

So if u_0 is not infinitely cohomologous to zero, by i. we can reduce the analysis of the topological dynamics of A_{λ,u_0} to the analysis of A_{λ,u_n}, where $\mathcal{L}^n(u_n) = u_0$ and u_n is not cohomologous to zero. Using our results, a similar analysis of attractors could potentially be achieved to far more general skew-products.

1.1. **Statement of results.** Let I be an interval. We say that $T\colon I\to I$ is a *piecewise monotone* map if there exists a partition by intervals $\{I_1,\dots,I_m\}$ of I such that for each $i\leq m$ the map T is continuous and strictly monotone in I_i. A piecewise monotone map is *onto* if furthermore $T(I_i) = I$ for every i. A piecewise monotone map is called *expanding* if T is differentiable on each I_i and

$$\inf_i \inf_{x\in I_i} |T'(x)| > 1.$$

In this work, we will consider mainly maps $T\colon I\to I$ satisfying the following conditions:

(D1) T is piecewise monotone, Lipschitz on each interval of the partition I_i, $i\leq m$. In particular T' is defined almost everywhere and it is an essentially bounded function. We also assume

$$ess \inf_m |T'| > 0. \tag{1}$$

Here $ess\inf_m$ denotes the essential infimum with respect to the Lebesgue measure m.

(D2) We have $T(I) = I$ and moreover for every interval $H\subset I$ there is a finite collection of pairwise disjoint open subintervals $H_1,\dots,H_k\subset H$ and n such that T^n is a homeomorphism on H_i and

$$int\ I \subset \cup_i T^n(H_i).$$

(D3) T has a horseshoe, that is, there are three open intervals $J_1, J_2 \subset J\subset I$, with $J_1\cap J_2 = \emptyset$, such that T is a homeomorphism on each J_i and $T(J_i) = J$, $i = 1,2$.

(D4) T has an invariant probability μ that is absolutely continuous with respect to the Lebesgue measure m, so

$$\mu(A) = \int_A h\ dm$$

for some $h \in L^1(m)$. We will denote $\mu = hm$, where $h \in L^1(m)$ and m is the Lebesgue measure on I. Moreover μ is exact and there exist a, b such that

$$0 < a \leq h(x) \leq b < \infty \tag{2}$$

for hm-almost every x and the support of μ is I.

Our main result is:

Theorem 1. *Let T be a transformation satisfying D1-D4 and let $u_0 \colon I \to \mathbb{R}$ be an observable with bounded p-variation. Then either u_0 is constant in I up to a countable set or there exist $M \geq 0$ and bounded p-variation functions $u_i : I \to \mathbb{R}$, with $i \leq M$, which are unique (in $L^1(hm)$ and $BV_{p,I}$) up to an addition by a constant, such that*

- *We have*
$$\mathcal{L}^i u_i = u_0,$$
in I up to a countable set, for every $i \leq M$.
- *For every function ρ with bounded p-variation and every $c \in \mathbb{R}$ we have $\mathcal{L}\rho \neq u_M + c$ in a nonempty open set in I.*

With somehow distinct, but related, assumptions on T and u_0, which are satisfied in many interesting situations, we can improve this result is such way that $\mathcal{L}\rho \neq u_M + c$ for every $\rho \in L^1(hm)$. In this direction A. Avila [2] contributed with improvements of the results in the original version of this work and we are grateful he agreed to include them here. Avila contribution is the following.

Theorem 2. [2] *Let $u_0 \in L^1(hm)$ be such that*

$$\int u_0\ h\ dm = 0$$

and such that for every $v \in L^\infty(hm)$ there exist $C > 0$ and $\lambda \in [0, 1)$ such that

$$\left|\int u_0 \cdot v \circ T^i \cdot h\ dm\right| \leq C\lambda^i.$$

Then either u_0 is constant hm-almost everywhere or there exist an unique $M \geq 0$ and functions $u_i : I \to \mathbb{R}$, with $i \leq M$, $u_i \in L^1(hm)$, which are unique in $L^1(hm)$, up to an addition by a constant, such that

- *We have*
$$\mathcal{L}^i u_i = u_0 \text{ in } L^1(hm)$$
for every $i \leq M$.
- *For every function $\rho \in L^1(hm)$ and every $c \in \mathbb{R}$ we have $\mathcal{L}\rho \neq u_M + c$ on $L^1(hm)$*

Let $(\mathbb{B}, |\cdot|_{\mathbb{B}})$ be a Banach space of real-valued, Lebesgue measurable functions defined on I such that

(D5) (i) T is a piecewise expanding map satisfying D1 and D4.
(ii) There exists C and $p_0 \geq 1$ such that

$$|f|_{L^1(hm)} \leq C|f|_{\mathbb{B}}$$

for every $f \in \mathbb{B}$.
(iii) the Perron-Frobenious operator Φ_T of T is a bounded operator on $\mathbb{B}$ and there exists $h \in \mathbb{B}$, $h > 0$, with $\int h \ dm = 1$, $\lambda \in [0,1)$ and an linear operator $\Psi \colon \mathbb{B} \to \mathbb{B}$ such that

$$\Phi_T(f) = \int f \ dm \cdot h + \Psi(f),$$

with

$$|\Psi^n(f)|_{\mathbb{B}} \leq C\lambda^n |f|_{\mathbb{B}},$$

for every $f \in \mathbb{B}$ and $n \in \mathbb{N}$. Moreover $\Psi(h) = 0$.
(iv) $1/h \in \mathbb{B}$.
(v) The multiplication

$$(f, g) \to f \cdot g$$

is a bounded bilinear transformation on $\mathbb{B}$.
(vi) The set $\mathbb{B}$ is dense in $L^1(hm)$.

Theorem 3. *Let T be a transformation satisfying D1 and D4 and suppose that the Banach space of functions $\mathbb{B}$ and T satisfy D5. Let $u_0 \in \mathbb{B}$ be an observable. Then either u_0 is constant hm-almost everywhere or there exist an unique $M \geq 0$ and functions $u_i : I \to \mathbb{R}$, with $i \leq M$, $u_i \in L^1(hm)$, which are unique in $L^1(hm)$, up to an addition by a constant, such that*

- *We have*

$$\mathcal{L}^i u_i = u_0 \ \text{in} \ L^1(hm)$$

  for every $i \leq M$.
- *For every function $\rho \in L^1(hm)$ and every $c \in \mathbb{R}$ we have $\mathcal{L}\rho \neq u_M + c$ on $L^1(hm)$*

Moreover u_i belongs to $\mathbb{B}$, for $i \leq M$.

Remark 1.1. In the first version of this work, Theorem 3 had additional assumptions. We assumed for instance that $\mathbb{B}$ was contained in the space of functions with p-bounded variation. This is not longer necessary due Avila's contribution (Theorem 2).

Remark 1.2. The finiteness result for the family of cohomological operators

$$\mathcal{L}_\lambda(v) = v \circ T - \lambda v,$$

with $\lambda \in (0,1]$, $T(x) = \ell x \mod 1$, for integers $\ell \geq 2$ and Lipschitz observables, obtained in [4, Main Lemma, page 225], can also be generalized for maps described in Remarks 1.3, 1.4 and 1.5, replacing Lipschitz observables by bounded variation observables. The methods to achieve this generalization are quite similar to those in [4], so we will not give a full proof here. It is necessary to use Theorem 3, and to replace in their argument the usual Fourier basis by the basis obtained in Section 3 and the compactness of closed balls centered at zero of the space of Lipschitz functions as subsets of the space of continuous functions by Helly's Theorem, that is, the compactness of closed balls centered at zero of the space of bounded variation functions as subsets of $L^1(hm)$.

Remark 1.3. There are plenty of examples of transformations $T\colon I \to I$ satisfying D1-D4. Let T be a piecewise monotone, expanding map, C^2 on each I_i. Consider the $m \times m$ matrix $A_T = (a_{ij})$ defined by $a_{ij} = 1$ if

$$\overline{T(int\ I_i)} \subset int\ I_j,$$

and $a_{ij} = 0$ otherwise. Here the closure and interior are taken with respect to the topology of $[0,1]$. Suppose that $A_T^k > 0$ for some k. Then T satisfies D1, D2 and D4 and some iteration of T satisfies D1-D4. If we add the assumption that T has a horseshoe, then T fulfills D1-D4. The space of bounded variation functions $BV(I)$ and T satisfy $D5$.

Remark 1.4. A class of examples satisfying D1-D4 are β-transformations $T(x) = \beta x \mod 1$, with $\beta \geq 2$, $\beta \in \mathbb{R}$, $I = [0,1]$. The space of bounded variation functions $BV(I)$ and T satisfy $D5$.

Remark 1.5. Let $T\colon [-1,1] \to [-1,1]$ be a continuous map with $T(-1) = T(1) = -1$, C^2 on the intervals $[-1,0]$ and $[0,1]$, with $T' > 0$ in $[-1,0]$ and $T' < 0$ in $[0,1]$ and $T(-x) = T(x)$ for every $x \in [-1,1]$. Define

$$\theta = \inf_x |T'(x)|.$$

If $\theta > 1$ then there exists an unique fixed point $p \in [0,1]$. Define $J = [-p,p]$. If $\theta > \sqrt{2}$ then T^2 has a horseshoe in J and satisfies D1-D4 with $I = [T^2(0), T(0)]$. The space of bounded variation functions $BV(I)$ and T satisfies $D5$.

Remark 1.6. Let $T\colon I \to I$ be a piecewise expanding and onto map, $C^{1+\alpha_0}$ in each I_i, $\alpha_0 \in (0,1)$. Then T satisfies $D1 - D4$. The space of Holder continuous functions $C^\alpha(I)$, for $\alpha \leq \alpha_0$, and T satisfy $D5$.

Remark 1.7. Let $T\colon I \to I$ be a piecewise expanding map, linear in each I_i. Suppose that T has a horseshoe and satisfies the conditions on the matrix A_T as in Remark 1.3. One can prove using the results of Wong [10] that T satisfies $D1 - D4$. The space of bounded p-variation functions $BV_p(I)$, with $p \geq 1$, and T satisfy $D5$.

Remark 1.8. The mixing assumptions on the invariant measure μ are necessary, as it is shown by the following example. Consider a piecewise C^2 expanding map $T\colon I \to I$, unimodal (continuous and only one turning point), and with a cycle of intervals, that is, there are open intervals $J_j \subset I$, $j < p$ pairwise disjoint, such that $f(\overline{J}_j) \subset \overline{J}_{j+1 \mod p}$ and $f(\partial J_j) \subset \partial J_{j+1 \mod p}$. Then T has an absolutely continuous invariant probability μ and its support is contained in $\cup_j \overline{J}_j$. Let $\delta \in \mathbb{C} \setminus \{1\}$ be a p-root of unit, $\delta^p = 1$. Define $u_i\colon I \to \mathbb{C}$, $i \geq 0$, as

$$u_i(x) = \frac{\delta^j}{(\delta - 1)^i},$$

for $x \in J_j$. Define u_i in an arbitrary way elsewhere. It is easy to see that $u_i = u_{i+1} \circ T - u_{i+1}$ on $L^1(hm)$. To obtain real-valued functions, we can consider the real and imaginary parts of u_i.

1.2. **Topological Results.** Replacing Lipschitzian by bounded p-variation observables has the advantage to allow us to obtain results similar to Theorems 1 and 3 to maps which are just *topologically conjugate* with maps satisfying the assumptions of those theorems.

We will say that two functions $f, g\colon W \to \mathbb{R}$ are equal except in a countable subset, $f = g$ on W (e.c.s.) if $\{x \in W\colon\ f(x) \neq g(x)\}$ is countable.

Theorem 4. *Let $H\colon I \to I$ be a homeomorphism, let T be a piecewise monotone map and $\tilde{T}$ satisfying D1-D4. Suppose that*

$$H \circ \tilde{T} = T \circ H$$

in I (e.c.s). Let $u_0\colon H(I) \to \mathbb{R}$ be an observable with bounded p-variation. Then either u_0 is constant in $H(I)$(e.c.s) or there exist an unique $M \geq 0$ and bounded p-variation functions $u_i : H(I) \to \mathbb{R}$, with $i \leq M$, which are unique up to an addition by a constant (e.c.s.), such that

- *We have*
$$\mathcal{L}^i u_i = u_0,$$
on $H(I)$ (e.c.s) for every $i \leq M$.
- *For every function ρ with bounded p-variation and every $c \in \mathbb{R}$ we have $\mathcal{L}\rho \neq u_M + c$ in a non-empty open subset in $H(I)$.*

Theorem 5. *Let $H\colon I \to I$ be a homeomorphism, let T be a piecewise monotone map and $\tilde{T}$ satisfying D1-D4. Suppose that*

$$H \circ \tilde{T} = T \circ H$$

in I (e.c.s.). Suppose that the space of functions with bounded p_0-variation $BV_{p_0,I}$ and $\tilde{T}$ satisfy D5. Let $u_0\colon H(I) \to \mathbb{R}$ be an observable with bounded p_0-variation. Then either u_0 is constant in $H(I)$(e.c.s) or there exist an unique $M \geq 0$ and continuous (e.c.s.) bounded borelian functions $u_i : H(I) \to \mathbb{R}$, with $i \leq M$, which are unique up to an addition by a constant (e.c.s.), such that

- *We have*
$$\mathcal{L}^i u_i = u_0,$$
on $H(I)$(e.c.s.) for every $i \leq M$.
- *We have $\mathcal{L}\rho \neq u_M + c$*
  - A. *in an uncountable subset of $H(I)$, if ρ is a Borel measurable, bounded function and $c \in \mathbb{R}$.*
  - B. *in a non-empty open subset of $H(I)$, if ρ is a Borel measurable, bounded function which is continuous in $H(I)$(e.c.s.) and $c \in \mathbb{R}$.*

Moreover u_i has bounded p_0-variation, $i \leq M$.

Remark 1.9. Let $T\colon [0,2] \to [0,2]$ be a piecewise monotone, C^1 in $[0,1]$ and $[1,2]$, $T[0,1] = T[1,2] = [0,2]$, with $T(0) = 0$, $T' \geq \lambda > 1$ in $[1,2]$ and $T'(x) > 1$ in $x \in (0,1)$ and $T'(0) = 1$. Then T is conjugate with $\tilde{T}(x) = 2 \cdot x \mod 1$, so T satisfies the assumptions of Theorems 4 and 5, considering $p_0 = 1$ in Theorem 5.

Remark 1.10. Let $T\colon [-1,1] \to [-1,1]$, $T(-1) = T(1) = -1$, C^3 in $[-1,1]$, $T'(0) = 0$, $T' > 0$ on $[-1,0)$, $T' < 0$ on $(0,1]$. If T has negative Schwarzian derivative and non-renormalizable then T is conjugate with a tent map $\tilde{T}_\beta\colon [-1,1] \to [-1,1]$, defined as $\tilde{T}(x) = -\beta|x| + \beta - 1$, with $\beta = exp(h_{top}(T))$. Here $h_{top}(T)$ denotes the topological entropy of T. If $h_{top}(T) \geq \ln(2)/2$ then $T^2\colon I \to I$, with $I = [T^2(0), T(0)]$, satisfies the assumptions of Theorems 4 and 5, considering $p_0 = 1$ in Theorem 5.

1.3. **Continuous observables infinitely cohomologous to zero.** A. Avila told us a nice argument showing the existence of continuous and non constant observables that are infinitely cohomologous to zero. He kindly agreed to include this result here.

Theorem 6. [2] *Let $T\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1 expanding map on the circle. Then there exists a non constant continuous observable u that is infinitely cohomologous to zero.*

2. Preliminaries

In this section we present some notations and definitions.

Definition 2.1. Given a function $f : I \to \mathbb{C}$ and $p \geq 1$, we define the p-variation of f by

$$v_{p,I}(f) = \sup \left(\sum_{i=1}^{n} |g(a_i) - g(a_{i-1})|^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all finite sequences $a_0 < a_1 < \cdots < a_n$, $a_i \in I$.

We say that f has bounded p-variation if

$$v_{p,I}(f) < \infty.$$

Since the Perron-Frobenious operator is not properly defined at points which are image of points where DT is not defined, to define Perron-Frobenious operator acting in the space of p-bounded variation functions it is convenient to identify functions u and v defined on I so that $u = v$ up to a countable subset of I. We write $u \sim v$. The set of equivalence classes $[f]$ with respect to the relation $\sim$ such that

$$v_{p,I}([f]) = \inf_{f\sim g} v_{p,I}(g) < \infty$$

will be called the space of the functions on I with bounded p-variation and denoted $BV_{p,I}$. The function $f \to v_{p,I}([f])$ is a pseudo-norm on $BV_{p,I}$. We can define a norm by

$$|[f]|_{BV_{p,I}} = \inf_{g\sim f} (\sup |g| + v_{p,I}(g)).$$

$(BV_{p,I}, |\cdot|_{BV_{p,I}})$ is a Banach space. As usual, from now on we will omit the brackets $[\cdot]$ in the notation of equivalence classes.

Note that $1/p$-Hölder continuous functions have bounded p-variation. When $p = 1$, we say that the function has bounded variation.

Remark 2.2. One of the greatest advantages of dealing with p-bounded variation observables, in opposition to either Hölder or Lipschitzian ones, for instance, is that the pseudo-norm $v_{p,I}$ is invariant by homeomorphisms, that is, if $h\colon J \to I$ is a homeomorphism and $f\colon I \to \mathbb{R}$ is an observable then

$$v_{p,I}(f) = v_{p,J}(f \circ h).$$

Definition 2.3. Given a piecewise monotone, expanding map T, satisfying $D1$, define the Perron-Frobenius operator associated to T by

$$\Phi_T f(x) = \sum_{j\in J} f(\sigma_j(x)) \frac{1}{|T'(\sigma_j x)|} 1\!\!1_{T(I_j)}(x),$$

where $\sigma_j : T(I_j) \to I_j$ stands for the inverse branch of T restricted to I_j and $1\!\!1_J$ denotes the characteristic function of the set J.

The main properties of Φ_T are (see for instance [5] and [3]):
$i)$ Φ_T is a continuous linear operator on $L^1(hm)$.
$ii)$ $\displaystyle\int_0^1 \Phi_T f \cdot g \, dm = \int_0^1 f \cdot g \circ T \, dm$, where $f \in L^1(m)$ and $g \in L^\infty(m)$.
$iii)$ $\Phi_T f = f$ if and only if the measure $\mu = fm$ is invariant by T.

3. A special basis of $L^2(hm)$

In this section we assume that T satisfies $D1$ and $D4$. Consider the Hilbert space $L^2(hm)$ with the inner product

$$\langle u, w \rangle_{hm} = \int uwh \; dm.$$

Indeed $\langle u, w \rangle_{hm}$ is well defined even for $u \in L^k(hm)$ and $w \in L^b(hm)$, with $k, b \in [1, \infty) \cup \{+\infty\}$ satisfying

$$\frac{1}{k} + \frac{1}{b} = 1.$$

Since the measure hm is T-invariant we have

$$\langle u \circ T, w \circ T \rangle_{hm} = \langle u, w \rangle_{hm}.$$

In this section we will built a special Hilbert basis for $L^2(hm)$. Consider the bounded linear operator $P : L^k(hm) \to L^k(hm)$, $k \geq 1$, defined by

$$P(u) = \frac{\Phi(uh)}{h}.$$

Due Eq. (2), the operator P is well defined. Indeed

$$\sum_{j \in J} \frac{h(\sigma_j(x))}{h(x)} \frac{1}{|T'(\sigma_j x)|} 1\!\!1_{T(I_j)}(x) = 1$$

for every x and z^k is convex, so we have

$$\int |Pu|^k h \; dm \leq \int \Big(\sum_{j \in J} \frac{h(\sigma_j(x))}{h(x)} \frac{1}{|T'(\sigma_j x)|} |u|(\sigma_j(x)) 1\!\!1_{T(I_j)}(x) \Big)^k h(x) \; dm$$

$$\leq \int \sum_{j \in J} \frac{h(\sigma_j(x))}{h(x)} \frac{1}{|T'(\sigma_j x)|} |u|^k(\sigma_j(x)) 1\!\!1_{T(I_j)}(x) h(x) \; dm = \int P(|u|^k) h \; dm$$

$$\leq \int \Phi(|u|^k) \; dm = \int |u|^k \; dm \leq \frac{1}{a} \int |u|^k \; hdm.$$

Note that for $k = 1$ we have

$$\int |Pu| h \; dm \leq \int \Phi(|u|h) \; dm = \int |u| h \; dm,$$

so $||P||_{L^1(hm)} \leq 1$.
Let $\mathcal{B} = \{\varphi_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for

$$\mathrm{Ker}(P) = \{u \in L^2(hm) \; s.t. \; P(u) = 0\}.$$

Define

$$\mathcal{W} = \{\varphi_i \circ T^j \; : \; \varphi_i \in \mathcal{B} \; \text{ e } \; j \in \mathbb{N}\} \cup \{1\!\!1_I\}.$$

Recall that $1\!\!1_A$ denotes the indicator function of a set A. The main result of this section is

Proposition 3.1. *Suppose that T satisfies D1 and D4. Then $\mathcal{W}$ is a Hilbert basis for $L^2(hm)$. Indeed we can choose $\mathcal{B}$ such that $\mathcal{W} \subset L^\infty(hm)$.*

Remark 3.2. A very interesting example of this theorem is given by the function $T\colon [0,1] \to [0,1]$ defined by $T(x) = \ell x \mod 1$, with $\ell \in \mathbb{N} \setminus \{0,1\}$. In this case the Ruelle-Perron-Frobenious operator is just

$$(\Phi_T \psi)(x) = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \psi(\frac{x+i}{\ell}).$$

The Lebesgue measure m is an invariant probability, so $P = \Phi_T$. Moreover

$$\mathcal{B} = \{\sin(2\pi n x), \cos(2\pi n x)\colon\ \ell \text{ does not divide } n\}$$

is a basis for $Ker\ P$. Note that

$$\sin(2\pi n T^j(x)) = \sin(2\pi n \ell^j x) \text{ and } \cos(2\pi n T^j(x)) = \cos(2\pi n \ell^j x),$$

so the corresponding set $\mathcal{W}$ is just the classical Fourier basis of $L^2([0,1])$.

By property ii. of the Perron-Frobenious operator, it is easy to see that the Koopman operator $U : L^k(hm) \to L^k(hm)$, $k \geq 1$, defined by

$$U(w) = w \circ T,$$

is the adjoint operator of P, that is

$$\langle P(u), w \rangle_{hm} = \langle u, U(w) \rangle_{hm} \tag{3}$$

for every $u \in L^k(hm)$ and $w \in L^b(hm)$. Note that U preserves $L^k(hm)$ because hm is invariant. Moreover

$$P \circ U(f) = f$$

for every $f \in L^1(hm)$.

Lemma 3.3. *$\mathcal{W}$ is an orthonormal set.*

Proof. Indeed

$$|\mathbb{1}_I|_{L^2(hm)} = 1,$$

$$|\varphi_i \circ T^j|^2_{L^2(hm)} = |\varphi_i|^2_{L^2(hm)} = 1.$$

Futhermore if

$$(i_1, j_1) \neq (i_2, j_2)$$

then either $j_1 = j_2$, so we have

$$\left\langle \varphi_{i_1} \circ T^{j_1}, \varphi_{i_2} \circ T^{j_2} \right\rangle_{hm} = \langle \varphi_{i_1}, \varphi_{i_2} \rangle_{hm} = 0,$$

or without loss of generality we can assume $j_1 < j_2$ and

$$\left\langle \varphi_{i_1} \circ T^{j_1}, \varphi_{i_2} \circ T^{j_2} \right\rangle_{hm} = \left\langle \varphi_{i_1}, \varphi_{i_2} \circ T^{j_2 - j_1} \right\rangle_{hm} = \left\langle P^{j_2 - j_1}(\varphi_{i_1}), \varphi_{i_2} \right\rangle_{hm} = 0,$$

and

$$\left\langle \varphi_{i_1} \circ T^{j_1}, \mathbb{1}_I \right\rangle_{hm} = \int \varphi_i \circ T^j h\ dm = \int \varphi_i h\ dm = \int P(\varphi_i) h\ dm = 0.$$

□

Lemma 3.4. *There exists a countable set of functions $\Lambda \subset L^\infty(hm) \cap Ker(P)$ with the following property: Let $w \in L^k(hm)$, with $k \geq 1$. If for all $\varphi \in \Lambda$ we have*

$$\int w\varphi h\, dm = 0,$$

then there exists $\beta \in L^k(hm)$ such that

$$w = \beta \circ T$$

hm-almost everywhere. Moreover $Ker(P)^\perp = U(L^2(hm))$.

Proof. We claim that for the existence of $\beta \in L^k(hm)$ such that $w = \beta \circ T$, it is necessary and sufficient that for hm-almost every $y \in I$ we have

$$\sharp\{w(x) \colon h(x) \neq 0 \text{ and } T(x) = y\} = 1. \tag{4}$$

Indeed, if the Eq. (4) holds then for every y satisfying (4), choosing x such that $T(x) = y$ and $h(x) \neq 0$ we can define

$$\beta(y) = w(x).$$

If y does not satisfy (4), define $\beta(y) = 0$. Of course $w = \beta \circ T$ hm-almost everywhere and, since hm is an invariant measure of T, β belongs to $L^k(hm)$.
On the other hand, suppose that there exists $\beta \in L^k(hm)$ is such that $w = \beta \circ T$. Then

$$K = \{x \colon w(x) = \beta(T(x))\}$$

has full hm-measure. Since the support of hm is I and $I \subset Im\ T$ it follows that for hm-almost every y we have $\sharp A_y \geq 1$, where

$$A_y = \{w(x) \colon h(x) \neq 0 \text{ and } T(x) = y\}.$$

Suppose there is Ω, with $hm(\Omega) > 0$ such that $\sharp A_y \geq 2$ for every $y \in \Omega$. Note that $D1$ implies that f and its inverse branches are absolutely continuous functions, so it is easy to see that there are X_1 X_2 such that $m(X_1), m(X_2) > 0$, $T(X_i) = \Omega$ and for each $y \in \Omega$ and $i = 1, 2$ there exists only one $x_i \in X_i$ such that $T(x_i) = y$. Furthermore $w(x_1) \neq w(x_2)$, $h(x_i) \neq 0$. The absolutely continuity of T and its inverses branches implies that

$$\tilde{\Omega} = T(X_1 \cap K) \cap T(X_2 \cap K) \subset \Omega$$

has positive measure. Let $y \in \tilde{\Omega}$ and x_i as above. Then $w(x_i) = \beta(T(x_i)) = \beta(y)$, which contradicts $w(x_1) \neq w(x_2)$. This concludes the proof of the claim.
Let C_i be the set of points $x_0 \in I$ such that the function

$$F_i(a) = \int_0^a w \circ \sigma_i(Tx) \cdot 1\!\!1_{T(I_i)}(T(x)) \cdot h(x) dm(x) \tag{5}$$

has derivative $w \circ \sigma_i(T(x_0)) 1\!\!1_{T(I_i)}(T(x_0)) h(x_0)$ at $a = x_0$. The function in the above integral belongs to $L^1(m)$, so by the Lebesgue diffentiation theorem the set

$$C = \cap_i C_i \setminus \cup_i \partial I_i$$

has full Lebesgue measure in I. Since T is piecewise Lipschitz we obtain that

$$m(T(I \setminus C)) = 0.$$

Suppose that Eq. (4) does not hold for hm-almost every $y \in I$. Then it is not true that Eq. (4) holds for hm-almost every $y \in I \setminus T(I \setminus C)$. Since hm-almost every point has at least one preimage x with $h(x) \neq 0$,we conclude that there exists

$y_0 \in I \setminus T(I \setminus C)$ and two inverse branches of T, denoted by σ_1 and σ_2 such that y_0 belongs to the interior of $T(I_1) \cap T(I_2)$ and furthermore

$$w \circ \sigma_1(y_0) \neq w \circ \sigma_2(y_0), h(\sigma_1(y_0)) \neq 0, h(\sigma_2(y_0)) \neq 0.$$

We can assume

$$w \circ \sigma_1(y_0) > w \circ \sigma_2(y_0),$$

so

$$w \circ \sigma_1 \circ T \circ \sigma_2(y_0) \mathbb{1}_{T(I_1)} \circ T \circ \sigma_2(y_0) h \circ \sigma_2(y_0) > w \circ \sigma_2 \circ T \circ \sigma_2(y_0) \mathbb{1}_{T(I_2)} \circ T \circ \sigma_2(y_0) h \circ \sigma_2(y_0). \tag{6}$$

Since $\sigma_2(y_0) \in C$, the derivatives of the functions F_1 and F_2 at $a = \sigma_2(y_0)$ are the left and right hand sides of Eq. (6) respectively, so there exists $\varepsilon > 0$ such that for every closed non degenerate interval $\tilde{I}_2$ satisfying

$$\sigma_2(y_0) \in \tilde{I}_2 \subset (\sigma_2(y_0) - \varepsilon, \sigma_2(y_0) + \varepsilon) \cap I_2 \tag{7}$$

we have

$$\int_{\tilde{I}_2} w \circ \sigma_1(Tx) \mathbb{1}_{T(I_1)} \circ T(x) \cdot h(x) dm(x) > \int_{\tilde{I}_2} w \circ \sigma_2(Tx) \mathbb{1}_{T(I_2)} \circ T(x) \cdot h(x) dm(x).$$

Choose an interval $\tilde{I}_2$ satisfying Eq. (7) and small enough such that $T(\tilde{I}_2) \subset T(I_1)$. We can assume without loss of generality that $\partial \tilde{I}_2 \subset \mathbb{Q}$. Then

$$\int_{\tilde{I}_2} w \circ \sigma_1(Tx) \cdot h(x) dm(x) > \int_{\tilde{I}_2} w \circ \sigma_2(Tx) \cdot h(x) dm(x).$$

Let $\tilde{I}_1 := \sigma_1(T(\tilde{I}_2)) \subset I_1$. Define φ as

$$\varphi(x) = \begin{cases} -\frac{|T'(x)|}{|T'(\sigma_2(Tx))|} \cdot \frac{h(\sigma_2(Tx))}{h(x)} & \text{if } \quad x \in \tilde{I}_1, \\ 1 & \text{if } \quad x \in \tilde{I}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Note that $\varphi \in L^\infty(hm)$ and $\Phi(\varphi h) = 0$.
Hence

$$\begin{aligned} \int w\varphi h \, dm = & \\ &= \int_{\tilde{I}_1} w\varphi h \, dm + \int_{\tilde{I}_2} w\varphi h \, dm \\ &= \int_{\tilde{I}_1} w \cdot \left(-\frac{|T'|}{|T' \circ \sigma_2 \circ T|} \frac{h \circ \sigma_2 \circ T}{h} \right) h \, dm + \int_{\tilde{I}_2} wh \, dm. \end{aligned}$$

Since $\sigma_2 \circ T : \tilde{I}_1 \to \tilde{I}_2$ is Lipschitzian and monotone increasing, we can make a change of variables to get

$$\begin{aligned}
&-\int_{\tilde{I}_1} w \frac{|T'|}{|T' \circ \sigma_2 \circ T|} \frac{h \circ \sigma_2 \circ T}{h} h\, dm + \int_{\tilde{I}_2} wh\, dm \\
&= -\int_{\tilde{I}_2} w \circ \sigma_1 \circ T \cdot h\, dm + \int_{\tilde{I}_2} w \circ \sigma_2 \circ T \cdot h\, dm \\
&< -\int_{\tilde{I}_2} w \circ \sigma_2 \circ T \cdot h\, dm + \int_{\tilde{I}_2} w \circ \sigma_2 \circ T \cdot h\, dm = 0.
\end{aligned}$$

Therefore

$$\int w\varphi h\, dm \neq 0.$$

Let Λ be the set of functions φ of the form in Eq. (8), with

- The intervals $\tilde{I}_j \subset I_{i_j}$, $j = 1, 2$, and $\sigma_2 \colon T(I_{i_2}) \to I_{i_2}$ is the inverse of $T \colon I_{i_2} \to T(I_{i_2})$.
- $T(\tilde{I}_2) = T(\tilde{I}_1)$.
- $\partial \tilde{I}_2 \subset \mathbb{Q}$.

Then it is easy to see that Λ is countable and $\Lambda \subset L^\infty(hm) \cap KerP$ and, by the argument above, Λ has the wished property.

In particular for $k = 2$ we obtain $Ker(P)^\perp \subset U(L^2(hm))$. The inclusion $U(L^2(hm)) \subset Ker(P)^\perp$ follows from Eq. (3). $\square$

Proposition 3.5. *Let Λ be as in Lemma 3.4. Let $u : I \to \mathbb{R}$ be a non constant function in $L^1(hm)$. Then there exists $\varphi \in \Lambda$, and an integer $p \geq 0$ such that*

$$\int u \cdot \varphi \circ T^j \cdot h\, dm = 0, \quad \textit{for all } 0 \leq j < p$$

and

$$\int u\varphi \circ T^p \cdot h\, dm \neq 0.$$

Proof. Suppose that, for all $\varphi \in \Lambda$ and for all $k \geq 0$

$$\int u\, \varphi \circ T^k \cdot h\, dm = 0. \tag{9}$$

We claim that for every n there exists $\beta_n \in L^1(hm)$ such that

$$u = \beta_n \circ T^n. \tag{10}$$

Indeed, choosing $k = 0$ in Eq. (9) we obtain that for all $\varphi \in \Lambda$

$$\int u\, \varphi h\, dm = 0.$$

By Lemma 3.4, there exists $\beta_1 \in L^1(hm)$ such that

$$u = \beta_1 \circ T.$$

Suppose by induction that $u = \beta_n \circ T^n$, with $\beta_n \in L^1(hm)$. By Eq. (9) when $k = n$, for all $\varphi \in \Lambda$ we have

$$\int \beta_n\, \varphi\, h\, dm = \int \beta_n \circ T^n \cdot \varphi \circ T^n \cdot h\, dm = \int u\, \varphi \circ T^n \cdot h\, dm = 0.$$

By Lemma 3.4, there exists $\beta_{n+1} \in L^1(hm)$ such that

$$\beta_n = \beta_{n+1} \circ T.$$

Hence one has $u = \beta_{n+1} \circ T^{n+1}$.

Since the measure hm is an exact measure, we can conclude that u is a constant function. So $u = 0$.

□

Corollary 3.6. *Let $u : I \to \mathbb{R}$ be a non constant function in $L^2(hm)$. Then there exist $\varphi_i \in \mathcal{B}$ and an integer $p \geq 0$ such that*

$$\left\langle u, \varphi_i \circ T^j \right\rangle_{hm} = 0 \ \text{ for all } \ 0 \leq j < p$$

and

$$\left\langle u, \varphi_i \circ T^p \right\rangle_{hm} \neq 0.$$

Proof. Suppose that for every $\varphi_i \in \mathcal{B}$ and every $j \in \mathbb{N}$

$$\left\langle u, \varphi_i \circ T^j \right\rangle_{hm} = 0.$$

Since $\mathcal{B}$ is a base for $Ker(P)$ and $U^j : L^2(hm) \to L^2(hm)$ is an isometry, it follows that

$$\int \varphi \circ T^j \cdot u \cdot h \; dm = 0$$

for every $\varphi \in Ker(P)$ and $j \in \mathbb{N}$. This contradicts Proposition 3.5. □

Proof of Proposition 3.1. It follows from Lemma 3.3 and Corollary 3.6 that $\mathcal{W}$ is a basis of $L^2(hm)$. To construct a basis $\hat{\mathcal{W}} \subset L^\infty(hm)$, consider an enumeration of the set $\Lambda = \{\psi_i\}$ defined in Lemma 3.4. Apply the Gram-Schmidt process in the sequence ψ_i to obtain a sequence $\tilde{\psi}_i$ of pairwise orthogonal functions. Discarding the null functions and normalizing the remaining functions, we obtain an orthonormal set of functions $\hat{\mathcal{B}}$. Due Lemma 3.4

$$\overline{span(\hat{\mathcal{B}})} = \overline{span(\Lambda)} = Ker\ P,$$

so $\hat{\mathcal{B}}$ is a basis of $Ker\ P$, and

$$\hat{\mathcal{W}} = \{\phi \circ T^j : \ \phi \in \hat{\mathcal{B}}, \ j \in \mathbb{N}\} \cup \{\mathbb{1}_I\}$$

is a basis of $L^2(hm)$. □

Corollary 3.7. *Let $u : I \to \mathbb{R}$ be a non constant function in $L^1(hm)$. Let $\hat{\mathcal{B}}$ as in the proof of Proposition 3.1. Then there exist $\varphi_i \in \hat{\mathcal{B}}$ and an integer $p \geq 0$ such that*

$$\left\langle u, \varphi_i \circ T^j \right\rangle_{hm} = 0 \ \text{ for all } \ 0 \leq j < p$$

and

$$\left\langle u, \varphi_i \circ T^p \right\rangle_{hm} \neq 0.$$

Proof. Suppose that for every $\varphi \in \hat{\mathcal{B}}$ and every $j \in \mathbb{N}$

$$\left\langle u, \varphi \circ T^j \right\rangle_{hm} = 0. \tag{11}$$

Let Λ be as in Lemma 3.4. Since $\hat{\mathcal{B}}$ was obtained applying the Gram-Schmidt process to Λ, it follows that Eq. (11) holds for every $\varphi \in \Lambda$. This contradicts Proposition 3.5. □

From now on we assume $\mathcal{W} \subset L^\infty(hm)$. Let $u \in L^1(hm)$ and consider the Fourier coefficients of u with respect to the basis $\mathcal{W}$

$$c_{i,j}(u) = \left\langle u, U^j(\varphi_i)\right\rangle_{hm} = \int u \cdot \varphi_i \circ T^j \cdot h\, dm.$$

Proposition 3.8. *The functionals $c_{i,j}$ have the following properties:*

1. *$c_{i,j}$ is linear on $L^1(hm)$*
2. *$c_{i,j}(U(u)) = c_{i,j-1}(u)$ for $j \geq 1$.*
3. *$c_{i,0}(U(u)) = 0$.*
4. *$c_{i,j}(P(u)) = c_{i,j+1}(u)$.*

Proof. We have

1. The proof is straightforward.
2. $c_{i,j}(u \circ T) = \left\langle u \circ T, \varphi_i \circ T^j\right\rangle_{hm} = \left\langle u, \varphi_i \circ T^{j-1}\right\rangle_{hm} = c_{i,j-1}(u)$.
3. $c_{i,0}(u \circ T) = \langle U(u), \varphi_i\rangle_{hm} = \langle u, P(\varphi_i)\rangle_{hm} = \langle u, 0\rangle_{hm} = 0$.
4. $c_{i,j}(Pu) = \left\langle P(u), U^j(\varphi_i)\right\rangle_{hm} = \left\langle u, U^{j+1}(\varphi_i)\right\rangle_{hm} = c_{i,j+1}(u)$.

□

Proposition 3.9. *For every $u \in L^1(hm)$ and $\varphi_i \in \hat{\mathcal{B}}$ we have*

$$\lim_j c_{i,j} = 0$$

Proof. Since hm is exact, it is mixing, so

$$\lim_j c_{i,j} = \lim_j \int u \cdot \varphi_i \circ T^j \cdot h\, dm = 0.$$

□

Remark 3.10. V. Baladi drew to our attention the method used by M. Pollicott [7] to built eigenvectors of transfer operators for eigenvalues inside its essential spectral radius in certain function spaces. In our setting the method is the following: pick $\varphi \in Ker(P)$ and $|\lambda| < 1$. Then

$$v = \sum_{j=0}^{\infty} \lambda^j \varphi \circ T^j$$

is a λ-eigenvector of P in $L^2(hm)$. Using Propositions 3.1 and 3.8 one can easily show that *all* λ-eigenvalues of P in $L^2(hm)$, for every $|\lambda| < 1$, can be built in this way.

4. Proof of Theorem 1

In this section we will study the linear operator

$$\mathcal{L}u = u \circ T - u$$

acting on functions with bounded p-variation $u : I \to \mathbb{R}$.

First, we will present some properties and then, at the end of this section, we will prove the theorems announced in introduction. The following results are well know.

Lemma 4.1. *Let $\mathcal{L}$ be the linear operator defined above acting on $L^1(hm)$. Then:*

1. *If $f \in \mathrm{Im}(\mathcal{L})$, then $\int f h\, dm = 0$.*
2. $\mathrm{Ker}(\mathcal{L}) = \{f \in L^1(hm) \ :\ f$ *is constant hm-almost everywhere*$\}$.

Corollary 4.2. *Let $u \in L^1(hm)$ and suppose that there exist functions $v, w \in L^1(hm)$ such that*

$$\mathcal{L}^n v = u = \mathcal{L}^n w.$$

Then $v = w + c$ on $L^1(hm)$, for some $c \in \mathbb{R}$. Moreover if v, w have bounded p-variation then $v = w + c$ on I(e.c.s.).

Proof. Define $v_i = \mathcal{L}^{n-i} v$, $w_i = \mathcal{L}^{n-i} w$. We will prove by induction on i that $v_i = w_i$, if $i < n$ and $v_n = w_n + c$, for some $c \in \mathbb{R}$. Indeed, for $i = 0$ we have $w_0 = v_0 = u$. Suppose that $v_i = w_i$, $i < n$. Then

$$\mathcal{L}(v_{i+1} - w_{i+1}) = v_i - w_i = 0,$$

so $v_{i+1} - w_{i+1}$ is hm-almost everywhere constant. If $i + 1 = n$ we are done. If $i + 1 < n$ then $\mathcal{L} v_{i+2} = v_{i+1}$ and $\mathcal{L} w_{i+2} = w_{i+1}$, so

$$\int v_{i+1} \; h \; dm = \int w_{i+1} \; h \; dm = 0,$$

which implies $c = 0$. Now assume that u, v and w have bounded p-variation. Since the support of hm is I and $v = w + c$ hm-almost everywhere, we have the $v = w + c$ on a set $\Lambda \subset I$ such that for every non-empty open subset O of I we have that $O \cap \Lambda$ is a dense and uncountable subset of O. Since v and w have just a countable number of discontinuities in I, it follows that $v = w + c$ in I(e.c.s.). □

Lemma 4.3. *Let J be an open interval as in D3 and $u \in BV_{p,I}$. Then*

$$v_{p,J}(\mathcal{L}u) \geq v_{p,J}(u)$$

for every $n \in \mathbb{N}$.

Proof. Let $J_1, J_2 \subset J$ be as in D3. Since T is a homeomorphism on J_1 and J_2, by Remark 2.2

$$v_{p,J}(u \circ T) \geq v_{p,J_1}(u \circ T) + v_{p,J_2}(u \circ T) = 2 v_{p,J}(u),$$

so

$$v_{p,J}(u \circ T - u) \geq v_{p,J}(u \circ T) - v_{p,J}(u) \geq v_{p,J}(u).$$

□

Lemma 4.4. *There exists C with the following property: Let $u_n : I \to \mathbb{R}$, $n \leq M + 1$, be observables with bounded p-variation, $p \geq 1$, such that for every $n \leq M$*

$$u_n = \mathcal{L} u_{n+1}.$$

Then

$$|u_n|_{L^\infty(hm)} \leq v_{p,I}(u_n) \leq C v_{p,I}(u_0)$$

for every $n \leq M$.

Proof. Let $J \subset I$ be an one interval as in D3. By Lemma 4.3

$$v_{p,J}(u_n) \leq v_{p,J}(u_0) \tag{12}$$

for every $n \geq 0$. By D2 there is a finite collection of pairwise disjoint open intervals $H_1, \dots, H_k \subset J$ and j such that T^j is a homeomorphism on each H_i and

$$int \; I \subset \cup_{i=1}^k T^j(H_i). \tag{13}$$

We claim that for every $\ell \leq j$ and n

$$v_{p,T^\ell(H_i)}(u_n) \leq 2^\ell v_{p,J}(u_0) \tag{14}$$

We will prove this by induction on ℓ. Of course since $H_i \subset J$, Eq. (12) implies that for every $i = 1, \dots, k$

$$v_{p,H_i}(u_n) \leq v_{p,J}(u_0), \tag{15}$$

So Eq. (14) holds for $\ell = 0$. Suppose by induction that Eq. (14) holds for $\ell < j$ and every n. Since T is a homeomorphism on $T^\ell(H_i)$ and $u_{n-1} = u_n \circ T - u_n$ we have

$$\begin{aligned} v_{p,T^{\ell+1}(H_i)}(u_n) = v_{p,T^\ell(H_i)}(u_n \circ T) &\leq v_{p,T^\ell(H_i)}(u_n) + v_{p,T^\ell(H_i)}(u_{n-1}) \\ &\leq 2^{\ell+1} v_{p,J}(u_0). \end{aligned}$$

By Eq (13)

$$v_{p,I}(u_n) = v_{p,int\ I}(u_n) \leq \sum_{i=1}^{k} v_{p,T^j(H_i)}(u_n) \leq k2^j v_{p,J}(u_0) \leq k2^j v_{p,I}(u_0).$$

Note that since $u_n = u_{n+1} \circ T - u_{n+1}$ it follows that

$$\int u_n h\ dm = 0,$$

Suppose that

$$ess\ sup_{hm}\ u_n = |u_n|_{L^\infty(hm)}.$$

Then

$$\begin{aligned} 0 = \int u_n h\ dm \geq\ ess\ inf\ u_n &= (ess\ inf\ u_n - ess\ sup\ u_n) + ess\ sup\ u_n \\ &\geq -v_{p,I}(u_n) + |u_n|_{L^\infty(hm)}. \end{aligned}$$

so $|u_n|_{L^\infty(hm)} \leq v_{p,I}(u_n)$. We can obtain the same conclusion for the case

$$-ess\ inf_{hm}\ u_n = |u_n|_{L^\infty(hm)},$$

replacing u_n by $-u_n$ in the argument above. □

Proof of Theorem 1. Define by induction the (either finite or infinite) sequence $u_n \colon I \to \mathbb{R}$ of functions in the following way: u_0 is given. If u_n is defined and there exists a function $v \colon I \to \mathbb{R}$ with bounded p-variation such that $\mathcal{L}v = u_n$ in $L^1(hm)$, then define

$$u_{n+1} = v - \int v\ h\ dm.$$

Otherwise the sequence ends with u_n. Note that

$$\mathcal{L}^n u_n = u_0.$$

Define

$$M_0 = \sup\{n \in \mathbb{N} \colon\ u_n \text{ is defined }\} \in \mathbb{N} \cup \{\infty\}.$$

We will show that $M_0 < \infty$. Let $M \in \mathbb{N}$, $M \leq M_0$. Recall the basis $\mathcal{W}$ defined in Section 3. By Corollary 3.6 if u_0 is not constant almost everywhere there exist i and $q \geq 0$ such that

$$c_{i,j}(u_0) = \int u_0\, \varphi_i \circ T^j \cdot h\, dm = 0, \quad \text{for all} \quad 0 \leq j < q$$

and

$$c_{i,q}(u_0) = \int u_0\, \varphi_i \circ T^q \cdot h\, dm \neq 0.$$

By Lemma 4.4 we have that $|u_n|_{L^2(hm)} \leq |u_n|_{L^\infty(hm)} \leq Cv_{p,I}(u_0)$, so since

$$|\varphi_i \circ \ T^i|_{L^2(hm)} \ = \ 1$$

we obtain

$$|c_{i,k}(u_n)| = \left| \int u_n \cdot \varphi_i \circ T^k \cdot h \, dm \right| \leq Cv_{p,I}(u_0).$$

Using Lemma 3.8, we can now use an argument quite similar to [4]. Observe that

$$c_{i,l}(u_{n-1}) = c_{i,l}(u_n \circ T - u_n) = c_{i,l}(u_n \circ T) - c_{i,l}(u_n) = c_{i,l-1}(u_n) - c_{i,l}(u_n),$$

for $l \geq 1$.

For $l = 0$,

$$c_{i,0}(u_{n-1}) = c_{i,0}(u_n \circ T - u_n) = c_{i,0}(u_n \circ T) - c_{i,0}(u_n) = -c_{i,0}(u_n),$$

for $0 < n \leq M$.

Therefore, for $0 < n \leq M$

$$c_{i,l}(u_n) = c_{i,l-1}(u_n) - c_{i,l}(u_{n-1}), \text{ for } \ l \geq 1. \tag{16}$$

$$c_{i,0}(u_{n-1}) = -c_{i,0}(u_n). \tag{17}$$

Since $c_{i,j}(u_0) = 0$ for $0 \leq j < q$, by equations (16) and (17), we can conclude that

$$c_{i,j}(u_n) = 0 \text{ for } \ 0 \leq j < q \ \text{ and } \ 0 \leq n \leq M. \tag{18}$$

Now, by equation (16), considering $l = q$, we have

$$c_{i,q}(u_{n-1}) = c_{i,q-1}(u_n) - c_{i,q}(u_n).$$

By equation (18), for every $n \leq M$

$$c_{i,q}(u_{n-1}) = -c_{i,q}(u_n). \tag{19}$$

By equation (19), we conclude that for $n \leq M$

$$c_{i,q}(u_n) = (-1)^n c_{i,q}(u_0).$$

Considering $l = q + 1$ in the equation (16)

$$c_{i,q+1}(u_n) = (-1)^n c_{i,q}(u_0) - c_{i,q+1}(u_{n-1}) \Rightarrow$$

$$c_{i,q}(u_0) = (-1)^n c_{i,q+1}(u_n) + (-1)^n c_{i,q+1}(u_{n-1}). \tag{20}$$

Putting $n = 1, \ldots, M$ in Eq. (20) and adding the resulting equations we obtain

$$M \cdot c_{i,q}(u_0) = (-1)^M c_{i,q+1}(u_M) - c_{i,q+1}(u_0). \tag{21}$$

Therefore,

$$\begin{aligned} M &= \frac{-c_{i,q+1}(u_0) + (-1)^M c_{i,q+1}(u_M)}{c_{i,q}(u_0)} \\ &\leq \frac{|c_{i,q+1}(u_0)| + |c_{i,q+1}(u_M)|}{|c_{i,q}(u_0)|} \\ &\leq \frac{|c_{i,q+1}(u_0)| + Cv_{p,I}(u_0)}{|c_{i,q}(u_0)|}. \end{aligned}$$

So M_0 is bounded. Note that by Corollary 4.2, if $v_n \in L^1(hm)$ satisfies $\mathcal{L}^n v_n = u_0$ then $v_n = u_n + c$ in $L^1(hm)$, for some $c \in \mathbb{R}$. This proves the uniqueness statements of Theorem 1. □

5. Proof of Theorem 2

Fix $\lambda < 1$. Denote by S_λ the linear space of the real sequences $x = (x^j)_{j\in\mathbb{N}}$ such that there exists C satisfying

$$|x^j| \leq C\lambda^j.$$

Here we use x^j to denote the j-th element of the sequence x. Consider the linear space $\ell_0(\mathbb{N})$ of real sequences $x = (x^j)_{j\in\mathbb{N}}$ such that

$$\lim_j x^j = 0.$$

We define the operator $U\colon \ell_0(\mathbb{N}) \to \ell_0(\mathbb{N})$ as

$$U(x) = y,$$

where $y^0 = 0$ and $y^{j+1} = x^j$ for $j \geq 0$.

We say that $x \in \ell_0(\mathbb{N})$ is infinitely cohomologous to zero with respect to U in $\ell_0(\mathbb{N})$ if there exists an infinite sequence $x_i \in \ell_0(\mathbb{N})$, with $x = x_0$, such that

$$x_i = U(x_{i+1}) - x_{i+1}. \tag{22}$$

for every $i \geq 0$.

Lemma 5.1. [2] *Let $x \in S_\lambda$. Suppose that there exists a finite sequence $x = x_0$, $x_1, \ldots, x_k \in \ell_0(\mathbb{N})$ such that $x_i = U(x_{i+1}) - x_{i+1}$ for every $i < k$. Then $x_i \in S_\lambda$, for every $i \leq k$. If x is infinitely cohomologous to zero with respect to U in $\ell_0(\mathbb{N})$ then $x = 0 = (0, 0, \ldots)$.*

Proof. Let $x_i \in \ell_0(\mathbb{N})$, $i \leq k$, with $x_0 = x$, satisfying Eq. (22) for $i < k$. One can see that

$$x^j_{i+1} = -\sum_{p\leq j} x^p_i.$$

Since $\lim_j x^j_{i+1} = 0$, it follows that

$$\sum_p x^p_i = 0,$$

consequently since $x_0 \in S_\lambda$ we can prove by induction on i that

$$|x^j_{i+1}| = |\sum_{p>j} x^p_i| \leq C_i\lambda^j$$

for some C_i. We concluded that $x_i \in S_\lambda$ for every $i \leq k$. For each $i \leq k$ we can associate the power series

$$f_i(z) = \sum_{j=0}^{\infty} x^j_i z^j.$$

Since $x_i = (x^j_i)_j \in S_\lambda$, the power series f_i converges to a complex analytics function on the disc with center at 0 and radius $1/\lambda > 1$. Note that the sequence $U(x_i)$ is the sequence of coefficients of the Taylor series (centered at 0) of the function $zf_i(z)$. So Eq. (22) yields

$$f_i(z) = zf_{i+1}(z) - f_{i+1}(z) = (z-1)f_{i+1}(z)$$

So if x_0 is infinitely cohomologous to zero we conclude that

$$f_0(z) = (z-1)^k f_k(z)$$

for every k, where f_k is defined in a disc strictly larger than the unit disc. It follows that $f_0^{(k)}(1) = 0$ for every k, so $f_0(z) = 0$ everywhere. So $x = x_0 = 0 = (0, 0, \ldots)$. □

Proof of Theorem 2. The Corollary 4.2 gives the uniqueness of the sequence u_i. Now suppose that u_0 is infinitely cohomologous to zero. So there exists a sequence $u_i \in L^1(hm)$ such that

$$u_i = u_{i+1} \circ T - u_{i+1}. \tag{23}$$

Consider $\hat{\mathcal{B}}$ as in Corollary 3.7. Fix $\varphi \in \hat{\mathcal{B}}$. Define the sequence $x_i = (x_i^j)_j$ as

$$x_i^j = \int u_i \cdot \varphi \circ T^j \cdot h \ dm$$

Since x_i^j are Fourier coefficients of $u_i \in L^1(hm)$ with respect to the Hilbert basis $\mathcal{W}$, by Proposition 3.9 we have that $\lim_j x_i^j = 0$. By Eq. (23) and Proposition 3.8 we have

$$x_i = U(x_{i+1}) - x_i,$$

so x_0 is infinitely cohomologous to zero in $\ell_0(\mathbb{N})$. Note that

$$|x_0^j| = \Big| \int u_0 \cdot \varphi \circ T^j \cdot h \ dm \Big| \leq C\lambda^j,$$

so $x_0 \in S_\lambda$. By Lemma 5.1 we have that $x_0 = 0$. That holds for every $\varphi \in \hat{\mathcal{B}}$, so by Corollary 3.7 the function u_0 is zero.

□

6. Proof of Theorem 3

We first make a couple of remarks on condition (D5).

Remark 6.1. Suppose that T and $\mathbb{B}$ satisfy $D5$. Let $\tilde{h} \in L^1(m)$ be a function satisfying $\Phi_T(\tilde{h}) = \tilde{h}$. Then

$$\tilde{h} = \int \tilde{h} \ dm \cdot h, \tag{24}$$

where h is as in D5.iii. Indeed, by D5.vi there exists a sequence $h_n \in \mathbb{B}$ such that $h_n \to_n \tilde{h}$ in $L^1(hm)$. Furthermore since h, $1/h \in \mathbb{B}$, due D5.ii there exist $a, b > 0$ such that

$$0 < a \leq h(x) \leq b < \infty \tag{25}$$

on I. So

$$\begin{aligned}
&\Big|\int \tilde{h} \ dm \cdot h - \tilde{h}\Big|_{L^1(m)} \\
&\leq |\tilde{h} - h_n|_{L^1(m)} + \Big|\int h_n \ dm \cdot h - \Phi_T^k(h_n)\Big|_{L^1(m)} + |\Phi_T^k(h_n) - \Phi_T^k(\tilde{h})|_{L^1(m)} \\
&\leq 2|\tilde{h} - h_n|_{L^1(m)} + \Big|\int h_n \ dm \cdot h - \Phi_T^k(h_n)\Big|_{\mathbb{B}} \\
&\leq 2|\tilde{h} - h_n|_{L^1(m)} + C\lambda^k |h_n|_{\mathbb{B}}.
\end{aligned}$$

Given $\epsilon > 0$, choose n_0 such that

$$|\tilde{h} - h_{n_0}|_{L^1(m)} \leq \frac{1}{a}|\tilde{h} - h_{n_0}|_{L^1(hm)} < \frac{\epsilon}{4},$$

and k_0 such that

$$C\lambda^{k_0}|h_{n_0}|_{\mathbb{B}} < \frac{\epsilon}{2}.$$

Then

$$|\int \tilde{h}\ dm \cdot h - \tilde{h}|_{L^1(m)} < \epsilon$$

for every $\epsilon > 0$, so Eq. (24) holds. In particular if T and $\mathbb{B}$ satisfy D1, D4 and D5 we have those functions h in D4 and D5 coincide.

Remark 6.2. Note that D5.ii implies that

$$\mathbb{B} \subset L^1(hm).$$

Moreover D5.iii-v implies that

$$\frac{1}{h}\Psi^j(vh)$$

converges exponentially to zero in $L^1(hm)$ and $\mathbb{B}$.

Lemma 6.3. *Let T be a transformation satisfying D1 and D4 and suppose that $\mathbb{B}$ and T satisfy D5. Let $u \in \mathbb{B}$ and suppose that there exists $v \in L^1(hm)$ such that*

$$u = \mathcal{L}v$$

on I. Then v coincides hm-almost everywhere with a function $v_1 \in \mathbb{B}$.

Proof. The method we are going to use here is very well known for specific kinds of dynamical systems and observables. See for instance [5] for the case of C^2 piecewise smooth expanding maps and bounded variation observables. Replacing v by

$$v - \int vh\ dm\ \mathbb{1}_I,$$

we may assume without loss of generality that

$$\int vh\ dm = 0.$$

Since

$$u = v \circ T - v,$$

Applying P^j, $j \geq 1$, we get

$$P^j u = P^{j-1}v - P^j v, \tag{26}$$

Putting $j = 1, \ldots, n$ in Eq. (26) and adding the resulting equations we obtain

$$v = P^n v + \sum_{j=1}^{n} P^j u$$

We claim that $|P^j v|_{L^1(hm)} \to_j 0$. Indeed, due D5.vi for every $\epsilon > 0$ there exists $w \in \mathbb{B}$ such that $\int w\ h\ dm = 0$ and $|v - w|_{L^1(hm)} < \epsilon$. Since $||P||_{L^1(hm)} \leq 1$, for every j

$$|P^j v - P^j w|_{L^1(hm)} < \epsilon.$$

Due D5 for every $w \in \mathbb{B}$

$$P^j(w) = \frac{1}{h}\Psi^j(wh),$$

and

$$|\Psi^j(wh)|_{L^1(hm)} \leq C|\Psi^j(wh)|_{\mathbb{B}} \leq C\lambda^j|wh|_{\mathbb{B}},$$

we have that for j large enough

$$|P^j v|_{L^1(hm)} \leq |P^j v - P^j w|_{L^1(hm)} + |P^j w|_{L^1(hm)} < 2\epsilon.$$

This proves our claim. In particular

$$v = \sum_{j=1}^{\infty} P^j u,$$

where the convergence of the series is in $L^1(hm)$. On the other hand, by Remark 6.2 this series converges in $L^1(hm)$ and $\mathbb{B}$ to a function $v_1 \in \mathbb{B}$. So $v = v_1$ hm-almost everywhere. □

Proof of Theorem 3. Since $u_0 \in \mathbb{B}$, by D5, for every $v \in L^\infty(hm)$ we have

$$\Big|\int u_0 \cdot v \circ T^j \cdot h\ dm\Big| = \Big|\int P^j(u_0) \cdot v \cdot h\ dm\Big| \leq C\lambda^j |u_0|_{\mathbb{B}} |v|_{L^\infty(hm)}.$$

By Theorem 2 we have that u_0 is not infinitely cohomologous to zero in $L^1(hm)$. Now suppose $\mathcal{L}u_i = u_0$. The uniqueness (up to a constant) of u_i follows from Corollary 4.2. By Lemma 6.3 we have $u_i \in \mathbb{B}$.

□

7. Topological results

Proof of Theorem 4. Define $\tilde{u}_0 = u_0 \circ H$. Then $\tilde{u}_0$ has bounded p-variation. By Theorem 1 there exist bounded p-variation functions $\tilde{u}_i$, $i \leq M$, unique up to a constant, such that

$$\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0 \ on\ L^1(hm),$$

and

$$\tilde{\mathcal{L}}\alpha \neq \tilde{u}_M + c \ on\ L^1(hm), \tag{27}$$

for every bounded p-variation function α. Here $\tilde{\mathcal{L}}v = v \circ \tilde{T} - v$. Since the support of hm is I, it follows that $\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0$ in I(e.c.s). Define $u_i = \tilde{u}_i \circ H^{-1}$. Then u_i has bounded p-variation and

$$\mathcal{L}^i u_i = u_0 \ on\ H(I)(e.c.s).$$

Suppose that there exists a function ρ with bounded p-variation such that $\mathcal{L}\rho = u_M + c$ (e.c.s). Define $\tilde{\rho} = \rho \circ H$. Then $\tilde{\rho}$ has bounded p-variation and $\tilde{\mathcal{L}}\tilde{\rho} = \tilde{u}_M + c$ on $L^1(hm)$. That contradicts Eq. (27). So $\mathcal{L}\rho \neq u_M + c$ in an uncountable subset of $H(I)$. Since the discontinuities of $\mathcal{L}\rho$ and $u_M + c$ are countable, it follows that there is a continuity point $x_0 \in H(I)$ of both functions such that $(\mathcal{L}\rho)(x_0) \neq u_M(x_0) + c$. So there is a non-empty open subset of $H(I)$ such that $\mathcal{L}\rho \neq u_M + c$. □

Proof of Theorem 5. The proof of this theorem is quite similar to the proof of Theorem 4. Define $\tilde{u}_0 = u_0 \circ H$. Then $\tilde{u}_0$ has bounded p_0-variation. By Theorem 3 there exist bounded p_0-variation functions $\tilde{u}_i$, $i \leq M$, unique up to a constant, such that

$$\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0 \ on\ L^1(hm),$$

and

$$\tilde{\mathcal{L}}\alpha \neq \tilde{u}_M + c \ on\ L^1(hm), \tag{28}$$

for every $\alpha \in L^1(hm)$. Here $\tilde{\mathcal{L}}v = v \circ \tilde{T} - v$. Since the support of hm is I, it follows that $\tilde{\mathcal{L}}^i \tilde{u}_i = \tilde{u}_0$ in I(e.c.s). Define $u_i = \tilde{u}_i \circ H^{-1}$. Then u_i has bounded p_0-variation and

$$\mathcal{L}^i u_i = u_0 \ on \ H(I)(e.c.s).$$

Now we show the uniqueness of u_i in the set of continuous (e.c.s.), bounded borelian functions. If continuous (e.c.s.) bounded borelian functions v_i satisfy $\mathcal{L}^i v_i = u_0$ then $\tilde{v}_i = v_i \circ H$ are also continuous (e.c.s.) and moreover they belong to $L^1(hm)$ and satisfies $\tilde{\mathcal{L}}^i \tilde{v}_i = \tilde{u}_0$, so by Theorem 3 we have that $\tilde{v}_i = \tilde{u}_i + c_i$ for some $c_i \in \mathbb{R}$, where this equality holds in $L^1(hm)$. Since both functions $\tilde{v}_i$, $\tilde{u}_i$ are continuous (e.c.s) it follows that $\tilde{v}_i = \tilde{u}_i + c_i(e.c.s.)$, so $v_i = u_i + c_i(e.c.s.)$.
To show conclusions A. and B., suppose that there exists a bounded borelian function ρ such that $\mathcal{L}\rho = u_M + c$ (e.c.s). Define $\tilde{\rho} = \rho \circ H$. Then $\tilde{\rho}$ is also a bounded borelian function, so it belongs to $L^1(hm)$ and $\tilde{\mathcal{L}}\tilde{\rho} = \tilde{u}_M + c$ (e.c.s), so since hm has no atoms it follows that this equality holds on $L^1(hm)$. That contradicts Eq. (28). So $\mathcal{L}\rho \neq u_M + c$ in an uncountable subset of $H(I)$. If ρ is continuous (e.c.s.) we can now finish the proof exactly as in the proof of Theorem 4. □

Remark 7.1. One can ask why the conclusions of Theorem 5 are weaker than those in Theorem 3. The problem is that the conjugacy between one-dimensional maps can be singular with respect to the Lebesgue measure. Indeed that is often the case even when the two one-dimensional maps T and $\tilde{T}$ are very regular, as expanding maps on the circle (see [8]). In particular the conjugacy H does not in general preserve either $L^1(hm)$, $L^1(m)$ or the space of Lebesgue measurable functions (see [6]). So note that if in the proof of Theorem 5 we pick ρ to be either in $L^1(m)$ or $L^1(hm)$ then it is not true in general that $\rho \circ H$ belongs to $L^1(hm)$. Moreover since composition with H does not in general preserve Lebesgue measurable functions, we need to assume that ρ is a Borel measurable function, so $\rho \circ H$ is also Borel measurable. Those are the reasons why we assume that ρ is bounded and borelian in Theorem 5.

8. Observables infinitely cohomologous to zero

Consider the Banach space of summable sequences $\ell^1(\mathbb{N})$. For a sequence $x = (x^j)_{j\in\mathbb{N}}$ denote

$$|x|_{\ell^1(\mathbb{N})} = \sum_j |x^j|.$$

We define the operator $U \colon \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ as the norm preserving map

$$U(x) = y,$$

where $y^0 = 0$ and $y^{j+1} = x^j$ for $j \geq 0$.

We say that $x \in \ell^1(\mathbb{N})$ is infinitely cohomologous to zero with respect to U if there exists an infinite sequence $x_i \in \ell^1(\mathbb{N})$, with $x = x_0$, such that

$$x_i = U(x_{i+1}) - x_{i+1}.$$

for every $i \geq 0$.

Lemma 8.1. [2] *There is a non vanishing sequence $x \in \ell^1(\mathbb{N})$ which is infinitely cohomologous to zero with respect to U.*

Proof. We claim that for every $k \in \mathbb{N}$ there exist

$$x_{0,k},\ x_{1,k},\ \dots,\ x_{k,k} \in \ell^1(\mathbb{N}),$$

all of them with compact support, such that $x^0_{0,k} = 1$,

$$x_{i,k} = U(x_{i+1,k}) - x_{i+1,k}$$

$$|x_{i,k+1} - x_{i,k}|_{\ell^1(\mathbb{N})} < 2^{-k-1}, \tag{29}$$

for every $i < k$.

The proof is by induction on k. Choose $x_{0,0} = (1, 0, 0, 0, \dots)$. Suppose by induction we found a finite sequence $x_{i,k}$, $i \leq k$, with the properties above. Fix $N > 0$. Define $x_{k,k+1}$ as $x^0_{k,k+1} = x^0_{k,k}$, $x^j_{k,k+1} = x^j_{k,k} - \delta/N$, for $1 \leq j \leq N$, and $x^j_{k,k+1} = x^j_{k,k}$ for $j \geq N + 1$. Here $\delta = \sum_j x^j_{k,k}$. Defining

$$x^j_{k+1,k+1} = -\sum_{p \leq j} x^p_{k,k+1},$$

we have that $x_{k+1,k+1}$ has compact support and $x_{k,k+1} = U(x_{k+1,k+1}) - x_{k+1,k+1}$. Now define by induction

$$x_{i,k+1} = U(x_{i+1,k+1}) - x_{i+1,k+1},\ i < k.$$

In particular $x^0_{i,k+1} = -x^0_{i+1,k+1}$ for $i \leq k$. Since $x^0_{i,k} = -x^0_{i+1,k}$ for $i < k$ and $x^0_{k,k+1} = x^0_{k,k}$ we have $x_{0,k+1} = 1$. Furthermore it is not difficult to see that if N is large enough then

$$|x_{i,k+1} - x_{i,k}|_{\ell^1(\mathbb{N})} < 2^{-k-1},$$

for every $i < k$. This completes the inductive step.

By Eq. (29), for every i there exists $x_i \in \ell^1(\mathbb{N})$ such that $\lim_k x_{i,k} = x_i$ on $\ell^1(\mathbb{N})$. It is easy to check that $x_i = U(x_{i+1}) - x_{i+1}$ and $x^0_0 = 1$. Pick $x = x_0$. □

Proof of Theorem 6. Since T is topologically conjugate with $T_\ell = \ell x \mod 1$, $\ell \in \mathbb{Z} \setminus \{-1, 0, 1\}$, it is enough to show the Theorem 6 for T_ℓ. Choose n such that ℓ does not divide n. Let $x = (x_j)_j \in \ell^1(\mathbb{N})$ as in Lemma 8.1. Define

$$u(x) = \sum_{j=0}^{\infty} x_j \sin(2\pi n \ell^j\ x).$$

The function u is continuous and non constant. Using Remark 3.2 and Proposition 3.8 one can easily show that u is infinitely cohomologous to zero. □

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